BCS versus Overhauser pairing in dense (2+1)-dimensional QCD

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We compare the BCS and Overhauser effect as competing mechanisms for the destabilization of the quark Fermi surface at an asymptotically large chemical potential, for the special case of 2 space and 1 time dimensions. We use the framework of perturbative one-gluon exchange, which dominates the pairing at $\mu/g^2 \gg 1$. With screening in matter, we show that in the weak coupling limit the Overhauser effect can compete with the BCS effect only for a sufficiently large number of colors. Both the BCS and the Overhauser gaps are of order g^4/μ in the Landau gauge.

I. INTRODUCTION

Quantum chromodynamics (QCD) at high density was first studied in the late 1970s [1,2] and has received renewed attention, especially in the past three years [3–6]. At large baryon density and small temperature, it is relevant to studies of neutron stars and heavy ion collisions in the baryon rich regime. Theoretical studies of dense quark matter have revealed the preferred phase to be a color superconductor with several novel features such as color-flavor locking, chiral symmetry breaking, color Meissner effect and the formation of a mass gap.

At asymptotic densities, the Fermi surface for quarks is well defined and the low energy excitations are weakly interacting quasiparticles and holes. In QCD, perturbative 1-gluon exchange can provide an arbitrarily weak attractive interaction in the color anti-symmetric $\bar{\mathbf{3}}$ channel that destabilizes the Fermi surface and causes particle-particle pairing. Another mechanism of destabilization is particle-hole pairing (Overhauser effect) [7], studied earlier in a variational method and for large N_c [8,9]. More recently, it has been shown that the equations driving the particle-hole instability resemble those of the BCS instability, modulo phase-space factors [10]. A novel mechanism for destabilizing the Fermi surface through a BCS crystal was also recently suggested in [11] and may also compete with the instabilities to be studied below. Its consideration is outside the scope of this work.

Screening affects the strengths of the BCS and Overhauser pairing. The authors of [10] have shown that in the regime of weak coupling in 3+1 dimensions, the BCS effect is dominant up to a large number of colors. In 1+1 dimensions, the drastic reduction in phase space for BCS pairing with no comparable reduction for Overhauser pairing means that the latter dominates whatever N_c. In this work, we wish to establish the case for 2+1 dimensions, taking into account the screening in this space-time dimension.

This paper is organized as follows. In Sec. II, we derive the quark 1-loop contribution to the gluon propagator in a covariant gauge, at finite temperature and density (T$\ll \mu$ for cold and dense matter). Section III motivates the symmetry of the particle-particle and particle-hole condensates in 2+1 dimensions, followed by formally differing derivations of the gap equations for these two objects. The resulting gap equa-

tions are solved in the presence of electric screening and Landau damping in dense quark matter. We present our results for the gaps in the Landau gauge. The conclusions of our analysis are in Sec. IV.

II. SCREENING IN 2+1 DIMENSIONS

We begin by evaluating the gluon polarization tensor in matter in the imaginary time formalism [12]

$$\Pi^{\mu\nu}(q_0, \vec{\mathbf{q}}) = g^2 T \sum_n \int \frac{d^2p}{(2\pi)^2} \text{Tr}\left(\gamma^\mu \frac{1}{\not{p}} \gamma^\nu \frac{1}{\not{p}+\not{q}} \right), \quad (1)$$

where q, p and $p+q$ denote respectively the 4-momentum of the gluon and the two internal quark lines. At finite temperature and density, one makes the usual identification $ip_3 = p_0 = (2n+1)i\pi T + \mu$ for fermions and $iq_3 = q_0 = 2ni\pi T$ for bosons (here, n labels the Matsubara frequency). In 2+1 dimensions, the Pauli matrices $\gamma_\mu \equiv \sigma_\mu \equiv (\sigma_3, i\vec{\sigma})$ generate a representation of the Clifford algebra $\{\gamma_\mu \gamma_\nu\} = 2g_{\mu\nu}$ with $g_{\mu\nu} = \text{diag}(1,-1,-1)$. In an arbitrary covariant gauge with gauge parameter λ, the inverse propagator in matter $(D^{-1})^{\mu\nu}$ and $\Pi^{\mu\nu}$ are related as [13]

$$(D^{-1})^{\mu\nu} = q^2 g^{\mu\nu} + \Pi^{\mu\nu} - q^{\mu\nu}\left(1 - \frac{1}{\lambda} \right). \quad (2)$$

Evaluating the Matsubara sum using contour integration, omitting the vacuum piece (T and μ independent), and analytically continuing $iq_3 = \omega + i\epsilon$, we get in the limit $T \rightarrow 0$ (with $\mathbf{q} = |\vec{\mathbf{q}}|$)

$$\Pi^{00} = g^2\left[\left(-\frac{\mu}{2\pi} + \frac{\mathbf{q}}{16} \right) + i\frac{\omega}{\mathbf{q}}\left(\frac{-\mu}{4\pi} \right) \right]\theta(\mathbf{q}-\omega)$$

$$+ g^2\left[\frac{\mathbf{q}^2}{\omega^2}\left(\frac{\mu}{4\pi} \right) + i\frac{\mathbf{q}^2}{\omega^2}\left(\frac{-\omega}{16} \right) \right]\theta(\omega-\mathbf{q}) \quad (3)$$

$$\Pi_\perp = g^2\left[\left(\frac{\mathbf{q}}{16} - \frac{\omega^2}{\mathbf{q}^2}\left(\frac{\mu}{2\pi} \right) \right) + i\frac{\omega}{\mathbf{q}}\left(\frac{3\mu}{8\pi} \right) \right]\theta(\mathbf{q}-\omega)$$

$$+ g^2\left[\left(-\frac{\mu}{4\pi} \right) - i\left(\frac{\omega}{8} \right) \right]\theta(\omega-\mathbf{q}) \quad (4)$$

$$\Pi_{\mathbf{L}} = g^2 \left[\frac{\omega^2}{\mathbf{q}^2} \left(\frac{\mu}{2\pi} \right) + i \frac{\omega^3}{\mathbf{q}^3} \left(\frac{\mu}{4\pi} \right) \right] \theta(\mathbf{q} - \omega)$$

$$+ g^2 \left[\left(-\frac{\mu}{4\pi} \right) + i \left(\frac{\omega}{16} \right) \right] \theta(\omega - \mathbf{q}), \qquad (5)$$

where $\Pi^{ij} = \Pi_\perp P_\perp^{ij} + \Pi_{\mathbf{L}} P_{\mathbf{L}}^{ij}$, with $P_\perp^{ij} = (\delta^{ij} - q^i q^j / \mathbf{q}^2)$ and $P_{\mathbf{L}}^{ij} = (q^i q^j / \mathbf{q}^2)$ $(i,j = 1$ to $3)$.

In deriving Eqs. (3)–(5), the approximations[1] $\omega, \mathbf{q} \ll \mu$ have been made since energy transfers in the scattering of qq (BCS) and $\bar{q}q$ (Overhauser) pairs are of the order of the gap, and momentum transfers can be taken to be small for forward scattering of quarks. In Euclidean space, the screened gluon propagator in Feynman gauge reads

$$D_{E,M}(x-y) = \int \frac{d^3 q}{(2\pi)^3} \frac{e^{-iq\cdot(x-y)}}{q^2 + m_{E,M}^2}, \qquad (6)$$

where $m_E^2 = m_D^2$ refers to the Debye mass and m_M^2 to the magnetic scale generated by Landau damping. Their values (generalized to N_f flavors) follow from Eqs. (3) and (4) respectively and are given by $m_E^2/g^2\mu = m_D^2/g^2\mu = N_f/2\pi$ and $m_M^2/m_D^2 = \frac{3}{4}|q_3|/\mathbf{q}$. Note that the coupling g^2 has mass dimensions. The results are similar to that in $3+1$ dimensions for which $m_E^2/g^2\mu^2 = m_D^2/g^2\mu^2 = N_f/2\pi^2$ and $m_M^2/m_D^2 = (\pi/4)|q_4|/\mathbf{q}$.

III. CONDENSATES AND GAP EQUATIONS IN 2+1 DIMENSIONS

A. Condensates

Usually, diquarks are taken to condense in the parity even 0^+ channel. In $2+1$ dimensions, parity is defined by inversion about 1 spatial coordinate only, as it is an improper transformation. For the sake of definiteness, we define the parity transformation in space as $x_1 \to x_1$, $x_2 \to -x_2$. The parity operator in spin space is then σ_1 which implies that a term like $\bar{q}q$ is not invariant under this transformation. Thus, for the particle-hole pseudoscalar condensate with 2 flavors, we have the action of the parity operator (P) as

$$\bar{q}\mathbf{1}^c \mathbf{1}^f q = \bar{u}u + \bar{d}d \xrightarrow{P} -\bar{u}u - \bar{d}d, \qquad (7)$$

where c and f denote color and flavor respectively. A scalar condensate may still be written down if we allow parity to change isospin as $u(d) \xrightarrow{P} +/- \sigma_1 d(u)$. Then we have

$$\bar{q}\mathbf{1}^c \sigma_3^f q = \bar{u}u - \bar{d}d \xrightarrow{P} -\bar{d}d + \bar{u}u. \qquad (8)$$

The $+/-$ indicates that there are 4 possible sign combinations we can choose in the combined transformation of the u and d quarks. Note that this new definition of parity does not affect the way the pseudoscalar transforms. Now, we will see that the relative sign is also determined for us by the particle-particle condensates.

The charge conjugated spinor transforms under parity as $q^c \xrightarrow{P} \det(\Lambda_{\mathrm{P}}) \sigma_1 q^c$, where Λ_{P} denotes the matrix corresponding to the improper Lorentz transformation P. There is an extra minus sign as compared to the parity transformation of the quark spinor. The charge conjugation matrix C is σ_2, which is antisymmetric. The attractive color channel $\bar{\mathbf{3}}$ is anti-symmetric and the overall wave function for the BCS pairing must also be anti-symmetric, therefore, one is forced to choose σ_2^f for the flavor matrix. Then, the scalar BCS condensate is

$$\bar{q}^c \lambda_2^c \sigma_2^f q = \bar{u}^c d - \bar{d}^c u. \qquad (9)$$

For this to be even-parity, both u and d should have the same sign in their transformation. On the other hand, a pseudoscalar BCS condensate will necessarily have a relative sign between the transformations for u and d. In this paper, it is the scalar which is studied.

These problems arise in odd space-time dimensions due to the lack of a consistent definition for a gamma matrix that anti-commutes with all the γ_μ's, namely γ_5. We proceed now to the gap equations for the scalar particle-particle and particle-hole condensates.

B. BCS gap equations

In order to derive the BCS gap equation, we will use the standard Nambu-Gorkov formalism, which introduces charge-conjugated spinors to enable the writing of the generating functional of QCD at finite chemical potential in a form identical to that at zero chemical potential [14]. With charge conjugation defined as $\psi^c = C\bar{\psi}^T$, the Nambu-Gorkov spinor is $\Psi = (\psi, \bar{\psi}^T)$ and can be expressed as a $4 \cdot N_c \cdot N_f$ column vector. Neglecting quark mass effects, the form of the gap matrix is

$$\Delta_{ij}^{ab}(q) = (\lambda_2)^{ab}(\sigma_2)_{ij} C \left\{ \Delta_1(q) \frac{1}{2}(1 + \vec{\sigma} \times \hat{q}) \right.$$

$$\left. + \Delta_2(q) \frac{1}{2}(1 - \vec{\sigma} \times \hat{q}) \right\}, \qquad (10)$$

where $\Lambda_\pm = \frac{1}{2}(1 \pm \vec{\sigma} \times \hat{q})$ are the particle and anti-particle projectors respectively in $2+1$ dimensions. (In the weak coupling limit $\alpha/\mu \ll 1$ with $\alpha = g^2/4\pi$, the equations of motion allow us to replace $\vec{\sigma} \times \hat{q}$ by the unit matrix so that only the gap Δ_1 remains and the anti-gap Δ_2 drops out. Moreover, the anti-gap is strongly gauge dependent.) The self-energy in the Nambu-Gorkov formalism obeys the Schwinger-Dyson (SD) equation

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[1]The gaps in $2+1$ dimensions are power law suppressed, and care is needed in introducing this approximation in our perturbative result.

$$\Sigma(k) = -ig^2 \int \frac{d^3q}{(2\pi)^3} \Gamma^a_\mu S(q) D^{ab}_{\mu\nu}(q-k) \Gamma^b_\nu. \quad (11)$$

Here, $\Sigma(k) = S^{-1}(k) - S_0^{-1}(k)$ is the proper self-energy with $S(k)$ as defined in [15]. Γ^a_μ is the quark-gluon vertex which we take to be the leading order result in perturbation theory,

$$\Gamma^a_\mu = \begin{pmatrix} \gamma_\mu \lambda^a/2 & 0 \\ 0 & -(\gamma_\mu \lambda^a/2)^T \end{pmatrix}. \quad (12)$$

We also neglect the wave-function renormalization to lead-

ing order in the weak coupling so that the diagonal part of the self-energy has only the free quark propagators. Therefore,

$$S^{-1}(q) = \begin{pmatrix} \slashed{q} + \slashed{\mu} & \bar{\Delta} \\ \Delta & (\slashed{q} - \slashed{\mu})^T \end{pmatrix}, \quad (13)$$

where $\bar{\Delta} = \gamma_0 \Delta^\dagger \gamma_0$. Determining S_{21} from Eq. (13), inserting the ansatz Eq. (10) for the gap, and projecting out the positive energy part, we obtain the integral equation

$$\Delta_1(k) = -\frac{N_c+1}{2N_c} ig^2 \int \frac{d^3q}{(2\pi)^3} \left[\frac{\Delta_1(q)}{q_0^2 - (|\vec{q}| - \mu)^2 - \Delta_1(q)^2} \left\{ \left(\frac{1}{2} + \frac{1}{2} \hat{k} \cdot \hat{q} \right) D_\perp(k-q) + \left(\frac{1}{2} + \frac{1}{2} \hat{k} \cdot \hat{q} \right) D_L(k-q) \right. \right.$$

$$+ (K^{0^2}(1 + \hat{k} \cdot \hat{q})) \frac{D_\lambda}{2} \right\} + \frac{\Delta_2(q)}{q_0^2 - (|\vec{q}| + \mu)^2 - \Delta_2(q)^2} \left\{ (1 - \hat{k} \cdot \hat{q}) D_\perp(k-q) + \left(\frac{1}{2} - \frac{1}{2} \hat{k} \cdot \hat{q} \right) D_L(k-q) \right.$$

$$\left. \left. + (K^{0^2}(1 - \hat{k} \cdot \hat{q}) - |\vec{K}|^2(1 + \hat{k} \cdot \hat{q})) \frac{D_\lambda}{2} \right\} \right], \quad (14)$$

where $K = q - k$, $D_\perp = 1/(K^2 - m_M^2)$, $D_L = 1/(K^2 - m_D^2)$ and $D_\lambda = \lambda/K^4$ from the gluon propagator in the covariant gauge. We note that only the term for the gap Δ_1 [henceforth referred to as $G(q)$] is relevant at the Fermi surface and that it's gauge dependence multiplies the factor K^{0^2}/K^2 which is essentially the amount by which the quarks are off mass shell. This is not sufficient to tame the infrared divergence from the $1/K^2$ of the gauge-fixing part of the propagator in $2+1$ dimensions, in contrast to $3+1$ dimensions (see below). An estimate of the relative contributions of the electric, magnetic and gauge-fixing terms can now be made.

1. Gauge fixing contribution (D_λ)

It is the momenta on the Fermi surface that make the most contribution to the integral, therefore $|\vec{q} - \vec{k}|^2 \cong 2\mu^2(1 - \cos\theta)$. Setting $|\vec{q}| - \mu \cong q_{||}$, the integration measure becomes $dq_3 \mu dq_{||} d\theta$, and the continuation to Euclidean space gives

$$G(k_3) = \frac{\lambda\kappa}{2} \int \frac{dq_3 \mu dq_{||} G(q_3)(q_3 - k_3)^2}{q_3^2 + q_{||}^2 + G(q_3)^2} \int \frac{d\theta(1 + \cos\theta)}{((q_3 - k_3)^2 + 2\mu^2(1 - \cos\theta))^2}, \quad (15)$$

where $\kappa = ((N_c+1)/2N_c)(g^2/(2\pi)^3)$. Performing the angular integration exactly, we find

$$G(k_3) = \lambda\kappa\pi \int \frac{dq_3 \mu dq_{||} G(q_3)}{q_3^2 + q_{||}^2 + G(q_3)^2} \left[\frac{1}{|q_3 - k_3|\sqrt{(q_3 - k_3)^2 + 4\mu^2}} \right]. \quad (16)$$

Integrating over $q_{||}$, the pole from the quasi-particle propagator gives

$$G(k_3) = \lambda\kappa\mu\pi^2 \int dq_3 \frac{G(q_3)}{\sqrt{q_3^2 + G(q_3)^2}} \left[\frac{1}{|q_3 - k_3|\sqrt{(q_3 - k_3)^2 + 4\mu^2}} \right]. \quad (17)$$

The region of interest is $q_3, k_3 \ll \mu$, since the quarks are almost on mass shell. In that case, we have

$$G(k_3) \cong \frac{\lambda\kappa\pi^2}{2} \int dq_3 \frac{G(q_3)}{\sqrt{q_3^2 + G(q_3)^2}} \frac{1}{|q_3 - k_3|}. \quad (18)$$

2. The electric contribution (D_L)

The angular integration yields

$$G(k_3) = \frac{\kappa\pi}{2\mu} \int \frac{dq_3 dq_{\parallel} G(q_3)}{q_3^2 + q_{\parallel}^2 + G(q_3)^2} \left(\frac{\sqrt{((q_3-k_3)^2+m_D^2)^2 + 4\mu^2((q_3-k_3)^2+m_D^2)}}{((q_3-k_3)^2+m_D^2)} - 1 \right). \tag{19}$$

As before, the integration over $q_{\parallel}$ picks up the pole in the diquark propagator, and the form factor is the physical regulator for the energy integral. Note that the integrand shuts off for $q_3 \gg \mu$, but this is an unphysical regulation since we are far from the Fermi surface physics. For the physical region of $q_3 \ll \mu$, we obtain

$$G(k_3) \cong \kappa\pi^2 \int \frac{dq_3 G(q_3)}{\sqrt{q_3^2 + G(q_3)^2}} \frac{1}{\sqrt{(q_3-k_3)^2 + m_D^2}}. \tag{20}$$

The value of the Debye mass suggests that it is much larger than the gap, so that we may approximate $\sqrt{(q_3-k_3)^2+m_D^2} \cong m_D$, with q_3 and k_3 on mass-shell being of the order of the gap. (We have checked the consistency of this approximation with the explicit solution for the gap.) The electric term is expected to be of order κ/m_D.

3. The magnetic contribution ($D_\perp$)

The magnetic scale is generated by Landau damping through

$$m_M^2 = \frac{3}{4} m_D^2 \frac{q_0 - k_0}{|\vec{q} - \vec{k}|}, \tag{21}$$

and the gap equation reads

$$G(k_3) = \frac{\kappa}{2} \int \frac{dq_3 \mu dq_{\parallel} G(q_3)}{q_3^2 + q_{\parallel}^2 + G(q_3)^2} \int \frac{d\theta(1+\cos\theta)}{(q_3-k_3)^2 + |\vec{q}-\vec{k}|^2 + \frac{3}{4} m_D^2 \frac{q_3-k_3}{|\vec{q}-\vec{k}|}}. \tag{22}$$

As $\vec{q}, \vec{k}$ are on the Fermi surface, we may write

$$G(k_3) = \frac{\kappa}{2} \int \frac{dq_3 dq_{\parallel} G(q_3)}{q_3^2 + q_{\parallel}^2 + G(q_3)^2} \int \frac{\mu d\theta(1+\cos\theta)}{(q_3-k_3)^2 + 2\mu^2(1-\cos\theta) + \frac{3}{4} m_D^2 \frac{q_3-k_3}{\sqrt{2\mu^2(1-\cos\theta)}}}, \tag{23}$$

The physics of Landau damping guides us in evaluating the angular piece. For forward scattering, θ is small, in which case the angular piece reads as follows:

$$\int \frac{2\mu d\theta}{(q_3-k_3)^2 + \mu^2\theta^2 + 3m_D^2(q_3-k_3)/4\mu\theta}, \tag{24}$$

where we have set $1-\cos\theta \cong \theta^2/2$ in the denominator and $1+\cos\theta \cong 2$ in the numerator. The denominator contains 3 pieces, which are, respectively, the gluon energy squared, the gluon momentum squared and the damping term. Since the scattering of BCS pairs costs little energy (typically of the order of the gap), the first piece may be dropped. This is equivalent to saying that the quarks are nearly on mass shell. The validity of this approximation rests on the smallness of the gap which will be checked explicitly.

The remaining 2 terms give us a bound on the angular region in which magnetic binding of the BCS pairs is disturbed by Landau damping. Setting the terms comparable, we find

$$\theta_{min} = \frac{\left(\frac{3}{4} m_D^2 |q_3-k_3|\right)^{1/3}}{\mu}. \tag{25}$$

If $\theta < \theta_{min}$, the Landau damping is active, and is ineffective once $\theta > \theta_{min}$. Energy transfer of the order of the gap implies that θ_{min} is small if

$$m_D^{2/3} G^{1/3} < \mu. \tag{26}$$

This also implies that ignoring $(q_3-k_3)^2$ in the denominator is valid when

$$\frac{G}{m_D} < \frac{m_D}{\mu}. \tag{27}$$

It is magnetic gluon interaction between the quasiparticles that builds up the gap, as in $3+1$ dimensions [16], so we work in the regime $\theta > \theta_{min}$. If we keep only the momentum exchange term in the denominator, we obtain for the angular integration

$$\frac{2}{\mu} \int_{\theta_{min}}^{\theta_{max}} \frac{d\theta}{\theta^2} = \frac{4}{6^{1/3} m_D^{2/3} |q_3 - k_3|^{1/3}}, \tag{28}$$

where $\mu \theta_{max} \cong (|\vec{q} - \vec{k}|)_{max}$ and we have picked the contribution from the lower limit. The contribution from the upper limit is subleading. The $dq_{\parallel}$ integration yields

$$G(k_3) \cong \frac{2\pi\kappa}{6^{1/3} m_D^{2/3}} \int \frac{dq_3 G(q_3)}{\sqrt{q_3^2 + G(q_3)^2}} \frac{1}{|q_3 - k_3|^{1/3}}. \tag{29}$$

Collecting Eqs. (18),(20),(29), the contributions of the electric, magnetic and gauge-fixing terms are respectively

$$G(k_3) \cong \kappa\pi \left(\int \frac{dq_3 G(q_3)}{\sqrt{q_3^2 + G(q_3)^2}} \left\{ \frac{\pi}{\sqrt{(q_3 - k_3)^2 + m_D^2}} \right. \right.$$
$$\left. \left. + \frac{2}{6^{1/3} m_D^{2/3} |q_3 - k_3|^{1/3}} + \frac{\lambda\pi}{2|q_3 - k_3|} \right\} \right). \tag{30}$$

The gauge-fixing contribution in covariant gauges is large suggesting that the ladder plus screening expansion of the SD equation in α/μ may receive additional contributions due to the enhanced infrared sensitivity in $2+1$ dimensions. A reorganization of the expansion may be needed, with perhaps an emphasis on a physical observable from the outset. Below, we will carry estimates of the gap in Landau gauge $\lambda = 0$. The electric contribution gives an exponentially small gap if $G < m_D$. The electric screening implies $G \sim \Lambda e^{-m_D/\kappa\pi^2}$, which is exponentially suppressed due to the large value of m_D/g^2. (Λ is the UV cutoff.)

The magnetic term drives the formation of the gap since $m_D^{2/3} G^{1/3} < m_D$. The gap equation becomes

$$G(k_3) = \kappa_* \int \frac{dq_3 G(q_3)}{\sqrt{q_3^2 + G(q_3)^2}} \frac{1}{|q_3 - k_3|^{1/3}}, \tag{31}$$

where $\kappa_* = 2\kappa\pi/(6^{1/3} m_D^{2/3})$. The integral may be split into two regions, $0 < q_3 < k_3$ and $k_3 < q_3 < \Lambda$, where in the first, $1/|q_3 - k_3|^{1/3} \approx 1/k_3^{1/3}$, and in the second $1/|q_3 - k_3|^{1/3} \approx 1/q_3^{1/3}$. Then, Eq. (31) may be recast as

$$G(k_3) = \kappa_* \left[\frac{1}{k_3^{1/3}} \int_{G_0}^{k_3} dq_3 \frac{G(q_3)}{q_3} + \int_{k_3}^{\Lambda} dq_3 \frac{G(q_3)}{q_3^{4/3}} \right], \tag{32}$$

with the scale $G(k_3 = G_0) = G_0$. Differentiating Eq. (32) twice with respect to k_3, we obtain

$$3k_3^2 G''(k_3) + 4k_3 G'(k_3) + \kappa_* \frac{G(k_3)}{k_3^{1/3}} = 0, \tag{33}$$

which has the solution

$$G(k_3) = \frac{1}{k_3^{1/6}} \left[c_1 J_{-1} \left(-6\sqrt{\frac{\kappa_*}{3}} k_3^{-1/6} \right) \right.$$
$$\left. + c_2 Y_{-1} \left(-6\sqrt{\frac{\kappa_*}{3}} k_3^{-1/6} \right) \right]. \tag{34}$$

Since the Neumann function is complex valued for positive k_3, $c_2 = 0$ (we are solving for a real gap). Using $G(k_3 = G_0) = G_0$ and the boundary condition $G'(k_3 = G_0) = 0$, we obtain

$$G(k_3) = \frac{G_0^{7/6}}{J_1 \left(6\sqrt{\frac{\kappa_*}{3}} G_0^{-1/6} \right)} \frac{J_1 \left(6\sqrt{\frac{\kappa_*}{3}} k_3^{-1/6} \right)}{k_3^{1/6}}, \tag{35}$$

with $G_0 = (\kappa_*/3)^3 (6/x_0)^6$, where $x_0 = 2.405$ is the minimum *finite* solution to $J_1(x) = -x J_1'(x)$. G_0 is therefore the maximum BCS gap. Note that the condition $G_0 < m_D$ is satisfied since

$$\frac{G_0}{m_D} \sim \left(\frac{\alpha}{\mu} \right)^{3/2}. \tag{36}$$

The conditions imposed by Eqs. (26) and (27) are also met because

$$\left(\frac{G_0}{m_D} \right) \left(\frac{m_D}{\mu} \right)^{-1} \sim \frac{\alpha}{\mu}, \quad \frac{m_D^{2/3} G_0^{1/3}}{\mu} \sim \frac{\alpha}{\mu}. \tag{37}$$

Therefore, the approximations made in evaluating the magnetic and electric pieces are justified. It is easy to check that the electric piece is down compared to the magnetic piece by

$$\frac{m_D^{2/3} G_0^{1/3}}{m_D} \sim \left(\frac{\alpha}{\mu} \right)^{1/2}. \tag{38}$$

As we work in the Landau gauge, we can repeat the preceding analysis using a simplified propagator

$$D_{\mu\nu}(q) = \frac{-g_{\mu\nu}}{2} [D_{\perp}(q) + D_L(q)], \tag{39}$$

and arrive at the same equation for the gap, which in Euclidean space is

$$G(k) = \frac{N_c + 1}{2N_c} g^2 \int \frac{d^3 q}{(2\pi)^3} D(k - q) \frac{G(q)}{q_3^2 + (|\vec{q}| - \mu)^2 + G(q)^2}. \tag{40}$$

As expected, the integral over q is dominated by the momenta $|\vec{q}| \simeq \mu$ and $q_3 \ll \mu$.

C. Overhauser gap equations

While it is possible in $3+1$ dimensions to follow the same formalism for the Overhauser pairing as in the BCS case, the method is involved in $2+1$ dimensions due to the absence of γ_5, which is crucial in inverting S^{-1}. Chiral projectors are required to pair particles and holes but since chirality cannot be consistently defined in $2+1$ dimensions, the arguments are more subtle. However, there is a straightforward way and that is to use the effective action formalism developed in [10].

The induced action in Euclidean space is

$$S_\psi = \frac{g^2}{2} \int d^3x \, d^3y \, J_\mu^a(x) D_{\mu\nu}(x-y) J_\nu^a(y) + \int d^3x \, \bar{\psi} \tilde{\partial}_\mu \gamma_\mu \psi, \tag{41}$$

where $\tilde{\partial}_\mu = (\partial_1, \partial_2, \partial_3 + \mu)$ and J_μ^a is the usual colored current $g\bar{\psi}\gamma_\mu(\lambda_a/2)\psi$. We may now Fierz the JJ term into the relevant particle-hole channel. The simplified propagator [Eq. (39)] is used as we are working in the Landau gauge. The Fierzing factors are

$$\text{Color:} \quad -\frac{1}{2}\left(1 - \frac{1}{N_c}\right),$$

$$\text{Flavor:} \quad \frac{1}{2}, \tag{42}$$

$$\text{Spin:} \quad \frac{3}{2}.$$

Following [10], we introduce a Hermitian bilocal field $\Sigma(x,y)$ to linearize the Fierzed form of the JJ term by using a Hubbard-Stratanovich transformation, as

$$\exp\left(\frac{3g^2}{16}\left(1 - \frac{1}{N_c}\right) \int d^3x \, d^3y \, [\bar{\psi}(x)\psi(y)] D(x-y)\right.$$

$$\left. \times [\bar{\psi}(y)\psi(x)]\right)$$

$$= \int d\Sigma(x,y) \exp\left(-S_\Sigma - \int d^3x \, d^3y \, \bar{\psi}(x)\Sigma(x,y)\psi(y)\right), \tag{43}$$

with

$$S_\Sigma = \frac{4}{3g^2}\left(1 - \frac{1}{N_c}\right)^{-1} \int d^3x \, d^3y \, \frac{(|\Sigma(x,y)|)^2}{D(x-y)}. \tag{44}$$

The action in the quark fields is now linear and the functional integration may be performed. Before doing so, however, we make a simplifying ansatz for the bilocal field

$$\Sigma(x,y) = 2\cos\left[P_\mu\left(\frac{x_\mu + y_\mu}{2}\right)\right]\sigma(x-y)$$

$$= 2\cos\left[P_\mu\left(\frac{x_\mu + y_\mu}{2}\right)\right] \int \frac{d^3q}{(2\pi)^3} e^{-iq\cdot(x-y)} F(q), \tag{45}$$

where $P_\mu = (\vec{P}_F, 0)$ and $|\vec{P}_F| = 2\mu$. $\vec{P}_F$ points in the original direction of one of the quarks. Σ characterizes a standing wave of total momentum 2μ. (The pairing is between a particle and a hole at the opposite edges of the Fermi surface.) Introducing fermion fields $\psi(\pm P/2 + q)$ as independent integration variables as in [8], and performing the functional integration over fermions, we obtain

$$\frac{3g^2}{8}\left(1 - \frac{1}{N_c}\right)\frac{S_\Sigma}{V_3} = \int d^3x \frac{|\sigma(x)|^2}{D(x)} - 2g^2 \int \frac{d^3q}{(2\pi)^3}$$

$$\times \ln\det\left\| \begin{array}{cc} -\sigma\cdot Q_+ & F \\ F & -\sigma\cdot Q_- \end{array} \right\|, \tag{46}$$

where $Q_\pm = [(\pm\vec{P}/2 + \vec{q}), (\pm P_3/2 + q_3 - i\mu)]$. The determinant is over an $(4 \cdot N_c \cdot N_f) \times (4 \cdot N_c \cdot N_f)$ matrix. The gap equation follows by variation and the result is

$$F(k) = \frac{3}{2}(N_c - 1)g^2 \int \frac{d^3q}{(2\pi)^3} D(k-q)$$

$$\times \frac{F(q)}{\left\{q_3 + \frac{|\vec{q}|^2}{2i\mu}\right\}^2 + q_\parallel^2 + F(q)^2}. \tag{47}$$

The measure is $d^3q = dq_\parallel dq_\perp dq_3$, where $q_\parallel = \vec{q}\cdot\hat{\mathbf{P}}$ and $\mathbf{q}_\perp = \vec{q} - q_\parallel\hat{\mathbf{P}}$. The form factor $F(q)$ decreases rapidly as we go further from the Fermi surface, therefore, dominant contributions to the integral over $q_\parallel$ come from the region where $q_\parallel \sim F(q_\parallel)$. The gap is approximately constant in the $\mathbf{q}_\perp$ direction. The $q_\perp$ integration extends to $\Lambda_\perp = \sqrt{2\mu F_0}$, where F_0 is the maximum Overhauser gap. Thus, it picks up contributions from a larger region than $q_\parallel$ does. As the particles will be placed on mass shell, q_3 and $q_\parallel$ are of the same order. The contour integration over q_3 is performed with the constraint $|\mathbf{q}_\perp|^2 \leq 2\mu\epsilon_q \equiv 2\mu\sqrt{q_\parallel^2 + F(q_\parallel)^2}$. With these observations, the gap equation becomes

$$F(k_\parallel, k_\perp) = \frac{3(N_c - 1)g^2}{2(2\pi)^3} \int dq_\parallel \frac{F(q_\parallel)}{\sqrt{q_\parallel^2 + F(q_\parallel)^2}} \int_0^{\Lambda_\perp} dq_\perp$$

$$\times D(k_\perp - q_\perp, k_\parallel - q_\parallel). \tag{48}$$

The electric and magnetic terms can be dealt with separately, as in the BCS case. The gap is approximately constant in the $\mathbf{k}_\perp$ direction, so we study the dependence on $k_\parallel$ alone.

With electric screening in matter, the gap equation reads

$$F(k_{||}) = \kappa \pi \int dq_{||} \frac{F(q_{||})}{\sqrt{q_{||}^2 + F(q_{||})^2}} \frac{1}{\sqrt{m_D^2 + (k_{||}-q_{||})^2}}$$

$$\times \tan^{-1}\left(\frac{\Lambda_\perp}{\sqrt{m_D^2 + (k_{||}-q_{||})^2}}\right), \qquad (49)$$

where $\kappa = 3(N_c-1)g^2/4(2\pi)^3$.

In case of Landau damping of the magnetic mode, the gap equation is

$$F(k_{||}) \cong \kappa \pi \int dq_{||} \frac{F(q_{||})}{\sqrt{q_{||}^2 + F(q_{||})^2}} \frac{c}{\left(\frac{3}{4}m_D^2 |k_{||}-q_{||}|\right)^{1/3}}, \qquad (50)$$

where $c = (6\ln 2 + 2\sqrt{3}\pi)/18$. In writing Eq. (50), we have expanded in $(\frac{3}{4}m_D^2|k_{||}-q_{||}|)^{1/3}/\Lambda_\perp$ and retained only the leading term. This ratio is N_c dependent and the expansion turns out to be valid only at sufficiently large N_c [see Eq. (53)]. Noting that m_D is typically much larger than the gap, the magnetic binding will drive the formation of the gap. Proceeding as in the BCS case, we convert the integral equation into a differential one,

$$3k_{||}^2 F''(k_{||}) + 4k_{||}F'(k_{||}) + \kappa_* \frac{F(k_{||})}{k_{||}^{1/3}} = 0, \qquad (51)$$

where $\kappa_* = (c\pi\kappa)/(\frac{3}{4}m_D^2)^{1/3}$. Equation (51) is analogous to Eq. (33) obtained for the BCS gap. The solution follows:

$$F(k_{||}) = \frac{F_0^{7/6}}{J_1\left(6\sqrt{\frac{\kappa_*}{3}}F_0^{-1/6}\right)} \frac{J_1\left(6\sqrt{\frac{\kappa_*}{3}}k_{||}^{-1/6}\right)}{k_{||}^{1/6}}, \qquad (52)$$

with $F_0 = (\kappa_*/3)^3 (6/x_0)^6$, the maximum Overhauser gap. Note that the condition $F_0 < m_D$ is satisfied since $F_0/m_D \sim (\alpha/\mu)^{3/2}$. As in the BCS case, $F_0/\mu \sim (\alpha/\mu)^2$. These observations justify the neglect of electric screening effects. The validity of the approximation in Eq. (50) implies that the ratio

$$\frac{\left(\frac{3}{4}m_D^2 F_0\right)^{1/3}}{\Lambda_\perp} = \left(\frac{2\pi x_0^2}{6c(N_c-1)}\right)^{1/2} \cong \left(\frac{7.24}{(N_c-1)}\right)^{1/2} \qquad (53)$$

should be small. So long as N_c is sufficiently large, the approximation and the resulting estimate of the Overhauser gap can be trusted. If N_c does not meet this condition, the binding is disturbed throughout the pairing region by Landau damping. With this caveat, we can make a comparison of the maximum BCS and Overhauser gaps (F_0 and G_0). We find

$$\frac{F_0}{G_0} \cong 2\left(N_c \frac{(1-1/N_c)}{(1+1/N_c)}\right)^3, \qquad (54)$$

which shows that in $2+1$ dimensions, the Overhauser gap is larger than the BCS gap for sufficiently large N_c. Finally, an assessment of the energy budget for the two competing mechanisms yields

$$\frac{\epsilon_{OV}}{\epsilon_{BCS}} = \frac{(\Lambda_\perp F_0)F_0}{(\mu G_0)G_0} \cong \left(\frac{F_0}{G_0}\right)^2 \left(\frac{F_0}{\mu}\right)^{1/2}. \qquad (55)$$

IV. CONCLUSIONS

In considering BCS and Overhauser pairing in $2+1$ dimensions, we have argued for the form of the condensates that we have chosen to study. The gap equations are written down and treated with the effects of electric screening and Landau damping in matter. For the case of $2+1$ dimensions, and to leading order in perturbation theory, we have derived modifications to the gluon propagator from screening by quarks, using the imaginary time formalism. The standard Nambu-Gorkov formalism is more subtle for the Overhauser case due to the absence of γ_5 and the effective action method is applied instead. A comparison of the BCS and Overhauser gap in Landau gauge, shows that the latter is preferred with increasing N_c. The energy budgeting shows that BCS pairing is negatively affected by the reduction of available phase space for pairing in the lower dimension. The gaps themselves are small, being of order (α^2/μ), although the enhanced infrared sensitivity in $2+1$ dimensions prevents us from a definitive conclusion on their gauge independence. Also, the coupling does not run in $2+1$ dimensions, so it is not possible to set a perturbative scale, and consequently to say when α/μ is small. Our perturbative analysis assumes that we can take α/μ small. It is interesting to note that unlike the case of $3+1$ dimensions or $1+1$ dimensions, the BCS and Overhauser gaps show power law dependence (g^4/μ) in Landau gauge. Therefore, they are more sensitive to higher order corrections from perturbation theory than their corresponding forms in $3+1$ dimensions. A numerical analysis of the gap equations in $2+1$ dimensions would complement our theoretical approach, and would also serve to test the validity of some of the approximations. Finally, it would be interesting to consider the effects of the crystalline superconducting phase discussed recently by others [11,17] as a competing instability in this space-time dimension.

Note added. While writing our paper, we were informed that a related analysis for the BCS case was being performed by V. A. Miransky, G. W. Semenoff, I. A. Shovkovy and L. C. R. Wijewardhana.

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