More remarks on the electromagnetic properties of a domain wall interacting with charged fermions

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The response to a magnetic flux is considered of the vacuum state of charged Dirac fermions interacting with a domain wall made of a neutral spinless field in 3+1 dimensions with the fermion mass term having a phase variation across the wall. It is pointed out that as a result of simple C parity arguments the spontaneous magnetization for this system is necessarily zero, thus invalidating some claims to the contrary in the literature. The cancellation of the spontaneous magnetization is explicitly demonstrated in a particular class of models. The same calculation produces a general formula for the electric charge density induced by the magnetic flux—an effect previously discussed in the literature for axionic domain walls. The distribution of the induced charge is calculated in specific models.

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I. INTRODUCTION

The possibility that domain walls that could have existed in the early universe could also be related to the generation of a primordial magnetic field correlated at large distances [1] has been recently discussed in the literature [2–5]. The models discussed are based on the idea that fermions, coupled to the field forming the wall, develop a spontaneous magnetization perpendicular to the wall. Although it is a simple exercise in general physics to show that a uniform magnetization of an infinite domain wall does not produce a magnetic field [6], the phenomenon of magnetization of the wall is interesting on its own.

The claims to a nonzero magnetic moment of the ground state of a fermion field coupled to the wall in 3+1 dimensions are inferred from the behavior in 2+1-dimensional QED of Dirac fermions with a definite sign of the mass term m. Namely, in certain calculations [2,7–9] of the energy of the ground state of the fermion field in an external magnetic field B, it is claimed that the total energy contains a linear in magnetic field term proportional to mB, which corresponds to a spontaneous magnetization proportional to the mass parameter m (including the sign). For a Dirac fermion field coupled to a domain wall in 3+1 dimensions the quantization of the motion perpendicular to the wall splits the fermion system into an infinite set of modes, each corresponding to a (2+1)-dimensional QED with its own parameter m. If the phase of the fermion mass term varies across the wall, the set of positive values of m differs from that of the negative values of m. Therefore in this picture it might at least be not obvious that the overall magnetization cancels after summation over modes corresponding to positive and negative values of m.

It is nevertheless quite easy to argue that the cancellation necessarily takes place and the magnetization of the fermion field ground state at the wall in 3+1 dimensions is strictly zero. Indeed, the Lagrangian density for the fermions with the phase of the mass μ depending on the coordinate z perpendicular to the wall can be generally written as

$$L=\overline{\psi}[i(\partial_\alpha-iA_\alpha)\gamma^\alpha-\mu_1(z)-i\mu_2(z)\gamma^5]\psi, \tag{1}$$

with μ_1 and μ_2 being, respectively, the real and imaginary parts of $\mu(z)$, and A standing for the electromagnetic field potential and absorbing the charge e in the normalization of the field. The variation of μ_2 breaks the P and CP parities. However, the C parity is manifestly conserved with both μ_1 and μ_2 being C even. Therefore after the fermions are integrated out the energy of the system as a function of A in the C-even background of $\mu(z)$ cannot contain odd powers of the C-odd field A. In particular the energy cannot be linear in the magnetic field B. The same argument holds also for the situation where the fermions are assumed to possess an anomalous magnetic and/or electric dipole moment since both such interactions also conserve the C parity.

This simple general C parity argument is clearly sufficient for excluding the possibility of spontaneous magnetization of the fermion vacuum in a domain wall background. However, as a result of the existence in the literature of claims to the contrary [3,9], it is quite instructive to demonstrate explicitly the vanishing of the linear in magnetic field $B_z=B$ term in the energy (as well as of all the odd terms) at least in a specific model. Moreover, the C parity argument certainly allows a dependence of the energy on *even* powers of the electromagnetic field, which gives rise to the most interesting phenomenon of the appearance of an electric charge density once a magnetic flux is applied across the wall. This phenomenon is related to the well-known coupling of a pseudoscalar field to the electromagnetic invariant $\mathbf{E}\cdot\mathbf{B}$. For a slowly varying with z mass term, one can approximate $\mu(z)$ around a given point z_0 as $\mu(z)=\mu(z_0)+\delta\mu(z)$ and, treating the varying part $\delta\mu(z)$ as a small perturbation, find the term proportional to $\mathbf{E}\cdot\mathbf{B}$ in the energy density w from the well-known triangle graph as

$$\delta w=\frac{1}{4\pi^2}\,\frac{\mu_2\,\delta\mu_1-\mu_1\,\delta\mu_2}{\mu_1^2+\mu_2^2}\,\mathbf{E}\cdot\mathbf{B}. \tag{2}$$

The charge density is then found from variation with respect to the potential A_0. For the magnetic field in the z direction, $B_z=B$, one finds

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$$\langle \overline{\psi}(z)\gamma^0\psi(z)\rangle = -\frac{\delta w}{\delta A_0} = \frac{B}{4\pi^2}\frac{d}{dz}\arctan\left(\frac{\mu_2(z)}{\mu_1(z)}\right). \quad (3)$$

The total charge is then given by the total flux of the magnetic field through the wall, $F=\int dx dy B_z$, and the difference of the phases of the mass term $\mu_1+i\mu_2$ at two infinities in z, $\Delta\Phi = \arctan(\mu_2/\mu_1)|_{z\to+\infty} - \arctan(\mu_2/\mu_1)|_{z\to-\infty}$:

$$Q=\int dx dy dz \langle \overline{\psi}(z)\gamma^0\psi(z)\rangle = \frac{F\Delta\Phi}{4\pi^2}. \quad (4)$$

The induced charge is a direct analogue of the (generally fractional) fermionic charge of a kink in (1+1)-dimensional models [10,11]. This phenomenon in 3+1 dimensions was considered [12] in connection with a magnetic monopole (dyon) traversing an axionic domain wall, in which process the net change (reversal) of the magnetic flux across the wall results in charge exchange between the dyon and the wall.[1]

In the specific model considered in the present paper, the calculation of the induced charge density automatically comes along with the calculation of the (eventually vanishing) spontaneous magnetization. It will be shown that, as expected on general grounds, the relation (4) between the total induced charge and the total magnetic flux does not depend on the specific shape of the dependence of the phase Φ on z. However the *distribution* of the induced charge density in z does depend on the specific rate of variation of the mass parameter $\mu(z)$, and generally differs from that given by Eq. (3), which is justified in the limit where the rate of variation can be considered as slow. One can notice in this connection that the charge distribution may be of greater physical relevance than the total charge, since barring the existence of monopoles it is physically impossible to produce a net magnetic flux through an infinite or closed wall. Therefore the total charge has to be zero, while the distribution of the density of charge can be nontrivial.

The further material in this paper is organized as follows. In Sec. II the class of models considered is described as well as some properties of the relevant operators corresponding to motion in the x-y plane in a magnetic field and to motion in the z direction in the domain wall background are discussed. In Sec. III the dependence on B of the energy of the ground state of the fermion field is calculated and the vanishing of all odd terms in the expansion in B is demonstrated, including the vanishing of the spontaneous magnetization. In Sec. IV a general expression for the induced charge density is presented, and Sec. V contains the calculation of this density in the limit of a slowly varying phase as well as a discussion of the topological nature of the total induced charge. An explicit calculation of the distribution of the induced charge in two particular sample situations is considered in Sec. VI. Finally, a general discussion and a summary of results are presented in Sec. VII.

[1]Reference [12] contains a reference to an unpublished communication with H. Georgi and J. Polchinski, who apparently had also interpreted the effect of the triangle graph in terms of an induced charge in a magnetic field. However, as they both kindly corresponded to me, they had never pursued this issue beyond an unwritten remark. The monopole-axion wall charge exchange was also later discussed by Kogan [13].

II. THE MODEL AND THE RELEVANT OPERATORS

The simplifying assumption in the class of models to be considered here is that the real part μ_1 of the mass term is fixed and nonzero, while the imaginary part μ_2 depends on z. To the best of the author's knowledge, a model of such type with μ_2 being proportional to a spinless field ϕ, $\mu_2 = g\phi$, with the field varying as $\phi(z)=v\tanh(m_\phi z/2)$ across the domain wall, was first suggested by Lee and Wick [14] as a model of spontaneous breaking of CP symmetry. The fermion spectrum and scattering states in the presence of the domain wall in this model were studied in Ref. [15].

In the generic case of a nonzero change of μ_2 between the infinities in z and a nonzero constant μ_1, one can assume for definiteness without further loss of generality that μ_1 is positive and that $\mu_2(z)$ changes from a negative value at $z\to-\infty$ to a positive one at $z\to+\infty$. Also for definiteness it is assumed here that a uniform positive magnetic field B is applied in the z direction. In what follows the gauge for the electromagnetic field is fixed in a standard way such that the vector potential for the field B is given by $A_x=0$, $A_y=Bx$. Adopting also the standard representation for the γ matrices, the one-particle Hamiltonian, corresponding to the Lagrangian (1), takes the following form:

$$H=\begin{pmatrix} \mu_1 & 0 & iP^\dagger & -iR \\ 0 & \mu_1 & iR^\dagger & iP \\ -iP & -iR & -\mu_1 & 0 \\ iR^\dagger & -iP^\dagger & 0 & -\mu_1 \end{pmatrix}, \quad (5)$$

and H^2 has the diagonal form

$$H^2=\mathrm{diag}(\mu_1^2+P^\dagger P+RR^\dagger, \mu_1^2+PP^\dagger + R^\dagger R, \mu_1^2+PP^\dagger+RR^\dagger, \mu_1^2+P^\dagger P+R^\dagger R). \quad (6)$$

These formulas make use of the following notation for the operators describing, respectively, the Landau quantization of the motion in the x-y plane and the quantization of motion along the z axis:

$$R=\partial_x+Bx+p_y, \quad R^\dagger=-\partial_x+Bx+p_y \quad (7)$$

and

$$P=\partial_z+\mu_2(z), \quad P^\dagger=-\partial_z+\mu_2(z). \quad (8)$$

The quantity p_y is the value of the conserved momentum in the y direction (as a consequence of the chosen gauge condition). The energy levels, determined by the eigenvalues of $R^\dagger R$ and $RR^\dagger$, do not depend on p_y and the degeneracy number is well known to be given by $BS/(2\pi)$, where S is the normalization area in the x-y plane. In what follows this degeneracy factor will be explicitly accounted for and the value of p_y set to $p_y=0$ in the definition of the operators (7).

According to Eq. (6) the spectrum of one-particle energies is determined by eigenvalues of the operators $R^\dagger R$, $RR^\dagger$, $P^\dagger P$, and $PP^\dagger$. These spectra exhibit two separate structures found in supersymmetric quantum mechanics, well known for the operators R and $R^\dagger$ and also recently used for P-type operators in connection with kinks in (1+1)-dimensional models [16]. Namely, if the issue of boundary conditions in z is ignored, one would naively conclude that the spectra of eigenvalues of $P^\dagger P$ and $PP^\dagger$ coincide except for an extra zero eigenvalue (under the adopted sign conventions) of $P^\dagger P$. Indeed, let v_k, $k=1, 2, \ldots$, be the normalized eigenfunction corresponding to the (necessarily positive) eigenvalue λ_k^2 of the positive operator $PP^\dagger$, so that

$$PP^\dagger v_k=\lambda_k^2 v_k . \tag{9}$$

Applying the operator $P^\dagger$ to both sides of this equation, one finds that the function

$$u_k=P^\dagger v_k/\lambda_k \tag{10}$$

is the normalized eigenfunction of the operator $P^\dagger P$ with the same eigenvalue λ_k^2. Applying the operator P to both sides of the latter relation and using Eq. (9), one finds the inverse of the relation (10):

$$v_k=Pu_k/\lambda_k . \tag{11}$$

This construction does not work, however, for the zero mode u_0 of $P^\dagger P$, satisfying the equation $Pu_0=0$ [Bogomol'nyi-Prasad-Sommerfield (BPS) state]. The explicit form of the normalizable function $u_0(z)$ is readily found from the definition (8):

$$u_0(z)=\text{const}\times\exp\left(-\int^z \mu_2(\tilde{z})d\tilde{z}\right). \tag{12}$$

The operators $R^\dagger$ and R coincide, up to normalization, with the creation and annihilation operators for a harmonic oscillator, and their spectra are the textbook ones: the spectrum of eigenvalues of $RR^\dagger$ is given by $2Bn$ with $n=1, 2, \ldots$, while that of $R^\dagger R$ is given by the same simple expression, however also including $n=0$.

This discussion of the properties of the operators involved in the Hamiltonian in Eq. (5) is helpful in considering the spectrum of the one-particle energies. In particular, one can separately consider each eigenmode of the motion in the z direction as a (2+1)-dimensional fermion system. Then the eigenmodes of the operator $P^\dagger P$ correspond to such systems with positive mass parameter m, $m=\sqrt{P^\dagger P+\mu_1^2}$, while the eigenmodes of $PP^\dagger$ correspond to negative m, $m=-\sqrt{PP^\dagger+\mu_1^2}$. For the former modes the negative energy spectrum of Landau levels is given by $-\sqrt{P^\dagger P+\mu_1^2+2Bn}$ and includes $n=0$, while for the latter ones the negative energies are given by $-\sqrt{PP^\dagger+\mu_1^2+2Bn}$ excluding $n=0$. Since all the nonzero eigenvalues of $P^\dagger P$ and $PP^\dagger$ coincide, one might very naively conjecture that their effects in magnetization cancel and the net result is given by only one ''unpaired'' zero mode of the operator $P^\dagger P$. Such a conjecture, however, would be false, since the summation over the modes is generally divergent, and one should perform a proper calculation with a proper regularization. Also the discretization of the continuum spectra of the operators $P^\dagger P$ and $PP^\dagger$ requires imposing conditions at the boundaries of a large but finite bounding box in z. These conditions generally are not satisfied by the relations (10) and (11), and there arises a splitting of the spectra of continuum modes of the operators $P^\dagger P$ and $PP^\dagger$ which should be accounted for in a proper calculation.

III. DEPENDENCE OF THE FERMION VACUUM ENERGY ON *B*

For a full regularized calculation of the energy of the ground state of the fermion field we use here the standard four-dimensional technique with the Pauli-Villars regularization procedure. The latter regularization preserves gauge invariance and, importantly for the discussed problem, Lorentz covariance.[2] This amounts to introducing the regulator fermion field Ψ with large but finite mass, so that the mass term for Ψ can be written as $M+i\mu_2\gamma^5$ instead of $\mu_1+i\mu_2\gamma^5$ for the ''physical'' fermion ψ, and treat the loop with the regulator field with an extra minus sign.[3] The regularized expression for the total energy W then reads as

$$\begin{aligned} W=&\, i\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\{\text{Tr}\ \ln[iD_\alpha\gamma^\alpha-\mu_1-i\mu_2(z)\gamma^5] \\ &-\text{Tr}\ \ln[iD_\alpha\gamma^\alpha-M-i\mu_2(z)\gamma^5]\} \\ =&\, i\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\,\text{Tr}\int_{\mu_1}^{M}dm[iD_\alpha\gamma^\alpha-m-i\mu_2(z)\gamma^5]^{-1}, \end{aligned} \tag{13}$$

where $iD_\alpha\gamma^\alpha=\gamma^0p_0-\boldsymbol{\gamma}\cdot(\mathbf{p}+\mathbf{A})$ with the spatial momentum $\mathbf{p}$ understood as the operator, and the trace running over the spinor indices and the spatial variables.

The inverse of the Dirac operator in the last expression in Eq. (13) can be readily found by the usual multiplication of the numerator and the denominator by $iD_\alpha\gamma^\alpha+m-i\mu_2(z)\gamma^5$. After taking into account the previously mentioned degeneracy in the momentum p_y, the result can be written as

[2] The vanishing of spontaneous magnetization can in fact be viewed as due to the possibility of using a regularization preserving the Lorentz symmetry.

[3] Only one regulator field is indicated here for simplicity of expressions, whereas the full regularization of the vacuum energy requires additional regulator fields. However, for the considered here effects in the energy one regulator is in fact sufficient.

$$W=i\frac{BS}{2\pi}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\mathrm{Tr}\int_{\mu_1}^{M}dm \times\begin{pmatrix} p_0+m & 0 & -iP^\dagger & iR \\ 0 & p_0+m & -iR^\dagger & -iP \\ -iP & -iR & -p_0+m & 0 \\ iR^\dagger & -iP^\dagger & 0 & -p_0+m \end{pmatrix} \times[p_0^2-H^2(m)]^{-1}, \tag{14}$$

where $H^2(m)$ is the same diagonal matrix as in Eq. (6) with μ_1 being replaced by m. The trace over the spinor variables leaves only the contribution of the diagonal factors in Eq. (14), which thus can be rewritten as

$$W=i\frac{BS}{2\pi}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\int_{\mu_1}^{M}dm\left\{\mathrm{Tr}\left[\frac{p_0\gamma^0}{p_0^2-H^2(m)}\right] +\mathrm{Tr}\left[\frac{m}{p_0^2-H^2(m)}\right]\right\}. \tag{15}$$

The first term in the braces is manifestly odd in p_0; thus the integration over p_0 from $-\infty$ to ∞ yields zero. (The integral can be verified to be finite; thus no further regularization, which potentially could invalidate the symmetry argument, is needed.) The second term after explicitly performing the trace over the spinor variables takes the form

$$W=-i\frac{BS}{2\pi}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\int_{\mu_1}^{M}dm\, m\, \mathrm{Tr}\left[\left(\frac{1}{-p_0^2+m^2+P^\dagger P+RR^\dagger}+\frac{1}{-p_0^2+m^2+P^\dagger P+R^\dagger R}\right)+\left(\frac{1}{-p_0^2+m^2+PP^\dagger+RR^\dagger}+\frac{1}{-p_0^2+m^2+PP^\dagger+R^\dagger R}\right)\right]. \tag{16}$$

Here in braces are grouped together the terms with the same ordering of P and $P^\dagger$ and different order of R and $R^\dagger$. Each of the two expressions in the braces has the general form

$$\mathrm{Tr}\left(\frac{1}{X+RR^\dagger}+\frac{1}{X+R^\dagger R}\right), \tag{17}$$

where the operator X does not depend on the magnetic field. It is now easy to show that the expression (17) contains only odd powers in its expansion in B and thus that the expansion in B of the vacuum energy described by Eq. (16) contains only even powers, as required by the C invariance. Indeed, applying the formula $x^{-1}=\int_0^\infty \exp(-\beta x)d\beta$ and using the described spectra of the operators $RR^\dagger$ and $R^\dagger R$, one can perform the trace over the space of the latter operators and find the expression (17) in the form

$$\int_0^\infty d\beta\, \coth(\beta B)\mathrm{Tr}\, \exp(-\beta X), \tag{18}$$

which is manifestly odd in B.[4]

This concludes the proof by an explicit calculation of the absence of odd powers of the magnetic field B in the expansion of the energy of the fermion vacuum.

IV. FERMION CHARGE DENSITY INDUCED BY MAGNETIC FIELD

We now proceed to a calculation of the electric charge induced by the external magnetic field B. The general formula for the charge density $\rho(z)=\langle\bar\psi(z)\gamma^0\psi(z)\rangle$ is obtained from the variational derivative of the energy in Eq. (13) with respect to $A_0(z)$ at $A_0=0$ and reads as

$$\rho(z)=-\frac{i}{S}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi} \times\langle z|\mathrm{Tr}\{\gamma_0[iD_\alpha\gamma^\alpha-\mu_1-i\mu_2(z)\gamma^5]^{-1}\}|z\rangle. \tag{19}$$

The charge density is finite, thus eliminating the need for considering the contribution of the Pauli-Villars regulator. The trace here runs over the spinor indices and over the space of the operators R and $R^\dagger$. The z dependence is thus left ''untraced,'' as is indicated in Eq. (19) by the diagonal matrix element of the remaining (after ''tracing out'' the spinor variables and the motion in the x-y plane) z-dependent operator in the coordinate representation. Performing also the trace over the z dependence, i.e., calculating the integral over z, would yield the total induced charge Q. Using the inverse of the Dirac operator as in Eq. (14) one can calculate the trace over the spinor indices and rewrite the latter formula for ρ as

$$\rho(z)=\mu_1\frac{B}{2\pi}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi} \times\left\langle z\left|\mathrm{Tr}\left[\left(\frac{1}{p_0^2+\mu_1^2+P^\dagger P+R^\dagger R}-\frac{1}{p_0^2+\mu_1^2+P^\dagger P+RR^\dagger}\right)-\left(\frac{1}{p_0^2+\mu_1^2+PP^\dagger+R^\dagger R}-\frac{1}{p_0^2+\mu_1^2+PP^\dagger+RR^\dagger}\right)\right]\right|z\right\rangle. \tag{20}$$

[4]Clearly the first term of the expansion of the expression in Eq. (18), proportional to B^{-1}, results in a divergent expression, which corresponds to the divergence of the vacuum energy at $B=0$. This is the only place where additional Pauli-Villars regulators are formally required. The subsequent terms of the expansion, however, give a finite difference $W(B)-W(0)$ that is even in B.

In the latter expression also the Wick rotation $p_0 \to i p_0$ is done and the part of the integrand that is odd in p_0 is discarded. The trace over the space of the operators R and $R^\dagger$ in Eq. (20) is finite and trivial:

$$\mathrm{Tr}_{(R)}\left(\frac{1}{X+R^\dagger R}-\frac{1}{X+RR^\dagger}\right)=\frac{1}{X}. \tag{21}$$

Using this formula in Eq. (20) one finds

$$\rho(z)=\mu_1\frac{B}{2\pi}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi} \times\left\langle z\left|\frac{1}{p_0^2+\mu_1^2+P^\dagger P}-\frac{1}{p_0^2+\mu_1^2+PP^\dagger}\right|z\right\rangle. \tag{22}$$

One can notice the absence of higher than linear terms in the expansion of the charge density in powers of B, which is a consequence of the relation (21). It can be also observed that up to the factor $B/(2\pi)$ the expression (22) coincides with similarly calculated fermionic charge in a (1+1)-dimensional theory with z-dependent mass term $\mu_1 + i\mu_2(z)\gamma^5$. This relates the present calculations to the problem of the fermion charge of kinks [10,11].

V. CHARGE DENSITY FOR SLOWLY VARYING PHASE AND THE TOPOLOGICAL NATURE OF THE TOTAL CHARGE

We now consider a few specific cases of the dependence of μ_2 on z, in which the charge density can be calculated from Eq. (22) either in a closed form or, at least, ''in quadratures.'' Also the topological nature of the total charge is to be addressed in this section.

The first case to be considered is the one where the rate of variation of the parameter μ_2 can be considered as slow. In this case in order to calculate the charge density at a point z_0, one can approximate $\mu_2(z)$ near z_0 by the linear dependence: $\mu_2(z)\approx\mu_2(z_0)+(z-z_0)\mu_2'(z_0)$ and also expand the difference of the Green's functions in Eq. (22) to linear order in $\mu_2'(z_0)$:

$$\begin{aligned}\rho(z_0)&=2\mu_1\mu_2'(z_0)\frac{B}{2\pi}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\\&\quad\times\langle z_0|[-\partial_z^2+p_0^2+\mu_1^2+\mu_2^2(z_0)]^{-2}|z_0\rangle\\&=2\mu_1\mu_2'(z_0)\frac{B}{2\pi}\int_{-\infty}^{\infty}\frac{dp_0dp_z}{(2\pi)^2}\\&\quad\times[p_z^2+p_0^2+\mu_1^2+\mu_2^2(z_0)]^{-2}\\&=\frac{B}{4\pi^2}\,\frac{\mu_1\mu_2'(z_0)}{\mu_1^2+\mu_2^2(z_0)}\\&=\frac{B}{4\pi^2}\,\frac{d}{dz}\,\arctan\left(\frac{\mu_2(z)}{\mu_1}\right)\bigg|_{z=z_0},\end{aligned} \tag{23}$$

where the final expression matches that in Eq. (3) in the considered case of a constant μ_1. Clearly, for this calculation to be justified the characteristic length of variation of $\mu_2(z)$ should be much longer than the ''local'' Compton wavelength $[\mu_1^2+\mu_2^2(z_0)]^{-1/2}$.

Let us address now the statement that the total integral over the density, $Q=\int_{-\infty}^{\infty}\rho(z)dz$, is in fact topological; i.e., it is determined only by the limiting values of $\mu_2(z)$ at the infinities and does not depend on the specific shape of the function $\mu(z)$ at finite z. From Eq. (22) the total integral can be written as a trace:

$$\begin{aligned}Q&=\mu_1\frac{BS}{2\pi}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\mathrm{Tr}\left(\frac{1}{p_0^2+\mu_1^2+P^\dagger P}-\frac{1}{p_0^2+\mu_1^2+PP^\dagger}\right)\\&=\mu_1\frac{BS}{4\pi}\,\mathrm{Tr}\left(\frac{1}{\sqrt{\mu_1^2+P^\dagger P}}-\frac{1}{\sqrt{\mu_1^2+PP^\dagger}}\right).\end{aligned} \tag{24}$$

In the latter expression the traces of the individual operator terms in the braces are infinite. For this reason one cannot calculate them separately, each in its own basis. If each trace were finite, one could calculate the trace of the first operator in the basis of the eigenfunctions u_k of the operator $P^\dagger P$ and the trace of the second operator in the basis of the eigenfunctions v_k of the operator $PP^\dagger$ [cf. Eqs. (9)–(11)]. In that case the difference would arise only from the ''extra'' zero mode (12) of the operator $P^\dagger P$. In view of the divergence of each of the traces only the trace of the whole operator in braces in Eq. (24) can be calculated in an arbitrary complete basis. Also the continuum spectrum of the operators should be discretized by choosing a large bounding box in z: conventionally from $-L/2$ to $+L/2$, where L is sufficiently large for the asymptotic values of $\mu_2(z)$ to set in, and by imposing boundary conditions at $z=\pm L/2$. Consider, for instance, the case where the trace in Eq. (24) is calculated in the basis of the eigenfunctions v_k of the operator $PP^\dagger$ with an (anti)periodic condition at the boundaries of the box. Then the eigenfunctions u_k of the operator $P^\dagger P$ found from Eq. (10) do not satisfy the same condition since μ_2 takes different values at the infinities. This effect thus gives rise to a splitting of the eigenvalues in the two terms in the braces in Eq. (24). The latter effect, however, vanishes if $\mu_2(z)$ takes the same value at both infinities. Also in this case the operator $P^\dagger P$ no longer has the zero mode described by Eq. (12), and the spectra of the two discussed terms in Eq. (24) coincide in the same basis of (anti)periodic eigenfunctions. Thus the total charge vanishes if there is no net variation of μ_2: $\mu_2(-\infty)=\mu_2(\infty)$.

The latter observation in combination with Eq. (23) is in fact sufficient to conclude that the total charge depends only on the difference of the limiting values of μ_2. Indeed, consider a large box, defined by the length L, and let the difference $\Delta=\mu_2(L/2)-\mu(-L/2)$ be nonzero. Consider now an *extended* system, where the box is continued from $z=L/2$ to $z=\tilde{L}$, in such a way that the length of the extension $\tilde{L}-L/2$ is arbitrarily large and the behavior of $\mu_2(z)$ in the extension is chosen ''by hand'' such that it is a slow function smoothly interpolating μ_2 from its fixed value at $z=L/2$

back to $\mu_2(-L/2)$ at $z=\tilde{L}$. In other words, the net variation over the extension $\tilde{\Delta}=\mu_2(\tilde{L})-\mu_2(L/2)$ is $\tilde{\Delta}=-\Delta$. The net change of μ_2 over the entire extended box is then zero. Thus the total charge of the extended system is also zero. The total charge, however, is a sum of the one inside the original box Q and that in the extension $\tilde{Q}$. Thus $Q=-\tilde{Q}$. On the other hand, the charge $\tilde{Q}$ can be calculated in the limit of slowly varying $\mu_2(z)$ from the formula (23). Therefore one arrives at Eq. (4) for charge Q, independently of the details of the behavior of $\mu_2(z)$ at intermediate z within the original bounding box.

VI. CHARGE DENSITY IN SAMPLE MODELS

Here we consider explicit calculations of $\rho(z)$ from the formula (22) in sample models of the dependence of $\mu_2(z)$ on z in order to illustrate that the distribution of charge generally does not follow the slow variation limit described by Eq. (23) while the total charge is of course given by the universal formula (4).

A limit that is maximally opposite to that of a slow variation is where μ_2 changes as a step function at a point in z. Here we consider the situation where μ_2 changes at $z=0$ from a negative constant value $\mu_2(z)=-\mu_2$ at $z<0$ to a positive one, $\mu_2(z)=\mu_2$ at $z>0$. The Green's functions in Eq. (22) can be found explicitly by the standard method of matching at $z=0$:

$$\left\langle z\left|\frac{1}{p_0^2+\mu_1^2+P^\dagger P}\right|z\right\rangle=\frac{1}{2q}\left[1+\frac{\mu_2}{q-\mu_2}\exp(-2q|z|)\right],$$

$$\left\langle z\left|\frac{1}{p_0^2+\mu_1^2+PP^\dagger}\right|z\right\rangle=\frac{1}{2q}\left[1-\frac{\mu_2}{q+\mu_2}\exp(-2q|z|)\right],\tag{25}$$

where $q=\sqrt{p_0^2+\mu_1^2+\mu_2^2}$. The charge density is then given ''in quadratures'' by the formula

$$\rho(z)=\mu_1\mu_2\frac{B}{2\pi}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\frac{\exp(-2\sqrt{p_0^2+\mu_1^2+\mu_2^2}|z|)}{(p_0^2+\mu_1^2)},\tag{26}$$

which, as one could expect, describes an exponential distribution around the discontinuity point $z=0$ over the range $(\mu_1^2+\mu_2^2)^{-1/2}$, which is the Compton wavelength of the fermion. For the total charge the integration over z and then over p_0 is elementary, and the result matches the general formula (4).

It can be also noted, in connection with this example, that the nonexponential term in individual Green's functions in Eq. (25) would lead to a divergence upon integration over p_0. However, this term cancels in the difference, leaving the expression for $\rho(z)$ finite. This illustrates the behavior discussed in general terms in the previous section.

Another class of specific situations where the Green's functions are readily calculable in a closed form and the charge density can be found explicitly is presented by the Lee-Wick model [14]. In this model the term μ_2 in the fermion mass arises from the interaction with a neutral field ϕ: $\mu_2=g\phi$, with g being the Yukawa coupling. The ϕ^4 self-interaction of the field ϕ is assumed to lead to the domain wall solution

$$\phi(z)=v\tanh(\lambda vz/2),\tag{27}$$

which interpolates between the two vacuum states (domains) with $\phi=\pm v$, and where λ is the constant of the self-interaction, such that the mass of the ϕ bosons in either of the vacua is $m_\phi=\lambda v$.

The behavior of the fermion states in such a background depends on the ratio g/λ [15]. Clearly, the case $g\gg\lambda$ corresponds to the limit of a slow variation of μ_2, while the opposite case, $g\ll\lambda$, corresponds to the approximation of an abrupt change in μ_2, provided that in the latter case μ_1 is also assumed to be small in comparison with m_ϕ. For a generic value of the ratio g/λ the operators $P^\dagger P$ and $PP^\dagger$ correspond to the solvable potentials of the form $\mathrm{const}/\cosh^2(\lambda vz/2)$, and the Green's functions can be found in terms of hypergeometric functions. In the situation where $2g/\lambda$ is an integer; $2g/\lambda=N$, both potentials are nonreflecting, and the algebra is greatly simplified, since the relevant hypergeometric functions collapse to (Jacobi) polynomials in $\tanh(\lambda vz/2)$ of the power N for $P^\dagger P$ and $N-1$ for $PP^\dagger$. It can be also noted that the case of $N=2$ in the considered class of models would correspond to a supersymmetric model, albeit with $\mu_1=0$, which would take us beyond the assumptions adopted in the present paper. Here for illustrative purposes we consider only the most algebraically simple case of $N=1$.

For $\lambda=2g$, the operator $PP^\dagger$ corresponds to a constant potential, $PP^\dagger=-\partial_z^2+g^2v^2$, and the corresponding Green's function is especially simple:

$$G_{(PP^\dagger)}(z_1,z_2;p_0^2+\mu_1^2)\equiv\left\langle z_1\left|\frac{1}{p_0^2+\mu_1^2+PP^\dagger}\right|z_2\right\rangle=\frac{\exp(-q|z_1-z_2|)}{2q},\tag{28}$$

with $q=\sqrt{p_0^2+\mu_1^2+g^2v^2}$. The Green's function for the operator $P^\dagger P$ can then be found [16] using relations (10) and (12) for its eigenfunctions[5]:

[5]The individual Green's functions are finite; thus each can be expanded in the basis of the corresponding eigenfunctions.

$$
\begin{aligned}
G_{(P^\dagger P)}&(z_1,z_2;p_0^2+\mu_1^2)\\
&\equiv\left\langle z_1\left|\frac{1}{p_0^2+\mu_1^2+P^\dagger P}\right|z_2\right\rangle\\
&=\sum_{k=0}^{\infty}\frac{u_k(z_1)u_k^*(z_2)}{p_0^2+\mu_1^2+\lambda_k^2}\\
&=\sum_{k=1}^{\infty}P_{z_1}^\dagger\frac{v_k(z_1)v_k^*(z_2)}{\lambda_k^2(p_0^2+\mu_1^2+\lambda_k^2)}P_{z_2}+\frac{u_0(z_1)u_0(z_2)}{p_0^2+\mu_1^2}\\
&=\frac{1}{p_0^2+\mu_1^2}[P^\dagger G_{(PP^\dagger)}(z_1,z_2;0)P\\
&\quad -P^\dagger G_{(PP^\dagger)}(z_1,z_2;\ p_0^2+\mu_1^2)P]\\
&\quad +\frac{u_0(z_1)u_0(z_2)}{p_0^2+\mu_1^2}.
\end{aligned}
\tag{29}
$$

Applying in this formula the operators $P^\dagger$ and P to the Green's function (28) and using the zero mode from Eq. (12), $u_0(z)=\sqrt{gv/2}/\cosh(gvz)$, one finds that at coinciding points the Green's function (29) is given by

$$
G_{(P^\dagger P)}(z,z;p_0^2+\mu_1^2)=\frac{1}{2q}+\frac{g^2v^2}{2q(p_0^2+\mu_1^2)\cosh^2(gvz)},
\tag{30}
$$

where, again, $q=\sqrt{p_0^2+\mu_1^2+g^2v^2}$.

Upon substitution of the results from Eqs. (28) and (30) to Eq. (22) for the charge density, and after the integration over p_0, the final expression for the distribution of charge in this model is found in a remarkably simple explicit form:

$$
\rho(z)=\frac{gv}{2\cosh^2(gvz)}\frac{Q}{S},
\tag{31}
$$

where $Q=BS\arctan[gv/(\mu_1^2+g^2v^2)]/(2\pi^2)$ is the total induced charge, in agreement with the general formula (4).

VII. DISCUSSION

The explicit calculation described in Secs. II and III illustrates the implementation of the general statement about the absence of spontaneous magnetization based on C parity. In the concrete calculation the expected cancellation occurs due the fact that it is possible to regularize the theory while manifestly preserving charge conjugation symmetry. Alternatively, as already noted, this cancellation can be viewed as the manifest Lorentz invariance of the regularized theory, since a nonzero spontaneous magnetization would certainly imply a breaking of the Lorentz symmetry [8]. In other words, neither the C symmetry nor the Lorentz one are anomalous. The same symmetries preclude the appearance of a spontaneous magnetization also in models where the fermions are assumed to have anomalous magnetic and/or electric dipole moments, including the cases [4] where the fermions are neutron like, i.e., with zero charge and nonzero dipole moments.

The really existing effect, allowed by both these symmetries, is the fermion charge density induced by an external magnetic field applied across the domain wall, which can be traced to the well-known anomaly in the axial current. The total induced charge is of a topological nature and is determined, according to Eq. (4), by the total magnetic flux through the wall and the asymptotic values of the fields in the model far from the wall. The distribution of the induced charge, however, given by the general formula (22), depends on the details of the profile of the wall, as discussed in Secs. V and VI. This effect, in principle, can be relevant in detailed analyses of phenomena at the walls in the early universe—either for flat domain walls, which could exist during some epoch, or for dynamics near the walls of bubbles during a first order phase transition.

The arguments, based on C parity and/or Lorentz symmetry, do not directly apply to an asymmetric state with a net overall fermion charge, e.g., to the early universe with the baryon asymmetry, that is neither C nor Lorentz symmetric. In this case a spontaneous magnetization of the wall is generally allowed and is proportional to the asymmetry parameter of the considered state [6]. It should, however, be clearly understood that even a magnetized domain wall does not produce a magnetic field, which is a simple consequence of the classical Maxwell equations [6]. Therefore domain walls could not be a source of a magnetic field correlated at cosmological distances in the early universe, although their electromagnetic properties, like the charge distribution induced by a magnetic field, could generally be of importance in other aspects of dynamics of the early universe.

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