

Bilocal expansion of the Borel amplitude and the hadronic tau decay width

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The singular part of the Borel transform of a QCD amplitude near the infrared renormalon can be expanded in terms of higher order Wilson coefficients of the operators associated with the renormalon. In this paper we observe that this expansion gives nontrivial constraints on the Borel amplitude that can be used to improve the accuracy of the ordinary perturbative expansion of the Borel amplitude. In particular, we consider the Borel transform of the Adler function and its expansion around the first infrared renormalon due to the gluon condensate. Using the next-to-leading order (NLO) Wilson coefficient of the gluon condensate operator, we obtain an exact constraint on the Borel amplitude at the first IR renormalon. We then extrapolate, using judiciously chosen conformal transformations and Padé approximants, the ordinary perturbative expansion of the Borel amplitude in such a way that this constraint is satisfied. This procedure allows us to predict the $O(\alpha_s^4)$ coefficient of the Adler function, which gives a result consistent with the estimate by Kataev and Starshenko using a completely different method. We then apply this improved Borel amplitude to the tau decay width and obtain the strong coupling constant $\alpha_s(M_z^2) = 0.1193 \pm 0.0007_{\text{exp.}} \pm 0.0010_{\text{EW+CKM}} \pm 0.0009_{\text{meth.}} \pm 0.0003_{\text{evol.}}$. We then compare this result with those of other resummation methods.

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I. INTRODUCTION

The ordinary perturbative expansion in quantum chromodynamics (QCD) gives a divergent series with rapidly increasing perturbative coefficients. Having higher order corrections, thus does not automatically mean better accuracy. A further step should be taken to properly handle the divergent series. For this purpose, the Borel resummation technique is often invoked.

The Borel resummation of the perturbation series in QCD, however, is not straightforward because of the nonperturbative effects that cause singularities on the Borel plane. Generally, the Borel transform of a QCD amplitude has singularities [1,2]: the ultraviolet (UV) renormalons on the negative real axis and the infrared (IR) renormalons on the positive real axis. There are also singularities caused by instanton–anti-instanton pairs, but these are irrelevant to our discussion and shall be ignored.

In a Borel resummation the UV renormalons are not a serious problem, since they can be transformed far away from the Borel integration contour using a proper conformal mapping, but the IR renormalons, which are located on the integration contour, cause a real problem. First of all, the IR renormalons cause ambiguities in taking a proper contour at their positions. The IR renormalons can be associated with certain operator condensates [1] appearing in operator product expansion, and these ambiguities are known to arise from the ambiguities in defining the renormalized condensates in the continuum limit [3]. Because of the ambiguities there arises a mixing between “perturbative” and “nonperturba-

tive” effects, rendering it impossible to separate them in a unique way. Thus, the straightforward Borel resummation defined on a proper contour must be augmented by nonperturbative effects, which, in general, are impossible to calculate.

There can be, however, situations where the Borel resummation of the perturbation series alone can be useful. For example, in hadronic tau decay the nonperturbative effects are known to be small, and so the ambiguities are small or even ignorable. In this case, roughly speaking, the true amplitude is mostly of a perturbative nature and can be well-described by the Borel resummation. Then, the most important thing to do is to describe the Borel amplitude as accurately as possible in the interval between the origin and the first IR renormalon using the first few perturbative coefficients that are known.

To achieve this purpose, a few techniques were developed. One is to use conformal transformation to map the UV renormalons far away from the origin, which helps accelerate the convergence of the perturbative expansion of the Borel amplitude. Another is to use the Padé approximant for the Borel amplitude, either alone or combined with the conformal mapping. We introduce in this paper a new technique, which we believe to be powerful enough to predict higher order loop corrections, which combines the conformal mapping method with a perturbative expansion of the Borel amplitude in the neighborhood of the IR renormalon.

Since the ambiguities caused by IR renormalons can be associated with certain operator condensates, it is possible to expand the singular part of the Borel amplitude near the renormalon in terms of the Wilson coefficients and anomalous dimensions of the associated operators. For simplicity, and because we have hadronic tau decay in mind as an application of our technique, we shall confine ourselves to the

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Adler function of the current correlators, and the expansion around its first IR renormalon caused by gluon condensate. In Sec. II we then show that this expansion gives rise to two exact constraints on the Borel amplitude that need to be satisfied at the IR renormalon. Since one of the constraints involves the uncalculated next-to-next-to-leading (NNLO) order Wilson coefficient of the gluon condensate operator, we have only one constraint available, which depends only on the calculated next-to-leading order Wilson coefficient. In Secs. III and IV we then use this constraint to extrapolate, using judiciously chosen conformal transformations and Padé approximants that involve the unknown $\mathcal{O}(\alpha_s^4)$ coefficient of the Adler function, the perturbative Borel amplitude in such a way that the constraint is satisfied at the renormalon. This yields a prediction of the uncalculated $\mathcal{O}(\alpha_s^4)$ coefficient, which we compare with the estimate by Kataev and Starshenko [4] using the method of Stevenson's minimal scale dependence and find it to be consistent with the latter. We call our method bilocal expansion because the constraint is derived by using expansion of the Borel amplitude around the renormalon (around $b=2$) and the evaluation of the constraint is carried out by resummations based on the perturbative expansion of the Borel amplitude (around $b=0$).

With this prediction of the amplitude up to $\mathcal{O}(\alpha_s^4)$, we turn in Sec. V to the hadronic tau decay width without the massive component from $\Delta S \neq 0$ decays. The width is calculated from the Adler function by the contour approach in the complex momentum plane. We use for the Borel transform of the Adler function the ansatz which explicitly incorporates the structure of the first IR renormalon, and we perform the Borel integration by using an optimal conformal transformation to map away the effects of the UV renormalons and the higher IR renormalons. In Sec. VI, based on the stringent experimental results obtained by the ALEPH, OPAL, and CLEO Collaborations, we extract with our method of resummation the following values of the strong coupling parameter: $\alpha_s(M_Z^2) = 0.1193 \pm 0.0007_{\text{exp.}} \pm 0.0010_{\text{EW+CKM}} \pm 0.0009_{\text{meth.}} \pm 0.0003_{\text{evol.}}$. We compare this result with those of other variants of our resummation method and with the present world average, and subsequently in Sec. VII with the results of resummation methods applied previously by others. Section VIII contains a summary and conclusions.

II. BILOCAL EXPANSION

For definitiveness, we shall consider the current-current correlation function in the Euclidean region:

$$\int e^{-iqx} \langle T J^\mu(x) J^\nu(0)^\dagger \rangle d^4x = -i(q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi(q^2), \quad (1)$$

where $J^\mu(x) = \bar{u} \gamma^\mu d(x)$ is the current of up and down quarks. The canonically normalized massless Adler function $D(Q^2)$ is defined by

$$D(Q^2) \equiv -4\pi^2 Q^2 \frac{d}{dQ^2} \Pi(-Q^2) - 1, \quad (2)$$

with $Q^2 = -q^2 > 0$.¹

The Borel transform $\tilde{D}(b)$ of the Adler function is defined, formally, by

$$D(Q^2) = \frac{1}{\beta_0} \int_0^\infty db e^{-b/\beta_0 a(Q^2)} \tilde{D}(b), \quad (3)$$

where $a(Q^2) = \alpha_s(Q^2)/\pi$, with $\alpha_s(Q^2)$ being the strong coupling constant. The $\tilde{D}(b)$ is analytic around the origin at $b=0$ and can be expanded in the power series

$$\tilde{D}(b) = 1 + \sum_{n=1}^{\infty} \frac{d_n}{n!} \left(\frac{b}{\beta_0} \right)^n, \quad (4)$$

with d_n being the coefficients of the perturbation series for the Adler function

$$D(Q^2) = a(Q^2) \left[1 + \sum_{n=1}^{\infty} d_n a(Q^2)^n \right]. \quad (5)$$

The constant β_0 is the first coefficient of the QCD β function

$$\mu^2 \frac{d}{d\mu^2} a(\mu^2) \equiv \beta(a(\mu^2)) = -\beta_0 a(\mu^2)^2 [1 + c_1 a(\mu^2) + c_2 a(\mu^2)^2 + \dots], \quad (6)$$

where μ denotes the renormalization scale and $c_j \equiv \beta_j/\beta_0$ ($j \geq 2$) parametrize the renormalization scheme. The Borel transform $\tilde{D}(b)$ is known to have singularities, the UV renormalons on the negative real axis at $b = -n$, and the IR renormalons on the positive real axis at $b = n+1$ with $n = 1, 2, 3, \dots$. The renormalon resummation of $D(Q^2)$ and of the hadronic τ decay width in the large β_0 limit has been performed in [5–7]. While the UV renormalons do not cause any direct problem, the IR renormalons on the integration contour cause ambiguities in the Borel integral. For simplicity, we shall confine ourselves to the first IR renormalon at $b=2$. The Borel transform around the singularity can be written in the form

$$\tilde{D}(b) = \frac{C}{(1-b/2)^{1+\nu}} [1 + \tilde{c}_1(1-b/2) + \tilde{c}_2(1-b/2)^2 + \dots] + (\text{analytic part}), \quad (7)$$

with ν given by

$$\nu = 2c_1/\beta_0 \approx 1.580 \quad (8)$$

for the number of active quark flavors $N_f=3$. The convergence radius of the series within the bracket is bounded by the second IR renormalon at $b=3$, and so the series is expected to be convergent for $|1-b/2| < 1/2$.

¹For normalization convention, see the discussion after Eq. (A6) in the Appendix.

The part analytic at $b=2$ as well as the exact value of the residue C are not known, although the latter can be calculated in perturbation theory [8]. The coefficients \tilde{c}_i in the expansion of the singular part are calculable (see Refs. [9,10] for related discussions), and depend on the β function and the Wilson coefficients of the gluon condensate operator.

To use this expansion around the renormalon singularity in improving the Borel resummation, we consider the function $R(b)$,

$$R(b) \equiv (1-b/2)^{1+\nu} \tilde{D}(b), \quad (9)$$

which was introduced in [8] in the perturbative calculation of the renormalon residue and also in [11] to soften the renormalon singularity. Around the singularity, $R(b)$ is given by

$$R(b) = C[1 + \tilde{c}_1(1-b/2) + \tilde{c}_2(1-b/2)^2 + \dots] \\ + (1-b/2)^{1+\nu} (\text{analytic part}), \quad (10)$$

which shows $R(b)$ is singular but bounded at the first IR renormalon. Should the analytic part vanish, $R(b)$ would be analytic at the renormalon position, but since there is no reason to expect this to happen, we should regard $R(b)$ to be singular at $b=2$. With Eq. (10) we now obtain a set of constraints on $R(b)$ for $N_f=3$, and accordingly on the Borel transform $\tilde{D}(b)$, at the singularity

$$\left. \frac{R'(b)}{R(b)} \right|_{b=2} = -\frac{\tilde{c}_1}{2}, \quad \left. \frac{R''(b)}{R(b)} \right|_{b=2} = \frac{\tilde{c}_2}{2}. \quad (11)$$

In the next section we will exploit one of these equations to constrain the functional behavior of the Borel transform in the interval between the origin and the first IR renormalon singularity.

We now turn to the calculation of the coefficients \tilde{c}_1, \tilde{c}_2 . Because of the singularity, $\tilde{D}(b)$ has a branch cut beginning at $b=2$, and consequently, $D(Q^2)$ obtains an imaginary part from the Borel integral

$$\text{Im}[D(Q^2)] \propto \pm a(Q^2)^{-\nu} e^{-2/\beta_0 a(Q^2)} [1 + \frac{1}{2} \tilde{c}_1 \nu \beta_0 a(Q^2) \\ + \frac{1}{4} \tilde{c}_2 \nu(\nu-1) \beta_0^2 a(Q^2)^2 + O(a^3)], \quad (12)$$

which is obtained by plugging Eq. (7) into Eq. (3). The sign of the imaginary part depends on whether the contour along the positive real axis is on the upper or the lower half plane. Because $D(Q^2)$ must be real, this imaginary part should be canceled by something else. It has been suggested in [3] that this imaginary part is canceled by the imaginary part arising from the ambiguity in defining renormalized gluon condensate in the operator product expansion (OPE) of $D(Q^2)$,

$$D(Q^2) = C_0(a(Q^2)) + C_4(a(Q^2)) \frac{\langle O_4 \rangle}{Q^4} \\ + (\text{higher dimension terms}), \quad (13)$$

where $\langle O_4 \rangle$ is the scale-invariant matrix element (gluon condensate) of the anomalous-dimension free, dimension-four gluon operator

$$\langle O_4 \rangle = \left\langle \frac{\beta(a)}{a} G_{\mu\nu}^a G^{a\mu\nu} \right\rangle, \quad (14)$$

with $G_{\mu\nu}^a$ denoting the gluon field strength tensor.

The Wilson coefficient C_0 for the unit operator has the perturbative expansion given in Eq. (5) while the coefficient C_4 for the gluon condensate operator is known to the next-to-leading order (NLO) in the modified minimal subtraction ($\overline{\text{MS}}$) renormalization scheme [12]:

$$C_4(a(Q^2)) = -\frac{2\pi^2}{3\beta_0} [1 + w_1 a(Q^2) + w_2 a(Q^2)^2 + O(a^3)], \quad (15)$$

with

$$w_1 = \frac{7}{6} - c_1 (= -\frac{11}{18} \text{ for } N_f=3). \quad (16)$$

The next-to-next-to-leading order coefficient w_2 is not known yet.

Because the gluon condensate as well as its ambiguity should satisfy the homogeneous renormalization group (RG) equation, the ambiguous, imaginary part from the gluon condensate can be written as

$$\text{Im}[D(Q^2)_{\text{con.}}] = \pm C_4(a(Q^2)) \frac{\Lambda^4}{Q^4}, \quad (17)$$

with Λ being a RG-invariant and Q -independent constant. Therefore,

$$\frac{\Lambda^4}{Q^4} \propto \exp \left[-2 \int^{a(Q^2)} \frac{dx}{\beta(x)} \right] \\ \propto a(Q^2)^{-\nu} e^{-2/\beta_0 a(Q^2)} [1 + v_1 a(Q^2) + v_2 a(Q^2)^2 \\ + O(a^3)], \quad (18)$$

where the proportionality constants are Q independent, and v_j 's are obtained by expanding $1/\beta(x)$ in powers of x

$$v_1 = \frac{2}{\beta_0} (-c_2 + c_1^2), \\ v_2 = \frac{1}{2} v_1^2 + \frac{1}{\beta_0} (-c_3 + 2c_1 c_2 - c_1^3). \quad (19)$$

Thus,

$$\text{Im}[D(Q^2)_{\text{con.}}] \propto \pm a(Q^2)^{-\nu} e^{-2/\beta_0 a(Q^2)} [1 + (v_1 + w_1) a(Q^2) \\ + (v_2 + v_1 w_1 + w_2) a(Q^2)^2 + O(a^3)]. \quad (20)$$

Because the imaginary parts in Eqs. (12) and (20) should cancel each other, we have

$$\tilde{c}_1 = \frac{2}{\nu\beta_0}(v_1 + w_1) (\approx -0.9990 \text{ for } N_f=3), \quad (21)$$

$$\tilde{c}_2 = \frac{4}{\nu(\nu-1)\beta_0^2}(v_2 + v_1 w_1 + w_2). \quad (22)$$

III. AN OPTIMAL CONFORMAL MAPPING

To impose the constraints (11) on the Borel transform defined in series form (4), $\tilde{D}(b)$ needs to be analytically continued beyond its convergence radius $|b|=1$ which is set by the first UV renormalon. This cumbersome, analytic continuation, however, can be conveniently avoided by using a conformal mapping that pushes the UV renormalons away from the origin while mapping the first IR renormalon to be the closest singularity to the origin. Since, in practice, only the first few coefficients are known, choosing an optimal mapping can help accelerate convergence of the series (4). Even though several conformal mappings, optimal or not, were discussed in the literature [14,8,15], we introduce a new mapping which is especially well suited for our purpose.

Our criterion for an optimal mapping is simple; with an optimal mapping

$$w = w(b), \quad (23)$$

the function $R(b(w))$ should be as smooth as possible within the disk $|w| \leq w_0$, where $w_0 = |w(b=2)|$, so that $R(b(w))$ within the radius of convergence can be well approximated by the first terms of its perturbation series in w

$$R(b(w)) = \sum_{n=0}^{\infty} r_n w^n. \quad (24)$$

With this criterion our strategy for an optimal mapping is to send all the renormalon singularities save the unavoidable first IR renormalon as far away as possible from the origin. As a candidate for an optimal mapping we propose

$$w = \frac{\sqrt{1+b} - \sqrt{1-b/3}}{\sqrt{1+b} + \sqrt{1-b/3}}, \quad (25)$$

which is obtained by combining the mapping [8]

$$z = \frac{b}{1+b}, \quad (26)$$

which sends all the UV renormalons to the positive real axis, with the mapping [14]

$$w = \frac{1 - \sqrt{1 - z/z_0}}{1 + \sqrt{1 - z/z_0}}, \quad (27)$$

where $z_0 \equiv z(b=3) = 3/4$, which sends all renormalon singularities except for the first IR renormalon to the unit circle. With the conformal mapping (25) the first IR renormalon is mapped to $w = 1/2$ while all other renormalons are mapped to the unit circle (see Fig. 1). Since we are especially interested

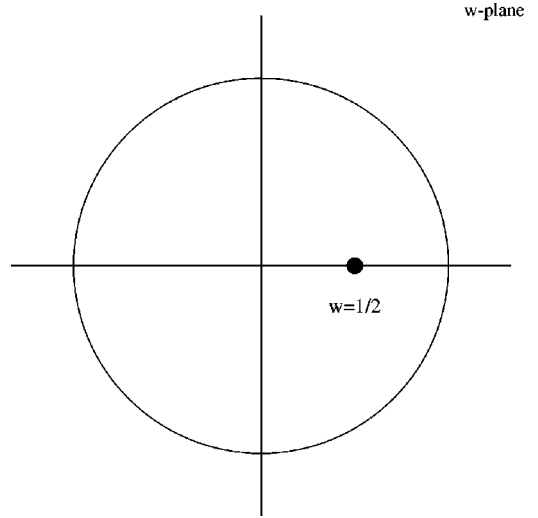


FIG. 1. The conformal mapping (25) maps the first IR renormalon to $w = 1/2$, and all other renormalons to the unit circle.

in the functional behavior of $R(b(w))$ within the radius of convergence $w_0 = 1/2$, we expect the mapping is well suited for our purpose because the divergence by the renormalon singularities is suppressed due to their relatively large distance from the origin.

Now on the w plane the first of the constraints (11) becomes

$$\left[\frac{dR(b(w))}{dw} + \frac{\tilde{c}_1}{2} \frac{db}{dw} R(b(w)) \right] \Big|_{w=1/2} = 0. \quad (28)$$

In the next section we will impose this constraint on the truncated perturbation series (TPS) of Eq. (24) to obtain a higher order correction of the Adler function. By noticing that the constraint (28) is set up at the first IR renormalon, which is exactly at the radius of convergence of the series (24), one may question the validity of applying the constraint directly on the TPS. However, it should be emphasized that the series (24) is convergent at the renormalon singularity $w = 1/2$ because $R(b(w))$, even though singular there, is bounded. Therefore, the constraint can be imposed on the perturbation series.

IV. PREDICTION FOR THE NNNLO COEFFICIENT OF THE ADLER FUNCTION

The NLO and NNLO coefficients d_1 and d_2 of the expansion of the canonical Adler function (5) have been calculated exactly in the $\overline{\text{MS}}$ scheme in [16,17]: $d_1 = 1.6398$, $d_2 = 6.3710$ (at $N_f=3$). The Borel transform $\tilde{D}(b)$ (4) and the function $R(b)$ (9) are thus also known up to NNLO in b . Upon subsequently applying the conformal transformation (25) to $R(b)$, and expanding in w , we obtain the power expansion of $R(b(w))$ (24) up to NNLO in w . On the other hand, if we assumed that the $N^3\text{LO}$ coefficient d_3 were known, we would obtain the power expansion (24) up to $N^3\text{LO}$

$$R(b(w)) = 1 - 1.68394w + 0.104w^2 + (-9.64395 + 0.395062d_3)w^3 + O(w^4). \quad (29)$$

The corresponding derivative dR/dw would then be known up to NNLO ($\sim w^2$). If we apply to the constraint (28) the above N³LO TPS of R and the corresponding NNLO TPS for dR/dw , we obtain $d_3 \approx 34$. This prediction, however, is not sufficiently precise, because, as mentioned before, the point $w = 1/2$ is at the border of the convergence disk of $R(b(w))$ and we are dealing with a strongly truncated series. Therefore, we apply at this stage yet another efficient mechanism of analytic continuation which would bring us beyond the $w = 1/2$ circle—Padé approximants (PA's)² that are either diagonal or near diagonal [19]. To the N³LO TPS (29) of $R(b(w))$ we can then either apply the $[1/2]$, $[2/1]$, or $[1/1]$ PA and to the NNLO TPS of dR/dw the $[1/1]$ PA. Then the constraint (28) predicts the values $d_3 \approx 24.7$ – 24.8 , virtually independent of the three PA choices for $R(b(w))$.³ Another practical approach is to construct, at a given fixed $d_3^{(0)}$, the NNLO TPS of $d \ln R/dw$ and thus the PA $[1/1]$ of $d \ln R/dw$. Employing this PA in the constraint (28) leads to the prediction $d_3^{(0)} \approx 30.4$. In the latter approach, however, higher order PA's ($[2/1]$, $[1/2]$) cannot be employed.

As a cross check, we carried out the same procedure but with a different conformal transformation

$$w = \frac{\sqrt{1+b} - \sqrt{1-b/4}}{\sqrt{1+b} + \sqrt{1-b/4}}. \quad (30)$$

This mapping also removes all the UV renormalons to the unit circle, as well as all the IR renormalons except for the first ($b=2$) and the second ($b=3$) one: $w(b=2) \approx 0.42$, $w(b=3) = 0.6$. This mapping apparently suppresses even more strongly than Eq. (25) the UV renormalon contributions, but probably less strongly the next-to-leading IR renormalon ($b=3$) contributions. The predictions are in this case $d_3 \approx 24.3$ – 24.5 , in good agreement with the aforementioned predictions. The use of the PA $[1/1]$ of $d \ln R/dw$ predicts in this case $d_3^{(0)} \approx 30.3$.

We further note another interesting feature of the expansion (29). Looking at the first three terms that are known, it appears reasonable to expect that the N³LO coefficient r_3 at w^3 is not very large, say, $|r_3| < 2$. Varying r_3 between -2 and 2 results in the variation of d_3 between 19.3 and 29.5 , i.e., only about 20% around the value $d_3 \approx 24.7$. Thus, the predictions of the described method, using the conformal transformation (25), are remarkably robust under the varia-

tion of the N³LO coefficient of $R(b(w))$. Similar robustness is observed when the conformal transformation (30) is used instead of (25).

If we applied to the relation (28) the PA's, but no conformal transformation, the predictions would vary more strongly ($d_3 \approx 26$ – 33) with the various choices for PA's of $R(b)$. Furthermore, this method does not possess the ‘‘robustness’’ under the variation of the N³LO coefficient of $R(b)$.

Thus, our prediction of the N³LO coefficient d_3 , in the $\overline{\text{MS}}$ scheme, of the Adler function $D(Q^2)$ is

$$d_3 \approx 25 \pm 5 \quad (\text{at } N_f = 3) \quad (31)$$

based on the simultaneous use of relation (28), the conformal mapping (25), and the Padé approximants. However, a larger uncertainty (± 10) of the predicted values of $d_3^{(0)}$ cannot be excluded, and we will use these more conservative uncertainty estimates in the next sections [see Eq. (52)].

Our predictions can be compared, for example, with those of Ref. [4]. They used the method of effective charge (ECH) [20–22] and the TPS principle of minimal sensitivity (PMS) [23,24] for the NNLO TPS of the Adler function $D(Q^2)$. The obtained approximants were then re-expanded back in powers of $a_0 \equiv a(Q^2; \overline{\text{MS}})$ up to $\sim a_0^4$, under the assumption $c_3^{\text{ECH}} - c_3^{\overline{\text{MS}}} \approx 0$ and $c_3^{\text{PMS}} - c_3^{\overline{\text{MS}}} \approx 0$. The resulting prediction was $d_3 \approx 27.5$, which is consistent with our prediction (31).

V. ANALYSIS OF THE HADRONIC TAU DECAY

In this section we will apply elements of the previous sections to the numerical study of the τ inclusive hadronic decay ratio

$$R_\tau \equiv \frac{\Gamma(\tau^- \rightarrow \nu_\tau \text{hadrons}(\gamma))}{\Gamma(\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e(\gamma))}. \quad (32)$$

Here, (γ) represent possible additional photons or lepton pairs. This inclusive decay ratio has been extensively studied in the literature, theoretically and numerically [25–31,13]. The ratio can be expressed, via the application of a variant of the optical theorem, with the two-point correlation functions of the vector (V) and axial-vector (A) currents, or equivalently, with the Adler functions $D^{\text{L+T}}(Q^2)$ and $D^{\text{L}}(Q^2)$ —we refer to the Appendix for some details. The theoretical/numerical resummation methods for evaluation of QCD observables are most efficient in the limit of massless quarks. When excluding hadrons with s (strange) quarks, and approximating the u and d quarks to be massless, the expression can be written as a contour integral (A13) in the complex momentum plane [28–31,13]

²The authors of Ref. [18] showed that combining the conformal transformations with the PA type of resummations can lead to significantly improved results, at least when a sufficient number of terms in the power expansion are known.

³In the procedure, we further require that $[1/1]$ PA of dR/dw not possess clearly unphysical poles (i.e., poles well below $w=0.5$).

$$r_\tau \equiv r_\tau^{V+A}(\Delta S=0; m_{u,d}=0) \\ \equiv \frac{R_\tau^{V+A}(\Delta S=0; m_{u,d}=0)}{3|V_{ud}|^2(1+\delta'_{EW})} - (1+\delta'_{EW}) \quad (33)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} dy (1+e^{iy})^3 (1-e^{iy}) D(Q^2) \\ \equiv -s = m_\tau^2 e^{iy}, \quad (34)$$

where the (minimal standard model) electroweak correction (EW) factors δ_{EW} and δ'_{EW} have been calculated in [32,33], and $D(Q^2)$ is the massless canonical Adler function (2), (5). The superscript $V+A$ in the above formulas emphasizes the fact that the quantities are inclusive in the sense of including the vector and axial-vector hadronic currents.

The experimental value of the observable $R_\tau^{V+A}(\Delta S=0)$ can be extracted from the values of the leptonic branching ratios $B_e \equiv B(\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau)$ and $B_\mu \equiv B(\tau^- \rightarrow \mu^- \bar{\nu}_\mu \nu_\tau)$, as obtained from the constrained fit derived from a set of basis modes [34] (see also [35]). The basis modes form an exclusive set of leptonic and hadronic decays whose branching ratios are normalized so that their sum is exactly one. The set of basis modes does not include the decays with photons in the final state, i.e., the right-hand side of Eq. (32) is for them without (γ). The only leptonic branching ratios in the set of basis modes are B_e and B_μ . Therefore, $R_\tau = (1 - B_e - B_\mu)/B_e$. The present values of B_e and B_μ , as determined from the constrained fit [34], based on the high precision measurements of the basis modes of the τ^- decay by the ALEPH, OPAL, and CLEO Collaborations [35–39], are $B_e = (17.83 \pm 0.06) \times 10^{-2}$, $B_\mu = (17.37 \pm 0.07) \times 10^{-2}$. The updated value of the strangeness-changing ratio is [40] $R_\tau(\Delta S \neq 0) = 0.1630 \pm 0.0057$. This implies

$$R_\tau^{V+A}(\Delta S=0) = \frac{(1 - B_e - B_\mu)}{B_e} - R_\tau(\Delta S \neq 0) \quad (35)$$

$$= 3.4713 \pm 0.0171. \quad (36)$$

The canonical τ^- decay ratio (33), at the moment still without the massless quark condition $m_{u,d} \rightarrow 0$, i.e., the reduced decay ratio (A1) in the Appendix, can then be obtained from the experimental values (36) by inserting the known values of the electroweak correction parameters δ_{EW} and δ'_{EW} and of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{ud}|$. Here we have to deal with additional uncertainties.

The main EW correction parameter has the value $\delta_{EW} = 0.0194 \pm 0.0050$ [32], while the residual correction parameter is $\delta'_{EW} = 0.0010$ [33]. In calculating δ_{EW} , the additional contributions from low scales ($< m_\tau$), dependent on the hadronic structure, although not enhanced by large logarithms, cannot be calculated and were estimated [32] to lead to the significant uncertainties ± 0.0050 .

The values of $|V_{ud}|$ from the (SM) unitarity constraint fit are 0.9749 ± 0.0008 [34]. On the other hand, the values extracted from the decays of mirror nuclei lead to lower values

$|V_{ud}| = 0.9740 \pm 0.0010$. This extraction is, however, fraught with theoretical uncertainties (see [34] for further references). Further, the values extracted from neutron decays $|V_{ud}| = 0.9728 \pm 0.0012$ ([34] and references therein) are even lower, but appear to have smaller theoretical uncertainties. For all these reasons, we will adopt the value range

$$|V_{ud}| = 0.9749 \pm 0.0021, \quad (37)$$

where the central value is the one from the unitarity constrained fit, but the uncertainty has been increased so that the values now include all the values from the decays of mirror nuclei and the upper half of the interval of values from neutron decays.

This now allows us to extract the values of the canonical τ^- decay ratio (A1)

$$r_\tau^{V+A}(\Delta S=0) \equiv \frac{R_\tau^{V+A}(\Delta S=0)}{3|V_{ud}|^2(1+\delta'_{EW})} - (1+\delta'_{EW}) \\ = 0.1933 \pm 0.0059_{\text{exp.}} \pm 0.0059_{EW} \\ \pm 0.0051_{\text{CKM}}, \quad (38)$$

where the uncertainty $\pm 0.0059_{EW}$ originates from the aforementioned ± 0.0050 uncertainty in δ_{EW} , and $\pm 0.0051_{\text{CKM}}$ from the ± 0.0021 uncertainty in $|V_{ud}|$ (37).

The QCD observable (38), as defined, has the non-QCD effects factored out. However, it still contains the problematic, though small, quark mass effects ($m_{u,d} \neq 0$). In the Appendix, we calculated the numerical strength of the quark mass contributions (A10),(A11). Subtracting these effects as in Eq. (A12), we end up with the following values for the massless QCD observable (33)

$$r_\tau \equiv r_\tau^{V+A}(\Delta S=0; m_{u,d}=0) \\ = 0.1960 \pm 0.0059_{\text{exp.}} \pm 0.0059_{EW} \\ \pm 0.0051_{\text{CKM}} \quad (39)$$

$$= 0.1960 \pm 0.0098. \quad (40)$$

In Eq. (39) we neglected the small contributions ~ 0.0001 from the corrections of the type $\sim m_{u,d}^2/m_\tau^2$ to Eq. (A11). In Eq. (40), the three uncertainties of Eq. (39) were added in quadrature.

The values (39) will be the starting point for our massless QCD resummation analyses of the hadronic τ decay. The experimental uncertainty (39) in the massless QCD observable r_τ is 3%, representing a high experimental precision when compared to many other QCD observables. This fact can be regarded at present as our main motivation to investigate theoretically and numerically this observable. Unfortunately, as we can see from Eqs. (39),(40), the total precision is worse (5%), due to the present uncertainties in the values of the electroweak corrections and of $|V_{ud}|$.

By adjusting the numerical (resummed) predictions for r_τ to the experimental ones (39), our main goal will be to predict the QCD coupling parameter $\alpha_s(m_\tau^2)$ with the high pre-

cision, i.e., with the resummation method uncertainty $(\delta\alpha_s)_{\text{meth.}}$ of the prediction being comparable to, or smaller than, the experimental uncertainty $(\delta\alpha_s)_{\text{exp.}}$ stemming from $(\delta r_\tau)_{\text{exp.}} = 0.0059$ (39). The starting point for our resummation method will be the contour integral representation (34) of r_τ in terms of the massless Adler function $D(Q^2)$.

As in the previous sections, we express $D(Q^2)$ as the Borel integral of its Borel transform $\tilde{D}(b) = R(b)/(1 - b/2)^{1+\nu}$, with the correct first IR renormalon singularity explicitly enforced in the ansatz

$$D(Q^2) = \frac{1}{\beta_0} \text{Re} \left[\int_{0+i\varepsilon}^{\infty+i\varepsilon} db e^{-b/\beta_0 a(\xi^2 Q^2)} \frac{R(b; \xi^2)}{(1-b/2)^{1+\nu}} \right], \quad (41)$$

where the integration contour is chosen to be on the upper half plane to avoid the singularity at $b=2$ (Cauchy principal value prescription). By explicitly enforcing the renormalon singularity, the Borel transform around the singularity can be more accurately described, and also the validity of the perturbative Borel transform can be extended beyond the first IR renormalon. The Borel transform $\tilde{D}(b)$ as well as $R(b)$ depend on the renormalization scheme and on the renormalization scale parameter $\xi^2 = \mu^2/Q^2$ through the ξ^2 dependence of the perturbative coefficients d_n in Eq. (5) when the running coupling $a(Q^2)$ is replaced by $a(\xi^2 Q^2)$. While we choose the $\overline{\text{MS}}$ scheme throughout this paper, the renormalization scale parameter ξ^2 will be kept arbitrary for the time being. When inserting Eq. (41) into Eq. (34), and exchanging the order of integrations, we obtain

$$r_\tau = \frac{1}{2\pi\beta_0} \text{Re} \left[\int_{0+i\varepsilon}^{\infty+i\varepsilon} db \frac{R(b; \xi^2)}{(1-b/2)^{1+\nu}} \times \int_{-\pi}^{\pi} dy (1+e^{iy})^3 (1-e^{iy}) e^{-b/\beta_0 a(\xi^2 m_\tau^2 \exp(iy))} \right]. \quad (42)$$

Since the integrand is exponentially suppressed at large b , it is convenient and reasonable to integrate over the Borel variable b just to a certain value b_{max} lying beyond the first IR renormalon. The contribution from the region beyond the first IR renormalon is expected to be smaller or comparable to the nonperturbative effect by the gluon condensate, which is known to be small [13]. If we know the perturbation series of the Adler function $D(Q^2)$ up to N³LO then we know automatically $R(b; \xi^2)$ up to N³LO, i.e., including the term $\sim b^3$. Further, $R(b; \xi^2)$ has no singularities on the positive axis for $b < 2$, and only a soft singularity at $b=2$, but it has some UV renormalons on the negative axis rather close to the origin: $b = -1, -2$. These UV renormalons make the power expansion of $R(b; \xi^2)$ in powers of b divergent for $|b| \geq 1$, which signals that the use of the (N³LO) TPS in powers of b for $R(b; \xi^2)$ in Eq. (42) may run into serious trouble already at $b \geq +1$. An efficient solution to this problem was already constructed in Sec. III, in the form of an optimal conformal transformation $b = b(w)$ (25), which

pushes all the UV renormalons (and all the higher IR renormalons at $b \geq 3$) onto a unit circle in the plane of the new variable w . The first IR renormalon at $b=2$ now corresponds to $w=1/2$, i.e., within the unit circle. Then, the expansion $R(b(w); \xi^2)$ in powers of w represents a convergent series for $w \leq 1/2$, i.e., for the corresponding $b(w) \leq 2$. Thus, the use of the corresponding N³LO TPS of $R(b(w); \xi^2)$, which is also explicitly known, will have a much better chance to describe reasonably well the true $R(b(w); \xi^2)$ within the interval between the origin and the first IR renormalon. Therefore, the double integral (42) will be rewritten in terms of the variable w

$$r_\tau \approx \frac{1}{2\pi\beta_0} \text{Re} \left[\int_0^{w_{\text{max}}} dw \frac{db(w)}{dw} \frac{R(b(w); \xi^2)}{[1-b(w)/2]^{1+\nu}} \times \int_{-\pi}^{\pi} dy (1+e^{iy})^3 (1-e^{iy}) e^{-b(w)/\beta_0 a(\xi^2 m_\tau^2 \exp(iy))} \right] \quad (43)$$

$$= \frac{3}{2\pi\beta_0} \text{Re} \left[e^{i\phi} \int_0^1 dx (1-w^2)(1-w+w^2)^{\nu-1} \times \frac{R(b(w); \xi^2)}{(1/2-w)^{1+\nu}(2-w)^{1+\nu}} \int_{-\pi}^{\pi} dy (1+e^{iy})^3 \times (1-e^{iy}) e^{-b(w)/\beta_0 a(\xi^2 m_\tau^2 \exp(iy))} \right]_{w=x e^{i\phi}}, \quad (44)$$

where we can choose in Eq. (43) $w_{\text{max}} \gg 1/2$, corresponding to $b_{\text{max}} \gg 2$. In practice, we can go in the dw integration in Eq. (43) beyond $w=1$, where the w contour then follows the unit circle arc into the first quadrant—for example up to a complex $w_{\text{max}} = \exp(i\phi)$ with $0 < \phi < \phi_\infty$, where $w(b=\infty) = \exp(i\phi_\infty)$, $\phi_\infty = \pi/3$. The fact that in this way we reach the $b \approx 3$ region, where the true $R(b)$ has an IR renormalon, and even go beyond it, does not change the result of Eq. (43) in practice. This is so because the contributions from the arc $|w|=1$ (corresponding to $b \geq 3$) turn out to be extremely suppressed in Eq. (43) (see also footnote 6 of the next section). This integration can be implemented in practice most easily, if we follow the ray $w = x \exp(i\phi)$, with x from 0 to 1 (see Fig. 2), because the integration over the corresponding closed contour yields zero since no singularities are enclosed. This practical ‘‘ray’’-integral implementation is denoted in Eq. (44).

The first two coefficients d_1 and d_2 of the expansion of the Adler function (5), which determine the expansions of $\tilde{D}(b)$ and $R(b)$ up to NNLO, have been calculated exactly in the literature [16,17]. For the choice $\mu^2 = Q^2$ and in the $\overline{\text{MS}}$ scheme, with $N_f = 3$, they are $d_1^{(0)} = 1.6398$, $d_2^{(0)} = 6.3710$. In the previous section, the arguments were presented suggesting the value of the N³LO coefficient $d_3^{(0)} \approx 25$. When the renormalization scale $\mu^2 = \xi^2 Q^2$ is changed ($\xi^2 \neq 1$), these coefficients change accordingly:

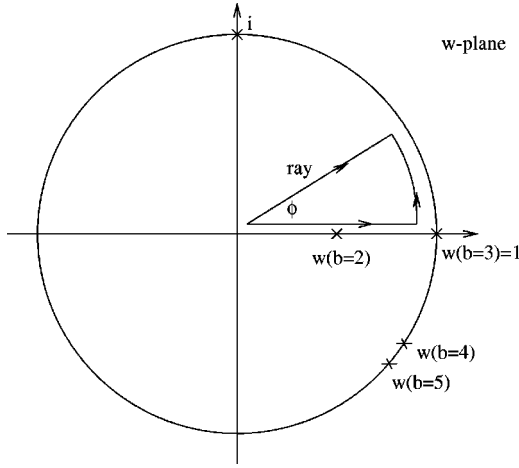


FIG. 2. Integration along the ray $w = x \exp(i\phi)$ ($0 < x < 1$, ϕ fixed) gives the same result as the integration parallel to the positive real axis ($0 < w < 1$) and arc $w = \exp(i\phi')$ ($0 < \phi' < \phi$).

$$d_1 = d_1^{(0)} + \beta_0 \ln \xi^2, \quad (45)$$

$$d_2 = d_2^{(0)} + 2\beta_0 \ln \xi^2 d_1^{(0)} + \beta_1 \ln \xi^2 + (\beta_0 \ln \xi^2)^2, \quad (46)$$

$$d_3 = d_3^{(0)} + 3(d_1 d_2 - d_1^{(0)} d_2^{(0)}) - 2(d_1^3 - d_1^{(0)3}) - (c_1/2)(d_1^2 - d_1^{(0)2}) + c_2(d_1 - d_1^{(0)}). \quad (47)$$

These relations follow from the expressions for the renormalization scheme and scale invariants ρ_1 , ρ_2 , ρ_3 , as given, e.g., in [23]. As an example, at $\xi^2 = 2$ they imply $d_1 = 3.1994$, $d_2 = 16.6908$, $d_3 = 97.4436$. The corresponding N³LO Borel transform is

$$R(b(w); \xi^2) = 1 - 1.68394w + 0.104w^2 + 0.232591w^3 \quad (\xi^2 = 1), \quad (48)$$

$$= 1 + 0.395499w + 3.30834w^2 + 5.13735w^3 \quad (\xi^2 = 2). \quad (49)$$

The apparently quite strong ξ^2 dependence of the Borel transform function $R(b(w); \xi^2)$ in Eq. (44) is combined with the strong ξ^2 dependence of the coupling parameter $a(\xi^2 Q^2)$ in the exponent (44) in such a way that the entire double integral is ξ^2 independent. However, since we know just the first few terms of $R(b(w); \xi^2)$, the ξ^2 dependence of Eq. (44) will appear. If the method is good this dependence should be weak, at least locally in a renormalization scale region $\xi^2 \sim 1$. Further, there should be some dependence on the choice of the renormalization scheme, but the scheme dependence is, in general, weaker than the ξ^2 dependence and we choose the MS scheme throughout.

At first sight, one may argue that the first IR renormalon of the Adler function has no significant bearing on the quantity r_τ because the singularity at $b=2$ is formally suppressed by a power of α_s due to the contour integration (34) (see Ref. [13]). We can see this, for example, if we consistently ignore

all effects beyond the one loop in Eq. (42) ($\beta_j \rightarrow 0$ for $j \geq 1$, $\nu \rightarrow 0$). In this approximation, the contour integration over y can be carried out explicitly and it yields an oscillating function of b which has a zero at $b=2$, thus erasing the singularity there. However, we wish to stress that this effect implies only that the nonperturbative power term $\sim 1/m_\tau^4$ contribution to r_τ is suppressed. This effect does not imply that the behavior of the Borel transform $\tilde{D}(b)$ near $b=2$ is not important for the determination of the value of r_τ . In fact, if we did not factor out the first IR renormalon singularity in Eqs. (42)–(44), the contributions from the $b \sim 2$ region would be very imprecise, thus adversely affecting our analysis. On the other hand, the higher IR renormalons, e.g., at $b=3$, which are not suppressed by powers of α_s , contribute insignificantly to the integral (43), as will be shown below.

VI. PREDICTIONS OF α_s FROM THE HADRONIC TAU DECAY

For the evaluation of Eq. (44), we will employ, at any given choice of ξ^2 , the corresponding N³LO TPS of $R(b(w); \xi^2)$, where we will use for $d_3^{(0)}$ the values around $d_3^{(0)} = 25$ suggested in Sec. IV. The double integral (44) then yields, for any given values of ξ^2 and $a_0 \equiv a(m_\tau^2)$, a specific prediction for r_τ . We then have to adjust, at a given ξ^2 , the value of $a_0 \equiv a(m_\tau^2)$ in such a way that the prediction is within the experimental limits (39). The renormalization scale parameter ξ^2 is then chosen according to the principle of minimal sensitivity,

$$\frac{\partial r_\tau(\xi^2)}{\partial \xi^2} = 0, \quad (50)$$

i.e., at the point in which the unphysical ξ^2 dependence disappears locally.⁴

There is still one minor technical detail that we might worry about; we have only a limited knowledge of the MS beta function $\beta(a)$ that governs the running of the coupling parameter a —its power expansion in a is known only up to the four-loop term $-\beta_0 c_3 a^5$ ($\sim a^5$) [41]. In the region with the low $\mu^2 = \xi^2 m_\tau^2 \exp(iy)$ ($|\mu^2| \sim m_\tau^2 \approx 3 \text{ GeV}^2$) where the contour integration in Eq. (44) is applied, the values of

⁴It is instructive to see why our method should fail at small and large values of ξ^2 . At small ξ^2 the running coupling $a(\xi^2 Q^2)$ becomes large, and so the Borel integral (44) will receive significant contribution from the region far beyond the first IR renormalon, in which the Borel transform cannot be well described by the first few terms of the perturbation theory. On the other hand, at large ξ^2 , the coupling $a(\xi^2 Q^2)$ becomes small, and for the integral (44) to be ξ^2 independent the Borel transform $R(b(w); \xi^2)$ should increase rapidly as ξ^2 increases [in fact, it can be shown that $R(b(w); \xi^2)$ increases approximately as ξ^{2b}]. This means that the Borel transform becomes steeper as ξ^2 increases, making the perturbation theory less efficient. It is therefore reasonable to expect an optimal ξ^2 for our method, and we expect it to be given by the PMS principle.

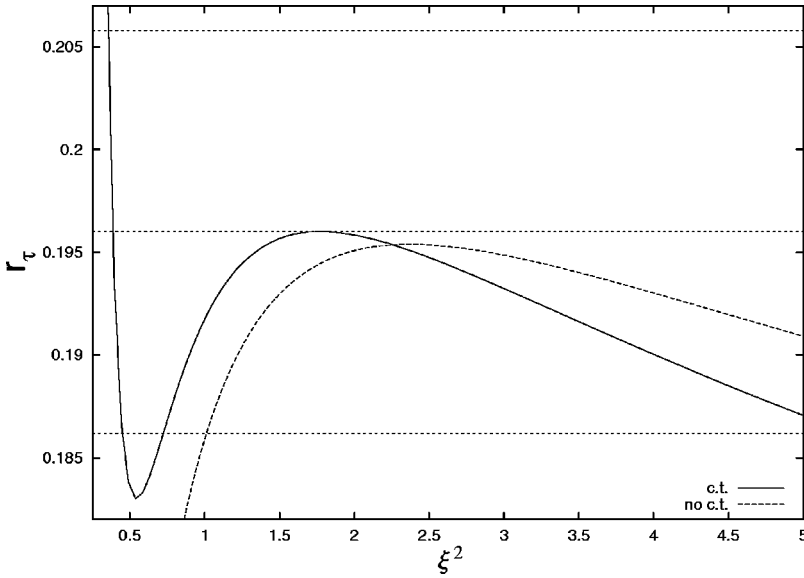


FIG. 3. The value of the predicted ratio $r_\tau \equiv r_\tau^{V+A}(\Delta S=0; m_{u,d}=0)$ of Eq. (44) as a function of the renormalization scale parameter ξ^2 for the choice $\alpha_s(m_\tau^2; \overline{\text{MS}}) = 0.3265$ and $d_3^{(0)} = 25$, when the conformal transformation (25) is employed (full curve; $\phi = 0.1$, i.e., $b_{\text{max}} \approx 3$), and when none is employed (dotted curve; $b_{\text{max}} = 3$). The measured values (40) are included as dotted horizontal lines.

$|a|$ ($\equiv |\alpha_s|/\pi$) are no longer very small ($|a| \approx 0.1$), and expansion terms with powers higher than a^5 may become significant in the resummed value of $\beta(a)$. To be specific, we chose the [2/3] Padé approximant for the resummed $\beta(a)$ in the RG evolution of a , above all because of the reasonable singularity structure of this beta function ($a_{\text{singularity}} = 0.311$).⁵ Later we will show how the results change when (N³LO)TPS β functions are used instead. Further, we chose in Eq. (44) $\phi = 0.1$, i.e., $w_{\text{max}} = \exp(i \times 0.1)$, corresponding to $b_{\text{max}} \approx 3.03$, i.e., well beyond the first IR renormalon.⁶ It turns out that the ξ^2 values as determined by the PMS principle (50) of the expression (44) are $\xi^2 \approx 1.75$ – 1.80 when $d_3^{(0)} = 25$. In Fig. 3 we show the numerical predictions of Eq. (44) as functions of the parameter ξ^2 , for the choice $\alpha_s^{(0)} \equiv \alpha_s(m_\tau^2) = 0.3265$ (and $d_3^{(0)} = 25$). The central experimental value (39) $r_\tau = 0.1960$ is then achieved at the PMS (50) value $\xi^2 \approx 1.77$. We see that the unphysical ξ^2 dependence is really quite weak in a large interval $1 < \xi^2 < 5$, indicating that the method is reliable. In the figure, we include for comparison the analogous predictions for the case (42), i.e., when no conformal transformation $b \mapsto b(w)$ is carried out in Eq. (42) [we used $\varepsilon = 0.005$ and $b_{\text{max}} = 3$ in Eq. (42)]. The latter method has a somewhat different ξ^2 dependence and a slightly different value at the PMS point. As argued after Eq. (42), the predictions of the curve(s) involving the conformal transformation are expected to be more reliable.

The predictions for $\alpha_s(m_\tau^2)$, obtained by matching the results of the described resummation (44) with the experimental results (39), for the choice $d_3^{(0)} = 25$, are

$$\alpha_s(m_\tau^2) = 0.3265 \pm 0.0062_{\text{exp.}} \pm 0.0062_{\text{EW}} \pm 0.0053_{\text{CKM}} \quad (d_3^{(0)} = 25). \quad (51)$$

The perturbative QCD part of the information incorporated in the prediction (51) was the N³LO TPS for the $N_f = 3$ Adler function $D(Q^2)$, with the N³LO coefficient $d_3(\xi^2 = 1) \equiv d_3^{(0)}$ set equal to $d_3^{(0)} = 25$, as obtained by the arguments of Sec. IV. Of course, the exact value of $d_3^{(0)}$ is not yet known. The authors of Ref. [4], using the effective charge (ECH) [20–22] and the TPS principle of minimal sensitivity (PMS) [23,24] methods, predicted $d_3^{(0)} = 27.5$. When considering a one-parameter subgroup $Q^2 \mapsto e^\gamma Q^2$ of the renormalization group, which of course leaves the coefficients of the (ECH) β function $d[D(Q^2)]/d[\ln Q^2] = -\beta_0 d^2(1 + \rho_1 d + \rho_2 d^2 + \dots)$ invariant, the authors of Ref. [44] obtained an estimate $d_3^{(0)} = 30.9$ using a variant of the PMS, and the authors of Ref. [45] obtained $d_3^{(0)} = 28.7$ using a so-called G scheme. Further, when employing the simple [2/1] PA estimate for the NNLO TPS $D(Q^2) = a_0(1 + d_1^{(0)} a_0 + d_2^{(0)} a_0^2)$, at $\mu^2 = Q^2$ ($= m_\tau^2$), the prediction is $d_3^{(0)\text{pr.}} = d_2^{(0)2}/d_1^{(0)} = 24.75$. If using the simple [3/1] PA estimate for the ECH TPS β function $-\beta_0 d^2(1 + \rho_1 d + \rho_2 d^2)$, the prediction is $\rho_3^{\text{pr.}} = \rho_2^2/\rho_1 = 5.39$ and thus⁷ $d_3^{(0)\text{pr.}} = 22.4$. Keeping all these estimates for $d_3^{(0)}$ in mind, as well as the estimate (31) of our approach, it appears reasonable and safe to allow for the following variation of the values of $d_3^{(0)}$ around the value 25 from Sec. IV

$$d_3^{(0)\text{est.}} = 25 \pm 10. \quad (52)$$

⁵This PA choice for $\beta(a)$ was motivated and used in Refs. [42], where a renormalization–scheme- and scale-invariant method was developed and employed for the resummations of NNLO TPS’s of Euclidean massless QCD observables, a generalization of the renormalization-scale-invariant Padé-related method of Refs. [43].

⁶When $w_{\text{max}} = \exp(i\phi)$ in Eq. (44) is varied between $w(b=3)$ ($\phi \approx 0$) and $w(b=4)$ ($\phi \approx 0.505$ rad), the values of Eq. (44) change insignificantly (relative change is about 2.5×10^{-6}).

⁷Note that $\rho_1 = -d_1^{(0)} + \beta_0 \ln m_\tau^2/\bar{\Lambda}^2 = 5.094$ is obtained here by using the (unsubtracted) Stevenson equation [23], with $\alpha_s(m_\tau^2) = 0.33$ and with the [2/3] PA for the $\overline{\text{MS}}$ β function; $\rho_2 = d_2^{(0)2} - d_1^{(0)2} - c_1 d_1^{(0)} + c_2^{\overline{\text{MS}}} = 5.238$. Further, $d_3^{(0)\text{pr.}} = \rho_3^{\text{pr.}} + d_1^{(0)} [2d_2^{(0)} - d_1^{(0)2} + c_1 d_1^{(0)}/2 + \rho_2] - c_3^{\overline{\text{MS}}}/2$.

The ± 10 variation in $d_3^{(0)}$ results in the variation of ∓ 0.0039 for $\alpha_s(m_\tau^2)$, respectively. The ξ^2 values as determined by the PMS principle (50) vary as well $\xi_{\text{PMS}}^2 \approx 2.10, 1.75, 1.35$, for $d_3^{(0)} = 15, 25, 35$, respectively.

In order to obtain an estimate of the various uncertainties in $\alpha_s(m_\tau^2)$ due to the use of the method itself, we proceed in the following way.

One of the major uncertainties is connected with our truncation of $R(b(w), \xi^2)$ to the N³LO TPS. One way of estimating these uncertainties would be to repeat the analysis using N⁴LO TPS for $R(b(w), \xi^2)$ in Eq. (44). For this, we need also the value of the coefficient $d_4^{(0)}$ in $D(Q^2)$. We note that the coefficients $d_j^{(0)}$ in $D(Q^2)$ follow roughly the geometric series pattern with $d_2^{(0)}/d_1^{(0)} \approx d_3^{(0)}/d_2^{(0)} \approx 4$. Therefore, we may estimate $d_4^{(0)} \approx 4d_3^{(0)}$. Using these values of $d_4^{(0)}$ with the $d_3^{(0)}$ values (52), our method gives predictions for $\alpha_s(m_\tau^2)$ which differ from the original (N³LO TPS) method by 0.0012, 0.0007, 0.0003, when $d_3^{(0)} = 15, 25, 35$, respectively. The PMS-determined ξ^2 are in the N⁴LO TPS case $\xi_{\text{PMS}}^2 \approx 3.2, 2.7, 2.15$, respectively. However, if we fix ξ^2 to the PMS-determined values of the original (N³LO TPS) method (2.10, 1.75, 1.35, respectively), then the differences in the predictions for $\alpha_s(m_\tau^2)$ are 0.0035, 0.0025, 0.0020, respectively. Choosing the largest difference here, this would suggest that the truncation uncertainty in our prediction of $\alpha_s(m_\tau^2)$ is about 0.0035.

We may obtain another estimate of the truncation error in the following way. We use for $R(b(w))$ in Eq. (44), instead of the N³LO TPS of the type (48),(49), the corresponding Padé approximant $[2/1](w)$. We expect the most reasonable pole of this PA to be $w_{\text{pole}} \approx 1$, corresponding to $b \approx 3$ (i.e., the second IR renormalon pole). We vary $d_3^{(0)}$, at two fixed values of ξ^2 parameter, $\xi^2 = 1.75$ and 1.95 (i.e., $\approx \xi_{\text{PMS}}^2$ for $d_3^{(0)} = 25, 20$, respectively) in such a way that w_{pole} varies between $w_{\text{pole}} = 1$ and $w_{\text{pole}} = 0.64$. The latter value corresponds to the location of the b pole half way between the first and second renormalon $[b(w=0.64) \approx 2.5]$. The variations of $d_3^{(0)}$ needed for this are $d_3^{(0)} = 23.5-27.0$, and 21.0–25.5, respectively. The variation of the predictions of $\alpha_s(m_\tau^2)$ for such variation of $d_3^{(0)}$, with the use of N³LO TPS and $[2/1]$ PA for $R(b(w), \xi^2)$, is then $\delta\alpha_s(m_\tau^2) = 0.0042, 0.0048$, for the two aforementioned choices of ξ^2 , respectively. This variation (e.g., the larger one, 0.0048) can be regarded as an estimate of the truncation error of our method, especially since the PA $[2/1]_R(w)$ represents a specific realization of the resummation of $R(b(w), \xi^2)$. Since this estimate is larger than the previous one (0.0035), we will use it: $\delta\alpha_s(m_\tau^2)_{\text{tr.}} = 0.0048$.

There is also an uncertainty in the predictions of our method due to possible ambiguities in the choice of the renormalization scale parameter ξ^2 . Our choice was to fix ξ^2 by the local PMS principle (50). Somewhat similarly as we estimated the truncation error, we may now vary ξ^2 instead and keep $d_3^{(0)}$ fixed ($=25$). If we vary ξ^2 from $\xi^2 \approx 1.55$ to $\xi^2 \approx 2.0$, the aforementioned PA $[2/1]_R(w)$ changes its pole from $w_{\text{pole}} = 1$ to $w_{\text{pole}} = 0.64$. The resulting variation in the

predicted (central) values of $\alpha_s(m_\tau^2)$, with the use of N³LO TPS and $[2/1]$ PA for $R(b(w), \xi^2)$, is then about 0.0033. Alternatively, the change of -0.0033 in $\alpha_s(m_\tau^2)$ would correspond to the variation of the renormalization scale parameter $\xi^2 \approx 1.50-4.10$ around its PMS (50) value $\xi^2 \approx 1.77$ when the N³LO TPS approach of Eq. (44) is applied. Very similar results are obtained if $d_3^{(0)} = 20$ is used instead. We will take for the uncertainty due to the ξ^2 ambiguity the value $\delta\alpha_s(m_\tau^2)_{\delta\xi^2} = 0.0033$.

Further, the predictions change when the renormalization scheme parameters c_2 and c_3 change. The leading scheme parameter is c_2 . We have $c_2^{\overline{\text{MS}}} = 4.471$ and $c_3^{\overline{\text{MS}}} = 20.99$ ($N_f = 3$). For comparison, for the N³LO TPS Adler function (with $d_3^{(0)} = 25$) in the TPS PMS scheme [23] we have $c_2^{\text{PMS}} = 6.584$, $c_3^{\text{PMS}} = 36.80$ (and $\xi^2 \approx 0.55$), and in the ECH scheme [20–22] $c_2^{\text{ECH}} = 5.238$, $c_3^{\text{ECH}} = 16.06$ (and $\xi^2 \approx 0.48$). This would indicate that it is reasonable to allow for the variation of the leading scheme parameter c_2 from its $\overline{\text{MS}}$ value by about 50%, i.e., $c_2 = 4.471(1 \pm 0.5)$, while adjusting the renormalization scale parameter ξ^2 according to the PMS condition (50). The central prediction in Eq. (51) then varies by about ± 0.0019 . On the other hand, changing the NNLO scheme parameter c_3 by 50% around its $\overline{\text{MS}}$ value changes the central prediction for $\alpha_s(m_\tau^2)$ by about ± 0.0006 . Adding in quadrature, the uncertainty due to the change of the scheme parameters⁸ is about ± 0.0020 .

Hence, our result is

$$\begin{aligned} \alpha_s(m_\tau^2) &= 0.3265 \pm 0.0062_{\text{exp.}} \pm 0.0062_{\text{EW}} \pm 0.0053_{\text{CKM}} \\ &\quad \pm 0.0039_{\delta d_3} \pm 0.0048_{\text{tr.}} \pm 0.0033_{\delta \xi^2} \pm 0.0020_{\delta c_j}, \end{aligned} \quad (53)$$

$$\begin{aligned} &= 0.3265 \pm 0.0062_{\text{exp.}} \pm 0.0082_{\text{EW} + \text{CKM}} \\ &\quad \pm 0.0073_{\text{meth.}} \end{aligned} \quad (54)$$

In the last line, we added the corresponding uncertainties in quadrature; the method uncertainty contains the uncertainties due to the variation of $d_3^{(0)}$, truncation error, and renormalization scale and scheme ambiguities. If we use for the $\overline{\text{MS}}$ β function the N³LO TPS, instead of $[2/3]_\beta$, the predictions in Eqs. (53),(54) decrease by 0.0006–0.0007, indicating that those nonperturbative effects which originate in the behavior of the β function are not strong in the applied resummation method. This has to do with relatively large values of the PMS-fixed (50) renormalization scale parameters $\xi^2 \equiv \mu^2/m_\tau^2$ (≈ 1.77 when $d_3^{(0)} = 25$). The uncertainty due to the

⁸The problem of the renormalization scale and scheme dependence in the determination of $\alpha_s(m_\tau^2, \overline{\text{MS}})$ from the y -contour representation (34) of r_τ was discussed by Rączka [46]. Using the NNLO TPS for $D(Q^2)$, he showed that a change from the $\overline{\text{MS}}$ scheme (with $\xi^2 = 1$) to the TPS PMS scheme and scale results in the change of $\alpha_s(m_\tau^2, \overline{\text{MS}})$ by 0.01, which is significant in the view of the new precise experimental data.

variation of b_{\max} in Eq. (43) ($3 < b_{\max} < 4$, i.e., $0 < \phi < 0.505$ rad) turns out to be insignificant, as mentioned before. For example, if changing from $\phi = 0.1$ (corresponding to $b_{\max} \approx 3.03$) to $\phi = 0.505$ ($b_{\max} \approx 4$) the central value of Eq. (54) increases by less than 10^{-6} .

One may argue that the method uncertainty as given above is too nonconservative, i.e., too small. Therefore, we carried out an additional cross check. We performed the resummation for r_τ by the double integration of the type (44), but this time taking into account explicitly the first UV renormalon (at $b = -1$) in the ansatz for the Borel transform $\bar{D}(b) = \bar{R}(b)(1+b)^{-\gamma_1}(1-b/2)^{-(1+\nu)}$, with $\gamma_1 = 2.589$ [47,15]. We performed again the conformal mapping $b = b(w)$ (25), expansion of $\bar{R}(b(w))$ in powers of w up to and including the N³LO ($\sim w^3$), and subsequently performed the double integration analogous to Eq. (44), with $\phi = 0.1$. The scale parameter ξ^2 was again fixed by the PMS principle (50), resulting, for the choices $d_3^{(0)} = 15, 25, 35$ [and $\alpha_s(m_\tau^2) \approx 0.32-0.34$] in considerably lower values $\xi^2 \approx 0.91, 0.88, 0.85$, respectively. We used for the $\overline{\text{MS}}$ β functions again the PA [2/3]. The predicted values of the QCD coupling parameter turned out to be very close to those (53),(54) of the method (44): $\alpha_s(m_\tau^2) = 0.3257 \pm 0.0062_{\text{exp.}} \pm 0.0062_{\text{EW}} \pm 0.0054_{\text{CKM}} \pm 0.0014_{\delta d_3}$. For example, the prediction for $\alpha_s(m_\tau^2)$ corresponding to the central experimental value of Eq. (39) with this method differed from the prediction of the method (44) by -0.0027 , -0.0008 , $+0.0018$, when $d_3^{(0)} = 15, 25, 35$, respectively. This would indicate again that the resummation method uncertainty does not surpass 0.0073 , i.e., in accordance with the method uncertainty estimate in Eq. (54).

If we apply the resummation (43) *without* the conformal transformation [using $b_{\max} \approx 3$, $\varepsilon = 0.005$ in Eq. (42)], for $d_3^{(0)} = 15, 25, 35$, the PMS-fixed (50) renormalization scale parameters are $\xi_{\text{PMS}}^2 \approx 3.00, 2.35, 1.25$, respectively,⁹ and the prediction is $\alpha_s(m_\tau^2) = 0.3271 \pm 0.0062_{\text{exp.}} \pm 0.0062_{\text{EW}} \pm 0.0053_{\text{CKM}} \pm 0.0060_{\delta d_3}$, which¹⁰ is only slightly different from the one with the conformal transformation (51)–(53). Although not using the conformal transformation is not so well motivated [see also the discussion after Eq. (42)], this result may represent yet another justification for the small estimate of the method uncertainty in Eqs. (53),(54).

The result (54) was then evolved from the scale $\mu = m_\tau \approx 1.777$ GeV to the canonical scale $M_z = 91.19$ GeV, by using the RG equation with the $[2/3]_{\overline{\text{MS}}}$ Padé approximant (based on the four loop $\beta_{\overline{\text{MS}}}$ [41]) and the three loop matching conditions [48] for the flavor thresholds. We used the matching at $\mu(N_f) = \kappa m_q(N_f)$ with the choice $\kappa = 2$, where $\mu(N_f)$ is the scale above which N_f flavors are assumed active, and $m_q(N_f)$ means the running quark mass $m_q(m_q)$ of

the N_f th flavor [we assumed $m_c(m_c) = 1.25$ GeV and $m_b(m_b) = 4.25$ GeV]. This leads to our final result

$$\alpha_s(M_z^2) = 0.1193 \pm 0.0007_{\text{exp.}} \pm 0.0010_{\text{EW+CKM}} \pm 0.0009_{\text{meth.}} \pm 0.0003_{\text{evol.}}, \quad (55)$$

$$= 0.1193 \pm 0.0015. \quad (56)$$

In Eq. (56), we added the uncertainties of Eq. (55) in quadrature. In Eq. (55), we included the uncertainty $\pm 0.0003_{\text{evol.}}$ due to the RG evolution from m_τ to M_z . This uncertainty estimate is obtained in the following way: keeping $\kappa = 2$, if we vary the mass $m_c(m_c) = 1.25 \pm 0.10$ GeV, the resulting uncertainty is ± 0.0002 ; if we vary the mass $m_b(m_b) = 4.25 \pm 0.15$ GeV, the uncertainty is ± 0.0001 . If we vary the flavor threshold parameter κ around its central value $\kappa = 2$ from 1.5 to 3, the uncertainty is ± 0.0001 . Furthermore, if we use for the ($m_\tau \rightarrow M_z$) RG evolution, instead of the PA $[2/3]_{\overline{\text{MS}}}$, the corresponding four-loop TPS $\beta_{\overline{\text{MS}}}$ function, the resulting $\alpha_s(M_z^2)$ changes by 0.0001 . Adding all these uncertainties in quadrature gives us approximately the uncertainty ± 0.0003 given in Eq. (55).

If we repeat the entire calculation of $\alpha_s(m_\tau^2)$ and $\alpha_s(M_z^2)$ by using throughout the four loop TPS $\overline{\text{MS}}$ β function instead of the PA $[2/3]_{\overline{\text{MS}}}$, the predictions for $\alpha_s(M_z^2)$ remain the same as in Eqs. (55),(56), up to the displayed digits. The reason for this is that the aforementioned change of β functions predicts the values of $\alpha_s(m_\tau^2)$ by about $0.0006-0.0007$ lower than those of Eqs. (53),(54), but then the RG evolution to $\mu = M_z$ pushes the results up, thus approximately neutralizing this effect.

Due to the high precision experimental data Eqs. (36)–(39) on the inclusive hadronic decay of τ , the experimental uncertainty in the extracted strong coupling constant is low. By incorporating a wealth of known theoretical information (perturbative as well as renormalon) on the related Adler function $D(Q^2)$, we were able to extract the strong coupling constant with the method uncertainty not significantly surpassing the experimental uncertainty. Further, the analysis by the ALEPH Collaboration [35,37] showed that those power (nonperturbative) contributions in the observable $R_\tau^{\text{V+A}}(\Delta S = 0)$ which do not originate from the nonzero quark masses are consistent with zero (see also the next section), and these OPE-type contributions were not included in our resummation either.

The experimental situation with other low energy QCD observables is not so favorable, and the experimental uncertainties of the extracted strong coupling constants appear to dominate over the theoretical uncertainties. This is reflected in the present world average (over various measured QCD observables) $\alpha_s^{\overline{\text{MS}}}(M_z^2) = 0.1173 \pm 0.0020$ by Ref. [49] and 0.1184 ± 0.0031 by Ref. [50], where the extracted (combined experimental and theoretical) uncertainties are significantly higher than those in Eq. (55).

In this context, we mention that the question of the violation of the quark(gluon)-hadron duality for correlation functions has been raised and investigated by the authors of Refs. [51]. They argued that the corrections to the correlation func-

⁹For $d_3^{(0)} = 35$, no strict stationarity is achieved, but at $\xi^2 \approx 1.25$ the slope (50) is almost zero: $\partial r_\tau(\xi^2)/\partial \xi^2 \approx -2.3 \times 10^{-4}$.

¹⁰The variation $\pm 0.0060_{\delta d_3}$ for $d_3^{(0)} = 25 \pm 10$ is in fact $+0.0043$ for $d_3^{(0)} = 15$ and -0.0060 for $d_3^{(0)} = 35$.

tions due to the duality violation could be significant (up to a few percent). However, no quantitative analyses are available at present. This violation could possibly affect many QCD (quasi)observables, including the Adler function $D(Q^2)$ and R_τ .

Further, the authors of Refs. [52,53] analyzed the possibility that the operator product expansion (13) contains an additional $1/Q^2$ term (other than the $d=2$ quark mass terms), whose origin would be an effective tachyonic gluon mass reflecting short-distance nonperturbative QCD effects. The authors of Ref. [52] suggested that such terms would decrease the value of $\alpha_s(m_\tau^2)$ extracted from hadronic τ decays by about 10%. However, the authors of Ref. [54] showed that the coefficient of the $1/Q^2$ term is consistent with zero. They did this by fitting a dimension-two finite energy sum rule to the new ALEPH data on the vector and axial-vector spectral functions extracted from measured τ decays. The type of sum rule used by the authors of Ref. [54] in ruling out the aforementioned $1/Q^2$ term are well satisfied at the continuum threshold scales $s_0 \approx 2-3 \text{ GeV}^2$ relevant for r_τ , as has been shown independently by two groups [55,56].

VII. COMPARISON WITH OTHER ANALYSES

We may compare our result (54) with that of an independent analysis of the hadronic τ decays by the ALEPH Collaboration [35,37], who used for $R_\tau^{\text{V+A}}(\Delta S=0)$ instead of the values (36), the different values available in 1998 (3.492 ± 0.016),

$$\alpha_s(m_\tau^2) = 0.334 \pm 0.007_{\text{exp.}} \pm 0.021_{\text{th.}} \quad (\text{ALEPH}), \quad \Rightarrow \quad (57)$$

$$\alpha_s(M_Z^2) = 0.1203 \pm 0.0008_{\text{exp.}} \pm 0.0025_{\text{th.}} \pm 0.0003_{\text{evol.}} \quad (\text{ALEPH}). \quad (58)$$

They used slightly smaller uncertainties for $\delta_{\text{EW}} (\pm 0.0040)$ and drastically smaller uncertainties in the CKM element $|V_{ud}| = 0.9752 \pm 0.0007$. They used different methods which involved, in addition, the analysis of moments of (their own measured) spectral functions $\text{Im} \Pi_{ud,VA}^{(J)}(s)$ ($s \leq m_\tau^2$) as proposed by [57]. The ALEPH's V + A analysis of the mentioned moments showed that those nonperturbative (OPE) contributions which do not originate from the nonzero quark masses were consistent with zero, $\delta r_\tau(\text{NP}, m_{u,d}=0) = 0.000 \pm 0.004$; and their quark mass nonperturbative contributions basically agree with ours (A11) (compare Table 8, fourth column, of Ref. [35]). Further, the central value of Eq. (57) was obtained by taking the arithmetic average of the predictions of two methods (i) the y -contour integration approach (34), with just the N³LO TPS for the Adler $D(Q^2)$ function, with $d_3^{(0)} = 50 \pm 50$, in $\overline{\text{MS}}$, i.e., the approach of [30]; (ii) the simple N³LO TPS of the power expansion of r_τ . The large theoretical uncertainty in Eq. (57) originates primarily from the difference of the predictions of the two aforementioned methods, from the ambiguities of the choice of $d_3^{(0)}$ renormalization scheme and scale, and the electroweak parameter δ_{EW} .

The y -contour integration approach, where we just use the N³LO TPS for the Adler function $D(Q^2)$ (with $d_3^{(0)} = 25$) in the contour integral (34), i.e., the approach of [30], in $\overline{\text{MS}}$ scheme with N³LO TPS $\beta_{\overline{\text{MS}}}$, achieves the minimal sensitivity condition (50) at $\xi^2 \approx 0.4$ and reproduces there the central experimental value of Eq. (39) at $\alpha_s(m_\tau^2) = 0.3399$ (0.3416 if using $[2/3]_{\overline{\text{MS}}}$), corresponding to $\alpha_s(M_Z^2) = 0.1209$ (0.1210), significantly higher¹¹ than our central value (55). However, if taking instead the simple TPS $r_\tau = a[1 + (d_1^{(0)} + 3.563)a + (d_2^{(0)} + 19.99)a^2 + (d_3^{(0)} + 78.)a^3]$ (see [30]), at renormalization scale $\mu = m_\tau$, the predictions change significantly: $\alpha_s(m_\tau^2) = 0.3211 \pm 0.0056_{\text{exp.}} \pm 0.0056_{\text{EW}} \pm 0.0048_{\text{CKM}} \pm 0.0011_{\delta d_3}$ (for $d_3^{(0)} = 25 \pm 10$), corresponding to $\alpha_s(M_Z^2) = 0.1188 \pm 0.0007_{\text{exp.}} \pm 0.0010_{\text{EW+CKM}} \pm 0.0002_{\delta d_3}$. These values are lower than our values (55). The arithmetic average of the central values of these two methods $\alpha_s(m_\tau^2)_{\text{arithm.}} = (0.3399 + 0.3211)/2 = 0.3305$ is close to the central value of the ALEPH analysis (57), the difference being that the ALEPH used the higher values for the hadronic τ decay ratio available in 1998. This value is close to the upper bounds of our prediction (54).

In Fig. 4 we present the predictions of three methods for r_τ as functions of $\alpha_s(m_\tau^2)$: (i) our method (44) [Borel transform approach (BTA), in $[2/3]_\beta \overline{\text{MS}}$ scheme]; (ii) the aforementioned method of the y -contour approach for the N³LO TPS Adler function (CATPS) (with ‘‘PMS’’ $\xi^2 = 0.40$, in N³LO TPS $\beta \overline{\text{MS}}$ scheme); (iii) the N³LO TPS of r_τ (with $\mu^2 = m_\tau^2$). The values (52) were taken for the N³LO coefficient $d_3^{(0)} \equiv d_3(\mu^2 = Q^2, \overline{\text{MS}})$. On the x axis, we denoted the corresponding values of $\alpha_s(m_\tau^2)$ when the predicted values of the three methods, with $d_3^{(0)} = 25$, reach the central experimental value (39) $r_\tau = 0.1960$.

In addition, we included on the x axis of the figure the central value of $\alpha_s(m_\tau^2)$ predicted by the (NNLO) ECH approach [20–22] applied directly to the NNLO TPS of r_τ (using the NNLO TPS β functions of the ECH and $\overline{\text{MS}}$ schemes $c_3^{\text{ECH}}, c_3^{\overline{\text{MS}}} \rightarrow 0$), i.e., the approach applied in Ref. [59]. The authors of Ref. [59] used the input values $r_\tau = 0.2030 \pm 0.0070_{\text{exp.}}$, which yield $\alpha_s(m_\tau^2) = 0.3184$

¹¹This contour approach, in $\overline{\text{MS}}$ scheme, was also applied in Ref. [58]. On the other hand, if we apply in this approach the N³LO TPS Adler function in the ECH renormalization scale and scheme ($\xi_{\text{ECH}}^2 = 0.482483$; $c_2^{\text{ECH}} = 5.23783$; $c_3^{\text{ECH}} = 16.0613 \pm 10$ for $d_3^{(0)} = 25 \pm 10$), and using the N³LO TPS (or $[2/3]$ Padé approximants) for the β functions, we obtain from Eq. (39) similar results:

$$\alpha_s(m_\tau^2, \overline{\text{MS}}) = 0.3400 \pm 0.0080_{\text{exp.}} \pm 0.0080_{\text{EW}} \pm 0.0069_{\text{CKM}} \pm 0.0034_{\delta d_3}$$

$$(0.3413 \pm 0.0083_{\text{exp.}} \pm 0.0083_{\text{EW}} \pm 0.0071_{\text{CKM}} \pm 0.0034_{\delta d_3}),$$

corresponding to

$$\alpha_s(M_Z^2, \overline{\text{MS}}) = 0.1210 \pm 0.0009_{\text{exp.}} \pm 0.0012_{\text{EW+CKM}}$$

$$\pm 0.0004_{\delta d_3} \pm 0.0003_{\text{evol.}}$$

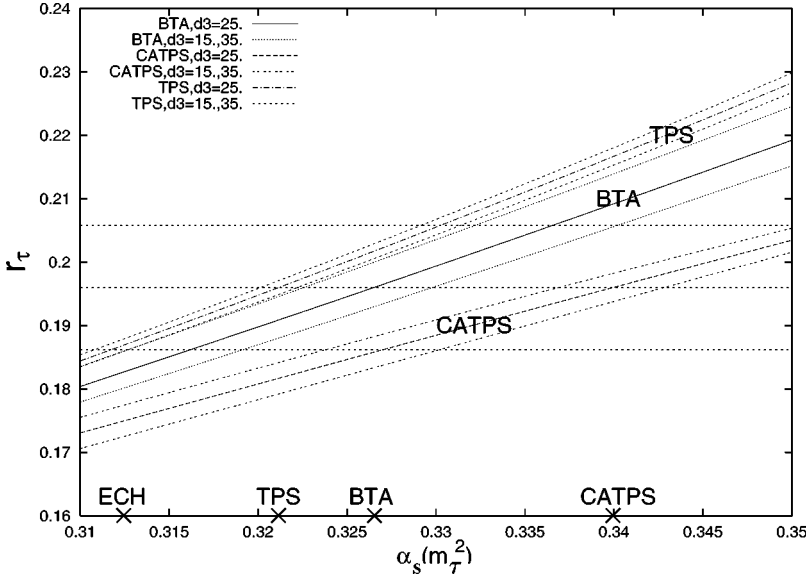


FIG. 4. The values of the predicted ratio $r_\tau \equiv r_\tau^{V+A}(\Delta S=0; m_{u,d}=0)$ as functions of $\alpha_s(m_\tau^2, \overline{\text{MS}})$ from various methods: our Borel transform approach of Eq. (44) [BTA, with $\phi = 0.1$ and PMS condition (50)]; the contour approach method using the N³LO TPS for the Adler function (CATPS, with the “PMS” $\xi^2=0.40$); and the fixed N³LO TPS evaluation of r_τ (TPS, at $\mu^2=m_\tau^2$). The uncertainties due to $d_3^{(0)}=25 \pm 10$ are included. The measured values (40) are included as dotted horizontal lines. On the x axis, we denoted the values of $\alpha_s(m_\tau^2, \overline{\text{MS}})$ of these three methods (with $d_3^{(0)}=25$), for which the central measured value $r_\tau=0.1960$ is obtained. In addition, we included the analogous prediction of the (NNLO) ECH method when applied to the fixed NNLO TPS of r_τ .

$\pm 0.0060_{\text{exp.}}$ and $\alpha_s(M_Z^2)=0.1184 \pm 0.0007_{\text{exp.}}$ ¹² Using our updated input values (39) for r_τ , this method predicts $\alpha_s(m_\tau^2)=0.3124 \pm 0.0052_{\text{exp.}} \pm 0.0052_{\text{EW}} \pm 0.0045_{\text{CKM}}$ (independent of $d_3^{(0)}$ since it is NNLO method), corresponding to $\alpha_s(M_Z^2)=0.1177 \pm 0.0007_{\text{exp.}} \pm 0.0009_{\text{EW+CKM}}$. However, this ECH method, applied directly to r_τ , appears to be unstable under the inclusion of the N³LO information, above all because the ECH renormalization scale parameter for r_τ is dangerously low: $\xi_{\text{ECH}}^2 \equiv \mu^2/m_\tau^2 \approx 0.10$. The N³LO ECH approach to r_τ , with $d_3^{(0)}=25 \pm 10$ (and using N³LO TPS β functions of the ECH and $\overline{\text{MS}}$ schemes) thus gives very different results: $\alpha_s(m_\tau^2)=0.3373 \pm 0.0079_{\text{exp.}} \pm 0.0079_{\text{EW}} \pm 0.0068_{\text{CKM}} \pm 0.0192_{\delta d_3}$, corresponding to $\alpha_s(M_Z^2)=0.1207 \pm 0.0009_{\text{exp.}} \pm 0.0012_{\text{EW+CKM}} \pm 0.0020_{\delta d_3}$.

The authors of [60] used the diagonal [2/2] Padé approximation to resum the N³LO TPS of r_τ (with $\xi^2=1$), where the N³LO coefficient r_3 of the series was determined by the asymptotic Padé approximant method (APAP) [61]. They obtained $\alpha_s(m_\tau^2)=0.314 \pm 0.010$. The central value is significantly lower than our prediction (53), although they used for the input the values $r_\tau=0.2048 \pm 0.0129$ where the central value is considerably higher than that of our input values (39).

The results of the methods by both groups of authors [59,60] thus give in general lower predictions for α_s than our method and the central value of the ALEPH method, as also seen in Fig. 4. We wish to point out, however, that both groups of authors of Refs. [59,60] applied resummation methods directly to the observable r_τ , which is Minkowskian ($q^2=m_\tau^2 > 0$). We applied our resummation to the (Borel transform of the) predominantly non-

Minkowskian quantity $D(Q^2)$, i.e., we used the y -contour representation (34).¹³ Various authors [4,62,63] have suggested that resummation techniques to (quasi)observables be used in the non-Minkowskian regions, because the physical singularities appear on the Minkowskian axis ($q^2 \equiv -Q^2 > 0$).

The reason that the predictions of our method differ significantly from those of the simpler CATPS y -contour approach lies in the apparently important role of the $b=2$ IR renormalon singularity of $\tilde{D}(b)$ [see the ansatz in Eqs. (41)–(44)] for the quasianalytic continuation of $\tilde{D}(b)$ and of the Adler function $D(Q^2)$, and consequently for the resummation of r_τ via the y -contour integration. This is so despite the fact that the y -contour integration leads to a suppression of the contributions from $b \approx 2$ (see also the last paragraph of Sec. V).

VIII. SUMMARY

We presented a new method of determination of the N³LO coefficient $d_3^{(0)}$ of the Adler function $D(Q^2)$. The method makes use of the known radiative correction to the $1/Q^4$ term in the operator product expansion of $D(Q^2)$. By requiring that the Q dependence of the ambiguity induced by the first nonzero infrared renormalon of $D(Q^2)$ ($b=2$) is the same as the Q dependence of the OPE term $\mathcal{C}_4(a(Q^2))/Q^4$, the exact condition (28) is obtained, which involves the Borel transform of $D(Q^2)$. This condition, in principle, would determine the coefficient $d_3^{(0)}$. However, this condition has to be evaluated at a relatively large value $b=2$ of the Borel variable, and the present knowledge of only two terms beyond the leading order leads to significant uncertainties in the evaluation of this condition. We solved this practical problem by applying judicious conformal transformations $b=b(w)$ and Padé resummation techniques, thus

¹²The evolution uncertainty $\pm 0.0006_{\text{evol.}}$ given in Ref. [59] is larger than ours in Eq. (55), possibly because they used a lower threshold parameter $\kappa=1$, while we used $\kappa_{\text{central}}=2$ and $\kappa=1.5-3$.

¹³The problematic Minkowskian region contribution $q^2 \equiv -Q^2 > 0$ ($y = \pm \pi$) in the contour integral (34) is suppressed by the third power, i.e., by the factor $(1 + e^{iy})^3$.

improving the convergence properties. The resulting value is then $d_3^{(0)} \approx 25 \pm 5$ (at $N_f=3$), but uncertainties of up to $\delta d_3^{(0)} = \pm 10$ cannot be entirely excluded and were used in the subsequent analyses of the τ inclusive hadronic decay ratio.

We wish to emphasize that our determination of $d_3^{(0)}$ is fundamentally different from the previous estimates in the literature. The latter estimates were mainly based on re-expanding the resummed (quasianalytically continued) expressions for $D(Q^2)$ in powers of the coupling parameter, thus relying on the assumption that a quasianalytic continuation of the NNLO truncated perturbation series of $D(Q^2)$ was efficient. However, this may only be true if the main contribution to the coefficient $d_3^{(0)}$ comes from those higher order Feynman diagrams which do not have new topological structures in comparison with the lower order diagrams contributing to $d_2^{(0)}$ [4,62,64]. In contrast, our relation (28), and its evaluation, are not based just on the knowledge of the NNLO truncated perturbation series, but also on the knowledge of the first nonzero infrared renormalon including its first radiative correction. Therefore, it is possible that the resummations of the expressions of Eq. (28) do not suffer from the uncertainties about the topologies of the Feynman diagrams.

We then used the obtained $d_3^{(0)}$, and the structure of the Borel transform $\tilde{D}(b)$ of $D(Q^2)$ near the first infrared renormalon at $b=2$, and an optimal conformal transformation, to evaluate the τ inclusive hadronic decay ratio R_τ , or more specifically its massless QCD reduced version r_τ , via the contour integration method. Comparing the obtained predictions with the precise experimental data available now, we obtained the prediction (55) for $\alpha_s(M_Z^2)$, where the estimated uncertainties from the method (and RG evolution) do not surpass significantly the uncertainties from the experimental data. All the uncertainties in Eqs. (55),(56) are significantly lower than the uncertainties in the present world average $\alpha_s^{\overline{\text{MS}}}(M_Z^2) = 0.1173 \pm 0.0020$ by Ref. [49] and 0.1184 ± 0.0031 by Ref. [50]. Furthermore, the central value (55) is by 0.0020 and 0.0009 higher than these two world averages.

In view of the present high precision experimental data for the R_τ decay ratio, we believe that the values of $\alpha_s(M_Z^2)$ deduced from it should eventually serve as the reference value for future tests of QCD via the experimental measurements and theoretical analyses of other QCD observables. For this, the theoretical EW correction factor to R_τ should be investigated further, and the present uncertainties in the value of the CKM element $|V_{ud}|$ should be reduced.

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APPENDIX: SUBTRACTING THE QUARK MASS EFFECTS

In order to be able to apply the massless QCD approach to our analysis, we have to subtract the quark mass ($m_{u,d} \neq 0$)

contributions from the reduced hadronic τ decay width $r_\tau^{V+A}(\Delta S=0)$. The general expression for the latter quantity, including the quark mass effects, is (see, for example, Refs. [13,31])

$$\begin{aligned} r_\tau^{V+A}(\Delta S=0) &\equiv \frac{R_\tau^{V+A}(\Delta S=0)}{3|V_{ud}|^2(1+\delta_{\text{EW}})} - (1+\delta'_{\text{EW}}) \\ &= (-\pi i) \int_{|s|=m_\tau^2} \frac{ds}{s} \left(1 - \frac{s}{m_\tau^2}\right)^3 \\ &\quad \times \left[\left(1 + \frac{s}{m_\tau^2}\right) D^{L+T}(-s) + \frac{4}{3} D^L(-s) \right] \\ &\quad - 1, \end{aligned} \quad (\text{A1})$$

where the contour integration is counterclockwise in the complex s plane, and the general Adler functions D^{L+T} and D^L are expressed with the current-current correlation functions

$$D^{L+T}(-s) = -s \frac{d}{ds} \sum_{J=0,1} [\Pi_{ud,V}^{(J)}(s) + \Pi_{ud,A}^{(J)}(s)], \quad (\text{A2})$$

$$D^L(-s) = \frac{s}{m_\tau^2} \frac{d}{ds} \{s[\Pi_{ud,V}^{(0)}(s) + \Pi_{ud,A}^{(0)}(s)]\}, \quad (\text{A3})$$

where $\Pi_{ud,V/A}^{(J)}$ are components in the Lorentz decomposition

$$\begin{aligned} \Pi_{ud,V/A}^{\mu\nu}(q) &= (-g^{\mu\nu}q^2 + q^\mu q^\nu) \Pi_{ud,V/A}^{(1)}(q^2) \\ &\quad + q^\mu q^\nu \Pi_{ud,V/A}^{(0)}(q^2) \end{aligned} \quad (\text{A4})$$

of the two-point correlation functions $\Pi_{ud,V/A}^{\mu\nu}$ of the vector $V_{ud}^\mu = \bar{d}\gamma^\mu u$ and axial-vector $A_{ud}^\mu = \bar{d}\gamma^\mu \gamma_5 u$ (color-singlet) currents

$$-i\Pi_{ud,V}^{\mu\nu}(q) = \int d^4x e^{iq \cdot x} \langle 0 | T \{ V_{ud}^\mu(x) V_{ud}^\nu(0)^\dagger \} | 0 \rangle, \quad (\text{A5})$$

$$-i\Pi_{ud,A}^{\mu\nu}(q) = \int d^4x e^{iq \cdot x} \langle 0 | T \{ A_{ud}^\mu(x) A_{ud}^\nu(0)^\dagger \} | 0 \rangle. \quad (\text{A6})$$

In the massless quark limit ($m_{u,d} \rightarrow 0$), $D^L(s)$ vanishes, the perturbative vector and axial-vector contributions in D^{L+T} become equal and $D^{L+T}(-s) \rightarrow [1 + D(-s)]/(2\pi^2)$, where $D(Q^2)$ is the canonically normalized massless Adler function (2) with the perturbative expansion (5).¹⁴ In order to apply the massless QCD analysis to the measured observable (A1), we have to subtract from it the quark mass ($m_{u,d} \neq 0$) contributions. These are largely the $m_\pi \neq 0$ contribu-

¹⁴Usually in the literature (e.g., see Refs. [31]), the (ud) Adler functions D^{L+T} and D^L (A2),(A3) include by convention the additional CKM factor $|V_{ud}|^2$.

tions from the pion (π^-) pole. The pion pole contributes to the axial-current correlation functions $\Pi_{ud,A}^{(J)}(s)$. Using PCAC, these contributions can be obtained and they lead to the corresponding contributions in the Adler functions $D^L(-s; \pi) = 2f_\pi^2 m_\pi^2 s / (s - m_\pi^2)^2 / m_\tau^2$ and $D^{L+T}(-s; \pi) = -2f_\pi^2 s / (s - m_\pi^2)^2$. This leads, via Eq. (A1), to the following estimate of the pion pole contribution to $r_\tau^{V+A}(\Delta S=0)$:

$$r_\tau^{V+A}(\Delta S=0; \pi) = \frac{8\pi^2 f_\pi^2}{m_\tau^2} \left(1 - \frac{m_\pi^2}{m_\tau^2}\right)^2 \approx 0.2135 \times (1 - 0.0123) \approx 0.2109. \quad (\text{A7})$$

Here, we employed the known values [34] $f_\pi = 92.4 \pm 0.3$ MeV, $m_{\pi^-} = 139.6$ MeV, $m_\tau = 1777$ MeV. In order to check whether the framework leading to Eq. (A7) is realistic, we may calculate from here the branching ratio for $\tau^- \rightarrow \pi^- \nu_\tau$

$$B(\tau^- \rightarrow \pi^- \nu_\tau) = \frac{\Gamma(\tau^- \rightarrow \pi^- \nu_\tau)}{\Gamma(\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau)} B(\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau) = R_\tau(\pi) B_e \approx 3 |V_{ud}|^2 r_\tau^{V+A}(\Delta S=0; \pi) \approx 0.1072, \quad (\text{A8})$$

where we used for the branching ratio $B_e \equiv B(\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau)$ the middle value of the world average [34] $B_e = 0.1783$, and $|V_{ud}| = 0.9749$. On the other hand, the measured branching ratio for $\tau^- \rightarrow \pi^- \nu_\tau$ is $B(\tau^- \rightarrow \pi^- \nu_\tau) = 0.1109 \pm 0.0012$ [34]. The value (A8), obtained from the PCAC-motivated approach (A7), thus differs by less than 4% from the actual prediction.

We can now read from the expression (A7) the quark mass ($m_{u,d} \neq 0$, i.e., $m_\pi \neq 0$) contribution to $r_\tau^{V+A}(\Delta S=0)$,

$$\delta r_\tau^{V+A}(\Delta S=0)_{m_\pi \neq 0} = -\frac{16\pi^2 f_\pi^2 m_\pi^2}{m_\tau^4} \left(1 - \frac{m_\pi^2}{2m_\tau^2}\right) \approx -0.0026. \quad (\text{A9})$$

However, we can go somewhat beyond the approximation made so far in calculating this contribution. In the operator product expansion approach to R_τ ratio, as given in [13], the largest quark mass contributions are of dimension $d=4$ ($\propto 1/m_\tau^4$, quark condensate contributions)

$$\delta r_\tau^{V+A}(\Delta S=0)_{m_{u,d} \neq 0} \approx 16\pi^2 \frac{(m_u + m_d) \langle \bar{q}q \rangle}{m_\tau^4} \left[1 + \frac{23}{8} \left(\frac{\alpha_s(m_\tau^2)}{\pi}\right)^2\right] \quad (\text{A10})$$

$$\approx -\frac{16\pi^2 f_\pi^2 m_\pi^2}{m_\tau^4} \left[1 + \frac{23}{8} \left(\frac{\alpha_s(m_\tau^2)}{\pi}\right)^2\right] \approx -0.0027. \quad (\text{A11})$$

In Eq. (A10) we denoted $\langle \bar{q}q \rangle \equiv \langle \bar{u}u \rangle \approx \langle \bar{d}d \rangle$. The renormalization scale in this quantity and in m_u and m_d in Eqs. (A10),(A11) can be taken to be $\mu \approx m_\tau$. In Eq. (A11) we took into account the PCAC relation $(m_u + m_d) \langle \bar{q}q \rangle \approx -f_\pi^2 m_\pi^2$. There are corrections to the expression (A11) of the order $\sim m_{u,d}^2/m_\tau^2$, i.e., of the order of the OPE $d=2$ terms which can reach, at most, values $\sim 10^{-4}$. Comparing the previous pion pole expression (A9) with the OPE expression (A11), we see that the latter apparently represents a slight improvement since it includes the radiative corrections. In obtaining the number (A11), we further used the value $\alpha_s(m_\tau^2, \text{MS}) \approx 0.32$.

The OPE approach of [13] includes other nonperturbative terms contributing to r_τ , which do not stem from quark masses, the $d=4$ gluon condensate and the $d=6$ term. The latter term could be large, but it also has comparably large uncertainties [13]. The gluon condensate contribution to r_τ in the OPE approach is α_s suppressed. The ALEPH analysis [35] indicates that these $d=4,6$ nonperturbative contributions are consistent with the value zero.

When subtracting the quark mass contributions (A10),(A11) from Eq. (A1), we end up with the massless QCD observable

$$r_\tau \equiv r_\tau^{V+A}(\Delta S=0; m_{u,d}=0) = r_\tau^{V+A}(\Delta S=0) - \delta r_\tau^{V+A}(\Delta S=0)_{m_{u,d} \neq 0} \quad (\text{A12})$$

$$= -\frac{i}{2\pi} \int_{|s|=m_\tau^2} \frac{ds}{s} \left(1 - \frac{s}{m_\tau^2}\right)^3 \left(1 + \frac{s}{m_\tau^2}\right) D(-s), \quad (\text{A13})$$

where the integration is counterclockwise, and the canonically normalized massless Adler function $D(Q^2 \equiv -s)$ (2), (5) was introduced according to the aforementioned limiting procedure: $D^L(s) \rightarrow 0$ and $D^{L+T}(-s) \rightarrow [1 + D(-s)]/(2\pi^2)$ (when $m_{u,d} \rightarrow 0$).

- [1] G. Parisi, Phys. Lett. **76B**, 65 (1978); Nucl. Phys. **B150**, 163 (1979).
 [2] G. 't Hooft, in *The Whys of Subnuclear Physics*, Erice, 1977, edited by A. Zichichi (Plenum, New York, 1977).
 [3] F. David, Nucl. Phys. **B234**, 237 (1984); **B263**, 637 (1986).
 [4] A. L. Kataev and V. V. Starshenko, Mod. Phys. Lett. **A 10**,

- 235 (1995).
 [5] M. Beneke and V. M. Braun, Phys. Lett. B **348**, 513 (1995); P. Ball, M. Beneke, and V. M. Braun, Nucl. Phys. **B452**, 563 (1995).
 [6] M. Neubert, Nucl. Phys. **B463**, 511 (1996).
 [7] C. N. Lovett-Turner and C. J. Maxwell, Nucl. Phys. **B452**, 188

- (1995).
- [8] T. Lee, Phys. Rev. D **56**, 1091 (1997); Phys. Lett. B **462**, 1 (1999).
- [9] A. H. Mueller, Nucl. Phys. **B250**, 327 (1985).
- [10] M. Beneke, Phys. Lett. B **307**, 154 (1993).
- [11] D. Soper and L. R. Surguladze, Phys. Rev. D **54**, 4566 (1996).
- [12] K. G. Chetyrkin, S. G. Gorishny, and V. P. Spiridonov, Phys. Lett. **160B**, 149 (1985).
- [13] E. Braaten, S. Narison, and A. Pich, Nucl. Phys. **B373**, 581 (1992).
- [14] A. H. Mueller, Phys. Lett. B **308**, 355 (1993).
- [15] I. Caprini and J. Fischer, Phys. Rev. D **60**, 054014 (1999).
- [16] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachev, Phys. Lett. **85B**, 277 (1979); M. Dine and J. Sapirstein, Phys. Rev. Lett. **43**, 668 (1979); W. Celmaster and R. Gonsalves, *ibid.* **44**, 560 (1980).
- [17] S. G. Gorishny, A. L. Kataev, and S. A. Larin, Phys. Lett. B **259**, 144 (1991); L. R. Surguladze and M. A. Samuel, Phys. Rev. Lett. **66**, 560 (1991).
- [18] U. D. Jentschura and G. Soff, J. Phys. A **34**, 1451 (2001).
- [19] George A. Baker, Jr. and Peter Graves-Morris, *Padé Approximants*, Encyclopedia of Mathematics and Its Applications, 2nd ed., Vol. 59, edited by Gian-Carlo Rota (Cambridge University Press, Cambridge, England, 1996).
- [20] G. Grunberg, Phys. Lett. **95B**, 70 (1980); **110B**, 501(E) (1982); **114B**, 271 (1982); Phys. Rev. D **29**, 2315 (1984).
- [21] N. V. Krasnikov, Nucl. Phys. **B192**, 497 (1981); A. L. Kataev, N. V. Krasnikov, and A. A. Pivovarov, *ibid.* **B198**, 508 (1982).
- [22] A. Dhar and V. Gupta, Phys. Rev. D **29**, 2822 (1984); V. Gupta, D. V. Shirkov, and O. V. Tarasov, Int. J. Mod. Phys. A **6**, 3381 (1991).
- [23] P. M. Stevenson, Phys. Rev. D **23**, 2916 (1981).
- [24] P. M. Stevenson, Phys. Lett. **100B**, 61 (1981); Nucl. Phys. **B203**, 472 (1982).
- [25] C. S. Lam and T. M. Yan, Phys. Rev. D **16**, 703 (1977).
- [26] K. Schilcher and M. D. Tran, Phys. Rev. D **29**, 570 (1984).
- [27] S. Narison and A. Pich, Phys. Lett. B **211**, 183 (1988).
- [28] E. Braaten, Phys. Rev. Lett. **60**, 1606 (1988); Phys. Rev. D **39**, 1458 (1989).
- [29] A. Pich, in *Proceedings of the Workshop on Tau Lepton Physics*, Orsay, France, 1990, edited by M. Davier and B. Jean-Marie (Edition Frontières, Paris, 1991), p. 321.
- [30] F. Le Diberder and A. Pich, Phys. Lett. B **286**, 147 (1992).
- [31] A. Pich and J. Prades, J. High Energy Phys. **06**, 013 (1998); **10**, 004 (1999).
- [32] W. Marciano and A. Sirlin, Phys. Rev. Lett. **56**, 22 (1986); **61**, 1815 (1988).
- [33] E. Braaten and C. S. Li, Phys. Rev. D **42**, 3888 (1990).
- [34] Particle Data Group, D. Groom *et al.*, Eur. Phys. J. C **15**, 1 (2000), p. 695.
- [35] ALEPH Collaboration, R. Barate *et al.*, Eur. Phys. J. C **4**, 409 (1998).
- [36] ALEPH Collaboration, R. Barate *et al.*, Eur. Phys. J. C **11**, 599 (1999); **10**, 1 (1999).
- [37] M. Davier, hep-ex/9912044.
- [38] OPAL Collaboration, G. Abbiendi *et al.*, Eur. Phys. J. C **13**, 197 (2000).
- [39] CLEO Collaboration, S. J. Richichi *et al.*, Phys. Rev. D **60**, 112002 (1999).
- [40] M. Davier (private communication); and talk given at the Sixth International Workshop on Tau Lepton Physics (Tau'2000 Workshop), Victoria, Canada, 2000 (<http://tau2000.phys.uvic.ca/>).
- [41] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, Phys. Lett. B **400**, 379 (1997).
- [42] G. Cvetič, Phys. Lett. B **486**, 100 (2000); G. Cvetič and R. Kögerler, Phys. Rev. D **63**, 056013 (2001).
- [43] G. Cvetič, Nucl. Phys. **B517**, 506 (1998); Phys. Rev. D **57**, R3209 (1998); G. Cvetič and R. Kögerler, Nucl. Phys. **B522**, 396 (1998).
- [44] S. Groote, J. G. Körner, and A. A. Pivovarov, Phys. Lett. B **407**, 66 (1997).
- [45] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachev, Nucl. Phys. **B174**, 345 (1980).
- [46] P. A. Rączka, Phys. Rev. D **57**, 6862 (1998).
- [47] M. Beneke, V. M. Braun, and N. Kivel, Phys. Lett. B **404**, 315 (1997).
- [48] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, Phys. Rev. Lett. **79**, 2184 (1997).
- [49] I. Hinchliffe and A. V. Manohar, Annu. Rev. Nucl. Part. Sci. **50**, 643 (2000).
- [50] S. Bethke, J. Phys. G **26**, R27 (2000).
- [51] B. Chibisov, R. D. Dikeman, M. Shifman, and N. G. Uraltsev, Int. J. Mod. Phys. A **12**, 2075 (1997); I. Bigi, M. Shifman, N. Uraltsev, and A. Vainstein, Phys. Rev. D **59**, 054011 (1999).
- [52] K. G. Chetyrkin, S. Narison, and V. I. Zakharov, Nucl. Phys. **B550**, 353 (1999).
- [53] S. J. Huber, M. Reuter, and M. G. Schmidt, Phys. Lett. B **462**, 158 (1999).
- [54] C. A. Dominguez and K. Schilcher, Phys. Rev. D **61**, 114020 (2000).
- [55] K. Maltman, Phys. Lett. B **440**, 367 (1998).
- [56] C. A. Dominguez and K. Schilcher, Phys. Lett. B **448**, 93 (1999).
- [57] F. Le Diberder and A. Pich, Phys. Lett. B **289**, 165 (1992).
- [58] S. Groote, J. G. Körner, and A. A. Pivovarov, Mod. Phys. Lett. A **13**, 637 (1998).
- [59] J. G. Körner, F. Krajewski, and A. A. Pivovarov, Phys. Rev. D **63**, 036001 (2001).
- [60] T. G. Steele and V. Elias, Mod. Phys. Lett. A **13**, 3151 (1998).
- [61] J. Ellis, M. Karliner, and M. A. Samuel, Phys. Lett. B **400**, 176 (1997); J. Ellis, I. Jack, D. R. T. Jones, M. Karliner, and M. A. Samuel, Phys. Rev. D **57**, 2665 (1998).
- [62] A. L. Kataev and V. V. Starshenko, Phys. Rev. D **52**, 402 (1995).
- [63] K. G. Chetyrkin, B. A. Kniehl, and A. Sirlin, Phys. Lett. B **402**, 359 (1997).
- [64] S. J. Brodsky, G. P. Lepage, and P. B. Mackenzie, Phys. Rev. D **28**, 228 (1983).