Pair production via crossed lasers

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We discuss the intrinsically nonperturbative probability P_0 for electron-positron production in the overlap region of a pair of high-intensity lasers, by adapting the Fradkin representation for the logarithm of the fermion determinant to several models defined as approximations to the exact problem. In each case we find for P_0 an expression resembling Schwinger's 1951 expression for the vacuum persistence probability of pair production in an external electric field, proportional to an exponential factor that contains an essential singularity, and hence does not admit a perturbative expansion about zero coupling. Qualitative estimates of the best of these models suggest that realistic yields for e^+e^- production must await lasers of intensity 10^{29} W/m^2, roughly seven orders of magnitude more powerful than the highest intensity of currently known lasers. We comment on the possibility of producing a quark-antiquark pair in this way, and note the possibility of achieving temporary, but large separations of the produced q-$\bar{q}$.

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I. INTRODUCTION

There are certain processes in quantum theory that are intrinsically nonperturbative, and we here describe a new addition to that list. The most relevant example of that *genre* to our problem is surely Schwinger's 1951 calculation [1] of P_0, the vacuum persistence probability/vol sec for e^+e^- production in a constant electric field, given as a sum over discrete terms (labeled by the integer n) proportional to the factor $\exp[-n\pi m^2/eE]$, where e and m are the electron $|$charge$|$ and mass, respectively, and E is the constant electric field. This probability is clearly nonperturbative because of the essential singularity in e, which assures that no perturbative expansion in the coupling constant can ever produce a result other than zero for this quantity.

It is well known [1] that a single laser, however high its intensity, cannot extract from the vacuum an oppositely charged pair, in contrast with the pairs that can be generated in a constant electric field of sufficient strength. The situation changes qualitatively, however, in the overlap region of two lasers whose beams make a fixed angle to each other, for here the conservation of energy and of momentum does not rule out pair production. The relevant question is whether the predicted rates of production are sufficiently large to make this process of experimental interest; and to answer this question, we have undertaken a series of calculations of three models which simplify and qualify the essence of the new physics considered. At once, we state that in this paper we are interested only in the qualitative orders of magnitude that appear in each model, as a way of understanding whether current high-intensity lasers are capable of generating pairs; and, if not, just how many more orders of magnitude of laser intensity are required.

The present estimates are an outgrowth of previous work [2] in which new functional representations for the electron Green's function $G_c[A]$ and the logarithm of the fermion determinant $L[A]$ were invented and approximated. Here, however, we take the more direct path of approximating the exact, well known [3] Fradkin representation for $L[A]$, and write the vacuum-to-vacuum amplitude in the form

$$\langle 0|S|0\rangle = e^{-\Gamma T/2+i\phi} = e^{L[A_{ext}]}, \tag{1.1}$$

where A_{ext} is the vector potential corresponding to the overlapping laser beams, and T denotes the elapsed time (assuming the beams were turned on at $T=0$).

It should be noted immediately that Eq. (1.1) as written omits the radiative corrections of the quantized photon field, which are exactly incorporated as

$$\langle 0|S|0\rangle = e^{\mathcal{D}_A} e^{L[A+A_{ext}]}|_{A\to 0}, \tag{1.2}$$

where the linkage operator

$$\mathcal{D}_A = -(i/2)\int (\delta/\delta A_\mu) D_{c,\mu\nu} (\delta/\delta A_\nu),$$

where $D_{c,\mu\nu}$ is the (bare) photon propagator. The reason why such radiative corrections are neglected here is that any charged particle so produced will find itself in the presence of intense laser beams, and its subsequent motion may be expected to be essentially classical [4]. This is not true for the case of QCD, as discussed below.

Before presenting the models and their estimates, it will be useful to discuss the physics of the process we are considering. For a single laser, it is impossible to satisfy momentum-energy conservation requirements for lifting a pair out of vacuum fluctuations, because the conservation laws $nk_\mu = p_\mu + p'_\mu$ cannot be satisfied; here, the four-dimensional square of the left-hand side (LHS) is zero, for arbitrary integer *n*, while that of the RHS is necessarily nonzero for a real electron (p) and positron (p'). For an arrangement of crossed lasers, the equation is changed to read $n_1 k_{1\mu} + n_2 k_{2\mu} = p_\mu + p'_\mu$, and there are now solutions for a variety of integers $n_{1,2}$. For ease of calculation, we shall suppose that both lasers are composed of photons of the same $\simeq 1$ eV energy, that the lasers beams are oriented at a relative angle of 90°, with a zero angle between their polarization vectors, and with an arbitrary phase difference between their fields; for purposes of estimation, we assume the lasers to have identical intensities of $F = 10^{22}$ W/m^2, to produce a beam over a small area of length dimension $D = 10^{-5}$ m, and to have a pulsed duration of 10^2 fsec. These numbers define our ''ideal'' high-intensity laser and are used in obtaining our numerical estimates for Γ below.

From a Feynman graph point of view, we are asking for the amplitude for a total of at least *n* laser photons to be absorbed coherently, as in Fig. 1, where for leptons produced at rest in their c.m., $n = 2mc^2/\hbar\omega = 10^6$; this means a factor of e^n in the production amplitude, and a factor of α^n in the cross section. What could possibly compensate such a minuscule factor? The fact that in the overlap volume D^3 of the crossed lasers there can be N ''available'' photons, and the production probability must include a counting factor similar to $N!/n!(N-n)!$, the number of ways of selecting *n* photons out of *N* available photons. If $N/n = f \gg 1$, that factor is approximatively f^n; and in this way, as long as $f > \alpha^{-1}$, the multiple factors of α^n are effectively neutralized. By using a functional representation for $L[A]$, all such counting factors are automatically included.

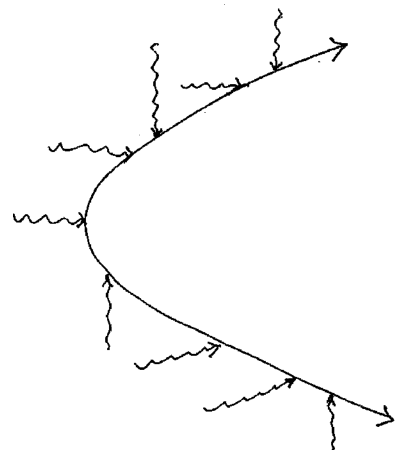

FIG. 1. A virtual e^+e^- pair absorbs photons from the two overlapping lasers.

The arrangement of these remarks is as follows. In the next section, we state the basic, functional formulas of the problem, with application to the external fields of crossed lasers, and we define the three models that, in increasing order of complexity, illustrate the physics contained in the exact problem. In Sec. III, we consider requirements for a viable experiment; and in Sec. IV the application of such experiments to QCD is briefly discussed. A short summary, Sec. V, completes the paper.

II. FUNCTIONAL FORMULATION

We begin by recalling [3] the exact Fradkin representation for the $L[A]$ of QED:

$$\begin{aligned} L[A] = -\frac{1}{2}\int_0^\infty \frac{ds}{s} & \\ &\times \exp(-ism^2)\int d^4x \int \frac{d^4p}{(2\pi)^4} \exp\left[i\int_0^s ds' \sum_\mu \frac{\delta^2}{\delta v_\mu^2(s')}\right] \\ &\times \exp\left(ip\cdot\int_0^s ds' v(s')\right)\left(\exp\left[-ie\int_0^s ds' v_\mu(s') A_\mu\left(x - \int_0^{s'} v\right)\right]\right. \\ &\times \mathrm{tr}\left\{ \exp\left[e\int_0^s ds' \sigma\cdot F\left(x - \int_0^{s'} v\right)\right]\right\}_+ \left. - (e\to 0)\right)\Bigg|_{v_\mu\to 0} \end{aligned} \tag{2.1}$$

or the equivalent form, obtained by the use of the convenient relation

$$\exp\left(i\int_0^s ds' \frac{\delta^2}{\delta v^2}\right)\exp\left(ip\cdot\int_0^s ds' v(s')\right)\mathcal{F}(v)\Bigg|_{v_\mu\to 0} = \exp(-isp^2)\exp\left(i\int_0^s ds'\frac{\delta^2}{\delta v^2}\right)\mathcal{F}(v-2p)\Bigg|_{v_\mu\to 0},$$

which gives

$$L[A]=-\frac{1}{2}\int d^4x\int\frac{d^4p}{(2\pi)^4}\int_0^\infty\frac{ds}{s}\exp[-is(m^2+p^2)]\exp\left(i\int_0^s ds'\frac{\delta^2}{\delta v^2}\right)\left(\exp\left[-ie\int_0^s ds'[v_\mu(s')-2p_\mu]\right.\right.$$
$$\left.\left.\times A_\mu\left(x+2s'p-\int_0^{s'}v\right)\right]\mathrm{tr}\left\{\exp\left[e\int_0^s ds'\sigma\cdot F\left(x+2s'p-\int_0^{s'}v\right)\right]\right\}_+-(e\to 0)\right)\Bigg|_{v_\mu\to 0}. \quad (2.2)$$

Here, $\sigma_{\mu\nu}=\frac{1}{4}[\gamma_\mu,\gamma_\nu]$, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, and tr indicates the trace over Dirac indices of the ordered exponential. The entire problem reduces to the estimation of the integrals of Eq. (2.2) for the case when the external field $A_\mu(x)=\epsilon_\mu^{(1)}\sin(k^{(1)}\cdot x)+\epsilon_\mu^{(2)}\sin(k^{(2)}\cdot x+\delta)$, corresponding to the vector potential of a pair of intersecting laser beams of well defined frequencies and polarizations. (We are using plane-wave solutions of transverse ''width'' D to represent each of the laser beams.) As stated in the previous section, and pictured in Fig. 1, we adopt the simplest theoretical and experimental setup, wherein two beams of the same frequency and polarization intersect perpendicularly, so that $\hat{\epsilon}^{(1)}=\hat{\epsilon}^{(2)}=\hat{\epsilon}$, $0=\hat{k}^{(1)}\cdot\hat{k}^{(2)}=\hat{\epsilon}\cdot\hat{k}^{(1)}=\hat{\epsilon}\cdot\hat{k}^{(2)}$, and $A_\mu(x)=\epsilon_\mu[\sin\delta_1+\sin\delta_2]$, with $\delta_1=k^{(1)}\cdot x$, $\delta_2=k^{(2)}\cdot x+\delta$, and $\epsilon_\mu\to\hat{\epsilon}\cdot\epsilon$. For definiteness, we choose the unit vector $\hat{\epsilon}$ to point in the $\hat{\imath}$ (or $\hat{x}$) direction, while $\hat{k}^{(1)}=\hat{\jmath}$, and $\hat{k}^{(2)}=\hat{k}$; hence, $\delta_1\to\omega(y-t)$ and $\delta_2\to\omega(z-t)+\delta$. Until the very last step, we shall use ''natural units,'' with $\hbar=c=1$; here, ω and ϵ have units of mass, and the average energy density U of each laser is given by $\epsilon^2\omega^2/8\pi$.

There are three operations that must be performed in Eq. (2.2)—the functional linkage operation, and the x and p integrations—and the complexity of the result can depend on the order in which these operations are arranged. One can begin by first performing the linkage operation (without approximation), which is the natural approach for the calculation of the corresponding $G_c[A]$ (where there is no $\int dx$ average over configuration space); but it will be somewhat more advantageous to consider these three operations in differing sequences. However, it is most useful, and simplest, to realize at once that the fundamental process we are trying to describe is such that a large number of (coherent) photons of energy $\omega\ll m$ are to be absorbed by the produced e^+ and e^- four-momenta p and p'. We therefore expect that the vacuum persistence probability $P_0=\exp(2\,\mathrm{Re}\,L[A])$ can be well approximated by treating the absorbed photons as ''soft'' compared to the lepton four-momenta; and this naturally suggests a simplifying, no-recoil approximation, of which several are available [5].

For our problem, perhaps the simplest such approximation is obtained by dropping the remaining v dependence inside the argument of the A_μ and $F_{\mu\nu}$ of Eq. (2.2), for the function of this dependence is to produce corrections to the p,p' fermion four-momenta as they absorb the soft laser photons. Hence, based on the reasonable expectation that soft corrections to hard e^+,e^- momenta are irrelevant, we here perform the first simplification of the exact Eq. (2.2), replacing the latter by

$$L[A]\Rightarrow-\frac{1}{2}\int_0^\infty\frac{ds}{s}\exp(-ism^2)\int d^4x\int\frac{d^4p}{(2\pi)^4}\exp(-isp^2)\exp\left(i\int_0^s ds'\frac{\delta^2}{\delta v^2}\right)$$
$$\times\left\{\exp\left(-ie\int_0^s ds'[v_\mu(s')-2p_\mu]A_\mu(x+2s'p)\right)\mathrm{tr}\left(\exp\left[e\int_0^s ds'\sigma\cdot F(x+2s'p)\right]\right)_+-(e\to 0)\right\}\Bigg|_{v_\mu\to 0}. \quad (2.3)$$

In the Schwinger model, the only function of the ordered exponential is to provide a contribution to the normalization of each of the sequence of essential singularities that comprise P_0; those singularities arise from the functional operation upon the A_μ dependence, followed by an appropriate $\int d^4p$. In the present problem, complicated by the necessity of spatial averaging, the essential singularity will also arise from the corresponding A_μ factor, with the $\sigma\cdot F$ term contributing to the normalization. Since we are interested only in the order of magnitude of Γ, generated by the essential singularity, and since we have every confidence that a complete calculation that includes the $\sigma\cdot F$ term will provide a positive Γ (that is, a negative $\mathrm{Re}\,L[A]$), we shall simply drop the $\sigma\cdot F$ ordered exponential, replacing its trace by $+4$. In principle, the entire analysis can be organized without this approximation; but this adds nothing but complication to the extraction of the order of magnitude of the essential singularity. Thus, we further simplify the expression for $L[A]$,

$$L[A] \Rightarrow -2\int_0^\infty \frac{ds}{s}\exp(-ism^2)\int d^4x\int \frac{d^4p}{(2\pi)^4}\exp(-isp^2)\exp\left(i\int_0^s ds'\frac{\delta^2}{\delta v^2}\right)$$
$$\times\left\{\exp\left(-ie\int_0^s ds'[v_\mu(s')-2p_\mu]A_\mu(x+2s'p)\right)-1\right\}\Bigg|_{v_\mu\to 0}, \tag{2.4}$$

and we now consider, in sequence, three models for its estimation, in order of increasing complexity.

These models are, in essence, approximations to a full, cluster decomposition/cumulant summation of the integrals and functional operation stated in Eq. (2.4). Model A is defined by first performing the elementary functional operation, inserting a useful representation of the momentum-space integrals, and subsequently approximating the configuration-space integrals in the simplest possible way. In model B, the spatial averaging and the functional operation are performed exactly, while the results of the exact momentum-space integrals are approximated in an analogous, first-cumulant way. The essential singularities that result in models A and B are the same. Model C attempts to include the most relevant contributions of all terms in the representation appearing in the analysis of model B, in contrast to the latter's retention of only its first, nonzero, cluster coefficient. Most interestingly, the essential singularity of model C is weaker than that of models A and B.

A. First-cumulant approximation

The linkage operation of Eq. (2.4) can be carried through immediately, yielding

$$\exp\left(i\int_0^s ds'\frac{\delta^2}{\delta v^2}\right)\exp\left(-ie\int_0^s ds' v_\mu(s')A_\mu(x+2s'p)\right)\Bigg|_{v_\mu\to 0}=\exp\left(-ie^2\int_0^s ds' A^2(x+2s'p)\right),$$

and one is then left with

$$\int d^4p\,\exp\left(-isp^2+2ie\int_0^s ds'\,p\cdot A(x+2s'p)-ie^2\int_0^s ds' A^2(x+2s'p)\right)$$

which has the form

$$\int d^4p e^{-isp^2}\mathcal{F}(p\cdot\epsilon^{(1)},p\cdot\epsilon^{(2)},p\cdot k^{(1)},p\cdot k^{(2)}) \tag{2.5}$$

where we (temporarily, and for maximum generality) reinstate $\epsilon_\mu^{(1)}$ and $\epsilon_\mu^{(2)}$ as independent and distinct polarizations. The integral of Eq (2.5) can be rewritten in the form

$$(2\pi)^{-4}\int du_a\int du_b\int du_c\int du_d\mathcal{F}(u_a,u_b,u_c,u_d)\int d\omega_a\int d\omega_b\int d\omega_c\int d\omega_d\exp[i(u_a\omega_a+u_b\omega_b+u_c\omega_c+u_d\omega_d)]$$
$$\times\int d^4p\exp(-isp^2)\exp(-ip\cdot[\omega_a\epsilon^{(1)}+\omega_b\epsilon^{(2)}+\omega_c k^{(1)}+\omega_d k^{(2)}]), \tag{2.6}$$

where the range of each $u_a,\ldots,\omega_d$ integration is from $-\infty$ to $+\infty$.

All the integrals of Eq. (2.6), except $\int du_c\int du_d$, can be performed immediately, leaving

$$\int d^4pe^{-isp^2}\mathcal{F}\to -\frac{i\pi^2}{s^2}\left(\frac{2s}{\omega^2}\right)\frac{1}{2\pi}\int du_c\int du_d e^{2isu_cu_d/\omega^2}e^{\mathcal{S}(x|u_c,u_d)}, \tag{2.7}$$

where $\mathcal{S}=-ie^2\epsilon^2 s\int_0^1 d\lambda[S^2-\langle S\rangle^2]$ and $S=\sin(\delta_1+2\lambda su_c)+\sin(\delta_2+2\lambda su_d)$. Here, the variable change $s'=\lambda s$ has been made, we have returned to the simplest case of identical polarization vectors, and the notation $\langle S\rangle=\int_0^1 d\lambda S$ has been used.

The configuration-space dependence of Eq. (2.7) must now be integrated, or averaged over the overlap region of the two laser beams. We assume that the linear dimension D of this volume D^3 is significantly larger (at least by a factor of 10) than the laser wavelength λ_γ, so that the averaging procedure adopted here and in the two subsequent models is sensible. The simplest sort of averaging replaces the spatial average of the exponential $(1/D^3)\int d^3xe^{\mathcal{S}(x)}$ by the exponential of the average: $\exp[(1/D^3)\int d^3x\mathcal{S}(x)]$, and is perhaps the simplest of the approximations used in statistical problems to estimate a full, cluster expansion.

It is straightforward to see that the present approximation generates

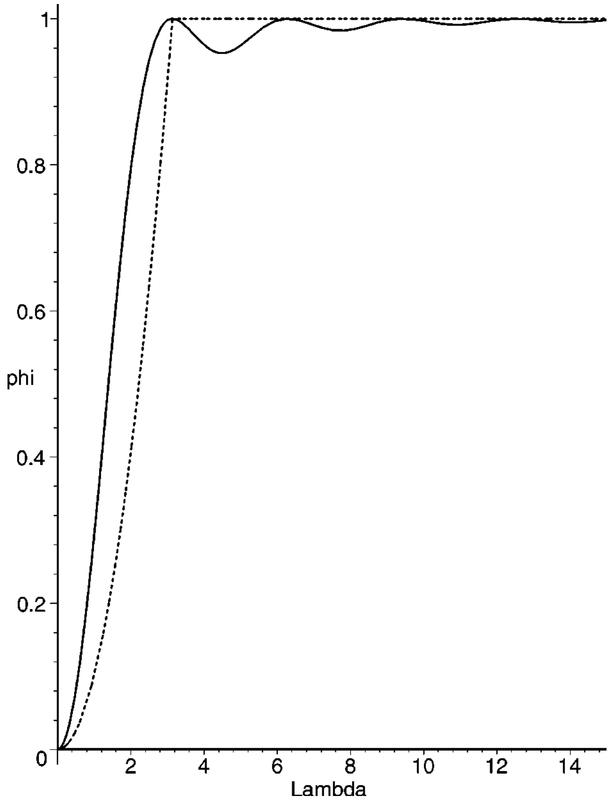

FIG. 2. Full line: Pictorial representation of the exact $\Phi(\Lambda)$. Dashed line: The approximation used in the text.

$$\frac{1}{D^3}\int d^3x\mathcal{S}(x) = -ie^2\epsilon^2 s\left\{1-\frac{1}{2}\int\int_0^1 d\lambda d\lambda'\{\times\cos[2(\lambda-\lambda')\Lambda]+\cos[2(\lambda-\lambda')\bar{\Lambda}]\}\right\}$$

where $\Lambda=su_c$ and $\bar{\Lambda}=su_d$. The integrals over λ and λ' can be done exactly, and yield

$$\frac{1}{D^3}\int d^3x\mathcal{S}(x)=-i\frac{e^2\epsilon^2 s}{2}[\Phi(\Lambda)+\Phi(\bar{\Lambda})]$$

with $\Phi(\Lambda)\equiv 1-(\sin\Lambda/\Lambda)^2$, so that, with $\int d^4x$ written as D^3cT,

$$L[A]\Rightarrow i\frac{m^4(D^3cT)}{(2\pi)^3}\left(\frac{m^2}{\omega^2}\right)\int_0^\infty\frac{dt}{t^4}e^{-it}\times\int\int_{-\infty}^{+\infty}d\Lambda d\bar{\Lambda}e^{2i\Lambda\bar{\Lambda}m^2/\omega^2 t}\times\left\{\exp\left(-it\frac{e^2\epsilon^2}{2m^2}[\Phi(\Lambda)+\Phi(\bar{\Lambda})]\right)-1\right\}, \quad (2.8)$$

where we have introduced the dimensionless variable $t=m^2 s$.

Figure 2 displays a graph of $\Phi(\Lambda)$, (full line) and a graph of the approximation we will use to simplify the $\Lambda,\bar{\Lambda}$ integrals of Eq. (2.8) (dashed line); this approximation corresponds to the replacement of $\Phi(\Lambda)$ by $\theta(\pi-|\Lambda|)\Lambda^2/\pi^2+\theta(|\Lambda|-\pi)$. One could use a more detailed approximation for $\Phi(\Lambda)$, and one could take into account the fact that $\Phi(\Lambda)\not\equiv 1$ for $|\Lambda|>\pi$; but our approximation should give the essence of the behavior of these integrals. A similar approximation will be employed for the more complex, and realistic, models B and C.

By breaking up the $\Lambda,\bar{\Lambda}$ integrations into the regions $(\int_{-\infty}^{-\pi}+\int_{+\pi}^{+\infty})$ and $(\int_{-\pi}^{+\pi})$, and adding (to $\int_{-\infty}^{-\pi}+\int_{+\pi}^{+\infty}$) and substracting (from $\int_{-\pi}^{+\pi}$) a contribution with $\Phi=1$ in the region $(\int_{-\pi}^{+\pi})$, with $\lambda_{max}=(\sqrt{2}\pi/\sqrt{t})(m/\omega)$, one obtains

$$L_A\Rightarrow i\frac{(D^3cT)}{2(2\pi)^2}m^4\int_0^\infty\frac{dt}{t^3}e^{-it}\left\{-\left(1-\exp\left[-i\frac{t}{2}\left(\frac{e\epsilon}{m}\right)^2\right]\right)^2+\frac{1}{2\pi}\int\int_{-\lambda_{max}}^{+\lambda_{max}}d\lambda_1 d\lambda_2 e^{i\lambda_1\lambda_2}\left(\exp\left[-i\frac{t^2\lambda_1^2}{12}\left(\frac{e\epsilon\omega}{m^2}\right)^2\right]-\exp\left[-i\frac{t}{2}\left(\frac{e\epsilon}{m}\right)^2\right]\right)\left(\exp\left[-i\frac{t^2\lambda_2^2}{12}\left(\frac{e\epsilon\omega}{m^2}\right)^2\right]-\exp\left[-i\frac{t}{2}\left(\frac{e\epsilon}{m}\right)^2\right]\right)\right\}. \quad (2.9)$$

Consider now the first term in the curly bracket of Eq. (2.9), independent of λ_{max}. This integrand is analytic in t in the lower half t plane, vanishing there as Im $t\to-\infty$, and as such may be rotated to a contour that runs down the imaginary t axis, from 0 to $-i\infty$. Then, simple inspection shows that its contribution is purely imaginary, and hence it cannot contribute to Re $L[A]$; and we discard it. The remaining λ_{max}-dependent terms of Eq. (2.9) define a function of two variables, $F_A(e\epsilon/m,\omega/m)$, variables that arrange themselves in three ways: as m/ω in λ_{max}, as $(e\epsilon\omega/m^2)^2$ in one set of exponential factors, and as $(e\epsilon/m)^2$ in another. As they stand, the $\lambda_{1,2}$ integrals generate the so-called error function $\Phi(x)$ combined with the Si(x) function, and it is tedious to write them down in detail. But it is surely not necessary, because—by inspection—the order of magnitude of the t variable can never be much larger than unity, while $(m/\omega)\simeq 10^6$, so that $\lambda_{max}\gg 1$. In other words, the physics should not be drastically changed if the limit $\lambda_{max}\to\infty$ is taken, in which case the integrals are trivial—the $\exp[-it(e\epsilon/m)^2/2]$ terms now cancel against the discarded terms in the curly bracket of Eq. (2.9)—and yield

$$L_A\Rightarrow i\frac{(D^3cT)}{2(2\pi)^2}m^4\int_0^\infty\frac{dt}{t^3}e^{-it}\left[\frac{1}{\sqrt{1-(\gamma t)^4}}-1\right] \quad (2.10)$$

where $\gamma=e\epsilon\omega/m^2\sqrt{6}$. Thus, in this limit of arbitrarily large m/ω, but fixed $e\epsilon\omega/m^2$, $L_A\to F_A(\gamma)$. The same behavior

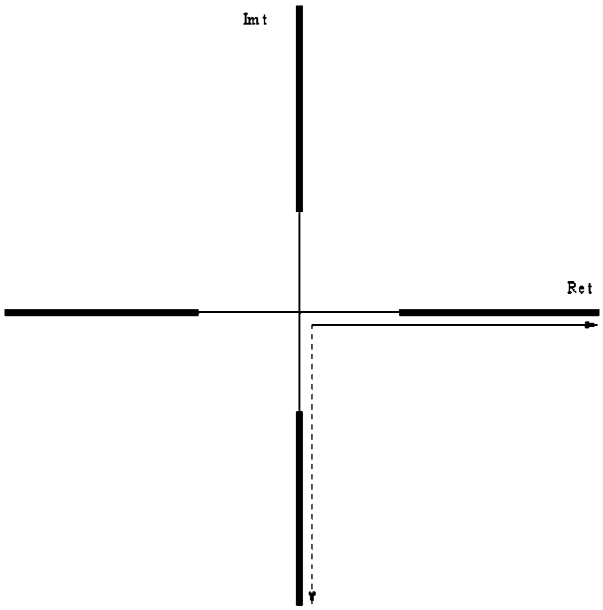

FIG. 3. Cuts of the function $[1-(\gamma\tau)^4]^{-1/2}$ in the τ plane, beginning at $\tau=\pm 1/\gamma$, $\pm i/\gamma$. The solid line represents the original direction of τ integration; the dashed line, the rotated-contour direction.

will be seen in all three models, and its relation to the Schwinger constant-field result will be discussed in the Summary.

One now sees that an (improper) perturbative expansion of the square root of Eq. (2.10) in powers of γ will generate a sequence of imaginary contributions to L_A, for the integral $\int_0^\infty dt t^{1+4n} e^{it}$ is real for every integer n. Hence, $\mathrm{Re}\, L_A$ does not have an expansion in powers of the coupling; it is intrinsically nonperturbative. Its value can be most easily obtained by remembering that the path of the t integration of the original Schwinger/Fradkin representation is to run ε below the positive t axis (because $m \to m - i\varepsilon$). Because $[1-(t/t_0)^4]^{-1/2}$, with $t_0 = 1/\gamma$, has a cut structure that may be expressed as in Fig. 3, a rotation of the integration contour to run as $t \to \varepsilon - i\tau$, $0 \leqslant \tau \leqslant \infty$, is permissible, so that

$$L_A \Rightarrow -i\frac{(D^3cT)}{2(2\pi)^2} m^4 \int_0^\infty \frac{d\tau}{\tau^3} e^{-\tau} \left[\frac{1}{\sqrt{1-(\tau/t_0)^4}} - 1 \right]$$

and

$$\mathrm{Re}\, L_A \Rightarrow -\frac{(D^3cT)}{2(2\pi)^2} m^4 \int_{t_0}^\infty \frac{d\tau}{\tau^3} \frac{e^{-\tau}}{\sqrt{(\tau/t_0)^4-1}}, \quad (2.11)$$

where the branch of the square root has been chosen to yield a negative value for $\mathrm{Re}\, L[A]$. Under the variable change $y = \tau/t_0 - 1$, the integral of Eq. (2.11) becomes

$$t_0^{-2} e^{-t_0} \int_0^\infty \frac{dy}{(1+y)^3} \frac{e^{-yt_0}}{\sqrt{(1+y)^4-1}}$$

$$\simeq t_0^{-2} \frac{e^{-t_0}}{2} \int_0^{t_0^{-1}} \frac{dy}{\sqrt{y}} = t_0^{-3} e^{-t_0}$$

since the y integrand effectively cuts off at $y \sim 1/t_0 \ll 1$. Hence

$$\mathrm{Re}\, L_A \simeq -\frac{(D^3cT)}{2(2\pi)^2} m^4 \gamma^3 e^{-1/\gamma}, \quad (2.12)$$

which clearly displays the essential singularity in γ.

B. A modified cumulant model

We now return to Eq. (2.4), denote the combination $v_\mu(s') - 2p_\mu$ by $V_\mu(s')$, and observe that if the spatial $\int d^3x$ is attempted before the linkage operation of the Fradkin representation, then the spatial averages over $\int dy$ and $\int dz$, for the individual laser beams, can be performed independently. We adopt the notation

$$\int d^3x \to D^3 \left(\frac{1}{D^3} \int dx \int dy \int dz \right) \to D^3 \langle \cdots \rangle_{\theta,\bar{\theta}}$$

where the symbol $\langle \cdots \rangle_{\theta,\bar{\theta}}$ signifies independent averages over the factors $\sin(k^{(1)}\cdot x + 2s'k^{(1)}\cdot p) = \sin[\theta + u(s')]$, and over the factors $\sin(k^{(2)}\cdot x + \delta + 2s'k^{(2)}\cdot p) = \sin[\bar{\theta} + \bar{u}(s')]$, where $\theta = k^{(1)}\cdot x \to \omega(y-t)$, $\bar{\theta} = k^{(2)}\cdot x + \delta \to \omega(z-t) + \delta$, $u(s') = 2s'k^{(1)}\cdot p$, $\bar{u}(s') = 2s'k^{(2)}\cdot p$.

We therefore consider $\langle \cdots \rangle_{\theta,\bar{\theta}}$ as given by

$$\frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\bar{\theta} \exp\left\{ -ie \int_0^s ds' V_\mu(s') \times \{ \epsilon_\mu^{(1)} \sin[\theta + u(s')] + \epsilon_\mu^{(2)} \sin[\bar{\theta} + \bar{u}(s')] \} \right\}$$

or

$$\sum_{n=0}^\infty \frac{(-ie)^n}{n!} \int_0^s ds_1 (V(s_1)\epsilon^{(1)}) \cdots \int_0^s ds_n (V(s_n)\cdot\epsilon^{(1)}) \times \langle \sin[\theta + u(s_1)] \cdots \sin[\theta + u(s_n)] \rangle_\theta \times \sum_{m=0}^\infty \frac{(-ie)^m}{m!} \int_0^s d\bar{s}_1 (V(\bar{s}_1)\cdot\epsilon^{(2)}) \cdots \int_0^s d\bar{s}_m \times (V(\bar{s}_m)\cdot\epsilon^{(2)}) \times \langle \sin[\bar{\theta} + \bar{u}(\bar{s}_1)] \cdots \sin[\bar{\theta} + \bar{u}(\bar{s}_m)] \rangle_{\bar{\theta}}.$$

Consider the first average; it can be rewritten as

$$\left(\frac{1}{2i}\right)^n \langle [e^{i[\theta+u(s_1)]} - e^{-i[\theta+u(s_1)]}] \cdots [e^{i[\theta+u(s_n)]} - e^{-i[\theta+u(s_n)]}] \rangle_\theta$$

and the only nonzero contributions to $\langle \cdots \rangle_\theta$ come from the $(2l)!/(l!)^2$ terms independent of θ; here, $n = 2l$. By symmetry, these terms can be rearranged into a "standard" form

$$\frac{(-)^l}{2^{2l}}\frac{(2l)!}{(l!)^2}(-2)^l\cos[u(s_1)-u(s_2)]$$
$$\times\cos[u(s_3)-u(s_4)]\cdots\cos[u(s_{2l-1})-u(s_{2l})]$$

to which one must append the $\int_0^s ds'[V(s')\epsilon^{(1)}]$ factors. The same analysis holds for the $\langle\cdots\rangle_{\bar\theta}$, with the result that the exact spatial averages can be written as

$$\sum_{l=0}^{\infty}(-)^l\frac{(e^2A_1/2)^l}{(l!)^2}\cdot\sum_{\bar l=0}^{\infty}(-)^{\bar l}\frac{(e^2A_2/2)^{\bar l}}{(\bar l!)^2}$$
$$\equiv J_0\left(e\sqrt{\frac{A_1}{2}}\right)J_0\left(e\sqrt{\frac{A_2}{2}}\right), \tag{2.13}$$

where

$$A_1=\int_0^s ds_a\int_0^s ds_b[V(s_a)\epsilon^{(1)}]\cos[u(s_a)-u(s_b)][V(s_b)\epsilon^{(1)}]$$

and

$$A_2=\int_0^s ds_a\int_0^s ds_b[V(s_a)\epsilon^{(2)}]\cos[u(s_a)-u(s_b)]$$
$$\times[V(s_b)\epsilon^{(2)}].$$

To perform the linkage operation, it is most convenient to introduce the representation [6]

$$J_0(z)=\frac{1}{2\pi i}\int_{-\infty}^{0^+}\frac{dt}{t}e^{t-z^2/4t}$$

where the contour is specified as approaching the origin from $-\infty$ underneath the negative t axis, swinging in a half circle around the origin, and moving out to $-\infty$ above the negative t axis. In this way,

$$\left\langle\exp\left\{-ie\int_0^s ds'V_\mu(s')\{\epsilon_\mu^{(1)}\sin[\theta+u(s')]\right.\right.$$
$$\left.\left.+\epsilon_\mu^{(2)}\sin[\bar\theta+\bar u(s')]\}\right\}\right\rangle_{\theta,\bar\theta}$$
$$=\left(\frac{1}{2\pi i}\right)^2\int_{-\infty}^{0^+}\frac{dt_1}{t_1}e^{t_1}\int_{-\infty}^{0^+}\frac{dt_2}{t_2}e^{t_2}$$
$$\times\exp\left(\frac{i}{2}\int_0^s ds_1\int_0^s ds_2V_\mu(s_1)K_{\mu\nu}(s_1,s_2)V_\mu(s_2)\right)$$

with

$$K_{\mu\nu}(s_1,s_2)=\frac{ie^2}{4}\left[\frac{\epsilon_\mu^{(1)}\epsilon_\nu^{(1)}}{t_1}\cos[2p\cdot k^{(1)}(s_1-s_2)]\right.$$
$$\left.+\frac{\epsilon_\mu^{(2)}\epsilon_\nu^{(2)}}{t_2}\cos[2p\cdot k^{(2)}(s_1-s_2)]\right].$$

For simplicity, we now return to the case of identical polarizations and frequencies, and perform the Fradkin linkages to obtain

$$L_B\Rightarrow-2\frac{(D^3cT)}{(2\pi)^4}\left(\frac{1}{2\pi i}\right)^2\int_{-\infty}^{0^+}\frac{dt_1}{t_1}e^{t_1}$$
$$\times\int_{-\infty}^{0^+}\frac{dt_2}{t_2}e^{t_2}\int_0^\infty\frac{ds}{s}e^{-ism^2}\int d^4p\,e^{-isp^2}$$
$$\times\left\{\exp\left(-\frac{1}{2}\mathrm{Tr}\ln(1+2K)\right)\exp(ip\cdot Q\cdot p)-1\right\} \tag{2.14}$$

where

$$Q_{\mu\nu}(s)=\int_0^s ds_1\int_0^s ds_2\left\langle s_1\left|\left(2K\frac{1}{1+2K}\right)_{\mu\nu}\right|s_2\right\rangle$$

and

$$\langle s_1|K_{\mu\nu}|s_2\rangle=i\frac{e^2}{4}\epsilon_\mu\epsilon_\nu\left[\frac{\cos[2p\cdot k^{(1)}(s_1-s_2)]}{t_1}\right.$$
$$\left.+\frac{\cos[2p\cdot k^{(2)}(s_1-s_2)]}{t_2}\right].$$

Here, both $K_{\mu\nu}$ and $Q_{\mu\nu}$ depend on p; and so we introduce

$$\mathcal{F}(2p\cdot k^{(1)},2p\cdot k^{(2)})$$
$$=\int\int_{-\infty}^{+\infty}du_1du_2\mathcal{F}(u_1,u_2)\times\frac{1}{(2\pi)^2}$$
$$\times\int\int_{-\infty}^{+\infty}d\Omega_1d\Omega_2e^{i(u_1\Omega_1+u_2\Omega_2)}$$
$$\times e^{-2ip\cdot k^{(1)}\Omega_1-2ip\cdot k^{(2)}\Omega_2},$$

where $\mathcal{F}(u_1,u_2)$ represents all the $2p\cdot k^{(1,2)}$ dependence in the curly bracket of Eq. (2.14).

Then we need

$$\int d^4p\,e^{-isp\cdot(1-Q/s)\cdot p}e^{-2ip\cdot(\Omega_1k^{(1)}+\Omega_2k^{(2)})}$$
$$=-i\frac{\pi^2}{s^2}e^{-(1/2)\mathrm{tr}\ln(1-Q/s)}e^{-2i\omega^2\Omega_1\Omega_2/s},$$

with $k^{(1)}\cdot k^{(2)}=-\omega^2$. The integrals over $\Omega_{1,2}$ are immediate, and yield

$$(2\pi)\left(\frac{s}{2\omega^2}\right)e^{iu_1u_2s/2\omega^2}$$

so that

$$L_B = i\frac{(D^3cT)}{16\pi^3}\left(\frac{1}{2\pi i}\right)^2\int_{-\infty}^{0^+}\frac{dt_1}{t_1}e^{t_1}\times\int_{-\infty}^{0^+}\frac{dt_1}{t_2}e^{t_2}\int_0^\infty\frac{ds}{s^3}e^{-ism^2}\left(\frac{s}{2\omega^2}\right)\times\int\int_{-\infty}^{+\infty}du_1du_2e^{iu_1u_2s/2\omega^2}\times\left\{\exp\left(-\frac{1}{2}\mathrm{Tr}\ln(1+2K)e^{-(1/2)\mathrm{tr}\ln(1-Q/s)}\right)-1\right\} \tag{2.15}$$

where

$$\langle s_1|2K_{\mu\nu}(u_1,u_2)|s_2\rangle = i\frac{e^2\epsilon^2}{2}\left(\frac{\epsilon_\mu\epsilon_\nu}{\epsilon^2}\right)\left[\frac{\cos[u_1(s_1-s_2)]}{t_1}+\frac{\cos[u_2(s_1-s_2)]}{t_2}\right] \equiv\left(\frac{\epsilon_\mu\epsilon_\nu}{\epsilon^2}\right)\langle s_1|2K(u_1,u_2)|s_2\rangle.$$

$Q_{\mu\nu}/s$ may be written as $q(s)(\epsilon_\mu\epsilon_\nu/\epsilon^2)$, and we henceforth suppress the factors $(\epsilon_\mu\epsilon_\nu/\epsilon^2)$.

We now estimate

$$\mathrm{Tr}\ln(1+2K)=\int_0^s ds_1\left[2K(s_1,s_1)-\frac{1}{2}\int_0^s ds_2 2K(s_1,s_2)2K(s_2,s_1)+\cdots\right]$$

and adopt the notation $2K(s_1,s_2)\equiv\xi(s_1-s_2)$. Note that $2K(s_1,s_1)\equiv\xi(0)$, which quantity would be the only one appearing were ω set equal to zero. Replacing each cosine term of Eq. (2.15) by 1 corresponds to the limit $\omega\to 0$. But, physically, for $\omega\to 0$ at fixed ϵ, L must vanish; and therefore we must find that the curly bracket of Eq. (2.15) will vanish when each $\xi(s_1-s_2)$ factor is replaced by $\xi(0)$. This suggests expanding Tr ln$(1+2K)$ and tr$[1-q(s)]$ in terms of the relevant quantity $\delta\xi(s_1-s_2)=\xi(0)-\xi(s_1-s_2)$. To first order in $\delta\xi$, one obtains

$$\mathrm{Tr}\ln(1+2K)\to s\xi(0)-\frac{1}{2}[s\xi(0)]^2+\frac{1}{3}[s\xi(0)]^3+\cdots+\left[\left(\frac{1}{2}2\right)[s\xi(0)]-\left(\frac{1}{3}3\right)\times[s\xi(0)]^2+\cdots\right]\Xi_{(1)}(s)$$

or

$$\mathrm{Tr}\ln(1+2K)\to\ln[1+s\xi(0)]+\frac{s\xi(0)}{1+s\xi(0)}\Xi_{(1)}(s), \tag{2.16}$$

where $\Xi_{(1)}(s)=(1/s)\int\int_0^s ds_1ds_2\delta\xi(s_1-s_2)$.

In a similar spirit,

$$\frac{Q_{\mu\nu}}{s}\to\left(\frac{\epsilon_\mu\epsilon_\nu}{\epsilon^2}\right)(s\xi(0)-[s\xi(0)]^2+\cdots-\{1-2s\xi(0)+3[s\xi(0)]^2+\cdots\}\Xi_{(1)}(s))$$

or

$$\frac{Q}{s}\to\left\{\frac{s\xi(0)}{1+s\xi(0)}-\frac{1}{[1+s\xi(0)]^2}\Xi_{(1)}(s)\right\},$$

so that

$$\mathrm{tr}\ln(1-Q/s)\to-\ln[1+s\xi(0)]+\frac{1}{1+s\xi(0)}\Xi_{(1)}(s). \tag{2.17}$$

Adding Eqs. (2.16) and (2.17), we see that that the terms independent of ω do indeed cancel, leaving

$$\exp\left(-\frac{1}{2}\mathrm{Tr}\ln(1+2K)-\frac{1}{2}\mathrm{tr}\ln(1-Q/s)\right.\Rightarrow\exp\left(-\frac{1}{2}\Xi_{(1)}(s)\right)\Big) =\exp\left\{-i\frac{e^2\epsilon^2}{4}\frac{1}{s}\int\int_0^s ds_1ds_2\times\left[\frac{1}{t_1}(1-\cos[u_1(s_1-s_2)])+\frac{1}{t_2}(1-\cos[u_2(s_1-s_2)])\right]\right\}.$$

The integrals over $s_{1,2}$ can be done exactly, and generate for this factor

$$\exp\left\{-i\frac{e^2\epsilon^2 s}{4}\left[\frac{1}{t_1}\Phi\left(\frac{\omega\Lambda_1\sqrt{s}}{\sqrt{2}}\right)+\frac{1}{t_2}\Phi\left(\frac{\omega\Lambda_2\sqrt{s}}{\sqrt{2}}\right)\right]\right\}, \tag{2.18}$$

where $\Phi(\Lambda)$ is the same function found in model A, and the $u_{1,2}$ have been rescaled according to $u_{1,2}=(\sqrt{2}\omega/\sqrt{s})\Lambda_{1,2}$. This "modified first-cumulant approximation" thus produces

$$L_B \to i\frac{(D^3cT)}{16\pi^3}\left(\frac{1}{2\pi i}\right)^2\int_{-\infty}^{0^+}\frac{dt_1}{t_1}e^{t_1}\int_{-\infty}^{0^+}\frac{dt_1}{t_2}e^{t_2}\int_0^\infty\frac{ds}{s^3}$$
$$\times e^{-ism^2}\int\int_{-\infty}^{+\infty}d\lambda_1 d\lambda_2 e^{i\lambda_1\lambda_2}$$
$$\times\left\{\exp\left(-i\frac{e^2\epsilon^2}{4}\left[\frac{1}{t_1}\Phi\left(\frac{\omega\Lambda_1\sqrt{s}}{\sqrt{2}}\right)+\frac{1}{t_2}\Phi\left(\frac{\omega\Lambda_2\sqrt{s}}{\sqrt{2}}\right)\right]\right)-1\right\}. \tag{2.19}$$

This is a ''modified first cumulant'' in the sense that the spatial averages are done exactly, and the ''first cumulant'' adjective refers to the simplest approximation (first order in $\delta\xi$) of the determinental factors' cluster expansion, involving the $s_{1,2}$ proper-time dependence. In contrast, model C will be defined by extracting and summing a portion of every cluster coefficient representing these factors.

Equation (2.19) can be put into a simpler form by again using the representation for J_0 introduced following Eq. (2.13), so that, with $m^2s=t$,

$$L_B \to \frac{im^4}{8\pi^2}(D^3cT)\int_0^\infty\frac{dt}{t^3}e^{-it}\frac{1}{2\pi}\int\int_{-\infty}^{+\infty}d\lambda_1 d\lambda_2 e^{i\lambda_1\lambda_2}$$
$$\times\left\{J_0\left[\frac{e\epsilon}{m}\sqrt{it\Phi\left(\frac{\omega\lambda_1}{\sqrt{2}m}\sqrt{t}\right)}\right]\times J_0\left[\frac{e\epsilon}{m}\sqrt{it\Phi\left(\frac{\omega\lambda_2}{\sqrt{2}m}\sqrt{t}\right)}\right]-1\right\}. \tag{2.20}$$

It immediately follows from Eq. (2.20) that an expansion in powers of $(e\epsilon/m)$ will generate a sequence of terms, each proportional to e^4, each of whose coefficients is purely imaginary. This is true because $\Phi(x)$ can be expressed as an infinite sequence of powers of x^2, while the $\lambda_{1,2}$ integrations require that the power l_1 of $\lambda_1^{2l_1}$ must be the same as the l_2 power of $\lambda_2^{2l_2}$; and because l_1 must equal l_2 for a nonzero result, the overall power of t in the resulting expansion is even: t^{2N+2}, $N\geqslant 1$, generating the imaginary coefficient of e^4.

To determine the nonperturbative form of L_B, we can proceed as for L_A, following Eq. (2.18), by rescaling the $\lambda_{1,2}$ variables into Λ and $\bar{\Lambda}$, and breaking up the Λ, $\bar{\Lambda}$ range of integration into the regions $(\int_{-\infty}^{-\pi}+\int_{+\pi}^{+\infty})$ and $(\int_{-\pi}^{+\pi})$, as one introduces the simplified $\Phi(x)=\theta(\pi-|x|)x^2/\pi^2+\theta(|x|-\pi)$. After changing back to the $\lambda_{1,2}$ variables, the result takes on a form analogous to that of Eq. (2.9):

$$L_B \to \frac{im^4}{8\pi^2}(D^3cT)\int_0^\infty\frac{dt}{t^3}\left\{-\left[1-J_0\left(\frac{e\epsilon}{m}\sqrt{it}\right)\right]^2+\frac{1}{2\pi}\int\int_{-\lambda_{max}}^{+\lambda_{max}}d\lambda_1 d\lambda_2 e^{i\lambda_1\lambda_2}\left[J_0\left(\frac{e\epsilon\omega}{m^2}\sqrt{i}\frac{t\lambda_1}{\sqrt{6}}\right)-J_0\left(\frac{e\epsilon}{m}\sqrt{it}\right)\right]\left[J_0\left(\frac{e\epsilon\omega}{m^2}\sqrt{i}\frac{t\lambda_2}{\sqrt{6}}\right)-J_0\left(\frac{e\epsilon}{m}\sqrt{it}\right)\right]\right\}. \tag{2.21}$$

We again make the same heuristic argument as that following Eq. (2.9), in which the limit $\lambda_{max}\to\infty$ was taken, so that

$$L_B \to \frac{im^4}{8\pi^2}(D^3cT)\int_0^\infty\frac{dt}{t^3}e^{-it}\frac{1}{2\pi}\times\int\int_{-\infty}^{+\infty}!d\lambda_1 d\lambda_2 e^{i\lambda_1\lambda_2}\times\{J_0(\gamma\sqrt{it}\lambda_1)J_0(\gamma\sqrt{it}\lambda_2)-1\}, \tag{2.22}$$

where, again, $\gamma=(e\epsilon\omega/m^2\sqrt{6})$, and $L_B\to L_B(\gamma)$. Because the J_0 are even functions of $\lambda_{1,2}$, the replacement $(1/2\pi)\int\int_{-\infty}^{+\infty}d\lambda_1 d\lambda_2\to(2/\pi)\int\int_0^\infty d\lambda_1 d\lambda_2$ can be made; and then a change of contour of the t integration is allowed, where t is to run from the origin along a straight line at an angle of $\pi/4$ below the positive real t axis; this corresponds to the replacement $t\to(-i)^{1/2}\tau$, with τ running from 0 to $+\infty$. In this way, Eq. (2.22) can be rewritten as

$$L_B \to -\frac{m^4}{8\pi^2}(D^3cT)\int_0^\infty\frac{d\tau}{\tau^3}e^{-\tau/\sqrt{2}-i\tau/\sqrt{2}}\times\left\{\frac{2}{\pi}\int\int_0^{+\infty}d\lambda_1 d\lambda_2\cos(\lambda_1\lambda_2)\times J_0(\gamma\tau\lambda_1)J_0(\gamma\tau\lambda_2)-1\right\}. \tag{2.23}$$

We next use the integrals [7]

$$\int_0^{+\infty}d\lambda_2\cos(\lambda_1\lambda_2)J_0(\gamma\tau\lambda_2)=\frac{\theta(\gamma\tau-\lambda_1)}{\sqrt{(\gamma\tau)^2-\lambda_1^2}}$$

and

$$\int_0^{\gamma\tau}d\lambda_1\frac{J_0(\gamma\tau\lambda_1)}{\sqrt{(\gamma\tau)^2-\lambda_1^2}}=\frac{\pi}{2}J_0^2\left(\frac{(\gamma\tau)^2}{2}\right)$$

and the convenient rescaling $\tau\to\sqrt{2}\tau$ to obtain

$$\mathrm{Re}\,L_B \to \frac{m^4}{(4\pi)^2}(D^3cT)\int_0^\infty \frac{d\tau}{\tau^3}e^{-\tau}\cos\tau[1-J_0^2((\gamma\tau)^2)]. \tag{2.24}$$

One can see directly from the perturbative expansion of Eq. (2.24) that the coefficient of every term (proportional to e^{4n}) is identically zero

$$\int_0^\infty \frac{d\tau}{\tau^3}\tau^{4n}e^{-\tau}\cos\tau \equiv 0, \quad n=1,2,\ldots,$$

so that the nonperturbative aspect of $\mathrm{Re}\,L_B$ has been maintained.

We are able to evaluate Eq. (2.24) only in an approximate way. The method we use takes advantage of the dissimilarity of scales inherent in any $f(\tau)$ and $g(\gamma\tau)$, when $\gamma \ll 1$, as is the case here. The function $g(x)=1-J_0(x^2)$ resembles $\Phi(x)$, Fig. 3, and our simple approximation will be similar: for $\tau<1/\gamma$, $g(\gamma\tau)\sim\frac{1}{2}(\gamma\tau)^4$. We then separate the integral of Eq. (2.24) into two parts, that contribution obtained by integrating τ over the region from 0 to $\tau_0=1/\gamma$,

$$\mathrm{Re}\,L_B^{(1)} \to \frac{m^4}{(4\pi)^2}(D^3cT)\frac{\gamma^4}{2}\int_0^{\tau_0} d\tau\,\tau e^{-\tau}\cos\tau, \tag{2.25}$$

and that part obtained by integrating τ from τ_0 to ∞, where we employ the asymptotic form of $J_0^2<1$ to write

$$\mathrm{Re}\,L_B^{(2)} \to \frac{m^4}{(4\pi)^2}(D^3cT)\int_{\tau_0}^\infty \frac{d\tau}{\tau^3}e^{-\tau}\cos\tau. \tag{2.26}$$

Equation (2.26) can certainly be improved upon, e.g., by making a Gaussian approximation to $g(x)$ so that the latter equals unity only at every (cusp) value of x_n, the square root of one of the zeros of this Bessel function. [This resembles the original Schwinger model, where there is a sequence of such essential singularities labeled by an integer n; here, $(x_n)^{1/2}$ replaces that n.] In effect, we are using only the first zero, the simplest approximation to that sequence, of form $\exp[-(x_1)^{1/2}/\gamma]$, where x_1^2 is the first zero of $J_0(x^2)$; and our approximation will be sufficiently crude so that our result will replace x_1 by unity. We are, however, only interested in the order of magnitude of the largest possible contribution corresponding to the first essential singularity; this is contained in Eq. (2.25)—which can be evaluated exactly—and in Eq. (2.26), which can be exhibited in terms of an asymptotic expansion of the $\mathrm{Ei}(-x)$ function. To leading order (that is, proportional to γ^3, with higher-order γ^4 corrections), the sum of Eqs. (2.25) and (2.26) generates

$$\mathrm{Re}\,L_B \simeq \frac{m^4}{(4\pi)^2}(D^3cT)\frac{\gamma^3}{2}e^{-\tau_0}[\cos\tau_0-\sin\tau_0], \quad \tau_0=\gamma^{-1}. \tag{2.27}$$

The sign of Eq. (2.27) is apparently related to its oscillatory dependence, but this connection can be misleading. Basically, the sign of Eq. (2.27) must be determined in conjunction with the omitted $\sigma\cdot F$ dependence, as in Schwinger's model; and the overall sign must be negative. The appearance of the oscillatory factors of Eq. (2.27) is also a function of the crudeness of the approximation procedures used to evaluate Eq. (2.24), and such factors may or may not arise in a better evaluation. We now argue that an alternative estimation of Eq. (2.24) can indeed yield a result of correct sign and no oscillatory dependence on τ_0, by noting that the value of τ at which the integrand of Eq. (2.24) is replaced—discontinuously—by the sum of those of Eqs. (2.25) and (2.26) is really not well defined. We have $\gamma^{-1} > \gg 1$, but instead of simply choosing $\tau_0=\gamma^{-1}$ let us set the upper limit of Eq. (2.25) and the lower limit of Eq. (2.26) equal to $2\pi N$, where N is that (very large) integer closest to $(2\pi\gamma)^{-1}$. [In other words, the limits of these integrals need not be (arbitrarily) set equal to γ^{-1} but can be very close to that value.] The result will be that the square bracket of Eq. (2.27) is replaced by $+1$. But if we use $\tau_0=2\pi(N+1/2)$ the bracket of Eq. (2.27) becomes -1. Which form should be used? If we write $\tau_0=2\pi N+x$, what should be the value assigned to x? One does not know; and therefore one might be tempted to average over all values of x between 0 and π, thereby replacing $[\cos x-\sin x]$ by $-2/\pi$, a result that has no oscillation and the correct sign. But the deeper answer to these questions is simply that if one has any choice (for an exact evaluation of these integrals, supposing that such were possible), that choice (as in model A, where there are two possible branches of a square root) must be taken so that $\mathrm{Re}\,L_B$ is negative. Further, because we have neglected the $\sigma\cdot F$ contributions, we cannot know whether or not a real choice is possible. What we must accept, in this approximate calculation which is concerned only with the order of magnitude of Γ, is that there is, according to our approximations, an inability to derive the correct sign—although there is always at least one path to the correct sign—but that the final sign of $\mathrm{Re}\,L$ must be negative. We therefore write

$$\Gamma_B \simeq \frac{m^4(D^3c)}{(4\pi)^2}\gamma^3 e^{-1/\gamma} \tag{2.28}$$

and, again, we emphasize that it is only the order of magnitude of this result which we believe to be a correct prediction of model B. Comparison with Eq. (2.12) shows that this is essentially the same result as obtained for model A.

C. An approximate nonperturbative cluster sum

We now return to Eq. (2.15) and ask if a better approximation can be found for

$$\mathrm{Tr}\ln(1+2K)+\mathrm{tr}\ln(1-Q/s),$$

an approximation that contains all the powers of the coupling, rather than just its quadratic dependence, as in the equation following Eq. (2.17). Further, any such approximation must be simple enough to permit its evaluation.

We begin by calculating $\langle s_1|(2K)^2|s_2\rangle$, a straightforward computation; with $\chi=ie^2\epsilon^2/2$ and $s_{12}=s_1-s_2$, one finds

$$\langle s_1|(2K)^2|s_2\rangle = \chi^2\Bigg\{ \frac{s}{t_1^2}\left[\cos(u_1 s_{12}) + \frac{\sin(u_1 s)}{(u_1 s)}\right] + \frac{s}{t_2^2}\left[\cos(u_2 s_{12}) + \frac{\sin(u_2 s)}{(u_2 s)}\right] + \frac{s}{t_1 t_2}\left[\cos\left(\left[\frac{u_1+u_2}{2}\right]s_{12}\right)\frac{\sin\left(\left[\frac{u_1-u_2}{2}\right]s\right)}{\left[\frac{u_1-u_2}{2}\right]s} + \cos\left(\left[\frac{u_1-u_2}{2}\right]s_{12}\right)\frac{\sin\left(\left[\frac{u_1+u_2}{2}\right]s\right)}{\left[\frac{u_1+u_2}{2}\right]s}\right]\Bigg\}. \tag{2.29}$$

In comparison,

$$\langle s_1|2K|s_2\rangle = \chi\left[\frac{\cos(u_1 s_{12})}{t_1} + \frac{\cos(u_2 s_{12})}{t_2}\right]$$

where, as in Eq. (2.29), we have suppressed the trivial $(\epsilon_\mu \epsilon_\nu/\epsilon^2)$ factors. Upon subsequent $u_{1,2}$ integrations, the terms $\sin\{[(u_1\pm u_2)/2]s\}/[(u_1\pm u_2)/2]s$ will take appreciable values only for $u_1\pm u_2 \simeq 0$, where they become unity, so that the $\cos\{[(u_1\pm u_2)/2]s\}/2$ factors multiplying them can be replaced by $\cos(u_1 s)$ or $\cos(u_2 s)$, precisely the terms that appear in $\langle s_1|2K|s_2\rangle$. This suggests that we define a model by the statement

$$\langle s_1|(2K)^2|s_2\rangle = \mathcal{F}\langle s_1|2K|s_2\rangle \tag{2.30}$$

with $\mathcal{F}=\chi s[1/t_1 + 1/t_2]$ [which quantity was called $s\xi(0)$ in Eq. (2.16)]. This model neglects the $\sin(u_{1,2}s)/(u_{1,2}s)$ parts of $\langle s_1|(2K)^2|s_2\rangle$ above; but it does correspond to an order-by-order extraction of what are probably the most significant pieces of every perturbative term. Note that our neglect of such oscillatory behavior is not associated with the expectation that the $u_{1,2}s \sim O(\omega\sqrt{\tau}/m)$ are small, for any $\omega/m\to 0$ limit here will bring about the cancellations described in model A; rather, we here keep only the terms that are expected to be significant upon subsequent integration over fluctuating $u_{1,2}$ dependence.

With Eq. (2.30), and the arbitrary number of iterations that can be formed from it,

$$\langle s_1|(2K)^n|s_2\rangle = \mathcal{F}^{n-1}\langle s_1|2K|s_2\rangle,$$

one easily calculates

$$\mathrm{Tr}\ln(1+2K) = \ln(1+\mathcal{F})$$

and

$$\mathrm{tr}\ln(1-Q/s) = -\ln(1+\mathcal{F}) + \ln\left[1+\chi s\left(\frac{\phi(u_1 s/2)}{t_1} + \frac{\phi(u_2 s/2)}{t_2}\right)\right],$$

so that the combination $\exp[-\frac{1}{2}\mathrm{Tr}\ln(1+2K) - \frac{1}{2}\mathrm{tr}\ln(1-Q/s)]$ becomes

$$\left[1+\chi s\left(\frac{\phi(u_1 s/2)}{t_1} + \frac{\phi(u_2 s/2)}{t_2}\right)\right]^{-1/2} - 1,$$

which can be rewritten as

$$\frac{2}{\sqrt{\pi}}\int_0^\infty du\, e^{-u^2}\left\{\exp\left(-\chi s u^2\left[\frac{\phi(u_1 s/2)}{t_1} + \frac{\phi(u_2 s/2)}{t_2}\right]\right) - 1\right\}.$$

Upon again using the J_0 representation following Eq. (2.13), we obtain

$$L_C \to \frac{im^4}{8\pi^2}(D^3 cT)\int_0^\infty \frac{dt}{t^3} e^{-it}\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^\infty du\, e^{-u^2/2}\frac{1}{2\pi}\int\int_{-\infty}^{+\infty} d\lambda_1 d\lambda_2 e^{i\lambda_1\lambda_2} \times\left\{J_0\left[\frac{e\epsilon u}{m}\sqrt{it\Phi\left(\frac{\omega\lambda_1}{\sqrt{2m}}\sqrt{t}\right)}\right]J_0\left[\frac{e\epsilon u}{m}\sqrt{it\Phi\left(\frac{\omega\lambda_2}{\sqrt{2m}}\sqrt{t}\right)}\right] - 1\right\} \tag{2.31}$$

and comparison with Eq. (2.20) shows that the only difference between L_B and L_C is the latter's $\sqrt{2/\pi}\int_0^\infty du\, e^{-u^2/2}$ together with the u dependence inside each J_0. Reserving the u integration for the very last step, one can therefore write

$$\Gamma_C(\gamma)\to\sqrt{\frac{2}{\pi}}\int_0^\infty du\, e^{-u^2/2}\Gamma_B(u\gamma)$$
$$\to\frac{m^4(D^3c)}{(4\pi)^2}\gamma^3\sqrt{\frac{2}{\pi}}\int_0^\infty du e^{-u^2/2}u^3e^{-1/u\gamma}. \tag{2.32}$$

We now approximately evaluate this last integral by writing it as

$$\int_0^\infty du\,\exp[f(u)],\quad f(u)=-\frac{u^2}{2}-\frac{1}{\gamma u}+3\ln u,$$

searching for the maximum of $f(u)$ at that u_0 defined by $f'(u_0)=0$, and then calculate in the standard Gaussian manner, with $f(u)\sim f(u_0)+\frac{1}{2}(u-u_0)^2f''(u_0)$, and the $u-u_0=u'$ integration running from $-\infty$ to $+\infty$. The equation defined by $f'(u_0)=0$ is cubic; and because $\gamma\ll1$ its solution is to leading order (in γ) equivalent to that which would be obtained by considering

$$\int_0^\infty du u^3\, e^{f(u)},\quad f(u)=-\frac{u^2}{2}-\frac{1}{\gamma u}$$

and replacing the u^3 factor by u_0^3. Either procedure leads to $u_0\sim\gamma^{-1/3}$ and

$$\sqrt{\frac{2}{\pi}}\int_0^\infty du u^3\, e^{-u^2/2-1/u\gamma}\simeq\frac{2}{\gamma\sqrt{3}}e^{-3/(2\gamma^{2/3})},$$

so that

$$\Gamma_C\simeq\frac{2}{\sqrt{3}}\frac{m^4(D^3c)}{(4\pi)^2}\gamma^2e^{-3/(2\gamma^{2/3})}, \tag{2.33}$$

which represents a significant increase in magnitude over the result for Γ_B, because there is one less factor of γ in the numerator, but more importantly because the form of the essential singularity has been changed to $\exp[-3/(2\gamma^{2/3})]$, compared to the $\exp[-1/\gamma]$ of Γ_B.

This change of form, and magnitude, of the essential singularity has a physical interpretation that may be of interest. Elementary QED processes, such as e^+e^- pair creation, are usually thought of as taking place over distances on the order of λ_c, the electron's Compton wavelength. Here, however, we expect coherent absorption of the laser photons by the incipient, still virtual pair, over distances larger than λ_c, perhaps as large as some fraction of the laser photons' wavelength λ_γ, because there are so many photons that must be absorbed. This coherence is made explicit by the u integration, as the parameter u varies over distances centered about $u_0\sim\gamma^{-1/3}$, which is considerably larger than 1. In physical terms, the $\exp[-1/\gamma]=\exp[-\sqrt{6}(m/e\epsilon)(\lambda_\gamma/\lambda_c)]$ of Γ_B is here replaced by $\exp[-\sqrt{6}(m/e\epsilon)(\lambda_\gamma/u_0\lambda_c)]$, where $u_0\lambda_c$ is perhaps 10^2–10^3 times larger than λ_c, and which can be interpreted as the qualitative distance over which coherent laser photon absorption takes place.

III. MAGNITUDES

Predictions of the half-lives of the three models found in Sec. II tend to be quite small, due mainly to their essential singularities, and we here consider these quantities in numerical detail.

We begin by calculating the size of the dimensionless parameter $e\epsilon/m$:

$$\left(\frac{e\epsilon}{m}\right)^2=\left(\frac{e\epsilon\omega}{m}\right)^2\frac{1}{\omega^2}\to\frac{(4\pi\alpha)(4\pi U)}{mc^2}\lambda_c\lambda_\gamma^2,$$

where U represents the laser energy density $(1/4\pi)\epsilon^2\omega^2$, and we have reinserted all relevant dimensional constants. Since we are interested only in orders of magnitude, we replace $(4\pi)^2\alpha=158/137$ by unity. The average laser flux $F=cU$ then provides

$$\left(\frac{e\epsilon}{m}\right)^2\sim\frac{\lambda_c\lambda_\gamma^2}{c(mc^2)}F$$

and since F is conventionally quoted in MKS units as W/m^2 we adopt those units, so that, for our ''ideal'' laser of $F_0=10^{22}$ W/m^2, $(e\epsilon/m)^2\sim5F_0\times10^{-21}=50$, or $(e\epsilon/m)\sim7$. With $(\omega/m)=10^{-6}$, $\gamma=(e\epsilon\omega/m^2)(1/\sqrt{6})\sim10^{-6}$. Therefore, for a laser of flux intensity F, $\gamma\simeq10^{-6}(F/F_0)^{1/2}$.

The multiplicative factors m^4D^3c may be represented as D^3c/λ_c^4, a relatively large number with the dimensions of inverse seconds, of size $(10^{-5})^3(3\times10^8)/(10^{-12})^4=3\times10^{41}$. We arbitrarily decrease this number by two orders of magnitude, to take into account the 2π factors of each Γ, as well as possible weightings of the neglected $\sigma\cdot F$ terms, so that

$$\Gamma_{A,B}\sim10^{39}\gamma^3e^{-10^6(F_0/F)^{1/2}}\simeq10^{21}(F/F_0)^{3/2}e^{-10^6(F_0/F)^{1/2}}. \tag{3.1}$$

Further, if these lasers are pulsed, with a duration τ, then the vacuum persistence probability at the end of each laser burst is $P_0=\exp[-\tau\Gamma_{A,B}]$, and the probability P_1 of producing one or more pairs during each pulse is given by $P_1=1-P_0\simeq\tau\Gamma_{A,B}$, if $\tau\Gamma_{A,B}\ll1$.

Let us arbitrarily choose $P_1\sim0.1$ as a minimum realization of this process—roughly one pair produced for every ten pulses—for a typical $\tau\sim10^{-13}$ sec. This means $10^{-1}\sim10^{-13}10^{21}(F/F_0)^{3/2}e^{-10^6(F_0/F)^{1/2}}$, or $10^{-9}x^{3/2}=e^{-(10^6\sqrt{x})}$, where $x=(F_0/F)$. Setting $x=10^{-p}$, it is not difficult to see that $p\simeq8.6$ and

$$F/F_0=10^{8.6}\sim4\times10^8 \tag{3.2}$$

so that for pair production at the rate of one pair per ten pulses, as given by models A and B, the available laser flux must be increased by 8 to 9 orders of magnitude.

On the other hand, the model C requirement for the same rate is significantly less, for the above calculations are replaced by

$$\Gamma_C \sim 10^{39}\gamma^2 e^{-3/2\gamma^{2/3}} \tag{3.3}$$

and

$$\tau\Gamma_C \sim 0.1 \sim 10^{26}10^{-12}(F/F_0)e^{-(1.5\times 10^4)(F_0/F)^{1/3}}. \tag{3.4}$$

With the same definition of x and p, one here finds

$$F/F_0 = 10^{7.4} \sim 2.5\times 10^7, \tag{3.5}$$

which is at least an order of magnitude improvement over Eq. (3.2).

If further improvements in the model calculations could be found, one would expect a further increase in the pair production rate.

IV. APPLICATION TO QCD

If sufficiently high-intensity lasers can be made that achieve e^+e^- and even $\mu^+\mu^-$ pair production, there is then no reason why one cannot contemplate laser-induced quark-antiquark production. Here one cannot neglect the QCD radiative corrections, since it is the gluon clouds surrounding q and $\bar{q}$ that form a flux tube/string, and produce quark confinement. But one can idealize what may happen in terms of two extreme, and differing, possibilities: (1) q and $\bar{q}$ appear with their flux tube/string in place, so that we have produced, in effect, a π^0, which the laser fields are incapable of tearing apart; or (2) q and $\bar{q}$ materialize each surrounded by its virtual gluonic structure, which immediately begins to form itself into a tube/string joining q to $\bar{q}$. The formation of the tube/string is surely not an instantaneous effect, but one that can be characterized in terms of a ''string formation velocity'' v_f. As a physical process, one expects that v_f cannot be larger than c, while it is perfectly possible for the q and $\bar{q}$ to be accelerated away from each other by the crossed lasers, so that their relative velocity of separation v_s could equal or exceed v_f. This suggests that, by this mechanism, q and $\bar{q}$ might temporarily reach separation distances greater than a few fermis. (Of course, after the laser beams pass over the q and $\bar{q}$, deceleration occurs, and the tube/string wins.)

What could be a signal of this second possibility? Large energy deposition in a small spatial region, perhaps leading to a pair of hadronic jets, built around the outgoing q and $\bar{q}$ lines, and arranged so as to maintain an overall color-singlet property. Other structures are also possible, such as the q and $\bar{q}$ falling back together and annihilating like positronium, but with a relatively large energy (absorbed from the intersecting lasers when the beams pulled the q and $\bar{q}$ apart, and converted into potential energy of $q\bar{q}$ separation) converted into a few high-energy gammas, or into a ''fireball'' of x rays. Much more theoretical work needs to be done on this question; but the qualitative way in which the $q\bar{q}$ pair materialize should be amenable to experimental determination.

V. SUMMARY

These estimates of pair production, via a calculation of the vacuum persistence probability, are examples of intrisically nonperturbative quantities in quantum field theory. Our estimates for models A and B show an essential singularity analogous to that of the 1951 Schwinger calculation, while the improved model C generates a related, but weaker, essential singularity.

As noted above, these probabilities are strictly functions of two dimensionless variables, $e\epsilon\omega/m^2$ and ω/m; but for reasons of calculational practicability, we have dropped all dependence on the second variable. In so doing, we have arrived at forms quite similar to Schwinger's; and it is worth mentioning that—once such calculational approximations have been performed—there is a strong reason to expect a result similar to Schwinger's.

Suppose that, instead of intersecting at right angles, the two laser beams overlap in an antiparallel way; that is, they fire in equal and opposite directions, with $\hat{k}_1 = -\hat{k}_2$. (Such geometry would of course increase the yields, assuming that the machinery did not destroy itself.) In this case, it is also possible to find a nonzero $\mathrm{Re}\,L$, although the spatial averaging over the beams becomes more difficult. However, it is easy to see for model A that if a time average is included along with the spatial average the expression for the averaged L_A will be unchanged. This is significant because the oscillating magnetic fields of overlapping, in-phase laser beams moving in opposite directions, whose electric polarization vectors are in the same direction, exactly cancel, leaving only the enhanced electric fields, of magnitude proportional to $\mathcal{E} \sim \omega\epsilon$. In the limit of decreasing frequency, $\omega \to 0$ but $\epsilon \to \infty$ such that their product is finite, we are in the situation of a constant electric field. For such head-on laser beams we would expect that the limit of $\omega \to 0$ and finite $\omega\epsilon \sim \mathcal{E}$ should reproduce, at least qualitatively, the Schwinger result; and it does. Of course, our calculations are approximate, and cannot reproduce the precise forms of an exact result; but the correspondence is physically clear.

Finally, we comment on the shortcomings of this analysis, which invents a sequence of models to bypass the difficult task of solving for the relevant function $\langle s_1|2K(1+2K)^{-1}|s_2\rangle$. An effort should be made to obtain this quantity to an accuracy better than those of the models presented here, especially in light of the difference between the essential singularities of models B and C. And, it might be noted, there are other ways of approaching the construction of $L[A]$ for this two-laser situation, such as the ''infrared'' approximation appearing in Chapters 12 and 13 of the book in Ref. [3]. Nevertheless, we believe that our results, approximate and model dependent as they are, do point the way to a new and interesting method of particle production, which should become available with lasers of just a few more orders of magnitude of intensity [8].

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[8] One assumes here that a continued increase of laser intensity, roughly one or two orders of magnitude per year, is possible. Professor A. Klein (University of Melbourne) has, in a private communication, kindly pointed out certain experimental difficulties that could conceivably limit the (F/F_0) ratio.