2D anti–de Sitter gravity as a conformally invariant mechanical system

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(Received 28 September 2000; published 21 May 2001)

We show that two-dimensional (2D) AdS gravity induces on the spacetime boundary a conformally invariant dynamics that can be described in terms of a de Alfaro–Fubini–Furlan model coupled to an external source with conformal dimension 2. The external source encodes information about the gauge symmetries of the 2D gravity system. Alternatively, there exists a description in terms of a mechanical system with anholonomic constraints. The considered systems are invariant under the action of the conformal group generated by a Virasoro algebra, which occurs also as an asymptotic symmetry algebra of two-dimensional anti–de Sitter space. We calculate the central charge of the algebra and find perfect agreement between the statistical and thermodynamical entropies of AdS_2 black holes.

DOI: 10.1103/PhysRevD.63.125021 PACS number(s): 11.10.Kk, 04.50.+h, 04.70.Dy, 11.25.Hf

Recent investigations have brought evidence of a deep connection between two-dimensional (2D) gravitating systems and conformal mechanics [1]. The most natural context to test this conjecture is the anti–de Sitter/conformal field theory (AdS/CFT) correspondence [2] in two spacetime dimensions [3]. In fact, for $D=2$ this correspondence essentially states that gravity on AdS_2 should be described by a conformally invariant quantum mechanics.

Most of the progress in this direction has been achieved in the context of 2D dilaton gravity, mainly because of the simplicity of the model [4–6]. For these models the conformal symmetry (generated by a Virasoro algebra) has a natural interpretation in terms of the asymptotic symmetries of the gravitational system. Moreover, the central charge of the algebra, whose value is crucial for calculating the statistical entropy of 2D black holes, can be calculated using the deformation algebra of the boundary of AdS_2 [4].

Despite the simplicity of the model, the various attempts to identify the conformal quantum mechanics that should be dual to gravity on AdS_2 and to calculate the entropy of 2D black holes by counting states of the CFT met only partial success [4–6]. The conformal mechanics involved could not be identified and a mismatch of a $\sqrt{2}$ factor between thermodynamical and statistical entropy was found.

The puzzle has become even more intricate in view of the results of Ref. [7]. There it was shown that in $D=2$ the AdS/CFT correspondence has a realization in terms of a two-dimensional CFT, which is essentially a theory of open strings with Dirichlet boundary conditions. Moreover, it was shown that the degeneracy of states of the 2D CFT explains correctly the thermodynamical entropy of 2D black holes.

In this paper we show that the boundary dynamics induced by AdS_2 dilaton gravity can be described by a de Alfaro–Fubini–Furlan (DFF) model [8] coupled to an external source with conformal dimension 2. The external source encodes information about the gauge symmetries of the 2D gravity system. Alternatively, the dynamics of the boundary fields admits an equivalent description in terms of a mechanical system with anholonomic constraints. In both cases, the mechanical system is invariant under the action of the full one-dimensional conformal group generated by a Virasoro algebra, which also appears as asymptotic symmetry algebra of AdS_2. We compute the central charge of this algebra and find perfect agreement between statistical and thermodynamical entropy of AdS_2 black holes.

Our starting point is the Jackiw-Teitelboim (JT) model [9], with action

$$I=\frac{1}{2}\int d^2x\sqrt{-g}\,\eta[R+2\lambda^2], \tag{1}$$

where η represents the dilaton. Two-dimensional anti–de Sitter space, or more generally black holes in AdS_2, are solutions of this model [11]:

$$ds^2=-(\lambda^2r^2-a^2)dt^2+(\lambda^2r^2-a^2)^{-1}dx^2, \quad \eta=\eta_0\lambda x, \tag{2}$$

where a is given in terms of the black hole mass M, $a^2=2M/\lambda$. The thermodynamical black hole entropy is given by

$$S=4\pi\sqrt{\frac{\eta_0 M}{2\lambda}}. \tag{3}$$

The asymptotic symmetries of AdS_2 are defined as the transformations which leave the asymptotic form of the metric invariant; i.e., they preserve the large r behavior

$$g_{tt}=-\lambda^2r^2+\gamma_{tt}(t)+\mathcal{O}\left(\frac{1}{r^2}\right), \tag{4}$$

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$$g_{tr}=\frac{\gamma_{tr}(t)}{\lambda^3 r^3}+\mathcal{O}\left(\frac{1}{r^5}\right),$$

$$g_{rr}=\frac{1}{\lambda^2 r^2}+\frac{\gamma_{rr}(t)}{\lambda^4 r^4}+\mathcal{O}\left(\frac{1}{r^6}\right),$$

where the fields $\gamma_{\mu\nu}$ parametrize the first sub-leading terms in the expansion and can be interpreted as deformations of the boundary.

The asymptotic form (4) is preserved by infinitesimal diffeomorphisms $\chi^\mu(x,t)$ of the form [4]

$$\chi^t=\epsilon(t)+\frac{\ddot{\epsilon}(t)}{2\lambda^4 r^2}+\frac{\alpha^t(t)}{r^4}+\mathcal{O}\left(\frac{1}{r^5}\right),$$

$$\chi^r=-r\dot{\epsilon}(t)+\frac{\alpha^r(t)}{r}+\mathcal{O}\left(\frac{1}{r^2}\right), \tag{5}$$

where $\epsilon(t)$ and $\alpha^\mu(t)$ are arbitrary functions, the α^μ describing pure gauge diffeomorphisms. In Ref. [4] it was shown that the symmetries (5) generate a Virasoro algebra.

The asymptotic behavior of the scalar field η, compatible with the transformations (5), must take the form

$$\eta=\eta_0\left(\lambda\rho(t)r+\frac{\gamma_\eta(t)}{2\lambda r}\right)+\mathcal{O}\left(\frac{1}{r^3}\right), \tag{6}$$

where ρ and γ_η play a role analogous to that of the $\gamma_{\mu\nu}$. Introducing the new fields, invariant under the pure gauge diffeomorphisms parametrized by α^μ,

$$\beta=\frac{1}{2}\rho\gamma_{rr}+\gamma_\eta,$$

$$\gamma=\gamma_{tt}-\frac{1}{2}\gamma_{rr}, \tag{7}$$

the equations of motion following from the action (1) yield, in the limit $r\to\infty$,[1]

$$\lambda^{-2}\ddot{\rho}=\rho\gamma-\beta, \tag{8}$$

$$\dot{\rho}\gamma+\dot{\beta}=0. \tag{9}$$

Equations (8) and (9) determine a mechanical system with anholonomic constraint, since the one-form

$$\omega\equiv\gamma d\rho+d\beta \tag{10}$$

is not exact. The Lagrange equations of the first kind for the fields $\varphi_i=\{\rho,\beta,\gamma\}$ read

$$F_i-m_i\ddot{\varphi}_i+\Lambda\omega_i=0, \tag{11}$$

where F_i is the force that can be derived from a potential U, $F_i=-\partial_i U$, m_i denote the masses of the fields, Λ is a Lagrange multiplier, and the ω_i are the components of the one-form ω. If we choose

$$m_\rho=\lambda^{-1},\quad m_\beta=m_\gamma=0, \tag{12}$$

and

$$U=\lambda\beta\rho, \tag{13}$$

the Lagrange equations (11) yield Eq. (8), together with the Lagrange multiplier $\Lambda=\lambda\rho$. Before we proceed, we note that from Eqs. (8) and (9), one gets the conservation law

$$T+U=\frac{1}{2}\lambda^{-1}\dot{\rho}^2+\lambda\beta\rho=\text{const.} \tag{14}$$

Notice that $T+U$ is essentially the mass of the black holes considered in [4].

The boundary fields φ_i span a representation of the full infinite dimensional group generated by the Killing vectors (5). In fact, under the asymptotic symmetries (5), they transform as

$$\delta\rho=\epsilon\dot{\rho}-\dot{\epsilon}\rho,$$

$$\delta\beta=\epsilon\dot{\beta}+\dot{\epsilon}\beta+\frac{\ddot{\epsilon}\dot{\rho}}{\lambda^2}, \tag{15}$$

$$\delta\gamma=\epsilon\dot{\gamma}+2\dot{\epsilon}\gamma-\frac{\dddot{\epsilon}}{\lambda^2}.$$

The above transformations are easily recognized as (anomalous) transformation laws for conformal fields of weights $-1,1,2$ respectively. We are interested in the transformation laws of the equation of motion (8), the constraint (9), and the conserved charge (14). Using Eq. (15), we get

$$\delta[\gamma\dot{\rho}+\dot{\beta}]=\epsilon\frac{d}{dt}[\gamma\dot{\rho}+\dot{\beta}]+2\dot{\epsilon}[\gamma\dot{\rho}+\dot{\beta}]+\ddot{\epsilon}\left[-\gamma\rho+\beta+\frac{\ddot{\rho}}{\lambda^2}\right], \tag{16}$$

$$\delta\left[-\gamma\rho+\beta+\frac{\ddot{\rho}}{\lambda^2}\right]=\epsilon\frac{d}{dt}\left[-\gamma\rho+\beta+\frac{\ddot{\rho}}{\lambda^2}\right]+\dot{\epsilon}\left[-\gamma\rho+\beta+\frac{\ddot{\rho}}{\lambda^2}\right], \tag{17}$$

$$\delta[T+U]=\epsilon\frac{d}{dt}[T+U]. \tag{18}$$

[1]At first sight, it seems not necessary to require Eq. (9), since it comes from the leading term in the stress tensor component T_{rt}, which is of order $1/x^2$ [12]. However, $T_{\mu\nu}$, transforming as a tensor, is clearly not invariant under coordinate transformations. In fact, in the light-cone coordinates used below, Eq. (9) originates from an order 1 term, so requiring Eq. (9) is really necessary for consistency.

We see that the constraint transforms like a conformal field of weight 2 with anomaly term. The conformal weights of the equation of motion and the conserved charge are 1 and 0 respectively, and anomalies are absent. The above equations imply that on shell the constraint and the equation of motion are invariant under the transformations (15).

Alternatively, we may describe the dynamical system (8) in terms of the DFF model [8] of conformal mechanics, coupled to an external source. To this aim, we start from the conservation law (14), i.e., $T+U=c$. Introducing the new field $q=\sqrt{\rho/\lambda}$, which has conformal dimension $-1/2$, and eliminating β from Eq. (14) by means of Eq. (8), we arrive at the equation

$$\ddot{q}-\frac{g}{q^3}=\frac{\lambda^2}{2}\gamma q, \tag{19}$$

with $g=-c/(2\lambda)$, whereas from Eq. (14) follows

$$\frac{\dot{q}^2}{2}+\frac{g}{2q^2}=-\frac{\lambda^2}{4}\beta. \tag{20}$$

One can easily check the equivalence of the system (8),(9) with Eqs. (19),(20). Moreover, Eq. (19) is easily recognized as the equation of motion for the DFF model coupled to an external source γ.

At this point it is straightforward to write down an action for the dynamical system (19). It is given by

$$I=\int dt\left[\frac{1}{2}\dot{q}^2-\frac{g}{2q^2}+\frac{1}{4}\lambda^2\gamma q^2\right]. \tag{21}$$

Equation (21) resembles very much the IR-regularized action proposed by DFF [8], the only difference consisting of the fact that the external source γ, which couples to the field q, is not constant, but represents an operator of conformal dimension 2. Note that in the calculation of δI, γ, being an external source, is not varied. One can easily show that the action is (up to a total derivative) invariant under the conformal transformations (15).

The dynamical system described by Eqs. (8),(9) [or equivalently by Eq. (19)] defines a one-dimensional conformal field theory (CFT_1). The invariance group of the model coincides with the group of asymptotic symmetries of AdS_2 and can be realized as the $diff_1$ group describing time reparametrizations $\delta t=\epsilon(t)$. In analogy with 2D CFT, one would like to identify the stress-energy tensor T_{tt} associated with the CFT_1. This analogy suggests that T_{tt} is proportional to the constraints (9):

$$T_{tt}=\lambda(\dot{\rho}\gamma+\dot{\beta}). \tag{22}$$

(The constant of proportionality has been chosen to ensure that T_{tt} has the dimensions of a mass squared.) In fact, from Eqs. (8), (9) and (16) it is evident that T_{tt} plays the same role as the holomorphic T_{++} (and antiholomorphic T_{--}) stress-energy tensors play in 2D CFT. A more compelling argument leading to Eq. (22) relies on the identification of T_{tt} as the boundary value for the stress-energy tensor of a 2D CFT.

To do this, we choose the conformal gauge

$$ds^2=-e^{2\omega}dx^+dx^-. \tag{23}$$

Then the action (1) takes the Liouville-like form

$$I=-\int d^2x\left(\partial_+\omega\partial_-\eta+\partial_-\omega\partial_+\eta-\frac{1}{2}\lambda^2\eta e^{2\omega}\right). \tag{24}$$

The action must be complemented by the constraints (the equations of motion for the missing components of the metric)

$$T_{\pm\pm}=\partial_\pm^2\eta-2\partial_\pm\eta\partial_\pm\omega=0, \tag{25}$$

where $T_{\pm\pm}$ denote the components of the stress-energy tensor.

For $\eta\to\infty$ the potential term in the action (24) goes to zero and the model becomes an exact 2D CFT [7]. In fact, defining the new fields X,Y,

$$\omega=X-Y,\quad \eta=X+Y, \tag{26}$$

the action and the constraints become

$$I=-2\int d^2x(\partial_+X\partial_-X-\partial_+Y\partial_-Y), \tag{27}$$

$$T_{\pm\pm}=\partial_\pm^2X+\partial_\pm^2Y-2\partial_\pm X\partial_\pm X+2\partial_\pm Y\partial_\pm Y=0. \tag{28}$$

Taking $\eta\to\infty$ we reach the boundary of AdS_2, which in light-cone coordinates is located at $x^+=x^-$. One can show that $T_{++}|_{boundary}=T_{--}|_{boundary}=T_{tt}$, with T_{tt} given by Eq. (22).

Let us now show that T_{tt} generates the $diff_1$ group. Introducing the charges

$$\hat{J}=\int \epsilon T_{tt}, \tag{29}$$

and using the transformation law (16), one has

$$\delta_\epsilon T_{tt}=[\hat{J},T_{tt}]=\epsilon\dot{T}_{tt}+2\dot{\epsilon}T_{tt}. \tag{30}$$

Expanding in Fourier modes,

$$T_{tt}=\sum L_m e^{-im\lambda t},\quad \epsilon(t)=\sum a_m e^{-im\lambda t}, \tag{31}$$

and using Eq. (30), one finds that L_m generates a Virasoro algebra:

$$[L_m,L_m]=(m-n)L_{m+n}. \tag{32}$$

Equation (22) does not give the most general form of the CFT_1 stress-energy tensor. We have the freedom to add to it a constant term (which we choose proportional to the black hole mass M) and an improvement term:

$$T_{tt}|_{impr}=T_{tt}+\lambda M+b\ddot{\rho}, \tag{33}$$

where b is an arbitrary constant. The improvement term is a total derivative and does not affect the status of T_{tt} as generator of the diff$_1$ group. Though the constant b is at the CFT level undetermined, it can be computed using the underlying gravitational dynamics. Comparing Eq. (33) with the expression for T_{tt} found in Ref. [5], one gets $b=-2\eta_0$.

We can now calculate the central charge C associated with the Virasoro algebra (32). A naive computation will give the value $C=24\eta_0$ found in Ref. [5]. This can be shown explicitly using the general transformation law of a CFT$_1$ stress-energy tensor:

$$\delta T_{tt}=\epsilon\dot{T}_{tt}+2\dot{\epsilon}T_{tt}+\frac{C}{12}\dddot{\epsilon}. \tag{34}$$

Using Eqs. (15) and (16), one can easily show that T_{tt} given by Eq. (33) follows the transformation law (34), with $C=24\eta_0$. This value of the central charge, once inserted in the Cardy formula, $S=2\pi\sqrt{CL_0/6}$ [10], produces a black hole entropy, which differs by a $\sqrt{2}$ factor from the thermodynamical value (3).

However, in this computation (and in those of Refs. [4,5] as well), one only considers the contribution of the $r\to\infty$ boundary of AdS$_2$. The black hole solution (2) has also an inner boundary [11,4] (located at $r=0$ for the ground state or at the horizon for the generic black hole), which can give a contribution to the central charge. That this inner boundary can be crucial for understanding the black hole entropy has been shown in [13].

There is a simple way to compute this contribution. We need to change the coordinates, from the Schwarzschild (r,t) frame used in Eq. (2) to the conformal frame, where the vacuum and black hole solutions have, respectively, the form [11]

$$ds^2=\frac{1}{\lambda^2x^2}(-dt^2+dx^2), \tag{35}$$

$$ds^2=\frac{a^2}{\sinh^2(a\lambda\sigma)}(-d\tau^2+d\sigma^2). \tag{36}$$

The key point is that in the conformal frame the inner boundary (and the horizon) is pushed to $x=\infty$, whereas the timelike $r=\infty$ boundary of AdS$_2$ is now located at $x=0$. The information about the existence of the inner boundary is now encoded in the coordinate transformation

$$t=\frac{1}{a\lambda}e^{a\lambda\tau}\cosh(a\lambda\sigma),\quad x=\frac{1}{a\lambda}e^{a\lambda\tau}\sinh(a\lambda\sigma), \tag{37}$$

which maps the vacuum (35) onto the black hole solution (36) [11].

Evaluating these transformations on the $x=\sigma=0$ boundary, where our one-dimensional conformal field theory resides, we find

$$t=\frac{1}{a\lambda}e^{a\lambda\tau}. \tag{38}$$

Thus the vacuum and black hole solutions correspond to different time variables on the boundary. Moreover, for $-\infty<\tau<\infty$, $0<t<\infty$, part of the ''history'' seen by the vacuum observer cannot be seen by the black hole observer. It follows that there will be a term in the entropy describing the entanglement of states, which has the form of a contribution C_{ent} to the central charge. This contribution can be calculated using a method similar to that employed in Ref. [7] (C_{ent} is interpreted as a Casimir energy).

The transformation law of the stress-energy tensor under a general change of coordinates $t=t(\tau)$ is given by the transformation (34) in its finite form

$$T_{\tau\tau}=\left(\frac{dt}{d\tau}\right)^2T_{tt}-\frac{C_{ent}}{12}\left(\frac{dt}{d\tau}\right)^2\{\tau,t\}, \tag{39}$$

where $\{\tau,t\}$ is the Schwarzian derivative. Applied to the transformation (38), Eq. (39) gives

$$T_{\tau\tau}=(a\lambda t)^2T_{tt}-\frac{C_{ent}}{24}a^2\lambda^2. \tag{40}$$

From Eq. (33) it follows that by fixing the diffeomorphisms invariance on shell, we can always have $T_{\tau\tau}=\lambda M$. In fact, on shell the term proportional to the constraints is zero and we can always choose $\rho=$const. Because T_{tt} in Eq. (40) refers to the vacuum, we have $T_{tt}=0$, and Eq. (40) becomes

$$\lambda M=-\frac{C_{ent}}{24}a^2\lambda^2. \tag{41}$$

The coordinate transformation (38) maps the ground state into the black hole with mass $M=a^2\eta_0\lambda/2$ [11], which inserted into Eq. (41) gives

$$C_{ent}=-12\eta_0. \tag{42}$$

Notice that the entanglement contribution is negative, yielding a total central charge

$$C_{tot}=C+C_{ent}=12\eta_0. \tag{43}$$

Using this value of the central charge in Cardy's formula, one finds perfect agreement with the thermodynamical entropy of the 2D black hole (3).

Let us now discuss the physical interpretation of the dynamical system (19) and its relationship with 2D dilaton gravity. The equation of motion (19) describes a mechanical system coupled to an external source. Alternatively, one can think of γ as a time-dependent coupling constant, appearing in the harmonic oscillator potential.

Because γ is arbitrary [the only constraint on it is the transformation law (15)], the dynamical system is essentially non-deterministic. Moreover, because the external source is time dependent, the energy is not conserved and we have a time-dependent Hamiltonian, whose evolution is not completely fixed by the dynamics of the system.

Strictly speaking we have to deal with an ensemble of Hamiltonians. One has thus a strong analogy with disordered systems in statistical mechanics. The main difference is that, whereas in the case of disordered systems we have a probability distribution for the couplings, in our case they are arbitrary functions, for which we only give the transformation law under the conformal group. Moreover, in our case γ is a smooth function of t.

From the point of view of the 2D gravitational theory the meaning of the source γ is clear. The fields β and γ describe deformations of the boundary of AdS_2, generated by 2D bulk diffeomorphisms, whereas the field ρ describes deformations of the dilaton. Thus, the function γ encodes information about the gauge symmetry of the 2D gravity theory. The non-deterministic nature of the dynamical system (19) is a consequence of the gauge freedom of the gravitational dynamics. This indicates an interesting relationship between gauge symmetries and non-deterministic dynamical systems.

In Ref. [5], it was pointed out that the non-constant value of the dilaton ($\rho \neq 0$ in terms of boundary fields) breaks the $SL(2,R)$ isometry group of AdS_2 and that the origin of the central charge (and hence of the black hole entropy) can be traced back to this breaking. Different values of ρ represent different vacua of the 2D gravity theory which break $SL(2,R)$. Indeed the conformal transformations (15) (which are the boundary counterpart of the 2D diffeomorphisms) map all these vacua one into the other. Moreover, because the energy (14) is invariant (on shell) under conformal transformations, all these vacua are degenerate in energy.

Now, the crucial point is that there is a one-to-one correspondence between these vacua and the solutions of the dynamical system (19). Note that also the (IR-regularized) DFF model (21) breaks the $SL(2,R)$ symmetry of the original model if we take the source γ constant. Introducing a time-dependent external source, transforming as a conformal field of weight 2, we reinstate the full conformal symmetry. From the point of view of 2D gravity we are considering different ρ-dependent vacua.

The above considerations indicate a natural way to explain statistically the entropy of 2D black holes. This entropy can be interpreted in terms of the degeneracy of the ρ vacua. The energy (14) is invariant under conformal transformations, so that we can calculate the entropy by counting the independent excitations in the configuration space of vacua.

From the point of view of the dynamical system this degeneracy is encoded in the external source γ. The quadratic mass-temperature and mass-entropy dependence [7], which is typical of a 2D CFT and which, in principle, could be used to rule out the duality of AdS_2 gravity with a conformal quantum mechanics, is presumably related to the fact that the conformally invariant mechanical system in question is not a usual mechanical system but a DFF model coupled to an external source.

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