Possible extensions of the 4D Schwarzschild horizon in the 5D brane world

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We show that, in the absence of matter in the bulk, the Einstein equations and the Gauss-normal form of the metric place stringent restrictions on the form of the event horizon in a brane world. As a consequence, the off-brane extension of the standard four-dimensional (4D) Schwarzschild horizon in the Randall-Sundrum AdS_5 spacetime, as it is viewed from the brane can only be of a tubular shape, instead of a pancake shape. When it is viewed from the AdS_5 horizon, such a tubular horizon is absent.

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I. INTRODUCTION

A recent breakthrough in high-energy physics is the possibility of observing large extra space dimensions. It has been conjectured that the standard model particles are confined to propagate within a three-dimensional (3D) brane embedded in a space of $(4+d)$ dimensions. On the contrary gravitons can escape and propagate also in the bulk. The most important recognition is that the energy scale of the extra dimensions can be as low as few TeV and their signature may show up in future colliders (see, for example, Ref. [1]). Since gravity is formulated by $(4+d)$-D general relativity, the gravitational force within the brane may contain detectable corrections to Newton's law [2]. The recovery of general relativity on the *physical brane* has become an interesting issue to explore.

One implementation of such a brane world scenario was proposed by Randall and Sundrum [3]. The physical brane in their model is the junction of two pieces of 5D spacetime manifolds that are asymptotically anti–de Sitter spacetime. The Gauss-normal form of the metric in this space is

$$ds^2=e^{-2\kappa|y|}\overline{g}_{\mu\nu}dx^\mu dx^\nu+dy^2, \tag{1}$$

with the brane located at $y=0$ and $\kappa>0$ sets the energy scale of the extra space dimension. The metric $\overline{g}_{\mu\nu}$ is determined by the 5D Einstein equations

$$R_{mn}-\frac{1}{2}Rg_{mn}-\Lambda g_{mn}=-4\pi^2G_5T_{\mu\nu}\delta_m^\mu\delta_n^\nu\delta(y)+6\kappa g_{\mu\nu}\delta_m^\mu\delta_n^\nu\delta(y), \tag{2}$$

where the cosmological constant $\Lambda=-6\kappa^2$, G_5 is the 5D gravitational constant and $T_{\mu\nu}$ the energy-momentum tensor on the brane. Here and throughout the paper, we adopt the convention that the Greek indices take values 0–3 and the Latin indices 0–4. The 4D gravitational constant is given by $G\sim G_5/\kappa$. In the absence of matter, $T_{\mu\nu}=0$, $\overline{g}_{\mu\nu}=\eta_{\mu\nu}$ is a solution of Eq. (2). Subsequently, the metric in Eq. (1) becomes that of AdS_5. The solution to Eq. (2) can also be obtained from the solution to the sourceless equation, $T_{\mu\nu}=0$, subject to the appropriate Israel matching condition [4] determined by $T_{\mu\nu}$. The perturbative solution of Eq. (2) to the linear order in G for an arbitrary $T_{\mu\nu}$ [5] and to second order G^2 for a static spherical mass distribution on the brane [6] reveals no tangible disagreement with the classical tests of 4D general relativity at large κ.

It is generally believed that 4D general relativity is recovered beyond the weak coupling expansion on the physical brane for large κ [7]. By combining the Schwarzschild metric on the physical brane and the profile of the linear gravitational potential off the physical brane, the authors of Ref. [5] conjectured a pancake shaped horizon for the physical black hole, a gravitational field generated by a mass point on the brane. In this paper, we shall make some rigorous statements on the form of the horizon of a physical black hole, which when confined to the physical brane reproduces the standard 4D Schwarzschild horizon. We find that the Einstein equations together with a Gauss-normal form of the metric imply particular types of horizons but the pancake shape does not belong to these types.

II. EINSTEIN EQUATIONS IN 5D

The 5D Einstein sourceless equations can be rewritten as

$$R_{\mu\nu}-4\kappa^2g_{\mu\nu}=0,\quad R_{y\mu}=0,\quad R_{yy}-4\kappa^2=0. \tag{3}$$

The most general metric in $D=4+1$ dimensions produced by a static, spherically symmetric matter distribution on the physical brane has the following form:

$$ds^2=e^{-2\kappa|y|}(-e^adt^2+e^bdr^2+e^cr^2d\Omega^2)+dy^2, \tag{4}$$

where $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$ is the solid angle on S^2 and a, b, and c are functions of r and y. Let us pause for a moment to explain this particular form of the metric. The most general, axially symmetric and static metric in $D=4+1$ dimensions can be written as

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$$ds^2=-A(\rho,\eta)dt^2+B(\rho,\eta)d\rho^2+C(\rho,\eta)\rho^2 d\Omega^2 +2D(\rho,\eta)d\rho d\eta+E(\rho,\eta)d\eta^2. \tag{5}$$

This form can be simplified further if we set $B(\rho,\eta)d\rho^2+2D(\rho,\eta)d\rho d\eta+E(\rho,\eta)d\eta^2=\lambda_1^2\delta_1^2+\lambda_2^2\delta_2^2$ where λ_1^2,λ_2^2 are the positive eigenvalues of the matrix $\left(\begin{smallmatrix} B & D\\ D & E\end{smallmatrix}\right)$, where $\delta_1=\cos\chi d\eta-\sin\chi d\rho$ and $\delta_2=\sin\chi d\eta+\cos\chi d\rho$. $\lambda_1^2,\lambda_2^2,\chi$ are known functions of ρ, η. Furthermore lets introduce $\Theta=\Theta(\rho,\eta)$ and write $\lambda_1^2\delta_1^2+\lambda_2^2\delta_2^2=\Delta_1^2+\Delta_2^2$ with

$$\begin{aligned}\Delta_1&=\lambda_1\delta_1\cos\Theta-\lambda_2\delta_2\sin\Theta,\\ \Delta_2&=\lambda_1\delta_1\sin\Theta+\lambda_2\delta_2\cos\Theta.\end{aligned} \tag{6}$$

Next we choose Θ such that $\Delta_1=dy$. As a result Θ satisfies the following first order partial differential equation:

$$\frac{\partial}{\partial\rho}(\lambda_1\cos\chi\cos\Theta-\lambda_2\sin\chi\sin\Theta)+\frac{\partial}{\partial\eta}(\lambda_1\sin\chi\cos\Theta +\lambda_2\cos\chi\sin\theta)=0. \tag{7}$$

Since Δ_2 is a differential form of two variables, there always exists an integration factor μ such that $\Delta_2=\mu dr$. Upon expressing all functions in terms of r and y and by identifying $A=e^{-2\kappa|y|+a}, \rho^2C=r^2e^{-2\kappa|y|+c}, \mu^2=e^{-2\kappa|y|+b}$ we recover the Gauss-normal form of the metric, Eq. (4).

Substituting the metric (4) into Eq. (3), we obtain the following components of the Einstein equation outside the source:

$$\begin{aligned}R_{tt}+4\kappa^2e^{-2\kappa y+a}=&\frac{1}{2}e^{a-b}\left[-a''-\frac{2}{r}a'+\frac{1}{2}a'(-a'+b'-2c')\right]\\ &-\frac{1}{2}e^{-2\kappa y+a}\left[\ddot{a}-5\kappa\dot{a}-\kappa\dot{b}-2\kappa\dot{c}+\frac{1}{2}\dot{a}(\dot{a}+\dot{b}+2\dot{c})\right]=0,\end{aligned} \tag{8}$$

$$\begin{aligned}R_{rr}-4\kappa^2e^{b}=&\frac{1}{2}a''+c''-\frac{1}{r}b'+\frac{2}{r}c'+\frac{1}{4}a'(a'-b')-\frac{1}{2}c'(b'-c')\\ &+\frac{1}{2}e^{-2\kappa y+b}\left[\ddot{b}-5\kappa\dot{b}-\kappa\dot{a}-2\kappa\dot{c}+\frac{1}{2}\dot{b}(\dot{a}+\dot{b}+2\dot{c})\right]=0,\end{aligned} \tag{9}$$

$$\begin{aligned}R_{\theta\theta}-4\kappa^2r^2e^{c}=&\frac{1}{2}r^2e^{c-b}\left[c''+\frac{4}{r}c'+\frac{a'-b'}{r}+c'^2+\frac{1}{2}(a'-b')c'\right]\\ &+\frac{1}{2}r^2e^{-2\kappa y+c}\left[\ddot{c}-\kappa(\dot{a}+\dot{b})-6\kappa\dot{c}+\frac{1}{2}\dot{c}(\dot{a}+\dot{b}+2\dot{c})\right]+e^{c-b}-1=0,\end{aligned} \tag{10}$$

$$\begin{aligned}R_{yy}-4\kappa^2&=\frac{1}{2}(\ddot{a}+\ddot{b}+2\ddot{c})-\kappa(\dot{a}+\dot{b}+2\dot{c})+\frac{1}{4}(\dot{a}^2+\dot{b}^2+2\dot{c}^2)=0,\\ R_{ry}=R_{yr}&=\frac{1}{2}\left[\dot{a}'+2\dot{c}'-\frac{2}{r}(\dot{b}-\dot{c})+\frac{1}{2}a'(\dot{a}-\dot{b})-c'(\dot{b}-\dot{c})\right]=0,\end{aligned} \tag{11}$$

where the prime denotes a partial derivative with respect to r and the dot denotes a partial derivative with respect to y. These equations apply to the positive side of the brane, $y>0$, the corresponding equations to the negative side of the brane, $y<0$, are obtained by switching the sign of κ.

III. POSSIBLE EXTENSIONS OF THE 4D SCHWARSCHILD HORIZON

In this section we shall formulate the concept that would define an event horizon in the 5D brane world. We assume that there exists a solution of the 5D Einstein equations which satisfies the Israel matching condition across the brane and maintains the Lorentzian signature in a certain region $\mathcal{P}$ of the parametric r-y plane. We call $\mathcal{P}$ the physical region. The physical region may or may not cover the entire r-y plane. An example of a physical region that does not cover the entire parametric space is the AdS C metric discussed in Ref. [8]. We also assume that there exists an asymptotic region $\mathcal{A}$ within $\mathcal{P}$, where the metric components e^a, e^b, and e^c are well approximated by linear gravity and the functions a, b, and c are well behaved beyond $\mathcal{A}$. As we have seen, this is the case for both brane-based coordinates and for AdS_5 horizon-based coordinates. Starting from $\mathcal{A}$, we trace all possible light rays given by $ds^2=0$ or more specifically by

$$dt^2=e^{b-a}dr^2+e^{c-a}r^2d\Omega^2+e^{2\kappa y-a}dy^2 \tag{12}$$

until we come to a point from which the light cannot propagate forward in certain spatial direction. By continuity, the union of these points forms a 4D hypersurface $\mathcal{H}$, which we refer to as an event horizon. We shall consider the case that $\mathcal{H}$ lies within (not coincide with the border of) $\mathcal{P}$.

To prevent light propagation, some of the coefficients on the right hand side of Eq. (12) need to become sufficiently divergent so that an increment of the corresponding spatial coordinate would take infinite amount of time t. Therefore, we can determine the form of the horizon by finding the locus of the logarithmic singularities of the functions a, b, and c. This locus has to be consistent with the Einstein equations (8)–(11).

We denote the locus of the logarithmic singularities of a, b, and c by $H(r,y)=0$ and an arbitrary point on it by $P(r_0,y_0)$. The unit normal vector $\vec{n}$ and the unit tangent vector $\vec{t}$ at P on r-y plane are

$$\vec{n}=(\cos\alpha,\sin\alpha),\quad \vec{t}=(-\sin\alpha,\cos\alpha), \tag{13}$$

where

$$\cos\alpha=\frac{1}{\Delta}\left(\frac{\partial H}{\partial r}\right)_P,\quad \sin\alpha=\frac{1}{\Delta}\left(\frac{\partial H}{\partial y}\right)_P \tag{14}$$

and

$$\Delta=\sqrt{\left(\frac{\partial H}{\partial r}\right)_P^2+\left(\frac{\partial H}{\partial y}\right)_P^2}. \tag{15}$$

Let us consider a point $Q(r,y)$ in the neighborhood of P and let us transform the coordinates into

$$\begin{aligned}\xi&=(r-r_0)\cos\alpha+(y-y_0)\sin\alpha,\\ \eta&=-(r-r_0)\sin\alpha+(y-y_0)\cos\alpha.\end{aligned} \tag{16}$$

As $\xi\to 0$ and $\eta\to 0$, we expect that

$$e^a\simeq A\xi^{n_a},\quad e^b\simeq B\xi^{n_b},\quad e^c\simeq C\xi^{n_c}, \tag{17}$$

where n_a, n_b, and n_c are integers due to the reality requirement of the metric components on both sides of the horizon, and A, B, and C are numerical constants. The integers n_a, n_b, and n_c could not all be zero, otherwise there would be no singularity at the point P. The Lorentzian signature in the physical region forbids e^c and the metric determinant $e^{-8\kappa y+a+b+2c}r^4\sin^2\theta$ from changing their signs. These considerations restrict both n_c and $n_a+n_b+2n_c$ to be even. By continuity, these exponents are maintained along the entire H. Therefore,

$$a\simeq n_a\ln\xi,\quad b\simeq n_b\ln\xi,\quad c\simeq n_c\ln\xi. \tag{18}$$

Taking into account that

$$\begin{aligned}\frac{\partial}{\partial r}&=\cos\alpha\frac{\partial}{\partial\xi}-\sin\alpha\frac{\partial}{\partial\eta},\\ \frac{\partial}{\partial y}&=\sin\alpha\frac{\partial}{\partial\xi}+\cos\alpha\frac{\partial}{\partial\eta},\end{aligned} \tag{19}$$

their derivatives behave as

$$\dot{a}\simeq\frac{n_a}{\xi}\sin\alpha,\quad \dot{b}\simeq\frac{n_b}{\xi}\sin\alpha,\quad \dot{c}\simeq\frac{n_c}{\xi}\sin\alpha \tag{20}$$

and

$$a'\simeq\frac{n_a}{\xi}\cos\alpha,\quad b'\simeq\frac{n_b}{\xi}\cos\alpha,\quad c'\simeq\frac{n_c}{\xi}\cos\alpha. \tag{21}$$

We observe that the derivatives of a, b, c become singular as Q approaches P. As a result of substituting Eqs. (20) and (21) into Eqs. (8)–(11), the cancellation of the leading singularities in $R_{yy}-4\kappa^2$ and in R_{ry}, $O(1/\xi^2)$, leads to two conditions on n_a, n_b, n_c, and α:

$$E\sin^2\alpha=0,\quad F\sin\alpha\cos\alpha=0 \tag{22}$$

with

$$E=-n_a-n_b-2n_c+\frac{1}{2}(n_a^2+n_b^2+2n_c^2) \tag{23}$$

and

$$F=-n_a-2n_c+\frac{1}{2}n_a(n_a-n_b)-n_c(n_b-n_c). \tag{24}$$

There are three types of solutions to be analyzed.

Type I: $E\neq 0$. The only solution is $\sin\alpha=0$, which implies a tube shaped H:

$$\left(\frac{\partial H}{\partial y}\right)_P=0. \tag{25}$$

The standard Schwarszchild horizon of a physical black hole belongs to this type. In the coordinates straight with respect to the brane, the recovery of 4D general relativity on the brane for $\kappa GM\gg 1$ implies that

$$ds^2\simeq-\left(1-\frac{2GM}{r}\right)dt^2+\frac{dr^2}{1-\frac{2GM}{r}}+r^2d\Omega^2 \tag{26}$$

for $y=0$. The standard Schwarszchild horizon at $r\simeq 2GM$ corresponds to the exponents $n_a=1$, $n_b=-1$, and $n_c=0$. Therefore, since the integer combination $E\neq 0$, the off-brane extension of the horizon is a tube as is shown in Fig. 1(a). It

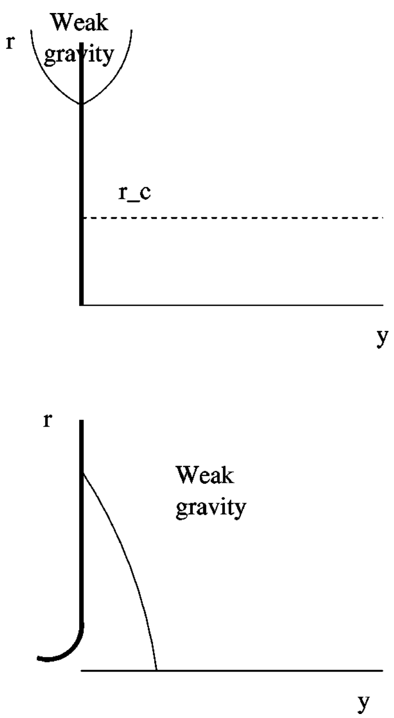

FIG. 1. A physical black hole viewed in two coordinate systems: (a) In the coordinates based on the brane, where the thick line denotes the brane and the dashed line the off-brane extension of the 4D Schwarzschild horizon. The Schwarzschild radius is $r_c = 2GM[1+O(1/\kappa^2 G^2 M^2)]$ and the mass point is located at $r=y=0$. (b) In the coordinates based on the AdS_5 horizon, where the thick line denotes the bent brane.

either extends to infinity in the y direction, similar to the horizon of the black cigar solution of Chamblin-Hawking-Reall [9] though the explicit forms of the solution are different, or it terminates at the border of the physical region, similar to the example in Ref. [8]. This rules out the possibility of a pancake shaped Schwarzschild horizon in 5D. On the other hand, in the coordinates straight with respect to the AdS_5 horizon, the validity of linear gravity for large r or large positive y excludes the tubular form of the horizon completely, as is shown in Fig. 1(b). Because of the brane bending in the negative y direction and the failure of the linear approximation there, we are unable to rule out the possibility of horizons of the subsequent two types in this coordinate system.

Before analyzing the next two types, we notice that the solutions to $E=0$ correspond to points with integer coordinates lying on an oblate spheroid with axis 2 and $\sqrt{2}$. There are only twelve of them, and we should exclude the solutions with odd n_c or odd $n_a+n_b+2n_c$ and the ones that make none of e^{b-a}, e^{c-a} and e^{-a} divergent. Consequently, we have the following.

Type II: $E=0$ but $F\neq 0$. The only solutions in this case are $(n_a, n_b, n_c)=(2,\ 2,\ 0)$ or $(2,\ 2,\ 2)$ with either $\sin\alpha=0$ or $\cos\alpha=0$. The latter implies a horizon parallel to the plane $y=0$ and it may exist in the coordinates straight with respect to the AdS_5 horizon for a physical black hole.

Type III: $E=0$ and $F=0$. The qualified solutions for (n_a, n_b, n_c) are (2, 0, 0) and (2, 0, 2). They are also consistent with the other components of the Einstein equations. We have not found yet any restrictions on the shapes of the corresponding horizons. The first solution of the integer triplet corresponds to the isotropic coordinates of a black hole ($n_a=2$, $n_b=n_c=0$), which can be obtained through a y-independent coordinate transformation from the standard metric. It can be shown, however, that a y-dependent and Gauss-normal form preserving transformation that leaves the brane intact does not exist. Therefore, this particular horizon cannot be transformed into a pancake shaped one.

For a general Gauss-normal form of the metric (1), we may define the 4×4 matrix of $\overline{g_{\mu\nu}}$ by $\mathcal{G}$, and the $R_{yy}-4\kappa^2$ equation can be written as

$$R_{yy}-4\kappa^2=\frac{1}{2}\left(\frac{\partial}{\partial y}e^{-2\kappa y}\frac{\partial}{\partial y}\ln(-\det\mathcal{G})\right)+\frac{1}{4}\operatorname{tr}\mathcal{G}^{-1}\frac{\partial\mathcal{G}}{\partial y}\mathcal{G}^{-1}\frac{\partial\mathcal{G}}{\partial y}=0. \tag{27}$$

Now our statements regarding the Schwarzschild horizon can be generalized to a spinning black hole located on the brane $y=0$. Assume that the 4D Kerr metric [10] is recovered on the brane for sufficiently large κ, then we have

$$ds^2\simeq-\frac{\Delta}{\rho^2}(dt-j\sin^2\theta d\phi)^2+\frac{\sin^2\theta}{\rho^2}[(r^2+j^2)d\phi-j dt]^2+\frac{\rho^2}{\Delta}dr^2+\rho^2 d\theta^2, \tag{28}$$

for $y=0$, where $\Delta=r^2-2GMr+j^2$ and $\rho^2=r^2+j^2\cos^2\theta$ with M the mass and j the angular momentum per unit mass. A horizon corresponds to the solutions of $\Delta=0$, that might have two solutions or none. It is straightforward to show that the metric determinant $\det\mathcal{G}=-(r^2+j^2\cos^2\theta)^2\sin^2\theta$ and its logarithm are free from the horizon singularities. On the other hand, the matrix $\mathcal{G}^{-1}(\partial\mathcal{G}/\partial r)$ contains a $1/\Delta$ singularity at the horizon. If the 5D extension of this horizon were bent towards or away from the y axis, this singularity would be shared by the matrix $\mathcal{G}^{-1}(\partial\mathcal{G}/\partial y)$ off the brane. This is again forbidden by the Einstein equation (27).

Having a pancake shaped horizon as the 5D extension of the Schwarszchild horizon is physically implausible. If this were the case, somewhere off the brane and near the horizon, we would expect that

$$e_a\sim y-y_c,\quad e^b\sim\frac{1}{y-y_c}, \tag{29}$$

with y_c a function of r. It follows then from Eq. (12) that it would take a finite amount of time for a light beam coming out of the horizon to propagate in the y direction. Consequently the black hole would appear to be luminating in the y direction.

The conclusions we have reached above depend on the form of the metric, Eq. (1), and the Einstein equations. We might try to relax the Gauss-normal form of the metric. In the case of a spherical black hole, we might consider the metric

$$ds^2 = e^{-2\kappa|y|}(-e^u dt^2 + e^v dr^2 + r^2 d\Omega^2) + e^f dy^2, \quad (30)$$

where u, v, f are functions of r and y. The equation $R_{yy} - 4\kappa^2 e^f = 0$ then becomes

$$\begin{aligned}&\ddot{u} + \ddot{v} - 2\kappa(\dot{u} + \dot{v} - 2\dot{f}) - \frac{1}{2}\dot{f}(\dot{u} + \dot{v}) + \frac{1}{2}(\dot{u}^2 + \dot{v}^2)\\ &\quad + e^{2\kappa y + f - v}\left[f'' + \frac{2}{r}f' + \frac{1}{2}f'(f' + u' - v')\right]\\ &\quad + 8\kappa^2(1 - e^f) = 0. \end{aligned} \quad (31)$$

Let us assume that an approximate Schwarszchild metric can be recovered on the brane, which is located at $y=0$ and that the 5D extension of the horizon is given by $H(r,y)=0$. If we consider an arbitrary point on it, $P(r_0, y_0)$, with the variable ξ defined as before, we have

$$u \sim \ln \xi, \quad v \sim -\ln \xi, \quad f \sim n \ln \xi \quad (32)$$

with n being an even integer. Then the leading singularity on the left hand side of Eq. (31) results from either

$$\ddot{u} + \ddot{v} + \frac{1}{2}(\dot{u}^2 + \dot{v}^2) \sim \frac{1}{\xi^2}\sin^2\alpha \quad (33)$$

or

$$e^{2\kappa y + f - v}\left(f'' + \frac{1}{2}f'^2\right) \sim \left(-n + \frac{1}{2}n^2\right)\xi^{n-1}\cos^2\alpha. \quad (34)$$

For $n < -1$, Eq. (34) represents the leading singularity, the only solution is $\cos\alpha = 0$ and this horizon will not join the Schwarszchild horizon on the brane. If $n > -1$, Eq. (33) represents the leading singularity, and the only solution is $\sin\alpha = 0$. Therefore, the 5D extension of Schwarszchild horizon is again a tube.

The existence of a 5D extension of a 4D Schwarzschild horizon for a physical black hole relies on the recovery of 4D general relativity on the brane for large κ. The rigorous statements we made on the possible shapes of the horizon, however, are independent of the value of κ, since the terms of the first derivative with respect to y in Eqs. (8)–(11) do not contribute to the leading singularity of a horizon. In the following, we illustrate various types of horizons for a 5D Schwarzschild metric $(\kappa = 0)$, i.e.,

$$ds^2 = -\left(1 - \frac{l^2}{R^2}\right)dt^2 + \frac{dR^2}{1 - \frac{l^2}{R^2}} + R^2(d\chi^2 + \sin^2\chi d\Omega^2), \quad (35)$$

where l is the Schwarzschild radius, R is the radial coordinate, and χ is a polar angle on S^3, $d\Omega$ is the solid angle on S^2. If we introduce cylindrical coordinates $r' = R\sin\chi$ and $y' = R\cos\chi$, we find

$$\begin{aligned} ds^2 = &-\left(1 - \frac{l^2}{R^2}\right)dt^2 + \frac{l^2}{1 - \frac{l^2}{R^2}}\frac{(r'dr' + y'dy')^2}{R^2} + dr'^2\\ &+ r'^2 d\Omega^2 + dy'^2, \end{aligned} \quad (36)$$

and we have a circular horizon on r'-y' plane at the expense of introducing off-diagonal terms of the metric. To enforce the Gauss-normal form of the metric, consider the transformation

$$R = R(r,y), \quad \chi = \chi(r,y). \quad (37)$$

The functions $R(r,y)$ and $\chi(r,y)$ have to satisfy the conditions

$$\begin{aligned} &\frac{R^2}{R^2 - l^2}\left(\frac{\partial R}{\partial y}\right)^2 + R^2\left(\frac{\partial \chi}{\partial y}\right)^2 = 1,\\ &\frac{1}{R^2 - l^2}\frac{\partial R}{\partial r}\frac{\partial R}{\partial y} + \frac{\partial \chi}{\partial r}\frac{\partial \chi}{\partial y} = 0. \end{aligned} \quad (38)$$

There is a simple solution with $R = \sqrt{y^2 + l^2}$ and χ an arbitrary function of r only, which results in a Gauss-normal form of the metric (36), i.e.,

$$\begin{aligned} ds^2 = &-\frac{y^2}{y^2 + l^2}dt^2 + (y^2 + l^2)\left(\frac{d\chi}{dr}\right)^2 dr^2\\ &+ (y^2 + l^2)\sin^2\chi d\Omega^2 + dy^2. \end{aligned} \quad (39)$$

The horizon, $y=0$, then becomes of type III. This transformation, however, is singular on the horizon, more specifically the Jacobian $\partial(R,\chi)/\partial(r,y)$ is zero. If a solution to Eq. (38) that is nonsingular at the horizon could be found, then we would have a horizon of type I, since $\lim_{R\to l}(\partial R/\partial y) = 0$ in any case in order to balance the first equation of Eq. (38) and the nonvanishing Jacobian demands that $\lim_{R\to l}(\partial R/\partial r) \neq 0$.

IV. CONCLUSIONS

In summary, we have found that the vacuum Einstein equations together with a Gauss-normal form of the metric place fairly stringent restrictions on the form of the event horizon. The suggested pancake shaped extension of the 4D Schwarzschild horizon cannot exist in the Gauss-normal form of the metric. In the coordinate system based on the brane [11], the event horizon has a tubular shape extending possibly to infinity in agreement with recent numerical studies [12], while in the coordinate system based on the AdS_5 horizon, the region of validity of linear gravity excludes this possibility. The horizon then might belong to either type II or type III. The technique of balancing the leading singularities in the Einstein equation that we have developed in the present work may be generalized to other singularities of the solution, e.g., the curvature singularity. Since our analysis does not depend on the value of κ, our classification of horizons applies to the large extra compact dimension scenario $M^4\times S^1$, which corresponds to $\kappa=0$ and periodicity in the y direction. Our rigorous statements on the shape of the horizon provide not only hints to the form of the exact solution, but may also impact on the stability of it. It is also interesting to explore the metric of a physical black hole numerically. We hope to report our progress towards these directions in the near future.

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