Noncommutative vortex solitons

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We consider the noncommutative Abelian-Higgs theory and investigate general static vortex configurations including recently found exact multivortex solutions. In particular, we prove that the self-dual Bogomol'nyi-Prasad-Sommerfield (BPS) solutions cease to exist once the noncommutativity scale exceeds a critical value. We then study the fluctuation spectra about the static configuration and show that the exact non-BPS solutions are unstable below the critical value. We have identified the tachyonic degrees as well as massless moduli degrees. We then discuss the physical meaning of the moduli degrees and construct exact time-dependent vortex configurations where each vortex moves independently. We finally give the moduli description of the vortices and show that the matrix nature of moduli coordinates naturally emerges.

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I. INTRODUCTION

The noncommutative solitons found in noncommutative scalar theory [1] do not even exist in the commutative version of the theory. This indicates that the characteristic properties of solitons in some noncommutative field theories may greatly differ from those of ordinary solitons. Of course, there are examples where the nature of noncommutative solitons and the corresponding ordinary solitons are quite similar to each other in the sense that the properties of noncommutative solitons are given by just smooth deformation governed by the noncommutativity scale θ.

One such example is the U(2) Bogomol'nyi-Prasad-Sommerfield (BPS) monopole discussed in Refs. [2–6]. The energy and the charge of the BPS monopole do not depend on the noncommutativity scale. The effect of the noncommutativity appears as a tilting of D strings in the transverse space giving the dipole nature of the magnetic charge distribution. It can be argued that the interactions of the U(2) BPS monopoles are independent of the noncommutativity scale θ within the moduli space description of their dynamics [5,7]. Contrary to the monopole case, the noncommutative scalar solitons found in Ref. [1] are genuinely noncommutative objects since they cannot exist in ordinary scalar theory. As discussed in Ref. [8], the shape deformation of the scalar soliton is quite peculiar when moving with a constant velocity. Specifically, their deformation is not simply dictated by the Lorentz contraction but described by an area preserving ellipse exhibiting the UV/IR mixing phenomena of noncommutative field theories.

We here pursue a similar issue on the recently found exact multivortex solutions [9] in the noncommutative Abelian-Higgs theory [10,11]. (For soliton solutions of some other models, see Refs. [6] and [12–18].) Certain apparent properties of the noncommutative vortices are striking even in their static properties. The multivortex solutions are in general not BPS saturated states but their energy, nevertheless, scales linearly in the number of vortices. This seems to imply that there are no interactions between vortices even in this non BPS case. We shall show that the self-dual BPS solutions exist only when $\theta v^2 \leqslant 1$ where v is the vacuum expectation value of the Higgs scalar. This property is also contrasted with the commutative Abelian-Higgs theory where the self-dual BPS vortices exist for all vacuum expectation values of the scalar. There is another aspect concerning the noncommutative vortex solitons; the theory allows exact time-dependent solutions of vortices, each of them moving in an arbitrary velocity from an arbitrary initial location. In view of generic complexity involved with soliton dynamics of field theory, the existence of such time-dependent solutions is quite peculiar.

In these respects, the systematic approach toward the understanding of the noncommutative vortex solutions seems imminent on the following issues. First, the possible static solitonic configurations need to be mapped out including the self-dual or anti-self-dual BPS branches. Second, the stability of the non-BPS multivortices is *a priori* unclear. This issue can be studied by turning on general perturbations around the static solutions. In case there are tachyonic degrees possessing a negative mass squared, the static configurations are necessarily unstable. Any small perturbations in this direction will make the vortices collapse to a stable configuration. On the other hand, when fluctuation spectra do not possess any tachyonic degrees, any individual vortex works as a stable solitonic object. The massless fluctuation is responsible for the moduli motions. Finally, one is interested in the interactions between vortices especially when they are stable. The interaction can be studied by adopting the scheme of the moduli space approximation. In fact, one may go beyond the moduli space dynamics by identifying quartic potential depending on the moduli coordinates in our present problem. Denoting the number of vortices by m, the U(m) matrix nature of the moduli coordinates emerges and the

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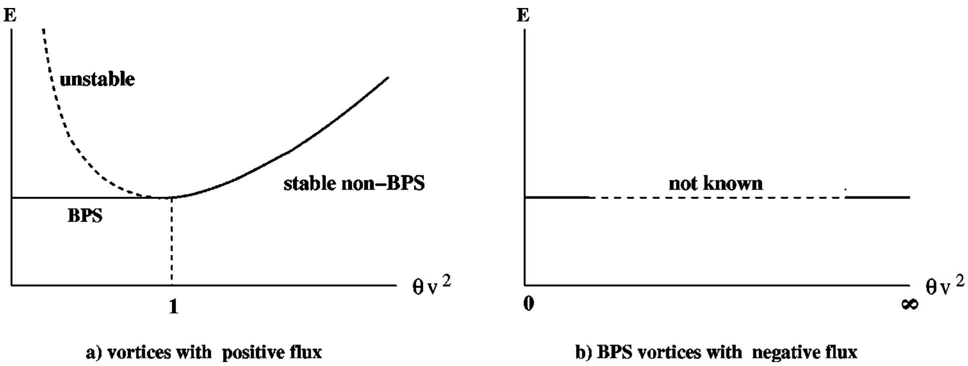

FIG. 1. The energy of (a) the self-dual vortices and (b) the anti-self-dual vortices. We also depict the energy of the non-BPS vortices that have a positive magnetic flux for $\lambda=1$.

dynamics turn out to be described by the matrix model of m $D0$ branes. We remark here that certain qualitative properties of this soliton are similar to those arising in the adjoint Higgs theories. These include the matrix nature of moduli coordinates, the fluctuation spectra, the tachyonic degrees, and time-dependent solutions [5,6,15].

In this paper, we review first the exact solutions of non-BPS multivortices. We also describe the exact solutions where vortices are positioned in arbitrary locations. In Sec. III, we study other static solutions focused on the self-dual BPS branch. Our study will be summarized in Fig. 1 where the anti-self-dual branch discussed in Ref. [11] is also included. In Sec. IV, we study the general fluctuation spectra around the static solutions identifying all the tachyonic modes and massless modes. Masses of the degrees connecting the vortex to the vacuum can be identified by diagonalizing the kinetic and quadratic potential terms simultaneously. The remaining degrees will be shown to be equivalent to the fluctuation spectra about the vacuum of the original Abelian-Higgs system. In Sec. V, we identify the moduli parameters appearing in the exact solutions by analyzing the translation and the moments (constructed with help of *covariant position operator*). We then construct exact time-dependent solutions describing vortices moving in arbitrary velocities. The moduli space description is then worked out and the relevant metric will be shown to be flat. We then describe how the matrix nature of the moduli coordinates emerges. The last section comprises the summary of our results and concluding remarks.

II. EXACT MULTIVORTEX SOLUTIONS

We begin by recapitulating the properties of the exact multivortex solutions of the noncommutative Abelian-Higgs theory found in Ref. [9]. The noncommutative Abelian-Higgs model in 2+1 dimensions is described by the Lagrangian

$$L=-\frac{1}{g^2}\int d^2x\left(\frac{1}{4}F_{\mu\nu}*F^{\mu\nu}+D_\mu\phi*(D^\mu\phi)^\dagger+\frac{\lambda}{2}(\phi*\phi^\dagger-v^2)^2\right), \tag{1}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-i(A_\mu*A_\nu-A_\nu*A_\mu),$$

$$D_\mu\phi=\partial_\mu\phi-iA_\mu*\phi. \tag{2}$$

The $*$ product is defined by

$$f(x)*g(x)\equiv[e^{-i(\theta/2)\epsilon^{ij}\partial_i\partial'_j}f(x)g(x')]|_{x=x'}, \tag{3}$$

where we take θ to be positive without loss of generality. The theory can be equivalently presented by operators on the Hilbert space defined by

$$[\hat{x},\hat{y}]=-i\theta, \tag{4}$$

where the $*$ product between functions becomes the ordinary product between the operators. For a given function

$$f(x,y)=\int\frac{d^2k}{(2\pi)^2}\tilde{f}(k)e^{i(k_x x+k_y y)}, \tag{5}$$

the corresponding operator can be found by the Weyl-ordered form of

$$\hat{f}(\hat{x},\hat{y})=\int\frac{d^2k}{(2\pi)^2}\tilde{f}(k)e^{i(k_x\hat{x}+k_y\hat{y})}. \tag{6}$$

One may then easily show that $\int d^2x f$ is replaced by $2\pi\theta\,\mathrm{tr}\,\hat{f}$ and $\partial_i f$ corresponds to $-(i/\theta)\epsilon_{ij}[\hat{x}_j,\hat{f}]$. With the operator-valued fields, the action can be written as

$$L=-\frac{2\pi\theta}{g^2}\mathrm{tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+D_\mu\phi(D^\mu\phi)^\dagger+\frac{\lambda}{2}(\phi\phi^\dagger-v^2)^2\right), \tag{7}$$

where hats are dropped for simplicity and the derivative notation is understood as $\partial_i f\equiv-(i/\theta)\epsilon_{ij}[x_j,f]$.

At this point, we introduce the creation and annihilation operators by $c^\dagger\equiv(1/\sqrt{2\theta})(x+iy)$ and by $c\equiv(1/\sqrt{2\theta})(x-iy)$, which satisfy $[c,c^\dagger]=1$. To represent arbitrary operators in the Hilbert space we shall use the occupation number basis by $G=\Sigma g_{kl}|k\rangle\langle l|$ with the number operator $c^\dagger c$. We will further denote $A=A_x-iA_y$, $\partial_- G\equiv(\partial_x-i\partial_y)G=\sqrt{2/\theta}[c,G]$, and $\partial_+ G\equiv(\partial_x+i\partial_y)G=-\sqrt{2/\theta}[c^\dagger,G]$.

The system is invariant under the gauge transformation

$$A'_\mu=U^\dagger A_\mu U+iU^\dagger\partial_\mu U,\quad \phi'=U^\dagger\phi, \tag{8}$$

where the gauge group element U satisfies

$$UU^{\dagger}=U^{\dagger}U=I. \tag{9}$$

We introduce a covariant quantity K defined by

$$A=-i\sqrt{\frac{2}{\theta}}(c-K), \tag{10}$$

which transforms as $K'=U^{\dagger}KU$ under the gauge transformation in Eq. (8). Later it will be interpreted as a covariant version of position operator up to numerical coefficient.

The Hamiltonian can be constructed as

$$H=\frac{2\pi\theta}{g^2}\mathrm{tr}\left(\frac{1}{2}(E^2+B^2)+D_t\phi(D_t\phi)^{\dagger}+D_i\phi(D_i\phi)^{\dagger}+\frac{\lambda}{2}(\phi\phi^{\dagger}-v^2)^2\right) \tag{11}$$

using the time translational invariance of the system. On the gauge choice $A_0=0$, the equations of motion read

$$\ddot{\phi}-D_iD_i\phi+\lambda(\phi\phi^{\dagger}-v^2)\phi=0,$$
$$\ddot{A}_i+\epsilon_{ij}D_jB=J_i\equiv i[\phi(D_i\phi)^{\dagger}-D_i\phi\phi^{\dagger}], \tag{12}$$

with the Gauss law constraint

$$D_i\dot{A}_i=J_0\equiv i[\phi\dot{\phi}^{\dagger}-\dot{\phi}\phi^{\dagger}]. \tag{13}$$

The exact multivortex solutions found in Ref. [9] are given by

$$K=S_mcS_m^{\dagger},\quad \phi=vS_m, \tag{14}$$

where S_m denotes the shift operator $S_m=\Sigma_{n=0}^{\infty}|n+m\rangle\langle n|$ $(m>0)$. The shift operator satisfies relations

$$S_m^{\dagger}S_m=I,\quad S_mS_m^{\dagger}=\overline{P}_m\equiv I-P_m, \tag{15}$$

with the projection operator P_m defined by

$$P_m=\sum_{a=0}^{m\text{-}1}|a\rangle\langle a|. \tag{16}$$

The magnetic field of the solitons reads

$$B=\frac{1}{\theta}P_m. \tag{17}$$

The flux defined by $\Phi\equiv\theta\,\mathrm{tr}\,B$ is m on the solution. Thus the solution describes m vortices of the Abelian-Higgs theory characterized by the topological quantity Φ. The energy of the vortices is evaluated as

$$M(v,\theta)=\frac{\pi m}{g^2}\left(\frac{1}{\theta}+\lambda\theta v^4\right)\geqslant\frac{2\pi m}{g^2}\sqrt{\lambda}v^2. \tag{18}$$

When $\lambda=1$, the theory allows so-called Bogomol'nyi bound as discussed in Ref. [11]. In fact it is straightforward to verify that the energy functional can be expressed as a complete squared form plus a topological term by

$$H=\frac{\pi\theta}{g^2}\mathrm{tr}\{[B\pm(\phi\phi^{\dagger}-v^2)]^2+2(D_{\pm}\phi)(D_{\pm}\phi)^{\dagger}\pm\epsilon_{ij}D_iJ_j\pm2v^2B\}$$
$$\geqslant\frac{2\pi v^2}{g^2}|\Phi|, \tag{19}$$

where we omitted the kinetic terms involving E_i and $D_t\phi$. The saturation of the bound occurs once the self-dual Bogomol'nyi equations

$$D_+\phi=0,\quad B=v^2-\phi\phi^{\dagger} \tag{20}$$

or the anti-self-dual equations

$$D_-\phi=0,\quad -B=v^2-\phi\phi^{\dagger} \tag{21}$$

are satisfied. When $\lambda=1$, the bound in Eq. (18) agrees with the Bogomol'nyi bound that is an absolute energy bound for m vortex solution. Hence when $v^2=1/\theta$ and $\lambda=1$, the solution should be a BPS solution. Indeed for the specific value of θv^2, one can check that the solution satisfies the self-dual BPS equations. This BPS solution is clearly stable because they saturate the energy bound set by the topological quantity.

Another obvious generalization of the static multivortex solution is given by [17]

$$K=S_mcS_m^{\dagger}+\frac{1}{\sqrt{2\theta}}\sum_{a=0}^{m-1}\lambda_a|a\rangle\langle a|,\quad \phi=vS_m, \tag{22}$$

where λ_a's are constant complex numbers. This solution has the same flux and energy as the solution in Eq. (14). Hence we see that λ_a is the moduli parameter of the multivortices. Later we shall clarify the stability of the vortex solutions, which is *a priori* not clear because they are not always BPS saturated solutions. But before discussing this matter, we will study the BPS solutions for $\lambda=1$ and $\theta v^2\neq1$ or other possible static solutions.

III. BPS SOLUTIONS OF MULTIVORTICES

In the last section, we have derived the BPS equations of the Abelian-Higgs theory with $\lambda=1$. The static multivortex solutions are in general not BPS saturated. However, they become self-dual BPS solutions for a special value of $\theta v^2=1$. In this section we focus on the BPS solutions. Some analysis on the anti-BPS solutions is carried out in Ref. [11] and the comparison will follow at the end of this section. In terms of K the BPS equations become

$$\frac{1}{\theta}(1-[K,K^{\dagger}])=v^2-\phi\phi^{\dagger},\quad \phi c^{\dagger}-K^{\dagger}\phi=0. \tag{23}$$

By virtue of the explicit form of $c^{\dagger}$ the latter can be solved,

$$\phi=\frac{1}{\sqrt{\theta}}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}K^{\dagger n}|\phi_0\rangle\langle n|, \tag{24}$$

where $|\phi_0\rangle=\sqrt{\theta}\phi|0\rangle$ is an arbitrary constant vector. Substituting this expression, the BPS equations are reduced to a single equation,

$$\theta v^2-1+[K,K^\dagger]=\sum_{n=0}^{\infty}\frac{1}{n!}K^{\dagger n}|\phi_0\rangle\langle\phi_0|K^n. \tag{25}$$

To solve this equation we take an ansatz for K as

$$K=\sum_{n=0}^{\infty}f_n|n\rangle\langle n+p|, \tag{26}$$

where p is any positive integer. Substituting this expression into Eq. (25), one can show the following: BPS solutions exist only when $1\geqslant\theta v^2$. Furthermore, finite energy or flux solutions exist only for $p=1$. Specifically we obtain

$$K=\sum_{a=1}^{m}\sqrt{a(1-\theta v^2)}|a-1\rangle\langle a|-\sum_{n=1}^{\infty}k_n|n+m-1\rangle\langle n+m|,$$

$$\phi=\frac{\zeta}{\sqrt{\theta}}\left(|m\rangle\langle 0|+\sum_{n=1}^{\infty}\frac{\bar{k}_1\bar{k}_2\cdots\bar{k}_n}{\sqrt{n!}}|n+m\rangle\langle n|\right), \tag{27}$$

where $\zeta\in\mathbb{C}$, m corresponds to the flux number which is a non-negative integer, and the sequence $k_n, n=1,2,\ldots$ satisfies the recurrence relation

$$q_{n+1}+q_{n-1}-2q_n=\frac{q_n}{n}(q_n-q_{n-1}+\theta v^2), \tag{28}$$

with $q_n\equiv|k_n|^2-n$. The initial data for the recurrence relation are

$$q_0=m(1-\theta v^2),\quad q_1=m(1-\theta v^2)+|\zeta|^2-\theta v^2, \tag{29}$$

where ζ is an adjustable parameter. The magnetic field is given by

$$B=\sum_{a=0}^{m-1}v^2|a\rangle\langle a|+\frac{1}{\theta}\sum_{n=0}^{\infty}(q_n-q_{n+1})|n+m\rangle\langle n+m|, \tag{30}$$

so that the flux is

$$\Phi=m-\lim_{n\to\infty}q_n. \tag{31}$$

Equation (27) satisfies the BPS equations for any value of ζ. Choosing $\zeta=0$ or $|\zeta|=\sqrt{\theta v^2}$ gives plus infinity or minus infinity flux solution, respectively. Furthermore, if q_n converges, the converging value must be zero. Thus by continuity, there exists ζ, $0<|\zeta|\leqslant\sqrt{\theta v^2}$, which makes q_n converge to zero, and hence BPS solutions have finite and quantized energy. The Appendixes contain our proof.[1] For $\theta v^2=1$, the choice $|\zeta|=1$ leads to the exact solution with $q_n=0$ or $k_n=\sqrt{n}$.

We have shown that, within the ansatz taken, there are no BPS solutions possessing a positive flux for $\theta v^2>1$. Without limiting the discussions to the specific form, one may prove that there are indeed no self-dual BPS solutions for $\theta v^2>1$ as an analytic perturbation of small parameter ω around the $\theta v^2=1$ BPS solution. For this purpose, we shall take a generic perturbation around the solution and expand it as a power series of the small parameter $\omega\equiv\sqrt{|\epsilon|}$ with $\epsilon\equiv\theta v^2-1$. (The choice $\omega=\epsilon$ will quickly lead to a contradiction.) Namely, we consider the fluctuation around the exact solution as

$$\phi=v(1+\varphi)S_m,$$

$$K=S_m c S_m^\dagger+h, \tag{32}$$

with the expansions

$$\varphi=\sum_{l=1}^{\infty}\omega^l\varphi_{(l)},\quad h=\sum_{l=1}^{\infty}\omega^l h_{(l)}, \tag{33}$$

and may show that there are no solutions for $\theta v^2>1$. The proof is relegated to the Appendixes.

One could also try an expansion with respect to a parameter ω_n defined by $|\epsilon|^{1/n}$ for arbitrary non-negative integers. Though a little complicated, one may show that the conclusion remains unchanged. Thus there are no solutions of the BPS equations for $\theta v^2>1$ that can be expanded in a power series of ω_n. Here we do not turn on the diagonal entry λ_a of K. As will be explained later, the effect of nonzero λ_a corresponds to locating each vortex at $\lambda_a=\lambda_x^a-i\lambda_y^a$ position. Considering the case of one vortex, one can easily turn off this value by using the translation symmetry of the system. Hence our proof above is strictly applicable to this case. Furthermore, m vortices are an assembly of individual vortices, one naturally expects that the above proof goes through even m vortices with generic values of λ_a.

Figure 1 summarizes our investigation of the static solutions in the Abelian-Higgs theory for $\lambda=1$. The self-dual BPS solutions exist only for $\theta v^2\leqslant 1$. In the range the non-BPS exact solutions are unstable due to their higher energies. When $\theta v^2>1$, the non-BPS branch alone continues to exist. For the solutions of a negative flux, it is shown in Ref. [11] that the anti-self-dual solutions exist for $\theta v^2\geqslant 1$ or $\theta v^2\leqslant 1$. In the intermediate values of θv^2, the existence of the self-dual solutions is not known. The exact vortex solutions with positive magnetic flux exist even for $\lambda\neq 1$. It will be shown later that they are also stable only when $\theta v^2\geqslant 1$.

[1]We here like to mention that we have also numerically verified that the value of $|\zeta|$, which makes the series to converge to zero up to a few hundred terms, approaches a unique value for a given value of $\theta v^2\in\{0.1,0.2,\ldots 0.9\}$ and for $m=1$.

IV. FLUCTUATION SPECTRA AROUND THE VORTICES

In the last section, we have identified the possible static vortex solutions including BPS and non-BPS cases. In the BPS case, the classical stability of the solution is quite clear because the energy is saturating the bound set by the topological quantity. For the case of non-BPS, however, it is not *a priori* clear whether the vortices are stable or not. When $\theta v^2<1$, we have shown that there exist solutions that have lower energies than the exact non-BPS solutions. Thus we expect naturally that there should be tachyonic modes. It is also shown that BPS solutions do not exist for $\theta v^2>1$. Hence in this case the issue of stability seems a different matter. To resolve these issues clearly, we shall study, in this section, the quadratic fluctuation spectra around the exact solutions identifying the signature of mass squared for all possible degrees. It turns out that the solutions $\theta v^2<1$ are indeed unstable by developing tachyonic modes in their spectra. In case of $\theta v^2=1$, the potential tachyonic degrees become massless and the solution is indeed stable. For $\theta v^2>1$, solutions are classically stable because the tachyonic degrees become massive. For all these three cases, the first m diagonal elements of the gauge field fluctuation are massless, which will be identified with the degrees of vortex positions.

Let us study first the quadratic fluctuation of the original theory about the vacuum $K=c+\mathcal{K}$ and $\phi=v(1+h)$ without any vortices. The Lagrangian is then reduced to

$$L_v=\frac{2\pi}{g^2}\mathrm{tr}\bigg[|\dot{\mathcal{K}}|^2+\theta v^2(|\dot h_R|^2+|\dot h_I|^2)-\frac{1}{2\theta}|[c,\mathcal{K}^\dagger]+[\mathcal{K},c^\dagger]|^2-2v^2|[c,h_R]|^2-2v^2|\mathcal{K}+i[c,h_I]|^2-2\lambda\theta v^4h_R^2\bigg], \tag{34}$$

with the Gauss law constraint

$$[c^\dagger,\dot{\mathcal{K}}]+[c,\dot{\mathcal{K}}^\dagger]-2i\theta v^2\dot h_I=0, \tag{35}$$

where $h_R\equiv 1/2(h+h^\dagger)$ and $h_I\equiv(1/2i)(h-h^\dagger)$. One may simplify this action by reintroducing A_0 field, which has a role of imposing the Gauss law constraint. We then choose a gauge $A_0=\dot h_I$, at which $\mathcal{K}+i[c,h_I]\to\mathcal{K}$. The Lagrangian becomes

$$L=\frac{2\pi}{g^2}\mathrm{tr}\bigg[|\dot{\mathcal{K}}|+\theta v^2|\dot h_R|^2-\frac{1}{2\theta}|[c,\mathcal{K}^\dagger]+[\mathcal{K},c^\dagger]|^2-2v^2(|[c,h_R]|^2+|\mathcal{K}|^2+\lambda\theta v^2h_R^2)\bigg], \tag{36}$$

with the gauge condition now

$$[c^\dagger,\dot{\mathcal{K}}]+[c,\dot{\mathcal{K}}^\dagger]-i[c^\dagger,[c,h_I]]-i[c,[c^\dagger,h_I]]-2i\theta v^2\dot h_I=0. \tag{37}$$

This can be solved in terms of h_I for arbitrary $\mathcal{K}$, on which the Lagrangian does not depend. It is now clear that all the degrees are massive; the components of $\mathcal{K}$ have a mass squared greater than $2v^2$ while h_R components a mass squared greater than λv^2. We see that the gauge field absorbs part of the scalar degrees and becomes massive. This corresponds to the so-called Higgs mechanism of the ordinary gauge theory when the gauge symmetry is broken spontaneously.

To study the quadratic fluctuation around the exact solutions, we turn on generic perturbation of the form

$$K=S_mcS_m^\dagger+\Lambda+\mathcal{K},\qquad \phi=v(1+\varphi)S_m, \tag{38}$$

with $\mathcal{K}$ and φ decomposed as

$$\mathcal{K}=\mathcal{A}+VS_m^\dagger+S_mW^\dagger+S_m\tilde{\mathcal{K}}S_m^\dagger=\begin{pmatrix}\mathcal{A} & V\\ W^\dagger & \tilde{\mathcal{K}}\end{pmatrix},$$

$$\varphi=XS_m^\dagger+S_m\tilde hS_m^\dagger=\begin{pmatrix}0 & X\\ 0 & \tilde h\end{pmatrix}. \tag{39}$$

Here we set $\varphi P_m=0$ with out loss of generality since an arbitrary $\delta\phi$ can be expressed by $v\varphi S_m$. Further introducing unitary operators

$$U_a\equiv e^{\frac{1}{\sqrt{2\theta}}(\bar\lambda_a c-\lambda_a c^\dagger)},\qquad (0\leqslant a\leqslant m-1), \tag{40}$$

we parametrize the components deliberately as

$$\mathcal{A}=\sum_{a=0}^{m-1}\sum_{b=0}^{m-1}\mathcal{A}_{ab}|a\rangle\langle b|,\qquad V=\sum_{a=0}^{m-1}\sum_{n=0}^{\infty}V_{an}|a\rangle\langle n|U_a,$$

$$W=\sum_{a=0}^{m-1}\sum_{n=0}^{\infty}W_{an}|a\rangle\langle n|U_a,\qquad X=\sum_{a=0}^{m-1}\sum_{n=0}^{\infty}X_{an}|a\rangle\langle n|U_a, \tag{41}$$

$$\tilde{\mathcal{K}}=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{K}_{kn}|k\rangle\langle n|,\qquad \tilde h=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\tilde h_{kn}|k\rangle\langle n|.$$

The unitary operators U_a satisfy

$$U_acU_a^\dagger=c+\frac{1}{\sqrt{2\theta}}\lambda_a,\qquad U_ac^\dagger U_a^\dagger=c^\dagger+\frac{1}{\sqrt{2\theta}}\bar\lambda_a, \tag{42}$$

which are helpful in identifying variables that diagonalize both the kinetic and potential terms.

Now we insert these into the original Lagrangian in the gauge $A_0=0$ and expand it to the quadratic terms of the fluctuation. We get

$$L_{\mathrm{quad}}=\frac{2\pi}{g^2}\bigg[\sum_{ab}\left(|\dot{\mathcal{C}}_{ab}|+|\dot{\mathcal{G}}_{ab}|^2-\frac{|\lambda_a-\lambda_b|^2}{\theta^2}|\mathcal{C}_{ab}|^2\right)+\sum_a\left(|\dot T_a|^2-\frac{\theta v^2-1}{\theta}|T_a|^2\right)\bigg]+\frac{2\pi}{g^2}\sum_{a,n}\bigg[|\dot H_{an}|^2+|\dot Y_{an}|^2+|\dot G_{an}|^2-\frac{2n+1+\theta v^2}{\theta}(|H_{an}|^2+|Y_{an}|^2)\bigg]+L_{\mathrm{D}}, \tag{43}$$

where $L_{\rm D}$ is same as the Lagrangian (34) but $\mathcal{K}$ and h and are replaced respectively by $\tilde{\mathcal{K}}$ and $\tilde{h}$. In this Lagrangian, we put

$$\mathcal{C}_{ab}\equiv\frac{1}{\sqrt{2}}[e^{-i\theta_{ab}}\mathcal{A}_{ab}-e^{i\theta_{ab}}\mathcal{A}^{\dagger}_{ab}],$$

$$\mathcal{G}_{ab}\equiv\frac{1}{\sqrt{2}}[e^{-i\theta_{ab}}\mathcal{A}_{ab}+e^{i\theta_{ab}}\mathcal{A}^{\dagger}_{ab}], \tag{44}$$

where θ_{ab} is the argument of $\lambda_a-\lambda_b$ or an arbitrary constant for $\lambda_a=\lambda_b$. We also set T_a, H_{an}, Y_{an}, and G_{an} as

$$T_a\equiv V_{a0},$$

$$H_{an}\equiv\frac{\sqrt{\theta v^2}}{\sqrt{\theta v^2+2n+1}}\left(X_{an}\sqrt{2n+1}-\frac{1}{\sqrt{2n+1}}(\sqrt{n+1}V_{a,n+1}+\sqrt{n}W_{a,n-1})\right),$$

$$Y_{an}\equiv\frac{1}{\sqrt{2n+1}}(\sqrt{n}V_{a,n+1}-\sqrt{n+1}W_{a,n-1}),$$

$$G_{an}\equiv\frac{1}{\sqrt{\theta v^2+2n+1}}\times(\theta v^2X_{an}+\sqrt{n+1}V_{a,n+1}+\sqrt{n}W_{a,n-1}). \tag{45}$$

To this order, the Gauss law constraints for $\mathcal{A}$ and the off diagonal degrees become

$$\dot{\mathcal{G}}_{ab}=0 \quad (\text{only for} \quad \lambda_a\neq\lambda_b), \quad \dot{G}_{an}=0, \tag{46}$$

and, for $L_{\rm D}$, it takes the same form in Eq. (35) where $\mathcal{K}$ and h are again replaced respectively by $\tilde{\mathcal{K}}$ and $\tilde{h}$.

From this it is clear that $\mathcal{C}_{ab}$ is massless when λ_a and λ_b coincide. In particular the diagonal components $\mathcal{C}_{aa}$ and $\mathcal{G}_{aa}$ are always massless; they are associated with the translational motion of the vortices. The nature of this motion will be exploited when we discuss the low energy dynamics of the vortices. When $\theta v^2<1$, T_a has a negative mass squared. Hence we see that the vortices are unstable even for the case of a vortex. On the other hand, for $\theta v^2\geq 1$, the instability disappears and the vortex solutions are stable. This is also quite consistent with the fact that there are no BPS solutions for $\theta v^2>1$. If there were such solutions, there must be tachyonic modes because the BPS solution should have lower energy than the non-BPS solutions.

Especially when $\theta v^2=1$, the potential tachyonic degrees become massless and may participate in the low energy dynamics as will be discussed later. The remaining of diagonal components are H_{an} and Y_{an}. The G_{an} degrees are dropped out of the physical space spectrum once the Gauss law constraint is imposed. Here we were be able to diagonalize these infinite dimensional degrees, which is in general not an easy task to achieve. The spectrum of these physical degrees is particularly simple; they are all massive with the same mass $v^2+(2n+1/\theta)$, which is independent of the index a. This spectrum can surely be understood from the underlying D-brane perspective.

Finally, $L_{\rm D}$ describes the fluctuation spectra of the original system around its trivial vacuum configuration. This is no coincidence because the degrees of the original system still remain around vortices. At this point, we like to emphasize again that they are all massive controlled by the mass scale v and $\sqrt{\lambda}v$.

V. LOW ENERGY DYNAMICS

From the analysis of the fluctuation spectra, it is clear that the vortices are unstable due to the tachyonic modes for $\theta v^2<1$. On the other hand, the vortices do not exhibit any tachyonic instabilities for $\theta v^2\geq 1$. For all ranges of parameter θv^2, the vortex solutions depend upon $2m$-dimensional free parameters where m is the topological number corresponding to the total number of vortices. We shall first consider the stable case where $\theta v^2\geq 1$ and begin by clarifying the physical interpretation of these parameters. In short, these parameters λ_a are positions of vortices on the plane where the noncommutative gauge theory is defined. For the gauge group element defined by

$$U_PU_P^{\dagger}=I \tag{47}$$

with $\bar{P}_mU_P=U_P\bar{P}_m=\bar{P}_m$, the corresponding gauge transformation affects only the first $m\times m$ and $m\times\infty$ component of K and ϕ. Utilizing this gauge freedom, we have diagonalized the $m\times m$ part of K by

$$P_mKP_m=\frac{1}{\sqrt{2\theta}}\mathrm{diag}[\lambda_0,\lambda_1,\ldots,\lambda_{m-1}] \tag{48}$$

in the solution (22). Any permutations of the eigenvalues λ_a and λ_b are achieved through the gauge transformation by the Weyl subgroup elements. So they are physically equivalent configurations. Thus the moduli space is in fact $(R^2)^m/\mathcal{S}_m$ where $\mathcal{S}_m$ is the permutation group.

In order to identify the meaning of the moduli parameters, let us first study the effect caused by the overall translation of the vortex solutions. For this, we note that the infinitesimal translation is given by

$$\delta A_i=-[\xi_j\partial_jA_i-D_i(\xi_jA_j)]=B\epsilon_{ij}\xi_j,$$

$$\delta\phi=-[\xi_j\partial_j\phi-i(\xi_jA_j)\phi]=-\xi_jD_j\phi, \tag{49}$$

where we have added the infinitesimal gauge transformation by the gauge function ξ_jA_j. On the solution, this produces

$$\delta A_i=\frac{1}{\theta}\epsilon_{ij}\xi_jP_m, \quad \delta\phi=0. \tag{50}$$

The magnetic field B and the Higgs gradient $D_i\phi$ are unchanged by the translation and, consequently, one may construct easily the fields translated by a finite amount. Namely, the Higgs change is $\Delta\phi=0$ while the change of gauge field in terms of the K variable is given by $\Delta K=(1/\sqrt{2\theta})\xi P_m$ where $\xi=\xi_1-i\xi_2$. Hence we see here that the total translation leads to a uniform shift of each λ_a by the amount ξ. This is of course quite consistent with the interpretation that the moduli parameters represent positions of the vortices. Of course due to the U(∞) gauge symmetry, the effect of translation does not quite look like a translation of profile in the case of ordinary field theory where a density $\rho(x)$, for example, is merely shifted by ξ as in $\rho(x-\xi)$ as a result of the translation. In this respect whether local informations such as positions of vortices is well defined in noncommutative gauge theory is not obvious at first sight. There is another way to get the above result of translation. The global translation generator can be alternatively expressed as

$$T=e^{-i\xi_i p_i}. \tag{51}$$

where p_i is the translation generator $p_i=-(1/\theta)[\epsilon_{ij}x_j,\cdot]$. In noncommutative field theory, the operation of translation on a field can be expressed as a similarity transformation,

$$Tf(x)=U_T f(x) U_T^\dagger, \tag{52}$$

where U_T is a unitary matrix defined by $e^{(1/\sqrt{2\theta})(\xi c^\dagger-\bar{\xi}c)}$. In our case, we add a gauge transformation by $U=U_T$ after the translation. Then the resulting gauge and scalar fields read

$$A'=A+\frac{i}{\theta}\xi,$$

$$\phi'=\phi U_T^\dagger. \tag{53}$$

The gauge field is shifted only by a constant piece. We see also that $\phi\phi^\dagger$ is invariant. If the scalar were in the adjoint representation, it would be invariant under the transformation. In order to obtain the previous result in Eq. (50), we further perform a gauge transformation by $U=e^{(1/\sqrt{2\theta})(\xi c_m^\dagger-\bar{\xi}c_m)}$ with $c_m\equiv S_m c S_m^\dagger$. Using the explicit expression of the solutions, one may easily check that results agree with Eq. (50).

One could also study the exact solutions moving in a constant velocity as discussed in Ref. [8]. The theory is not Lorentz invariant because the $*$ product does not respect the Lorentz symmetry. However, as discussed in Ref. [8], one may still construct moving soliton solutions once the static solution is given. The construction is achieved by Lorentz boosting of the static solution followed by the change of θ by $\gamma\theta$ where γ is the Lorentz dilation factor defined by $1/\sqrt{1-\beta^2}$ with velocity β. Constructed this way, the solution moving in x direction reads explicitly

$$A_0'=-\gamma\beta_x A_x(x',y';\gamma\theta),\quad A_x'=\gamma A_x(x',y';\gamma\theta),$$

$$A_y'=A_y(x',y';\gamma\theta),\quad \phi'=\phi(x',y';\gamma\theta), \tag{54}$$

assuming $A_0=0$ for the static solution. Here the arguments are given by $x'=\gamma(x-\beta_x t)$ and $y'=y$ and the fields without prime denote any static solutions. In the present case, one may further simplify the form of the moving solution again taking $A_0'=0$. This gauge choice is achieved from the above solution by the gauge transformation with

$$U=e^{-(1/\sqrt{2\theta})(\xi\tilde{S}_m\tilde{c}^\dagger\tilde{S}_m^\dagger-\bar{\xi}\tilde{S}_m\tilde{c}\tilde{S}_m^\dagger)}e^{(1/\sqrt{2\theta})(\xi\tilde{c}^\dagger-\bar{\xi}\tilde{c})}, \tag{55}$$

where we define

$$\tilde{c}=\frac{\sqrt{\gamma}x-iy(\sqrt{\gamma})^{-1}}{\sqrt{2\theta}},$$

$$\tilde{S}\equiv\sum_{n=0}^{\infty}|n+1\rangle'\langle n|'=\tilde{c}^\dagger(\tilde{c}\tilde{c}^\dagger)^{-1}, \tag{56}$$

and $\xi=\beta_x t$. Here $|n\rangle'$ is the number eigenstate constructed by the number operator $\tilde{c}^\dagger\tilde{c}$. The form of the solution becomes[2]

$$A'=-i\frac{\sqrt{2}}{\sqrt{\theta}}(c-\tilde{S}_m c\tilde{S}_m^\dagger)+\frac{i}{\theta}\beta_x t\tilde{P}_m,$$

$$\phi'=v\tilde{S}_m, \tag{57}$$

with $A_0'=0$. The map from $c=(x-iy)/\sqrt{2\theta}$ to the new basis $\tilde{c}=(1/\sqrt{2\theta})[x\sqrt{\gamma}-iy(\sqrt{\gamma})^{-1}]$ belongs to the area preserving diffeomorphism. Except for some overall numerical coefficients, the solution apparently represents a configuration that has an elliptic shape; for example, the magnetic field of a moving vortex appears in the function representation as $2e^{-(1/\theta)(x^2\gamma+y^2\gamma^{-1})}$. Utilizing the U($\infty$) gauge symmetry, the solution (57) can be further mapped to

$$A'=-i\frac{\sqrt{2}}{\sqrt{\theta}}(c-S_m cS_m^\dagger)+\frac{i}{\theta}\beta_x tP_m,$$

$$\phi'=vS_m, \tag{58}$$

by the gauge transformation with the unitary matrix

$$U_S=\tilde{S}_m S_m^\dagger+\sum_{a=0}^{m-1}|a\rangle'\langle a|. \tag{59}$$

Inserting Eq. (58) into the time-dependent field equations, one may directly check that it is indeed a solution. Actually, one may even construct solutions representing more general motion of vortices. The time-dependent solutions read

$$A'=-i\frac{\sqrt{2}}{\sqrt{\theta}}(c-S_m cS_m^\dagger)+\frac{i}{\theta}\sum_{a=0}^{m-1}[\lambda^a+\beta^a t]|a\rangle\langle a|,$$

$$\phi'=vS_m, \tag{60}$$

[2]The appearance of c instead of $\tilde{c}$ in the gauge field is not a typographical mistake.

with $\beta^a \equiv \beta^a_x - i\beta^a_y$. The motion of each vortex takes place independently to an arbitrary direction. The magnetic field and electric field are now

$$B' = \frac{1}{\theta} P_m, \quad E'_i = \epsilon_{ij} \sum_{a=0}^{m-1} \beta^a_j |a\rangle\langle a|. \tag{61}$$

The energy of the moving vortices is evaluated as

$$E(\beta) = \frac{1}{2}\left(\frac{2\pi}{g^2\theta}\right) \sum_{a=0}^{m-1} |\beta_a|^2 + \frac{\pi m}{g^2}\left(\frac{1}{\theta} + \lambda\theta v^4\right), \tag{62}$$

where no approximation is made. The energy behaves precisely as free nonrelativistic particles with a mass $2\pi/g^2\theta$.

One striking fact in the moving solutions lies in the fact that there seems to be no limit in the velocity (see Ref. [20] for the earlier investigation of this aspect). It can apparently exceed the light velocity.[3] On the other hand, in the original construction by the Lorentz boost followed by the change in the scale θ, the construction itself loses its validity when the velocity exceeds the light velocity. Specifically, the factor γ becomes imaginary. Nonetheless, the final form of the solution in this range of velocity does solve the time-dependent equations of motion. Our system lacks the Lorenz invariance and thus this seems not a serious problem. Without going into detail, we like to mention the fact that, when the velocity exceeds the light velocity, part of once stable degrees become tachyonic and instabilities are necessarily set in. Hence the solutions seems not to have much physical significance when the velocity exceeds the light velocity. Further investigation is required on this issue.

Let us now turn to the moduli dynamics of vortices. The study of translation justifies that λ's faithfully represent the overall position of vortices. Let us consider the following operator:

$$X_i \equiv x_i - \theta\epsilon_{ij} A_j, \tag{63}$$

which may be rewritten equivalently as $X = X_1 - iX_2 = \sqrt{2\theta}K$. This transforms covariantly under the gauge transformation, i.e., $X \to U^\dagger X U$. Since the operator reduces to x_i in the commutative limit and is gauge covariant, we shall call it the *covariant position operator*. Another justification for the terminology comes as follows. It transforms as

$$X'_i = X_i + \xi \tag{64}$$

under the translation of Eq. (52) followed by the gauge transformation by $U = U_T$. This is precisely the required property as a *position* operator under translation up to gauge freedom. It will be used to measure local properties of the noncommutative field theory. To show that the eigenvalues λ_a represent positions of vortices, let us consider the following moments:

$$I_{k,l} \equiv 2\pi\theta \,\mathrm{tr}[X^k (X^\dagger)^l H]. \tag{65}$$

These quantities are gauge invariant and measure the local distribution of matters in noncommutative gauge theory. For example, $I_{1,1}$ corresponds to the moment of inertia for the configurations of the ordinary field theory.

For the exact vortex solutions ($\theta v^2 \geqslant 1$), we have

$$I_{k,l} = M_{\rm one} \sum_{a=0}^{m-1} \lambda_a^k \bar{\lambda}_a^l. \tag{66}$$

In the case of commutative field theory limit, the same moments can be found only when the Hamiltonian density is the sum of the delta function as $H(\mathbf{x}) = \sum_{a=0}^{m-1} M_{\rm one} \delta^2(\mathbf{x} - \boldsymbol{\lambda}_a)$. Thus we show that the relatively local information of noncommutative gauge theory can be obtained from the moments defined above and that the eigenvalues λ_a are representing the positions of vortices up to the permutation symmetry. Considering, for example, vortices located at the origin, the size information of the vortex configuration can be extracted by the moment of inertia. The ''size'' (measured by the covariant position operator) is finite for the BPS vortices ($\theta v^2 < 1$). In fact it decreases within the BPS branch as θ gets larger and becomes zero for the stable non-BPS vortices ($\theta v^2 \geqslant 1$).

The moduli dynamics of the noncommutative solitons may be pursued in a similar manner as solitons in an ordinary field theory. As stated before, we shall consider first the case where $\theta v^2 \geqslant 1$. We proceed by giving the time dependence to the moduli parameters and adding an appropriate gauge freedom so that the motion respects the Gauss law constraint. But in our present case, it is enough to simply give the time dependence without adding any gauge degrees because they already satisfy the Gauss law constraint. Namely we insert

$$K = \bar{K}(\lambda_a(t)), \quad \phi = \bar{\phi}(\lambda_a(t)) \tag{67}$$

to the full Lagrangian where quantities with a bar denote the vortex solutions. (This ansatz is quite consistent with the moving solutions constructed before.) The resulting effective Lagrangian is given by

$$L_{\rm eff} = -mM_{\rm one} + \frac{\pi}{g^2\theta} \sum_{a=0}^{m-1} \dot{\lambda}_a \dot{\bar{\lambda}}_a. \tag{68}$$

Consequently, the moduli space metric on $(R^2)^m/S_m$ is flat, i.e.,

$$ds^2 = \sum_{a=0}^{m-1} d\lambda_a d\bar{\lambda}_a. \tag{69}$$

The inertia mass here is different from the rest mass but there is no physical reason why these two masses agree, not to

[3]If the moving solution were not exact, we would have easily missed this point. This is similar to the case of the noncommutative scalar field theory with a quartic interaction. The two particle bound state energy is unbounded from below, which was observed in the exact nonperturbative computation of the bound state energy [19].

mention that this effective Lagrangian can be easily quantized and wave functions are those of m free nonrelativistic bosons with mass $2\pi/g^2\theta$.

For the present model, one may in fact go beyond the moduli space description in discussing the relevant low energy dynamics. In the previous section, we studied full fluctuation spectra around the static solutions. We find that off diagonal degrees W_{aj}, V_{aj} $(j\neq 0)$, and $P_m\varphi\bar{P}_m$ [with $\phi=v(1+\varphi)S_m$] are massive with a mass squared $m_j^2=(2j+1/\theta)+v^2$ $(j\geqslant 0)$. Furthermore, $\bar{P}_m\mathcal{K}\bar{P}_m$ components have a mass squared at least order of v^2. The real part of $\bar{P}_m\varphi\bar{P}_m$ has a mass order of λv^2 while its imaginary part is a gauge degree of freedom that will be absorbed into the gauge field $\bar{P}_m\mathcal{K}\bar{P}_m$ by the Higgs mechanism. The alternative description of low energy dynamics is obtained by ignoring all these massive degrees of freedom and focusing on all the remaining fluctuations around the $\lambda_a=0$ solution. Namely we only consider the fluctuation of the gauge field in $m\times m$ sector and the potential tachyonic mode defined by

$$\mathcal{A}\equiv P_m\mathcal{K}P_m,$$

$$|\tau\rangle\equiv\sum_{a=0}^{m-1}T_a|a\rangle. \tag{70}$$

The full Lagrangian is then reduced to

$$L_{\rm eff}=\frac{2\pi}{g^2\theta}\bigg[\theta\,{\rm tr}\,\dot{\mathcal{A}}\dot{\mathcal{A}}^\dagger-\frac{1}{2}{\rm tr}[\mathcal{A},\mathcal{A}^\dagger]^2+\theta||\dot{\tau}\rangle|^2-\frac{1}{2}(|\mathcal{A}^\dagger|\tau\rangle|^2+|\mathcal{A}|\tau\rangle|^2)-\frac{3}{2}\langle\tau|[\mathcal{A},\mathcal{A}^\dagger]|\tau\rangle-(\theta v^2-1)||\tau\rangle|^2-||\tau\rangle|^4\bigg]+L_{\rm resa}, \tag{71}$$

where the Gauss law constraint

$$[\mathcal{A},\dot{\mathcal{A}}^\dagger]-[\dot{\mathcal{A}},\mathcal{A}^\dagger]+|\tau\rangle\langle\dot{\tau}|-|\dot{\tau}\rangle\langle\tau|=0,$$

$$\dot{\mathcal{A}}^\dagger|\tau\rangle-\mathcal{A}^\dagger|\dot{\tau}\rangle=0 \tag{72}$$

is still in effect on the Lagrangian. The residual part of the Lagrangian can be organized as follows. Denoting all the remaining massive modes collectively by Z_p, there are terms of $O(Z_p^2)$, $O(\mathcal{A}Z_p^2)$, $O(\tau Z_p^2)$, $O(\tau\mathcal{A}Z_p)$, $O(\tau^2 Z_p)$, $O(Z_p^3)$, and quartic terms including at least one massive degrees Z_p. One should note that there are no terms of order $O(\mathcal{A}^2Z_p)$.

When $\theta v^2>1$, the tachyonic modes become massive too. To truncate the Lagrangian consistently, we consider $\mathcal{A}\sim O(\epsilon)$. The the $\mathcal{A}^4$ terms contribute to the Lagrangian as $O(\epsilon^4)$. Now if one turns on any massive degrees, it should be $O(\epsilon^2)$ due to Z_p^2 or τ^2 terms in order to have a valid approximation of dropping the massive degrees. Then the interaction terms between the massive and the massless degrees are of higher order, i.e., $O(\epsilon^n)$ with $n\geqslant 5$. For example, we see that the terms of $O(\mathcal{A}Z_p^2)$ is of order $O(\epsilon^5)$. If there were terms of order $O(\mathcal{A}^2Z_p)$, these would contribute to the potential as $O(\epsilon^4)$. However, there are no such terms as stated previously. Hence the massive degrees are effectively decoupled from the massless degrees to the quartic order in the low energies. Hence we may consistently drop all the massive degrees consistently. Ignoring all the massive modes, we are led to

$$L_{\rm eff}=\frac{2\pi}{g^2}\left({\rm tr}\,\dot{\mathcal{A}}\dot{\mathcal{A}}^\dagger-\frac{1}{2\theta}{\rm tr}[\mathcal{A},\mathcal{A}^\dagger]^2\right) \tag{73}$$

with a constraint

$$[\mathcal{A},\dot{\mathcal{A}}^\dagger]+[\mathcal{A}^\dagger,\dot{\mathcal{A}}]=0. \tag{74}$$

This Lagrangian is precisely the matrix model, which coincides with the bosonic part of an effective Lagrangian for $mD0$ branes moving in two dimensional target space.

The vacuum moduli of this effective action is the vortex moduli described previously by the coordinate λ_a on $(R^2)^m/\mathcal{S}_m$. We see clearly that the singularity when vortices are overlapping is resolved in this description. Moreover, the commutative moduli coordinates are replaced by noncommutative matrix degrees whose structure is especially relevant when vortices are nearly coincident. Hence a legitimate approach toward the quantization of the low energy dynamics is also quite clear.

One might ask at this point about the nature of the coordinates of vortex positions. Since the noncommutative space underlies in defining the noncommutative field theories, one would also expect that the noncommutative solitons should see directly the noncommutative nature of the underling space through their forms of interactions. But the above description does not show directly the noncommutative nature. Namely, the interactions do not show any particular structure depending upon the noncommutativity scale θ. Stated again, nothing particular happens at the separation $\Delta\lambda\sim\sqrt{\theta}$. Nonetheless, the vortex positions are truly described not by c-number eigenvalues but by matrices. In this respect, the locations of vortices still possess a noncommutative nature that is originated from the matrix properties.

Next, we consider the case where $\theta v^2=1$. In this case the potential tachyonic modes become massless. But there exist quartic contributions, so it is not a moduli degree as defined by the configuration space of the constant energy. But we include it because its contribution is of the same order of $\mathcal{A}$ when $\mathcal{A}$ is small. Hence, to study interaction between the massless modes and the massive modes, we let $\mathcal{A}$ and τ be the order of ϵ as before. Then Z_p may be allowed to the order of ϵ^2 to have a well defined low energy description. But this time, there are interaction terms of the forms $\tau\mathcal{A}Z_p$ and $\tau^2 Z_p$, whose contribution to the Lagrangian is $O(\epsilon^4)$. Hence the massless degrees are not decoupling from massive degrees. One could write down the consistent effective Lagrangian for this case too. But it turns out that the effective Lagrangian involves an infinite number of massive degrees. Instead of giving a detailed analysis, we here briefly comment on the nature of the resulting motion involving the potential tachyonic modes. First note that one may effectively describe the motion by $\mathcal{A}$ and τ once all the massive modes are integrated out. One may then easily verify that, among $O(\mathcal{A}^2\tau^2)$ terms, only the term of $\langle\tau|[\mathcal{A},\mathcal{A}^\dagger]|\tau\rangle$ remains out of Eq. (71). This is quite consistent with the translational invariance of the underlying system, whose action is replacing $\mathcal{A}$ by $\mathcal{A}+\xi I_{m\times m}$. The motion along the τ direction is controlled by two terms, $\langle\tau|[\mathcal{A},\mathcal{A}^\dagger]|\tau\rangle$ and τ^4. This motion excites other components of the magnetic field out of the static solution $(1/\theta)P_m$, while the flux Φ is preserved. Hence the motion represents an oscillatory dispersion of magnetic field to other components. If the tachyonic modes are small enough, the part of matrix mechanics responsible for the vortex positions is little affected for fixed energies.

Now we turn to the case where $\theta v^2<1$. In this case the fluctuations include the tachyonic modes. Small fluctuations will trigger the vortex to run into a more stable lower energy configuration that corresponds to BPS states. As shown previously, the BPS state has the same flux as the original unstable static configuration. Thus during the process, the flux should be conserved while the difference in energy is eventually dissipated away. The tachyonic instability is present even for the case of a single vortex. So it can be interpreted as a collapse of each individual vortex to a more stable one, i.e., the BPS state. The detailed study of the collapse will be quite interesting in relation with a recent discussion of the tachyon condensation in string theory.

VI. CONCLUSIONS

In this paper, we have first investigated general static soliton solutions in the noncommutative Abelian-Higgs theory. There are exact multivortex solutions found in Ref. [3] for general values of parameters λ and θv^2. These are in general non-BPS except $\lambda=\theta v^2=1$. We extend these solutions by finding exact solutions describing vortices positioned at arbitrary locations. We have shown that these solutions are unstable only when $\theta v^2<1$. It is therefore expected that lower energy non-BPS solutions exist for $\theta v^2=1$ and $\lambda=1$. We confirm this by considering a self-dual BPS branch for $\lambda=1$. For $\theta v^2\leqslant 1$, the self-dual BPS branch develops, which has a lower energy than the exact unstable vortices. The BPS branch ended at the point θv^2 and there no longer exist BPS solutions for $\theta v^2>1$. Instead, the exact non-BPS configurations become stable configurations. We also illustrated the case of anti-self-dual BPS solutions that have a negative flux [11]. The solutions are shown to exist for $\theta v^2\ll 1$ or $\theta v^2\gg 1$. For the intermediate region, the existence of the BPS solutions are not clear yet.

We then discussed the general fluctuation spectra around the exact static vortices with general moduli parameters λ_a. It is shown that there are tachyonic instabilities only when $\theta v^2<1$. We have identified the massless degrees of freedom and masses of all the off diagonal degrees. With help of the covariant position operator and studying translation of vortices, we were able to identify the physical meaning of the moduli parameters; they are positions of the vortices. We were able to construct exact moving solutions of vortices, where each vortex is moving freely in a arbitrary constant velocity. We then show that the metric in the moduli space is indeed flat by evaluation of the low energy effective Lagrangian within the moduli space description. In fact, one may go beyond the moduli space description in this case by identifying quartic order interaction terms of the massless degrees of freedom. It is nothing but the matrix model of $mD0$ branes moving in a two-dimensional target space. Thus we have shown that the low energy dynamics are faithfully described not by positions of individual vortices but by matrices.

The exact time-dependent solutions describe vortices with constant velocity. What is striking in the solution is not that vortices are moving freely but that the velocity is not limited by the light velocity. The solution exists even for the velocity greater than the light velocity. We argued that the fluctuation becomes tachyonic when the velocity exceeds the light velocity. Therefore the solution seems not to have much physical meaning when the velocity exceeds the light velocity. Further detailed study is required on whether or not the solution in the region is consistent with special relativity. Though the system lacks the Lorentz symmetry, the special relativity should be still in effect because one may regard the system as a Lorentz invariant system with a specific background field (a constant NS-NS two form background field in string theory) is turned on.

We expect that our investigations can be generalized to the $N=2$ supersymmetric version of the noncommutative Abelian-Higgs theory. In particular, supersymmetries will not be preserved even partially for the sector of nonvanishing flux with $\theta v^2>1$. We like to finally mention that our investigations may be applicable to other exact solutions recently found [15–18].

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APPENDIX A: SELF-DUAL BPS SOLUTIONS

Here we demonstrate how to obtain the BPS solutions. Substituting the ansatz for K [Eq. (26)] into the master Eq. (25) gives

$$\sum_{n=0}^{\infty}(\theta v^2-1+|f_n|^2-|f_{n-p}|^2)|n\rangle\langle n| = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{s_i\overline{s_j}}{n!}\overline{f_i f_{i+p}\cdots f_{i+(n-1)p}} \times f_j f_{j+p}\cdots f_{j+(n-1)p}|i+np\rangle\langle j+np|, \tag{A1}$$

where we set $|\phi_0\rangle=\Sigma_{i=0}^{\infty}s_i|i\rangle$ and $f_j=0$ for any $j<0$. Comparing $|0\rangle\langle i|,\ i\geqslant 1$ components of the left and right sides we see $0=s_0\overline{s_i},\ i\geqslant 1$. Now by mathematical induction, one can show easily $0=s_i\overline{s_j},\quad i\neq j$. Hence we may put $|\phi_0\rangle=\zeta|m\rangle$ for some complex number ζ and a non-negative integer, m. This simplifies Eq. (A1) as

$$\sum_{n=0}^{\infty}(\theta v^2-1+|f_n|^2-|f_{n-p}|^2)|n\rangle\langle n| =|\zeta|^2\sum_{n=0}^{\infty}\frac{1}{n!}|f_m|^2|f_{m+p}|^2\cdots|f_{m+(n-1)p}|^2|m+np\rangle \times\langle m+np|. \quad (A2)$$

Hence

$$\theta v^2-1+|f_n|^2-|f_{n-p}|^2=0, \quad (A3)$$

for $0\leqslant n<m$ or $m<n, n\neq m \mod p$, and

$$\theta v^2-1+|f_{m+np}|^2-|f_{m+(n-1)p}|^2 =\frac{|\zeta|^2}{n!}|f_m|^2|f_{m+p}|^2\cdots|f_{m+(n-1)p}|^2 \quad (A4)$$

for $0\leqslant n$. With $q_n\equiv|f_{m+(n-1)p}|^2-n-\theta v^2(p-1)(n-1)$, the magnetic field is expressed as

$$B=v^2{\sum_n}'|n\rangle\langle n| +\frac{1}{\theta}\sum_{n=0}^{\infty}[q_n-q_{n+1}-\theta v^2(p-1)]|np+m\rangle\langle np+m|, \quad (A5)$$

where Σ' is the sum over $0\leqslant n<m$ and $m<n$, $n\neq m \mod p$. The flux is then given by

$$\Phi=m\theta v^2+q_0-\lim_{n\to\infty}q_n, \quad (A6)$$

where if we write $m=pk+r$, $0\leqslant k, 0\leqslant r\leqslant p-1$, $q_0=k(1-\theta v^2)+(p-1)\theta v^2$. In order to have a finite energy q_n ought to converge. If $\zeta=0$, we find that $|f_{np+r}|^2=(1-\theta v^2)(n+1)$, $0\leqslant n, 0\leqslant r<p$, and $q_n=-\theta v^2pn+\theta v^2(p-k-1)+k$. For this solution, the energy diverges. On the other hand, if $\zeta\neq 0$ then q_n satisfies the following recurrence relation:

$$\frac{q_{n+1}-q_n+p\theta v^2}{q_n-q_{n-1}+p\theta v^2}=1+(p-1)\theta v^2+\frac{1}{n}[q_n-(p-1)\theta v^2]. \quad (A7)$$

We take the $n\to\infty$ limit of the above equation and conclude that $p=1$ is a necessary condition for q_n to converge. Now for $p=1$ let us assume that $\lim q_n=\alpha$. This implies that, for any $\varepsilon>0$, there exists large N such that $\alpha-\varepsilon<q_n<\alpha+\varepsilon$ for $n\geqslant N$. Equation (A7) implies

$$\frac{\theta v^2}{q_N-q_{N-1}+\theta v^2}=\prod_{n=N}^{\infty}\left(1+\frac{q_n}{n}\right). \quad (A8)$$

Furthermore, we have

$$\prod_{n=N}^{\infty}\left(1+\frac{\alpha-\varepsilon}{n}\right)<\prod_{n=N}^{\infty}\left(1+\frac{q_n}{n}\right)<\prod_{n=N}^{\infty}\left(1+\frac{\alpha+\varepsilon}{n}\right). \quad (A9)$$

However, for any $\varrho\neq 0$

$$\prod_{n=N}^{\infty}\left(1+\frac{\varrho}{n}\right)=\exp\left(\sum_{n=N}^{\infty}\ln(n+\varrho)-\ln n\right), \quad (A10)$$

which is either infinity or zero depending on the signature of ϱ. Thus, Eqs. (A8)–(A9) implies that α must be zero.

With $p=1$, Eq. (A7) gives a recurrence relation

$$q_{n+1}-q_n=q_n-q_{n-1}+\frac{q_n}{n}(q_n-q_{n-1}+\theta v^2), \quad (A11)$$

with two initial data, $q_0=m(1-\theta v^2)$ and $q_1=m(1-\theta v^2)+|\zeta|^2-\theta v^2$. Therefore if $|\zeta|^2>\theta v^2$, then q_n is monotonically increasing. As the only possible converging value is zero, it must diverge. In case $|\zeta|=0$, it can be easily solved by $q_n=m(1-\theta v^2)-n\theta v^2$.

APPENDIX B: NONEXISTENCE OF SELF-DUAL BPS SOLUTIONS FOR $\theta V^2>1$

We shall work in a gauge $K_{ij}=0$ for $i>j$. The BPS equations can be written as

$$\varphi c_m^\dagger=h^\dagger\overline{P}_m+c_m^\dagger\varphi\overline{P}_m+h^\dagger\varphi\overline{P}_m, \quad (B1)$$

$$\epsilon P_m=(1+\epsilon)(\varphi\overline{P}_m+\overline{P}_m\varphi^\dagger+\varphi\overline{P}_m\varphi^\dagger) -([c_m,h^\dagger]+[h,c_m^\dagger]+[h,h^\dagger]), \quad (B2)$$

where $c_m\equiv S_mcS_m^\dagger$ and $\overline{P}_m=1-P_m$. The relevant part of the first order equations in ω reads

$$P_m\varphi_{(1)}\overline{P}_m=-P_mh_{(1)}^\dagger c_m+P_mh_{(1)}c_m^\dagger,$$

$$P_m\varphi_{(1)}c_m^\dagger=P_mh_{(1)}^\dagger\overline{P}_m. \quad (B3)$$

Since $P_mh_{(1)}^\dagger\overline{P}_m=0$ for our gauge choice, we find that $\varphi_{am}^{(1)}=h_{a,m+1}^{(1)}$, $h_{am}^{(1)}$ can be arbitrary but all the remaining components should vanish. Now we investigate the second order equations obtained from the perturbation equation (B2). Let us multiply P_m to the left and to the right of the equation at the same time. We obtain

$$-\frac{\epsilon}{|\epsilon|}P_m+P_m\varphi_{(1)}\overline{P}_m\varphi_{(1)}^\dagger P_m=[P_mh_{(1)}P_m,P_mh_{(1)}^\dagger P_m] +P_mh_{(1)}\overline{P}_mh_{(1)}^\dagger P_m. \quad (B4)$$

Using the result of the first order equations and taking trace of the above equation, one finds

$$-\frac{m\epsilon}{|\epsilon|}=\sum_{a=0}^{m-1}|h_{am}^{(1)}|^2. \quad (B5)$$

Hence we get a contradiction when $\epsilon>0$.

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