Quantum correlation functions and the classical limit

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We study the transition from the full quantum mechanical description of physical systems to an approximate classical stochastic one. Our main tool is the identification of the closed-time-path (CTP) generating functional of Schwinger and Keldysh with the decoherence functional of the consistent histories approach. Given a degree of coarse graining in which interferences are negligible, we can explicitly write a generating functional for the effective stochastic process in terms of the CTP generating functional. This construction gives particularly simple results for Gaussian processes. The formalism is applied to simple quantum systems, quantum Brownian motion, and quantum fields in curved spacetime. Perturbation theory is also explained. We conclude with a discussion on the problem of the back-reaction of quantum fields in spacetime geometry.

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I. CLASSICAL VS. QUANTUM PROBABILITY

A. Introduction

The emergence of classical behavior in quantum systems is a very important question on the foundations of quantum theory. An explanation of how the classical world emerges is absolutely essential for any scheme that has ambitions to go beyond the operational description of the Copenhagen interpretation. In recent years the program of decoherence has provided some insight into how this transition is effected and suggested branches of physics where relevant phenomena are important, such as quantum optics and mesoscopic physics.

From another perspective, the issue of classicalization is of significance in cosmology. We want to know how the perceived classical world is obtained from an underlying description that is of a (presumably) quantum nature. In the early universe, processes are assumed to be governed by quantum field theory, but later a classical hydrodynamics description suffices to capture all relevant physics. The same question is asked for quantum gravity; only now the focus is on the emergence of classical spacetime rather than on the matter fields. At a more technical level one is interested to know when the semiclassical gravity approximation (coupling classical metric variables to quantum fields) is valid.

In all such discussions, the first step is to establish what is meant by classical behavior. The notion of classicality can be defined in different ways, according to the context. For instance, the absence of interferences in a given basis: in other words an approximate diagonalization of the density matrix [1–3], determinism or approximate determinism or some form of predictability [4–6], the validity of a hydrodynamic or thermodynamic description for a many-body system [5,7,8], and the existence of exact or approximate superselection rules [9].

Whatever the definition of classicality might be, there is a consensus about how it appears. *Coarse-graining* is necessary. Since the underlying theory is assumed to be quantum theory (which is by definition nonclassical), one can get a different behavior only by examining a truncated version of the theory. The intuitive picture for emergent classicality is that of a random phase approximation; the coarser the description of the system, the more the interference phase cancels out when averaged within the coarse-grained observable. The general question is then, which types of coarse-graining can regularly lead to classical behavior.

In this paper, we take the attitude that a system exhibits classical behavior if it admits an approximate description in terms of classical probability theory. Since we are interested in systems changing in time, we ask that *the evolution of coarse-grained observables is described by probability theory*, in other words, that it should be modeled by a *stochastic process*.

Quantum processes have an important difference from stochastic processes, their correlation functions are complex valued rather than real valued. This is equivalent to the fact that quantum mechanical evolution cannot be described by a probability measure. In this paper, we focus on how classical correlation functions can be constructed from the quantum mechanical ones through coarse-graining, thus providing an effective stochastic description for a quantum system. A part of the relevant material has appeared previously in [10]. This presentation is simultaneously an elaboration and a simplification of the mathematical constructions performed in this reference with an eye to possible applications.

We shall then apply this formalism in various cases. We will show that in Gaussian systems, the classical limit is mostly determined by the *real part* of the quantum two-point function. We shall verify this in a number of examples: simple harmonic oscillators, the Caldeira-Leggett model of quantum Brownian motion, and scalar fields in curved spacetime. We shall then discuss the perturbation expansion from which we shall infer that a perturbation expansion of the quantum theory does not imply a perturbation expansion for the corresponding stochastic one. We conclude with a discussion of the validity of the semiclassical approximation in quantum gravity. This is a topic which our formalism is particularly adequate to address.

The first step is, however, a brief summary of classical probability theory.

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B. Classical probability

In classical probability one assumes that at a single moment of time the possible elementary alternatives lie in a space Ω, the *sample space*. Observables are functions on Ω, and are usually called *random variables*.

The outcome of any measurement can be phrased as a statement that the system is found in a given subset C of Ω. Hence, the set of certain well-behaved (measurable) subsets of Ω is identified with the set of all coarse-grained alternatives of the system. To each subset C, there corresponds an observable $\chi_C(x)$, the characteristic function of the set C. It is defined as $\chi_C(x)=1$ if $x\in C$ and $\chi_C(x)=0$ otherwise. It is customary to denote the characteristic function of Ω as 1 and of the empty set as 0.

Note that if an observable f takes values f_i in subsets C_i of Ω, we have that

$$f(x)=\sum_i f_i\chi_{C_i}(x). \tag{1.1}$$

A *state* is intuitively thought of as a preparation of a system. Mathematically it is represented by a measure on Ω, i.e., a map that to each alternative C it assigns its probability $p(C)$. It has to satisfy the following properties:

(i) for all subsets C of Ω, $0\leqslant p(C)\leqslant 1$.
(ii) $p(0)=0$; $p(1)=1$.
(iii) for all disjoint subsets C and D of ω, $p(C\cup D)=p(C)+p(D)$.

Because of (1.1) one can define $p(f)=\Sigma_i f_i p(C_i)$; $p(f)$ is then clearly the mean value of f. The usual notation for the mean value is $\overline{f}$; however the expression $p(f)$ is used when we want to stress the state with respect to which the mean value is taken. When Ω is a subset of $\mathbf{R}^n$, the probability measures are defined in terms of a probability distribution, i.e., a positive function on Ω, which we shall (abusingly) denote as $p(x)$,

$$p(f)=\int dx\, p(x)f(x). \tag{1.2}$$

There also exists the notion of *conditional probability*. Assume that in an ensemble described by a probability distribution $p(x)$, we measure whether the property corresponding to the set C is satisfied. The subensemble of all systems that have been found to satisfy this property is then described by the probability distribution $p(x)\chi_C(x)/p(C)$.

Assume now that we have prepared a system in a state p and we want to perform a series of measurements of an observable $f(x)=\Sigma_i f_i\chi_{C_i}(x)$ at time t_1 and of $g(x)=\Sigma_j g_j\chi_{D_j}(x)$ at time $t_2>t_1$. For simplicity, we shall ignore any self-dynamics of the physical systems as it evolves from t_1 to t_2. We can consider a number of measurement situations, labeled by i and j, corresponding to an arrangement where the filter C_i is placed at time t_1 and the filter D_j at time t_2. From a series of measurements one will establish the number of systems in the ensemble that pass both filters and hence identify the probability $p(i,t_1;j,t_2)$, that C_i has been found true at time t_1 *and then* D_j at time t_2

$$p(i,t_1;j,t_2)=\int dx p(x)\chi_{D_j}(x)\chi_{C_i}(x). \tag{1.3}$$

Performing this experiment for all different choices of i and j, we can construct the *statistical correlation function*

$$\langle f_{t_1}g_{t_2}\rangle=\sum_{ij} f_i g_j p(i,t_1;j,t_2)=p(BA). \tag{1.4}$$

By BA we mean the product of the observables B and A; hence $p(BA)$ stands for $\int dx\, p(x)B(x)A(x)$.

In general, the system may have intrinsic dynamics. This is implemented by a map τ_{t_1,t_2} that takes the state $p(x)$ at time t_1 to the state $\tau_{t_1,t_2}[p](x)$ at time t_2 in such a way as to preserve normalization and positivity. The correlation function then reads

$$\langle f_{t_1}g_{t_2}\rangle=\int dx g(x)\tau_{t_1,t_2}[fp](x). \tag{1.5}$$

Here fp stands for the state obtained from the multiplication of the function $f(x)$ with the probability distribution $p(x)$.

When we want to study properties of the system at more than one moment of time, we need to introduce a sample space for *histories*. If we denote by T the set of all possible time instants, we can identify the space of histories Ω^T as a suitable subset of $\times_{t\in T}\Omega_t$, where Ω_t is a copy of the system's sample space labeled by a moment of time t. The elements of Ω^T are *paths* $t\rightarrow x_t$ and will be denoted as $x(\cdot)$.

A history observable is a function on Ω^T. Given a function f on Ω, we can define a family of history observables F_t as

$$F_t[x(\cdot)]=f(x(t)). \tag{1.6}$$

The state is represented by a probability measure P on Ω^T. It contains information about both the initial condition and the dynamics; for any function F on Ω^T it gives its mean value $P(F)$ or simply $\overline{F}$. We can, abusingly, write it in terms of a probability distribution on Ω^T

$$P(F)=\int Dx(\cdot)P[x(\cdot)]F[x(\cdot)]. \tag{1.7}$$

The correlation functions $\langle f_{t_1}g_{t_2}\rangle$ can then be written as $P(F_{t_1}G_{t_2})$ in terms of the functions F_t and G_t defined by Eq. (1.6). The information of the correlation functions of a single observable f is contained in the generating functional

$$Z_f[J(\cdot)]=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int dt_1\ldots dt_n\langle f_{t_1}\ldots f_{t_n}\rangle J(t_1)\ldots J(t_n)$$
$$=\int Dx(\cdot)P[x(\cdot)]\exp\left(i\int dtF_t[x(\cdot)]J(t)\right), \tag{1.8}$$

in terms of a function of time $J(t)$, commonly referred to as the ''source.''

The generating functional is essentially the Fourier transform of the probability measures. The definition can be extended for families of observables f_i. Since correlation functions can be operationally determined, it is possible, in principle, to determine the probability measure with arbitrary accuracy.

C. Quantum correlations

In the previous section we gave a summary of classical probability theory, thus establishing our notation, and identified the operational meaning of correlation functions in classical probability.

The corresponding structures for a single moment of time are well known in standard quantum theory. Elementary alternatives are rays on a complex Hilbert space H, observables are self-adjoint operators on H, a general property corresponds to a projection operator, and a state to a density matrix.

Let us now consider an ensemble of quantum systems prepared in a state described by a density matrix $\hat{\rho}$ and try to operationally construct the correlation function of two observables $\hat{A}=\Sigma a_i \hat{P}_i$ and $\hat{B}=\Sigma_j b_j \hat{Q}_j$ at times t_1 and $t_2>t_1$, respectively. Here $\hat{P}_i$ refers to an exhaustive ($\Sigma_i \hat{P}_i=1$) and exclusive ($\hat{P}_i\hat{P}_j=\hat{P}_i\delta_{ij}$) set of projectors, and so does $\hat{Q}_j$.

Let the Hamiltonian of the system be $\hat{H}$ and $\hat{\rho}_0$ the state of the system at time $t=0$. Then a series of measurements will enable us to identify the probability that $\hat{P}_i$ is found true *and then* $\hat{Q}_j$ is found true. According to the rules of quantum theory this will be

$$p(i,t_1;j,t_2)=\mathrm{Tr}[\hat{Q}_j e^{-i\hat{H}(t_2-t_1)}\hat{P}_i e^{-i\hat{H}t_1}\hat{\rho}_0 e^{i\hat{H}t_1}\hat{P}_i e^{i\hat{H}(t_2-t_1)}]$$
$$=\mathrm{Tr}(\hat{Q}_j(t_2)\hat{P}_i(t_1)\hat{\rho}_0\hat{P}_i(t_1)), \tag{1.9}$$

where we used the Heisenberg picture notation for operators $\hat{A}(t)=e^{i\hat{H}t}\hat{A}e^{-i\hat{H}t}$. If we now vary over all possible values of i and j, we can construct the *statistical* correlation function between $\hat{A}$ and $\hat{B}$

$$\langle \hat{A}_{t_1}\hat{B}_{t_2}\rangle_S=\sum_{ij} a_i b_j p(i.t_1;j,t_2). \tag{1.10}$$

But this correlation function is *not* what one usually calls a correlation function in quantum theory. This name is usually employed for the expectation of a product of operators

$$\langle \hat{A}_{t_1}\hat{B}_{t_2}\rangle_Q=\mathrm{Tr}(\rho\hat{A}(t_1)\hat{B}(t_2))=\sum_{ij} a_i b_j \,\mathrm{Tr}(\rho\hat{P}_i(t_1)\hat{Q}_j(t_2)). \tag{1.11}$$

This is a complex-valued object in contrast to Eq. (1.10) that was constructed using probabilities of events and can only be real valued. Then, what does the quantum mechanical correlation correspond to? Clearly it is unlike classical correlations. The fact that it is complex-valued suggests that it has something to do with quantum mechanical quantities such as interference phases. This remark turns out to be accurate. In [11] a scheme was described in terms of which the quantum mechanical correlation functions can be operationally measured. It proceeds essentially by measuring interference phases between different states. It is a measurement procedure similar to ones used for the Aharonov-Bohm effect or the Berry phase [12]. This is natural in a sense, since the Berry phase is the irreducible element for which quantum theory necessitates the use of complex numbers [13]. However, in the present paper we are interested in the classical limit rather than the full structure of quantum theory and we shall not pursue this topic. The interested reader is referred to [11].

We now want to check the possibility that the quantum and the statistical correlation functions coincide. An easily discernible case is when $[\hat{A}(t_1),\hat{B}(t_2)]=0$, i.e., when the measured observables commute. More generally, it can be verified that a necessary and sufficient condition is

$$\mathrm{Re}\,\mathrm{Tr}(\hat{Q}_j(t_2)\hat{P}_i(t_1)\rho\hat{P}_{i'}(t_1))=0, \tag{1.12}$$

for all i,j and $i'\neq i$. In this case the following property is satisfied:

$$\sum_i p(i,t_i;j,t_2)=\mathrm{Tr}(\rho\hat{Q}_j(t_2))=p(j,t_2) \tag{1.13}$$

for all j. This implies that the probabilities assigned to the set of all possible histories satisfy the additivity condition. They, therefore, define a classical probability measure. It is evident that in this case the quantum and the statistical correlation functions coincide.

This condition for classicality is exactly the one upon which the formalism of consistent histories is based. This formalism is an indispensable part of our analysis and we therefore proceed to examine it next.

II. QUANTUM PROCESSES

A. Consistent histories

The consistent histories approach to quantum theory was developed by Griffiths [14], Omnés [4], Gell-Mann, and Hartle [15,5,6]. The basic object is a *history*, which corresponds to properties of the physical system at successive instants of time. A discrete-time history α will then correspond to a string $\hat{P}_{t_1},\hat{P}_{t_2},\ldots,\hat{P}_{t_n}$ of projectors, each labeled by an instant of time. From them, one can construct the class operator

$$\hat{C}_\alpha=\hat{U}^\dagger(t_1)\hat{P}_{t_1}\hat{U}(t_1)\ldots\hat{U}^\dagger(t_n)\hat{P}_{t_n}\hat{U}(t_n), \tag{2.1}$$

where $\hat{U}(s)=e^{-i\hat{H}s}$ is the time-evolution operator. The probability for the realization of this history is

$$p(\alpha)=\mathrm{Tr}(\hat{C}_\alpha^\dagger\hat{\rho}_0\hat{C}_\alpha), \tag{2.2}$$

where $\hat{\rho}_0$ is the density matrix describing the system at time $t=0$.

But this expression does not define a probability measure in the space of all histories, because the Kolmogorov additivity condition cannot be satisfied; if α and β are exclusive histories, and $\alpha \vee \beta$ denotes their conjunction as propositions, then it is not true that

$$p(\alpha \vee \beta) = p(\alpha) + p(\beta). \tag{2.3}$$

The histories formulation of quantum mechanics does not, therefore, enjoy the status of a genuine probability theory.

However, an additive probability measure *is* definable, when we restrict it to particular sets of histories. These are called *consistent sets*. They are more conveniently defined through the introduction of a new object, the decoherence functional. This is a complex-valued function of a pair of histories given by

$$d(\alpha,\beta) = \mathrm{Tr}(\hat{C}_\beta^\dagger \hat{\rho}_0 \hat{C}_\alpha). \tag{2.4}$$

A set of exclusive and exhaustive alternatives is called consistent if for all pairs of different histories α and β we have

$$\mathrm{Re}\, d(\alpha,\beta) = 0. \tag{2.5}$$

In that case one can use Eq. (2.2) to assign a probability measure to this set. The consistent histories interpretation then proceeds by postulating that any prediction or retrodiction we can make using probabilities *must always make reference to a given consistent set*. This leads to counterintuitive situations of getting mutually incompatible predictions, when reasoning within different consistent sets. The predictions of this theory are therefore contextual, but in any case, this is a general feature of all realist interpretations of quantum theory.

Except for trivial cases, it is only coarse-grained observables that satisfy an exact (or approximate) consistency condition. This means that the histories are constructed out of projectors $\hat{P}$, whose trace is much larger than unity.

B. The closed-time-path generating functional

We saw that in quantum theories probabilities and statistical correlations are contained in the decoherence functional, in fact, in its diagonal elements. We shall now show that the same is true for the quantum correlation functions.

Recall that in the decoherence functional projectors enter in a *time-ordered series*. This suggests that it would be best to use time-ordered correlation functions. Let $\hat{A}^a$ denote a family of commuting operators. Then the time-ordered two-point correlation function is defined as

$$G^{2,0}(a_1,t_1;a_2,t_2) = \theta(t_2-t_1)\mathrm{Tr}[\hat{\rho}_0 \hat{A}^{a_1}(t_1)\hat{A}^{a_2}(t_2)] + \theta(t_1-t_2)\mathrm{Tr}[\hat{\rho}_0 \hat{A}^{a_2}(t_2)\hat{A}^{a_1}(t_1)]. \tag{2.6}$$

Here we have denoted by $\theta(t)$ the step function.

One can similarly define time-ordered n-point functions, or anti-time-ordered

$$G^{0,2}(a_1,t_1;a_2,t_2) = \theta(t_1-t_2)\mathrm{Tr}[\hat{\rho}_0 \hat{A}^{a_1}(t_1)\hat{A}^{a_2}(t_2)] + \theta(t_2-t_1)\mathrm{Tr}[\hat{\rho}_0 \hat{A}^{a_2}(t_2)\hat{A}^{a_1}(t_1)]. \tag{2.7}$$

In general, one can define *mixed* correlation functions $G^{r,s}$, with r time-ordered and s anti-time-ordered entries, as for instance,

$$G^{2,1}(a_1,t_1;a_2,t_2|b_1,t_1') = \theta(t_2-t_1)\mathrm{Tr}[\hat{A}^{b_1}(t_1')\hat{\rho}_0 \hat{A}^{a_1}(t_1)\hat{A}^{a_2}(t_2)] + \theta(t_1-t_2)\mathrm{Tr}[\hat{A}^{b_1}(t_1')\hat{\rho}_0 \hat{A}^{a_2}(t_2)\hat{A}^{a_1}(t_1)]. \tag{2.8}$$

These correlation functions are generated by the closed-time-path (CTP) generating functional associated to the family A^a,

$$Z_A[J_+,J_-] = \sum_{n,m=0}^{\infty} \frac{i^n(-i)^m}{n!m!} \int dt_1 \ldots dt_n dt_1' \ldots dt_m', \quad G^{n,m}(a_1,t_1;\ldots a_n,t_n|b_1,t_1';\ldots;b_m,t_m'), \quad J_+^{a_1}(t_1)\ldots J_+^{a_n}(t_n) J_-^{b_1}(t_1')\ldots J_-^{b_m}(t_m'). \tag{2.9}$$

Here J_+^a and J_-^a are functions of time that play the role of sources similar to the ones in Eq. (1.8) for the classical stochastic processes.

The name closed time arose, because in the original conception (by Schwinger [16] and Keldysh [17]) the time path one follows is from some initial time $t=0$ to $t\to\infty$, thus covering all time-ordered points and then back from infinity to 0 covering the anti-time-ordered points. The total time path is in effect closed.

Conversely the correlation functions can be read from Z_A

$$G_A^{n,m}\Big(a_1,t_1;\ldots;a_n,t_n|b_1,t_1';\ldots;b_m,t_m'\Big) = (-i)^n i^m \frac{\delta^n}{\delta J_+^{a_1}(t_1)\cdots\delta J_+^{a_n}(t_n)} \frac{\delta^m}{\delta J_-^{b_1}(t_1)\cdots\delta J_-^{b_m}(t_m)} \times Z[J_+,J_-]\Big|_{J_+=J_-=0}. \tag{2.10}$$

C. Relation between the functionals

Clearly there must be a relation between the decoherence functional and the CTP one. One can see in the correlation functions if we assume a single operator $\hat{A}=\Sigma_i a_i \hat{P}_i$ and consider a pair of histories

$$\alpha(i_1,t_1;\ldots;i_n,t_n) = \{\hat{P}_{i_1},t_1;\ldots;\hat{P}_{i_n},t_n\}$$

and

$$\beta(i_1,t_1';\ldots;i_n,t_m')=\{\hat{P}_{j_1},t_1';\ldots;\hat{P}_{j_m},t_m\}.$$

Then one can easily verify that

$$G_A^{n,m}(t_1,\ldots,t_n;t_1',\ldots,t_m') = \sum_{i_1\cdots i_n}\sum_{j_1\cdots j_m} a_{i_1}\ldots a_{i_n}b_{j_1}\ldots b_{j_m} \times d[\alpha(i_1,t_1;\ldots;i_n,t_n),\beta(j_1,t_1;\ldots;j_m,t_m)]. \tag{2.11}$$

The straightforward relation is nonetheless not possible to show in an elementary fashion. One needs to consider correlation functions at all times t and this necessitates a description in terms of histories that can have temporal support over the whole of the real line or at least a continuous subset of it. This can be achieved in the framework of continuous-time histories [18–21]. However, this requires a significant upgrading of the formalism of quantum mechanical histories. The key idea is to represent histories by projectors on a tensor product of Hilbert spaces $\otimes_{t\in T}H_t$ [22] in analogy to the construction of the history sample space classically. A suitable Hilbert space (not a genuine tensor product) can be constructed [18] for the case that T is a continuous set and the decoherence functional can be defined as a bilinear, Hermitian functional on this space. It can then be shown that as a functional it is essentially a double ''Fourier transform'' of the CTP generating functional.

This proof is to be found in [10] and is elementary if one follows the logic of the construction. Here we shall restrict ourselves to a convenient statement of this result.

Let us assume that we have a family of commuting self-adjoint operators $\hat{A}^i$. Their spectrum is then a subset Ω of some vector space $\mathbf{R}^n$. Any operator that commutes with $\hat{A}^i$ is in one-to-one correspondence to functions $f(x)$ with $x\in\Omega$ and can be written as $f(\hat{A})$. Like the classical case we can construct a space of histories Ω^T as a suitable subset of $\times_{t\in T}\Omega_t$. Subsets of Ω^T are histories of the quantum mechanical observables $\hat{A}^i$.

The decoherence functional is then a map that to each pair of subsets C and C' of Ω^T it assigns a complex number in such a fashion that the following properties are satisfied [6,23]:

(i) $d(C',C)=d^*(C,C')$, Hermiticity,
(ii) $d(0,C)=0$, null triviality,
(iii) $d(1,1)=1$, normalization,
(iv) $d(C\cup C',D)=d(C,D)+d(C',D)$ for disjoint C and C', additivity.

Such a decoherence functional can be constructed as a continuum limit of the discrete-time expressions (2.2). Because of the additivity condition, one can *formally* write the decoherence functional as an integral over $\Omega^T\times\Omega^T$,

$$d(C,D)=\int Dx(\cdot)Dx'(\cdot)\Delta[x(\cdot)|x'(\cdot)] \times\chi_C[x(\cdot)]\chi_D[x'(\cdot)], \tag{2.12}$$

in terms of a function $\Delta:\Omega^T\times\Omega^T$ that plays the role of an integration kernel. This is in complete analogy to the stochastic probability measure $P[x(\cdot)]$ of Eq. (1.7).

One can view $\Delta[x(\cdot)|x'(\cdot)]$ as the decoherence functional between a pair of fine-grained histories $x(\cdot)$ and $x'(\cdot)$, only that such histories cannot be represented by projectors on a Hilbert space. For example, if these histories were defined on the configuration space for the time interval $[t_i,t_f]$, one could write the standard expression [6]

$$\Delta[x(\cdot)|x'(\cdot)]=\rho_0[x(t_i),x'(t_i)]\delta[x(t_f),x'(t_f)] \times e^{iS[x(\cdot)]-iS[x'(\cdot)]}, \tag{2.13}$$

in terms of the matrix elements of the initial density matrix ρ_0, the standard configuration space action, and a delta function for the final-time points of the paths.

If $Z_A[J_+,J_-]$ is the CTP generating functional associated to $\hat{A}^i$ we have

$$Z_A[J_+,J_-]=\int Dx(\cdot)Dx'(\cdot)e^{i\int dtJ_+^a(t)x^a(t)}e^{-i\int dtJ_-^a(t)x'^a(t)} \times\Delta[x(\cdot)|x'(\cdot)]. \tag{2.14}$$

In other words, viewed as a bi-functional over the functions on Ω^T, the decoherence functional is identical to the CTP generating functional. The only difference is on the type of functions upon which they take values—the first on characteristic functions and the second on complex-valued functions of unit norm. In fact, Eq. (2.12) amounts to

$$G^{n,m}(a_1,t_1;\ldots;a_n,t_n|b_1,t_1;\ldots;b_m,t_m) = \int\int Dx(\cdot)Dx'(\cdot)x^{a_1}(t_1)\ldots x^{a_n}(t_n)x'^{b_1}(t_1') \times x'^{b_m}(t_m)\Delta[x(\cdot)|x'(\cdot)]. \tag{2.15}$$

Hence there exists the following correspondence between classical and quantum probability.

	Quantum	$\rightarrow$	Classical
Probabilities	$d(C,C')$	$\rightarrow$	$p(C)$,
Correlations	$Z[J_+,J_-]$	$\rightarrow$	$Z^{cl}[J]$.

The probability measure is a real-valued functional on functions of Ω^T, while the decoherence functional is a Hermitian bilinear functional on the functions of Ω^T. In a given system, one goes from d to p when the decoherence condition (2.5) is satisfied, while in both cases one goes from probabilities to correlations through a Fourier transform.

What we will next show is how to effect the transition from the CTP generating functional to a stochastic process for the coarse-grained variables. Working at the level of the

correlation functions makes the construction of stochastic differential equations easier than working at the level of probabilities.

III. FROM QUANTUM TO CLASSICAL

A. The basic choice for coarse-graining

In order to study the transition from quantum to classical, we need to choose the variables on which we shall concentrate. This amounts to a choice of a family $\hat{A}^i$ of intercommuting operators. Now, a *maximal* family of intercommuting operators generically contains full information about the evolution of the quantum system. (Possible exceptions to this rule are trivial cases, as, for instance, when the Hamiltonian and the initial density matrix commutes with all $\hat{A}^i$).

One can implement a coarse-graining procedure even at this stage. It suffices to take for $\hat{A}^i$ a nonmaximal family of operators. This is the case, for instance, in quantum Brownian motion models. If we assume that the total system consists of a large number of harmonic oscillators, a maximal family of intercommuting operators consists of the position operators of all particles. When we choose to focus on a single one of them, we effectively coarse grain by treating the remaining degrees of freedom as an environment. This is the type of coarse-graining associated with the studies of environment-induced decoherence.

However, this type of coarse-graining does not suffice. One has usually to consider smeared values of the relevant observables. This is effected by considering projectors, which are sufficiently smeared over Ω. We shall take Ω to be $\mathbf{R}^s$ so its points will be vectors x^a.

In general, it is difficult to work with characteristic functions, so we will work with *smeared characteristic functions*. If we denote by $|x|$ their Euclidean distance, then a good choice for the projector is the function

$$f_{\bar{x}}(x)=\exp\left(-\frac{1}{2\sigma^2}|x-\bar{x}|^2\right). \tag{3.1}$$

This Gaussian is not a sharp projector; it is strongly peaked in a sphere of length σ around the point $\bar{x}$; hence it is a good approximation to a true projector for not very large values of σ.

We can construct discrete-time histories, consisting of projectors $f_{\bar{x}_{t_i}}$ centered around $\bar{x}_{t_i}$, at each time t_i. One such history can be viewed as a discretized approximation to a coarse-grained history in continuous time, centered around a path $t\to\bar{x}(t)$.

We now consider two such discretized histories, centered at the same time points t_i, each corresponding to a different path $\bar{x}(\cdot)$ and $\bar{x}'(\cdot)$. Let us denote them by $\alpha_{\bar{x}(\cdot)}$ and $\alpha_{\bar{x}'(\cdot)}$. If we expect our system to exhibit classical behavior, then the off-diagonal elements of the decoherence functional will fall rapidly whenever $\delta^2=||\bar{x}(\cdot)-\bar{x}'(\cdot)||^2:=\Sigma_i|\bar{x}_{t_i}-\bar{x}'(t_i)|^2$ is much larger than $N\times\sigma^2$. (Here N is the number of time steps). Typically one has

$$d(\alpha_{\bar{x}(\cdot)},\alpha_{\bar{x}'(\cdot)})=O(e^{-\delta^2/N\sigma^2}), \tag{3.2}$$

or some other type of rapid falloff. For pure initial states, this behavior is expected, *when σ^2 is much larger than the uncertainties of the initial state and the Hamiltonian evolution preserves this property* [24]. In this case, the diagonal elements are close to defining probabilities for coarse-grained histories centered around $\bar{x}(\cdot)$ and with a spread σ at each moment of time.

Now, we want to find a probability distribution that would give these values for the probabilities of these histories. A single-time projector is centered in a volume of Ω of size

$$\int dx f_{\bar{x}}(x)=(2\pi\sigma^2)^{r/2}. \tag{3.3}$$

A probability distribution on the space of (discretized) paths that reproduces these expressions for probabilities of these coarse-grained sets is

$$p[\bar{x}(\cdot)]=\frac{1}{(2\pi\sigma^2)^{rn/2}}d(\alpha_{\bar{x}(\cdot)},\alpha_{\bar{x}(\cdot)}), \tag{3.4}$$

where n is the number of time steps assumed. (Dividing by the volume turns the probabilities of events into a density.)

One can use Eq. (2.12) to write

$$\begin{aligned}d(\alpha_{\bar{x}(\cdot)},\alpha_{\bar{x}(\cdot)})=&\int Dx(\cdot)Dx'(\cdot)\\&\times\exp\left(-\frac{1}{2\sigma^2}||x(\cdot)-\bar{x}(\cdot)||^2\right.\\&\left.-\frac{1}{2\sigma^2}||x'(\cdot)-\bar{x}(\cdot)||^2\right)\Delta[x(\cdot)|x'(\cdot)].\end{aligned} \tag{3.5}$$

Note that our expressions are still defined with respect to discrete time.

From Eq. (2.14) we see that the kernel Δ can be obtained from the inverse Fourier transform of the CTP generating functional. This yields

$$\begin{aligned}p[\bar{x}(\cdot)]=&\int DJ_+(\cdot)DJ_-(\cdot)\\&\times e^{-1/4\pi(J_+\cdot J_++J_-\cdot J_-)-i/\sqrt{2\pi}\bar{x}\cdot(J_+-J_-)}\\&\times Z[J_+/(\sqrt{2\pi}\sigma),J_-/(\sqrt{2\pi}\sigma)].\end{aligned} \tag{3.6}$$

There is now no multiplicative term that depends on the number of time steps. Hence, one can safely go to the continuum limit from this expression. We have denoted as $J\cdot x=\Sigma_i J(t_i)x(t_i)$ and at the continuum limit this expression will become an integral.

To get the generating functional for the classical correlation functions one Fourier transforms the probability measure to get (we now drop the index that refers to our choice of variables for the correlation functions)

$$\tilde{Z}^{cl}[J]=e^{-\sigma^2/8J\cdot J}\int dR(\cdot)e^{-1/2\pi R\cdot R}\times Z\left[\frac{R}{\sqrt{2\pi\sigma}}+\frac{J}{2},\frac{R}{\sqrt{2\pi\sigma}}-\frac{J}{2}\right], \tag{3.7}$$

with $R=\frac{1}{2}(J_+ + J_-)$ and $J=J_+ - J_-$.

This generating functional needs to be normalized to unity by assuming that $\tilde{Z}^{cl}[0]=1$. The normalization condition is not kept because we have employed approximate characteristic functions. Had we used a sharp characteristic function, the construction would automatically guarantee normalization. Now there is a deviation from unity of the order of σ^2.

The expression (3.7) can be simplified. Assume that we have a classical stochastic process for the variables $x(\cdot)$ with a generating functional $Z_0[J]$. Let us follow the same procedure for coarse-graining as before, using the approximate projectors (3.1). The coarse-grained generating functional would be

$$\tilde{Z}^{cl}[J]=e^{-(\sigma^2/2)J\cdot J}Z_0[J]. \tag{3.8}$$

This means that we can consider the generating functional in Eq. (3.7) as coming from coarse-graining a classical stochastic process with twice the degree of coarse-graining as the one from quantum theory. One can then drop the term outside the integral in (3.7) as coming from coarse-graining of an underlying stochastic process given by

$$Z^{cl}[J]=\int dR(\cdot)e^{-1/2\pi R\cdot R}Z\left[\frac{R}{\sqrt{2\pi\sigma}}+\frac{J}{2},\frac{R}{\sqrt{2\pi\sigma}}-\frac{J}{2}\right]. \tag{3.9}$$

This equation gives the stochastic correlation functions in the classical limit of the quantum system described by the CTP generating functional $Z[J_+,Z_-]$. It should be always kept in mind that this process *gives reliable results only on scales much larger than* σ^2.

We should now pause for a minute and examine the assumptions we used in order to arrive here.

First, one should ask what is the meaning of the parameter σ. Is it arbitrary or not? In principle it is not. It is the degree of coarse-graining that is necessary in order that the falloff (2.5) of the off-diagonal elements of the decoherence functional is manifested. It therefore has to be much larger than the natural scale associated with microscopic processes; however, it ought to be small compared to macroscopic scales.

In principle, σ can be determined from a full study of the decoherence functional. Usually a measure of coarse-graining is the trace of the corresponding positive operator. However, we have considered here only commuting variables with a continuous spectrum, and the trace of the operators corresponding to Eq. (3.7) is infinite. This is related to the fact that there is no default *universal* scale by which to judge whether σ is large. This problem is remedied by considering phase space coarse-grainings as we shall see shortly. In this case the natural scale is $\hbar=1$ and one can say that $\sigma^2 \gg 1$ in order to have consistency of histories. It should nonetheless be much smaller than a macroscopic scale by which we observe phenomena.

More precisely, the stochastic approximation (3.9) is accurate within an order of $(l_{mic}/\sigma)^2$, where l_{mic} is the microscopic scale that is determined by the dynamics or the initial state. However, there is also an error proportional to $(\sigma/L_{mac})^2$, where L_{mac} is the macroscopic scale of *observation*, i.e., the scale of accuracy we are interested in having. This is due to the use of the Gaussian approximation for the projectors. Overall we have an error of the order of

$$c_1(l_{mic}/\sigma)^2+c_2(\sigma/L_{mac})^2, \tag{3.10}$$

where c_1 and c_2 are constants of the order of unity. It is therefore evident that a separation of scales is necessary, if the stochastic description is to make any sense.

Second, the general logic of this construction is to identify a stochastic process that adequately describes the evolution of the classicalized coarse-grained observables. There is a subtle difference from the consistent histories scheme in that we do not seek to construct consistent sets for the system and hence make statements about individual quantum systems. Our approach is more operational. Given that quantum theory is a model that provides the statistical behavior of physical systems, we ask to construct a different model based on probability theory that describes some regime of the same physical system. For this purpose we utilize the consistency condition in order to identify the validity of our approximation. Then, we build the probability distribution from the diagonal elements of the decoherence functional.

B. Phase space coarse-grainings

One does not have to restrict to the correlation functions of a family of commuting operators in order to construct the CTP generating functional. By considering correlation functions in both position and momentum, it is possible to generalize the definition (2.9). Indeed in this equation it is not necessary to assume that the operators $\hat{A}^a$ are commuting. We assumed commutativity because we wanted a description in terms of paths on the common spectrum of these operators.

However, in any Hilbert space that carries a representation of the canonical commutation relations

$$[\hat{q}^i,\hat{p}^j]=i\delta^{ij}, \tag{3.11}$$

it is possible to assign a function on the phase space $\Gamma=\{(q^i,p^i)\}$ for each operator by means of the Wigner transform

$$\hat{A}\rightarrow F_A(q,p)=\int d\xi d\chi e^{-iq\xi-ip\chi}\mathrm{Tr}(\hat{A}e^{i\hat{q}\xi+i\hat{p}\chi}):$$

$$=\mathrm{Tr}(\hat{A}\hat{\Delta}(q,p)). \tag{3.12}$$

An important property of this transform is that it preserves the trace

$$\mathrm{Tr}\,\hat{A}=\int dqdpF_A(q,p). \tag{3.13}$$

However, the Wigner transform does not preserve multiplication of operators. The defining condition $\hat{P}^2=\hat{P}$ for projectors is therefore not preserved and a projector is mapped into some general positive function, rather than a characteristic function of a subset of Γ.

Since any operator can be represented by a function of Ω, histories would be represented by functions on a space Γ^T which will be a suitable subspace of $\times_{t\in T}\Gamma_t$. In Ref. [10] it was shown that a decoherence functional can be constructed as an Hermitian bilinear functional on the space of functions on Γ^T. It is related to the CTP generating functional by means of a Fourier transform.

All the formulas in the previous paragraph can then be reinterpreted to fit the phase space context by allowing the variables x to denote both q and p. (The dimension of Γ is clearly even.) The main difference is that the Gaussian function f_x^- corresponds to an operator $\hat{F}$ with a *finite* trace. By virtue of Eqs. (2.1) and (2.13)

$$\mathrm{Tr}\,\hat{F}=(2\pi\sigma^2)^{r/2}. \tag{3.14}$$

The parameter σ has units of action and is an absolute measure of the degree of coarse-graining on phase space. Consistency occurs whenever $\sigma^2\gg\hbar$, where $\hbar$ provides the natural length scale on phase space. In fact, in the study of a large class of closed quantum systems, Omnés has showed [25] that the off-diagonal elements of the decoherence functional are of the order of $(\hbar/\sigma)^{r/4}$, where $r=2k$ is the dimension of Γ. Hence, even if $\hbar/\sigma\sim 10^{-8}$ there is a substantial degree of decoherence to justify the use of classical probability and σ *is still sufficiently small* compared to some external macroscopic scales to justify the use of the Gaussian approximation for the projector. From a macroscopic perspective it would be sufficient to consider the leading order in σ^2 of the correlation functions.

The study of phase space histories is more intricate because one has to choose proper units for position and momentum by which to write a Euclidean norm in the coarse-grained projector (2.1). For classicality it is not only necessary to have a large value of σ, but the *choice of units has to be preserved by the dynamical evolution* [25,26]. This is a nontrivial condition that largely depends on the system's Hamiltonian. For this we shall prefer to employ configuration space coarse-grainings.

Whenever we have a representation of the canonical commutation relations we can define the coherent states

$$|z\rangle=|\chi\xi\rangle=e^{i\hat{q}\xi+i\hat{p}\chi}|0\rangle, \tag{3.15}$$

where $|0\rangle$ is a fiducial vector, often taken to be the lowest energy eigenstate. The important point is that one can assign to a large class of density matrices $\hat{\rho}$ a function $f_\rho(\chi,\xi)$ (its P symbol) defined by

$$\hat{\rho}=\int d\chi d\xi f(\chi,\xi)|\chi\xi\rangle\langle\chi\xi|. \tag{3.16}$$

If one then denotes by $Z_{\chi_0\xi_0}[J_+,J_-]$ the CTP generating functional corresponding to an initial state given by $|\chi_0\xi_0\rangle$ then the CTP generating functional for the same system but a different initial state $\hat{\rho}$

$$Z[J_+,J_-]=\int d\chi_0 d\xi_0 f_\rho(\chi_0,\xi_0)Z_{\chi_0\xi_0}[J_+,J_-], \tag{3.17}$$

and a similar equation would hold for the classical limit, provided that the degree of coarse-graining necessary for decoherence is determined by the study of the state $\hat{\rho}$ rather than the coherent states.

C. Gaussian processes

Let us now consider a quantum system described by a Gaussian CTP generating functional. Its most general form would be

$$Z[J_+,J_-]=\exp\Big[-\frac{i}{2}J_+\cdot L\cdot J_+ +\frac{i}{2}J_-\cdot\bar{L}\cdot J_- +iJ_+\cdot K\cdot J_- +i(J_+-J_-)\cdot X\Big]. \tag{3.18}$$

Here we have denoted by L kernels of the form $L^{ab}(t,t')$ by

$$J\cdot L\cdot J'=\int dtdt' J^a(t)L^{ab}(t,t')J^b(t')$$

and the bar denotes complex conjugation. X denotes the one-point function $G^{10}=G^{01}$ and

$$iL^{ab}(t,t')=G^{2,0}(a,t;b,t')-X(a,t)X(b,t'), \tag{3.19}$$

$$iK^{ab}(t,t')=G^{1,1}(a,t|b,t')+G^{1,1}(b,t'|a,t) -2X(a,t)X(b,t'). \tag{3.20}$$

We can write $L=L_1-iL_2$ and $K=K_1-iK_2$ in terms of the real-valued kernels L_1,L_2,K_1,K_2. The Hermiticity condition on the CTP generating functional would then entail

$$L_1^T=L_1,\quad L_2^T=L_2, \tag{3.21}$$

$$K_1=0,\quad K_2=2L_2. \tag{3.22}$$

Evaluating the integral (3.7) yields

$$Z^{cl}[J]=e^{-J\cdot\Xi\cdot J+iJ\cdot X}, \tag{3.23}$$

where

$$\Xi=L_2+\frac{1}{4\sigma^2}L_1\cdot L_1. \tag{3.24}$$

It is worth noticing that whenever the term L_2 is dominant, the classical two-point function is independent of the

coarse-graining scale and equal to the real part of the quantum two-point function. However, this simplification can occur only in Gaussian systems.

IV. EXAMPLES

A. Harmonic oscillators

For a single harmonic oscillator with frequency ω and mass m in a thermal state, we have for the configuration space correlation functions

$$L_1(t,t')=-\frac{1}{2m\omega}\sin\omega|t-t'|, \tag{4.1}$$

$$L_2(t,t')=\frac{1}{2m\omega}\coth(\beta\omega/2)\cos\omega(t-t'), \tag{4.2}$$

and Eq. (3.24) gives $\Xi(t,t')$.

One should recall that the smearing scale σ is determined by the condition (2.5) on the falloff of the diagonal elements of the decoherence functional. Here σ^2 should be much larger than $(2m\omega)^{-1}$, the position uncertainty of the ground state. This can be verified by direct evaluation, but it is made plausible by the following observation: a thermal state has a positive P symbol, and hence its quantum behavior is identical to the one of the coherent states, which in a Gaussian system is identical with that of the vacuum.

The term $L_1\cdot L_1$ is proportional to $(\sigma^2 m\omega)^{-1}$, hence comparatively small. In particular, at high temperature $\beta\omega \ll 1$ the L_2 term is dominant, the correlation function is σ independent and one recovers the classical result.

Let us recall that this system does not describe a harmonic oscillator in contact with a heat bath; it describes a *closed* system, evolving unitarily and prepared in a thermal state (whatever that might mean). Physically more relevant is the case of an oscillator undergoing quantum Brownian motion, to be taken up later.

But we shall first examine the case where the system is initially prepared in a squeezed state. A squeezed state $|r,\phi\rangle$ is the zero eigenstate of the operator

$$\hat{b}=\cosh r/2\,\hat{a}+\sinh r/2\,e^{i\phi}\hat{a}^\dagger, \tag{4.3}$$

where $r\geqslant 0$. The correlation function L_1 is identical to the one for the vacuum case, while

$$L_2(t,t')=\frac{1}{2m\omega}[\cosh r\cos\omega(t-t') + \sinh r\cos\omega(t+t'-\phi)]. \tag{4.4}$$

Clearly it is necessary that

$$\sigma^2\gg(\cosh r/2m\omega) \tag{4.5}$$

in order to have decoherence. In that case the L_1^2 term is again negligible. For values of r at the order of unity, it is not different from the vacuum case, but for large r the degree of coarse-graining necessary for classicality might become too large to allow us to obtain any useful information. This is what is meant when we say that squeezed states are highly nonclassical states.

B. Quantum Brownian motion

We shall study here the Caldeira-Leggett model [27–29], i.e., a single harmonic oscillator of mass M and frequency ω coupled linearly to a bath of harmonic oscillators in a thermal state. More precisely the system is defined by the Hamiltonian

$$\hat{H}=\frac{\hat{p}^2}{2M}+\frac{1}{2}M\omega^2\hat{x}^2+\hat{x}\sum_i c_i\hat{q}_i+\sum_i\left(\frac{\hat{p}_i^2}{2m_i}+\frac{1}{2}m_i\omega_i^2\hat{q}_i^2\right). \tag{4.6}$$

From the Heisenberg equations of motion we get

$$M\frac{d^2}{dt^2}\hat{x}+M\omega^2\hat{x}^2=-\sum_i c_i\hat{q}_i, \tag{4.7}$$

$$\frac{d^2}{dt^2}\hat{q}_i+\omega_i^2\hat{q}_i^2=-\frac{c_i}{m_i}\hat{x}. \tag{4.8}$$

The second equation has a solution

$$\hat{q}_i(t)=\hat{q}_{0i}\cos\omega_i t+\frac{\hat{p}_{0i}}{m_i\omega_i}\sin\omega_i - \frac{c_i}{m_i\omega_i}\int_0^t ds\,\sin\omega_i(s-s')\hat{x}(s), \tag{4.9}$$

which when substituted into Eq. (4.7) yields

$$\frac{d^2}{dt^2}\hat{x}+\omega^2\hat{x}-\frac{2}{M}\int_0^t ds\,\eta(t-s)\hat{x}(s)=-\frac{1}{M}\sum_i c_i\hat{q}_i(t). \tag{4.10}$$

Here

$$\eta(s)=\sum_i\frac{c_i^2}{2m_i\omega_i}\sin\omega_i s \tag{4.11}$$

is known as the *dissipation kernel*. Let us denote by $u(t)$ the solution of the homogeneous equation corresponding to Eq. (4.10) with the initial conditions $u(0)=1$ and $\dot{u}(0)=0$. It can be identified as the inverse Laplace transform of the function

$$\tilde{u}(s)=\frac{1}{s^2+\omega^2-2/M\,\tilde{\eta}(s)}, \tag{4.12}$$

where $\tilde{\eta}$ is the Laplace transform of the dissipation kernel. We can then write the solution of Eq. (4.10) as

$$\hat{x}(t)=\hat{x}_0u(t)+\frac{\hat{p}_0}{M}\dot{u}(t)-\frac{1}{M}\sum_i c_i\left[\hat{q}_{0i}\int_0^t dsu(t-s)\cos\omega_i s+\frac{\hat{p}_{0i}}{m_i\omega_i}\int_0^t dsu(t-s)\sin\omega_i s\right]. \tag{4.13}$$

Now we assume that the initial state of the system is factorizable to a thermal state at temperature $T=\beta^{-1}$ for the environment and a density matrix $\widehat{\rho_0}$ for the distinguished oscillator. In this case, we can easily see that the expectation value

$$x(t)=\mathrm{Tr}[\hat{\rho}_0\hat{x}(t)]=x_0u(t)+\frac{p_0}{M}\dot{u}(t), \tag{4.14}$$

is a solution of the dissipative equations of motion, while the two-point function reads

$$\begin{aligned}\mathrm{Tr}[\hat{\rho}_0\hat{x}(t)\hat{x}(t')]=&(\Delta x_0)^2u(t)u(t')+\frac{(\Delta p_0)^2}{M^2}\dot{u}(t)\dot{u}(t')\\&+\frac{C_{pq}}{M}[u(t)\dot{u}(t')+u(t')\dot{u}(t)]\\&+\frac{i}{2M}[u(t)\dot{u}(t')-u(t')\dot{u}(t)]\\&+\frac{1}{M^2}\left[\int_0^t ds\int_0^{t'}ds'u(s)\nu(s-s')u(s')\right.\\&\left.+i\int_0^t ds\int_0^{t'}ds'u(s)\eta(s-s')u(s')\right].\end{aligned} \tag{4.15}$$

Here $\Delta x_0,\Delta p_0$, and C_{pq} are the uncertainties and correlation between position and momenta at $t=0$. Also,

$$\nu(s)=\sum_i\frac{c_i^2}{2m_i\omega_i}\coth\frac{\beta\omega_i}{2}\cos\omega_i s, \tag{4.16}$$

is known as the *noise kernel*.

From this equation, it is easy to determine the kernels L and K. If we write the last line in Eq. (4.15) as $(1/M^2)\times[N(t,t')+iH(t,t')]$, we have

$$\begin{aligned}L_1(t,t')=&\theta(t-t')\left[\frac{1}{M}[u(t)\dot{u}(t')-u(t')\dot{u}(t)]+\frac{1}{M^2}H(t,t')\right]+\theta(t'-t)\left[\frac{1}{M}[u(t')\dot{u}(t)\right.\\&\left.-u(t)\dot{u}(t')]+\frac{1}{M^2}H(t',t)\right]\end{aligned} \tag{4.17}$$

$$\begin{aligned}L_2(t,t')=&(\Delta x_0)^2u(t)u(t')+\frac{(\Delta p_0)^2}{M^2}\dot{u}(t)\dot{u}(t')\\&+\frac{C_{pq}}{M}[u(t)\dot{u}(t')+u(t')\dot{u}(t)]\\&+\frac{1}{M^2}N(t,t').\end{aligned} \tag{4.18}$$

Now we want to derive the stochastic limit to which these quantum correlation functions correspond. We shall use Eq. (3.24). Note, however, that the necessary degree of coarse-graining in order to achieve decoherence must be in a phase space region much larger than the one occupied by the initial state. Hence we take the coarse-graining scale to be much larger than the uncertainties and correlations of the initial state. This allows us to drop all terms but $N(t,t')$ in Eq. (4.15). The semiclassical equations are then largely independent of the details of the initial state.

To see this in more detail we need to specify a given distribution of modes and couplings in the environment. This information is encoded in the spectral density

$$I(k)=\sum_i\frac{c_i^2}{2m_i\omega_i}\delta(k-\omega_i). \tag{4.19}$$

If the spectral density is specified, then we can fully determine the noise and dissipation kernels as

$$\eta(s)=\int dkI(k)\sin ks, \tag{4.20}$$

$$\nu(s)=\int dkI(k)\coth\left(\frac{\beta k}{2}\right)\cos ks. \tag{4.21}$$

A large class of physically interesting choices for spectral density are

$$I(k)=\begin{cases}\dfrac{2M\gamma}{\pi}k\left(\dfrac{k}{\Theta}\right)^{s-1}, & k\leqslant\Lambda,\\ 0 & k>\Lambda,\end{cases} \tag{4.22}$$

where Λ is a high-frequency cutoff. The exponent s determines the infrared behavior of the bath. For $s=1$ the environment is called ohmic, for $s<1$ subohmic, and for $s>1$ supraohmic.

High temperature

For high temperature $\beta\Lambda\ll 1$, the L_2 term is proportional to β^{-1} and dominates the classical correlation function. The classical stochastic process will then be *independent* of the precise choice of coarse-graining. The two point function for x will then be

$$\Xi(t,t')=\frac{1}{M^2}\int_0^t dsu(t-s)\int_0^{t'}ds'u(t'-s')\nu(s-s'). \tag{4.23}$$

But this is the correlation function for the solution of the classical stochastic differential equation

$$M\frac{d^2}{dt^2}x(t)+M\omega^2x(t)-\int_0^t ds\,\eta(t-s)x(s)=f(t), \tag{4.24}$$

where $f(t)$ is a Gaussian process with two-point function

$$\langle f(t)f(t')\rangle=\eta(t-t'). \tag{4.25}$$

The noise kernel then gives the correlation function for an external noise perturbing the classical dissipative equations of motion. This justifies its name and recovers results suggested by the path integral techniques [28], or explicitly proved only in particular regimes [6,24]. *Note however, that this is true only when the L_2 term in Eq. (3.24) dominates, as in the case of high temperatures*. In the previous paragraph it is by necessity that the L_1^2 term is small. Here it is not the case, because L_1 also contains a contribution from the environment degrees of freedom that might give a substantial contribution in certain regimes. In the general case the L_1^2 term might be of importance, something that implies that the stochastic limit will not be given by such a simple expression in terms of an external force guided by the noise kernel.

C. Scalar field

Consider now the case of a free, massive scalar field in vacuum. The correlation functions read

$$L_1(\mathbf{x},t;\mathbf{x}',t')=-\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}}e^{-i\mathbf{k}\cdot\mathbf{x}}\sin\omega_{\mathbf{k}}|t-t'|, \tag{4.26}$$

$$L_2(\mathbf{x},t;\mathbf{x}',t')=-\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}}e^{-i\mathbf{k}\cdot\mathbf{x}}\cos\omega_{\mathbf{k}}(t-t'). \tag{4.27}$$

Since $\omega_{\mathbf{k}}\geqslant m$, coarse-grainings with $\sigma^2m^2\gg1$ for each mode will manifest a suppression of the interferences. As in the harmonic oscillator the L_1^2 term will be negligible and the L_2 will provide the classical correlation function.

For a massless field, there is no natural coarse-graining scale for all modes. One needs to take larger values of σ to adequately deal with the infrared modes. Configuration space coarse-graining is clearly bad in such a case. The analysis has to be performed in phase space and provides the same results as the $m\neq0$ case [30].

Consider now the case of a free scalar field in a general globally hyperbolic spacetime. The reader is referred to the standard treatments of Birrell and Davies [31] and Wald [32]. Let us by t denote the time coordinate in a spacelike foliation, by $\mathbf{x}$ the spatial coordinates, and let us use a collective index α to label the modes. The Heisenberg-picture field will read then

$$\hat\phi(\mathbf{x},t)=\sum_\alpha\,[\hat a_\alpha u_\alpha(\mathbf{x},t)+\hat a_\alpha^\dagger\overline{u}_\alpha(\mathbf{x},t)], \tag{4.28}$$

where $u_\alpha(\mathbf{x},t)$ are some complex-valued solutions to the Klein-Gordon equation and

$$[\hat a_\alpha,\hat a_\beta^\dagger]=\delta_{\alpha\beta}. \tag{4.29}$$

Let us assume that the system is found in a Gaussian state $|\Omega\rangle$, which is annihilated by the operator $\hat a_\alpha$. The action of $\hat a^\dagger$ on $|\Omega\rangle$ produces all states of the Fock space. It is easy to compute

$$\langle\Omega|\hat\phi(\mathbf{x},t)\hat\phi(\mathbf{x}',t')|\Omega\rangle=\sum_\alpha\,\overline{u}_\alpha(\mathbf{x}',t')u_\alpha(\mathbf{x},t). \tag{4.30}$$

Let us now consider a particular instant $t=0$ as reference time in which the mode functions $u_\alpha^0(\mathbf{x})=u_\alpha(\mathbf{x},t)$ form an orthonormal basis

$$\int d\mathbf{x}u_\alpha^0(\mathbf{x})u_\beta^0(\mathbf{x})=\delta_{\alpha\beta}, \tag{4.31}$$

$$\int d\mathbf{x}\overline{u}_\alpha^0(\mathbf{x})u_\beta^0(\mathbf{x})=0. \tag{4.32}$$

Here, we wrote as $d\mathbf{x}$ the volume element of the push-backed spacetime metric in the spacelike surface $t=0$. Any scalar function on a instant of time can be decomposed in modes

$$u_\alpha(\mathbf{x},t)=A_{\alpha\beta}(t)u_\beta^0(\mathbf{x})+B_{\alpha\beta}(t)\overline{u}_\beta^0(\mathbf{x}). \tag{4.33}$$

The matrices A and B are the Bogoliubov coefficients and satisfy the matrix identity

$$A^\dagger A-B^\dagger B=1, \tag{4.34}$$

which essentially means that time evolution is given (classically) by a symplectic transformation. The correlation function (4.30) reads then

$$\begin{aligned}\langle\Omega|\hat\phi(\mathbf{x},t)\hat\phi(\mathbf{x}',t')|\Omega\rangle&=\overline{u}^0(\mathbf{x}')A^\dagger(t')A(t)u^0(\mathbf{x})+u^0(\mathbf{x}')B^\dagger(t')B(t)\overline{u}^0(\mathbf{x})\\&\quad+\overline{u}^0(\mathbf{x}')A^\dagger(t')B(t)\overline{u}^0(\mathbf{x})+u^0(\mathbf{x}')B^\dagger(t')A(t)u^0(\mathbf{x}),\end{aligned} \tag{4.35}$$

where a matrix notation has been employed.

This gives for the kernels

$$\begin{aligned}L_1(t,\mathbf{x};t',\mathbf{x}')&=-2\,\theta(t-t')\,\mathrm{Im}\,\overline{u}^0(\mathbf{x}')[A^\dagger(t')A(t)\\&\quad-B^\dagger(t')B(t)]u^0(\mathbf{x})-2\,\theta(t'-t)\,\mathrm{Im}\,\overline{u}^0(\mathbf{x})\\&\quad\times[A^\dagger(t)A(t')-B^\dagger(t)B(t')]u^0(\mathbf{x}')\end{aligned} \tag{4.36}$$

$$\begin{aligned}L_2(t,\mathbf{x};t',\mathbf{x}')&=2\,\mathrm{Re}\,\overline{u}^0(\mathbf{x}')[A^\dagger(t')A(t)\\&\quad+B^\dagger(t')B(t)]u^0(\mathbf{x})\\&\quad+2\,\mathrm{Re}\,\overline{u}^0(\mathbf{x}')A^\dagger(t')B(t)\overline{u}^0(\mathbf{x}).\end{aligned} \tag{4.37}$$

The Bogoliubov transformation is a generalization of the squeezing transformation we studied earlier. For a single mode we had to coarse-grain in regions of configuration space much larger than the uncertainty. It is similar in this case. We can coarse-grain each mode separately at a scale σ^2. From Eq. (4.35) we can read the uncertainty for each mode (defined using the mode functions u^0_α). This will be equal to

$$(\Delta\phi_\alpha)^2=[A^\dagger A+B^\dagger B]_{\alpha\alpha}(t)+[A^\dagger B+B^\dagger A]_{\alpha\alpha}(t). \tag{4.38}$$

The first term is always larger in norm than the second due to Schwarz inequality. The degree of coarse-graining in each mode must be much larger than $(A^\dagger A+B^\dagger B)_{\alpha\alpha}(t)$ for all t. To have a uniform coarse-graining scale for all modes it is necessary that σ^2 has to be much larger than the norm of the matrix

$$||A^\dagger A+B^\dagger B||(t)=||2A^\dagger A-1||(t)<2||A||^2(t). \tag{4.39}$$

The matrix A has a finite norm by virtue of the Bogoliubov identity (4.34). Now, in order to have a meaningful coarse-graining procedure it is necessary that the norms of the Bogoliubov matrices be bounded in time. Hence one can write a sufficient condition for the possibility of a coarse-graining scale σ^2 valid for all modes,

$$\sigma^2\gg \sup_t||A(t)||^2. \tag{4.40}$$

We can then employ Eq. (3.24) to identify the correlation function for the classicalized field. Due to Eq. (4.40) the L_1^2 term will in general be smaller, as in the case of the squeezed system, hence the distribution (4.37) gives the correlation function of the classical stochastic process.

D. Perturbative expansion

So far we have considered only the case of Gaussian processes. We shall now study the case of nonquadratic systems through the use of perturbation theory. Let us by $Z_0[J_+,J_-]$ denote a CTP generating functional that can be exactly evaluated, e.g., a Gaussian. In general, a perturbative expansion around Z_0 will be of the form

$$Z[J_+,J_-]=\exp\left(iF\left[i\frac{\delta}{\delta J_+},-i\frac{\delta}{\delta J_-}\right]\right)Z_0[J_+,J_-], \tag{4.41}$$

in terms of a functional $F[x(\cdot),x'(\cdot)]$, which depends on some coupling constant. In the case that the CTP generating functional depends on configuration space variables, the Hamiltonian is of the form $\hat{H}_0+V(\hat{x})$, and the initial state is the vacuum, we have

$$Z[J_+,J_-]=\exp\left(i\int dtV\left[i\frac{\delta}{\delta J_+(t)}\right]-iV\left[-i\frac{\delta}{\delta J_-(t)}\right]\right)Z_0[J_+,J_-]. \tag{4.42}$$

Substituting $J_\pm=R/(\sqrt{2\pi}\sigma)\pm J/2$ in Eqs. (4.42) yields for the classical generating functional

$$\begin{aligned}Z^{cl}[J]=&\int dR(\cdot)e^{-(1/2\pi)R\cdot R}\\&\times\exp\left\{i\int dt\left(V\left[\frac{i}{2}\frac{\delta}{\delta J(t)}+i\sqrt{\pi/2}\sigma\frac{\delta}{\delta R(t)}\right]\right.\right.\\&\left.\left.-V\left[\frac{i}{2}\frac{\delta}{\delta J(t)}+i\sqrt{\pi/2}\sigma\frac{\delta}{\delta R(t)}\right]\right)\right\}\\&\times Z_0\left[\frac{R}{\sqrt{2\pi}\sigma}+\frac{J}{2},\frac{R}{\sqrt{2\pi}\sigma}-\frac{J}{2}\right].\end{aligned} \tag{4.43}$$

If we assume that the length scale by which the potential varies is much larger than σ we can keep the lower order term in σ in the exponential to get

$$\begin{aligned}Z^{cl}[J]=&\int dR(\cdot)e^{-(1/2\pi)R\cdot R}\\&\times\exp\left(-\int dt\sqrt{\pi/2}\sigma V'\left[\frac{i}{2}\frac{\delta}{\delta J(t)}\right]\frac{\delta}{\delta R(t)}\right)Z_0[R,J]\\\simeq&Z_0^{cl}[J]-\int dt\frac{1}{2\sqrt{2\pi}}\sigma V'\left[\frac{i}{2}\frac{\delta}{\delta J(t)}\right]\\&\times\int dR(t)R(t)e^{-(1/4\pi)R\cdot R}Z_0[R,J],\end{aligned} \tag{4.44}$$

where $Z_0[J]$ is defined from $Z_0[J_+,J_-]$ via Eq. (3.9) and we wrote for brevity $Z_0[R,J]=Z_0[R/\sqrt{2\pi}\sigma+J/2,R/\sqrt{2\pi}\sigma-J/2]$.

The above expression is the leading order in σ of the generating functional and is valid only *when the potential V is assumed to vary in macroscopic scales*.

Note that as we see from Eq. (4.44), a perturbation expansion of the quantum theory does *not* generically amount to a perturbative expansion in the corresponding classical limit. These results are of relevance for the study of perturbation theory in quantum Brownian motion [33,34].

V. CONCLUDING REMARKS

Let us now summarize our results. We first showed that the quantum mechanical correlation functions do not correspond to the statistical properties of a physical system and hence do not correspond to a classical stochastic process. Then we explained the relation of Schwinger and Keldysh's CTP generating functional to the decoherence functional of the consistent histories approach to quantum theory. This enabled us to use the decoherence condition for histories in order to develop a procedure of going from a quantum process to a stochastic process that corresponds to a given degree of coarse-graining for a class of observables. The end result was Eq. (3.9) that gives the relation between the classical generating functional and the CTP generating functional. But we should keep in mind that any results of the

stochastic description that give some detailed structure in scales smaller than σ are unreliable.

We then proceeded to study examples. We showed that in Gaussian processes the classical limit is a Gaussian process with a correlation function given by the *real part* of the quantum two-point function plus a term that depends on the coarse-graining scale. The second term is negligible in general, but this is not necessary in the case of quantum Brownian motion at low temperatures, because then it largely depends on the properties of the bath rather than the initial condition of the distinguished system.

Now, what is the use of these results? The answer is that a stochastic description might be more amenable to our intuition than a quantum one. For instance, given a stochastic process, one can simulate the evolution of individual systems and identify typical behaviors. This is something that *cannot be done* in quantum theory. We have no (uncontroversial) way to describe the random evolution of an individual system's observables, while in classical probability we can write equations of the Langevin type. In this sense, the stochastic approximation captures an aspect of the quantum mechanical randomness that might be manifested on macroscopic or even intermediate scales. Hence for the Brownian particle, the knowledge of the noise due to the environment might be sufficient for certain applications.

Substituting the full quantum behavior for an approximate description in terms of the Langevin equation is necessary in cases where the quantum system acts as a driving force term for another classical one. This is the case, for instance, of any detector coupled to a quantum field. While one often uses a quantum description for a detector, this is clearly inappropriate for realistic systems; a detector is a macroscopic system and the description of its effective behavior in terms of a simple Schrödinger equation is an idealization that is hardly justified from first principles. On the other hand, a detector can be treated as a classical system—e.g., with respect to its center of mass—that is coupled to a stochastic driving force, that arises from the fluctuation of the quantum system, with which it is coupled. In this case the stochastic description of a quantum system is not only convenient, but necessary.

The situation is similar in the case of quantum field theory in curved spacetime as far as the issue of back-reaction is concerned. The quantum field acts as a source for the classical gravitational field via its stress-energy tensor. One writes then the semiclassical gravity equation

$$G_{\mu\nu}=\kappa\langle T_{\mu\nu}\rangle. \tag{5.1}$$

There is an underlying assumption in any use of this equation. As it stands it is nonsensical; on the left hand side is an observable for an individual system and on the right hand side an ensemble average. What is implied is that (i) the behavior of the quantum field is in some regime approximately deterministic and (ii) the corresponding classical value of the stress energy tensor is equal to the expectation value over the field's state.

Of course these two assumptions need to be verified. In order for the quantum field to behave classically, coarse-graining is necessary. Even with coarse-graining it is not necessary that the system will classicalize. For instance, in the one dimensional case, time evolution might cause a continuous increase of the squeeze parameter in a given mode. In this case no fixed degree of coarse-graining is good at all times and there is very little we can do with a classical description. Let us assume, however, that this is not the case. According to our earlier analysis, this would mean that we can choose a coarse-graining scale satisfying Eq. (4.40). Then, one might expect that the system will exhibit stochastic behavior for at least some of its properties.

In order for assumption (i) to be valid, the stochastic process for the stress-energy tensor should have small deviations from its mean. This is rarely true, as in simple spacetimes one can show that the quantum fluctuations of the stress-energy tensor are of the same order of magnitude as its mean value [35]. This leads to the point that the semiclassical description of back reaction of quantum fields onto geometry ought to have a stochastic component [36–42].

This is where our results in Sec. IV C are of relevance. First, we argued that the condition (4.40) is sufficient to obtain classicality; then we showed that it is essentially the real part of the two-point function of the fields that gives the two-point function at the classical limit. Now the stress-energy tensor is a quadratic functional of the fields

$$T^{\mu\nu}(x)=\tfrac{1}{2}\nabla^{\mu}\phi(x)\nabla^{\nu}\phi(x)-\tfrac{1}{2}g^{\mu\nu}\phi(x)\times(-\nabla_{\rho}\nabla^{\rho}+m^{2})\phi(x). \tag{5.2}$$

Its expectation value can be read from the two-point function via point splitting and renormalization. This means that we can define

$$\langle T^{\mu\nu}(x)\rangle=\lim_{x'\to x}\tfrac{1}{2}[1/2(\nabla_{x}^{\mu}\nabla_{x'}^{\nu}+\nabla_{x}^{\mu}\nabla_{x'}^{\nu})-1/2[g^{\mu\nu}(x)+g^{\mu\nu}(x')](\nabla_{x}^{\rho}\nabla_{x'\rho}+m^{2})\Delta_{2}(x,x'). \tag{5.3}$$

Now, the imaginary point of the two-point function vanishes as $t\to t'$ as can be easily checked by Eq. (4.35). Since this part is antisymmetric the symmetrization of the derivatives in the definition (5.3) implies that it vanishes. Hence, the *quantum mechanical expectation of the stress-energy tensor equals the classical stochastic expectation*. However, as we said before, one needs to take the higher order correlations into account in order to have a consistent back reaction. In this case, the naive prescription of using the quantum mechanical correlation functions of the stress-energy tensor breaks down. The correct higher order correlation functions for the stress-energy tensor ought to be constructed from the classical correlations (4.37). Clearly, the stochastic process for the stress-energy tensor is *not* Gaussian, even though it is obtained from the Gaussian process for the field ϕ.

The general conclusion in this description is that the back-reaction of quantum fields to geometry can be described by a stochastic differential equation of the type

$$G_{\mu\nu} = \kappa T_{\mu\nu}[\phi], \tag{5.4}$$

where $T^{\mu\nu}$ is a random variable, a functional of the classicalized field $\phi(\cdot)$ that is defined by a Gaussian stochastic process with a generating functional given by Eq. (4.37). This result is conditional upon (4.40) holding that defines the possibility of having a robust coarse-graining.

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