

(De)stabilization of an extra dimension due to a Casimir force

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We study the stabilization of one spatial dimension in $(p+1+1)$ -dimensional spacetime in the presence of p -dimensional brane(s), a bulk cosmological constant and the Casimir force generated by a conformally coupled scalar field. We find general static solutions to the metric which require the fine-tuning of the inter-brane distance and the bulk cosmological constant (leaving the two brane tensions as free parameters) corresponding to a vanishing effective cosmological constant and a constant radion field. Taking these solutions as a background configuration, we perform a dimensional reduction and study the effective theory in the case of one- and two-brane configurations. We show that the radion field can have a positive mass squared, which corresponds to a stabilization of the extra dimension, only for a repulsive nature of the Casimir force. This type of solution requires the presence of a negative tension brane. The solutions with one or two positive tension branes arising in this theory turn out to have negative radion mass squared, and therefore are not stable.

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I. INTRODUCTION

The past few years witnessed a growing interest among particle physicists and cosmologists toward models with extra space-like dimensions. This interest was initiated by string theorists [1], who exploited a moderately large size of an external 11th dimension in order to reconcile the Planck and string/or grand unified theory (GUT) scales. Taking this idea further, it was shown that large extra dimensions allow for a reduction of the fundamental higher-dimensional gravitational scale down to the TeV scale [2]. An essential ingredient of such a scenario is the confinement of the standard model (SM) fields on field theoretic defects, so that only gravity can access the large extra dimensions. These models are argued to make contact with an intricate phenomenology, with a variety of consequences for collider searches, low-energy precision measurements, rare decays and astroparticle physics and cosmology. However, the mechanisms, responsible for the stabilization of extra dimensions, remain unknown. The fact that the size of extra dimensions is large as compared to the fundamental scale also remains unexplained. An alternative solution to the hierarchy problem was proposed in Ref. [3]. This solution appeals to the possibility of a strongly curved extra dimension limited by two branes with positive and negative tensions, with the scale factor in the bulk space between the two branes changing by orders of magnitude within a distance of several Planck lengths. In this case, the ratio of fundamental energy scales on the two branes is given by a large warping factor. However, the resolution of the hierarchy problem is possible only in the case where the observable brane (on which the SM fields are trapped) is the one with the negative tension.

The early studies of cosmologies in the brane models revealed an unusual dependence of the Hubble parameter H

$=\dot{a}/a$ on the energy density of matter on the brane [4–6]. In fact, as it was shown in Ref. [6], an abnormal behavior $H \sim \rho^{(4)}$ may persist in a later epoch, *if* the stability of the extra dimension is achieved via a fine-tuned cancellation between positive and negative energy densities on the two branes. As it was first shown in Ref. [7], a natural resolution to this problem comes from the stabilization of the extra dimension, which also removes the necessity for an unphysical fine tuning between energy densities on different branes. The smooth transition to the four-dimensional cosmology and Newtonian 4D interaction requires the presence of a non-vanishing (55)-component of the energy-momentum tensor. It was subsequently shown in Refs. [8,9] that the value of T_{55} is automatically adjusted to a value proportional to $\rho^{(4)} - 3\rho^{(4)}$ for a generic stabilization mechanism. It was also observed in [8] that the stabilization mechanism is crucial for getting a consistent solution to the gauge hierarchy problem on the negative tension brane in the set up of Ref. [3] (see also [10]). Subsequently, it was demonstrated that, in the presence of a stabilization mechanism, the solution to the hierarchy problem in the two positive tension brane models is also possible [11,12]. From a particle physics point of view, these models are more appealing than the models with negative tension branes, which cannot be realized as field theoretic solitonic solutions.

Since the nature of the stabilization mechanism is yet unclear, it is reasonable to study a *minimal* possibility, which introduces one scalar field in the bulk. Classical stabilization forces due to non-trivial background configurations of a scalar field along an extra dimension were first discussed by Gell-Mann and Zwiebach [13]. With the revived interest in extra dimensions and brane worlds, a modified version of this mechanism, which exploits a classical force due to a bulk scalar field with different interactions with the brane(s),

received significant attention [14–17]. However, as it was shown in Refs. [11,17], a classical scalar interaction is not useful for the stabilization of two positive tension branes.

In this paper we study another stabilization mechanism due to a Casimir force generated by the quantum fluctuations about a constant background of a conformally coupled massless scalar field. This effect was initially studied in Ref. [18] in the context of M theory, and subsequently in Ref. [19] for a background Randall-Sundrum geometry. The same effect was further investigated in [20]. Recently, Mukohyama [21] and our group [22] found an exact static solution in 5 dimensions in the presence of the Casimir force and the bulk cosmological constant. By imposing boundary conditions on the branes, one may transfer the breaking of the 5D Poincaré invariance into the bulk, naturally generating an anisotropic energy-momentum tensor with T_5^5 different from T_0^0 and T_i^i . It turns out that there are static solutions admitting one or two positive tension branes. The purpose of the present paper is to investigate the stability of these solutions.

In the next section, we present the theoretical framework of our analysis. We make an attempt to derive time-dependent solutions in a d -dimensional, conformally-flat spacetime, where there is a bulk cosmological constant and Casimir stress. In the same section we find an exact, static background in the case of positive, negative or zero bulk cosmological constant and for Dirichlet or Neumann boundary conditions for the scalar field. We formulate the fine-tuning conditions which allow for such a solution to exist. These conditions correspond to a vanishing effective four-dimensional cosmological constant and a constant radion field. In Sec. III, we study an effective field theory which is obtained after dimensional reduction in the presence of the static, gravitational background determined in the case of a negative bulk cosmological constant. The effective potential for the size of the extra dimension is calculated and, depending on the nature of the Casimir force, it is either positive or negative. Negative mass squared arises due to an attractive Casimir force and corresponds to an unstable extremum. Unfortunately, the models with single positive tension brane or two positive tension branes fall into this category. Positive m^2 , i.e. true stabilization, arises in the case of a repulsive Casimir force and *requires* the presence of the negative tension brane.

II. ONE OR TWO p -BRANES IN $(p+2)$ DIMENSIONS

A. Energy-momentum tensor due to the Casimir effect and the equations of motion in d dimensions

We start from the classical expression for the d -dimensional action describing a gravitational field, a conformally-coupled scalar field ϕ , and two p -branes embedded in $d=p+2$ dimensions. The branes have tensions Λ_1 and Λ_2 and are located at $z_1 < z_2$, respectively, where z denotes the coordinate along the compact extra dimension. The bulk between the branes is characterized by a nonzero cos-

mological constant Λ_B of an arbitrary sign. The action of the system is given by¹

$$S = - \int d^{d-1}x \int dz \sqrt{-g_{(d)}} \left(- \frac{1}{2\kappa_d^2} R_{(d)} + \Lambda_1 \delta(z-z_1) + \Lambda_2 \delta(z-z_2) + \Lambda_B + \frac{1}{2} \partial_M \phi \partial^M \phi + \gamma R_{(d)} \phi^2 \right), \quad (2.1)$$

where $M=0, \dots, d-1$, $x^0 \equiv t$, $x^{(d-1)} \equiv z$, and $\gamma \equiv (d-2)/(4(d-1))$. We also assume a Z_2 symmetry represented by mirror transformations about the branes in the z coordinate.

At the classical level the stress energy tensor in the bulk is determined only by Λ_B which makes it isotropic, $T_N^M = -\Lambda_B \delta_N^M$. At the quantum level the d -dimensional isotropy of T_N^M can be broken by the Casimir stress. Following Ref. [19], we exploit the relation between the quantum induced part of the energy-momentum tensor in flat spacetime and in conformally flat geometries. The vacuum average $\langle T_{MN} \rangle$ in a free field theory bounded in a d -dimensional flat spacetime can be inferred from the effective potential of a constant classical background, where the fluctuations are to obey the boundary conditions, by symmetry considerations or directly from the corresponding propagator [24].

In the case where the boundaries are two parallel p -branes, separated by a spatial distance L , the tracelessness of the (improved [25]) energy-momentum tensor implies the following result:

$$\langle (T_N^M) \rangle^f = \frac{\alpha}{L^d} \text{diag}(1, 1, \dots, 1, -(p+1)), \quad (2.2)$$

where α is dimensionless and depends on the fields of the theory, the boundary conditions, and d . The same form of T_N^M will hold for a single brane configuration, which can be obtained by identifying two branes of equal tensions.

It is common wisdom [19,26] that the Casimir effect $\langle (T_N^M) \rangle^{cf} \neq 0$ due to a conformally coupled, massless field in a conformally flat, odd-dimensional spacetime is related to the corresponding flat-space expression as follows:

$$\langle (T_N^M) \rangle^{cf} = a^{-d} \langle (T_N^M) \rangle^f, \quad (2.3)$$

where $a^2 = a^2(x) > 0$ denotes the conformal factor transforming the metric from flat to curved geometry. Note that there is no restriction to the dependence of a on the space-time coordinates $x = (t, x_1, \dots, x_p, z)$ other than smoothness and definiteness. This conversion formula for the vacuum averages of the energy-momentum tensor in curved and flat space can be obtained by an explicit calculation, using a field redefinition $\tilde{\phi} = a^{(d-2)/2} \phi$, and it is only valid for odd d [26].

¹Throughout this paper, we follow Wald's conventions [23]: The metric signature is $\eta_{MN} = (-, +, \dots, +)$ and the Riemann tensor is defined as $R^\sigma_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\sigma_{\rho\mu} + \dots$.

Our goal is to find whether this contribution to the total energy-momentum tensor of the bulk leads to a true stabilization of the extra dimension. To study that, we consider, for now, only a dependence of a on t and z :

$$ds^2 = a^2(t, z) (-dt^2 + dx_1^2 + dx_2^2 + \dots + dx_p^2 + dz^2). \quad (2.4)$$

As for the field ϕ the background configuration is $\phi_c = \text{const}$. Varying the action of Eq. (2.1) with respect to the metric, one obtains the following Einstein's equations $G_{MN} = \kappa_d^2 T_{MN}$:

$$G_{tt} = \frac{p}{2}(p+1) \frac{\dot{a}^2}{a^2} - p \frac{a''}{a} - \frac{p}{2}(p-3) \frac{a'^2}{a^2} = -\kappa_d^2 a^2 \left[-\Lambda_B + \frac{\alpha}{(aL)^{p+2}} \right], \quad (2.5)$$

$$G_{ii} = p \left(\frac{a''}{a} - \frac{\ddot{a}}{a} \right) + \frac{p}{2}(p-3) \left(\frac{a'^2}{a^2} - \frac{\dot{a}^2}{a^2} \right) = \kappa_d^2 a^2 \left[-\Lambda_B + \frac{\alpha}{(aL)^{p+2}} \right], \quad (2.6)$$

$$G_{tz} = p \left(2 \frac{a'}{a} \frac{\dot{a}}{a} - \frac{\dot{a}'}{a} \right) = 0, \quad (2.7)$$

$$G_{zz} = \frac{p}{2}(p+1) \frac{a'^2}{a^2} - p \frac{\ddot{a}}{a} - \frac{p}{2}(p-3) \frac{\dot{a}^2}{a^2} = \kappa_d^2 a^2 \left[-\Lambda_B - \frac{(p+1)\alpha}{(aL)^{p+2}} \right]. \quad (2.8)$$

Thereby, the dots and primes denote derivatives with respect to t and z , respectively.

Keeping the brane coordinates fixed, we now try to find general, time-dependent solutions of Eqs. (2.5)–(2.8). Taking the sum of Eqs. (2.5) and (2.6), an ordinary differential equation can be derived for $a(t, z)$ with respect to time. An easy integration yields the solution

$$a(t, z) = \frac{1}{f(z)(t-t_0) + g(z)}, \quad (2.9)$$

where $f(z)$ and $g(z)$ are associated with the initial data $a(t_0, z)$ and $\dot{a}(t_0, z)$. Substituting the solution (2.9) into Eq. (2.7), gives $f(z) = c_0 \equiv \text{const}$. With this information, insertion of Eq. (2.9) into Eq. (2.8) gives

$$\begin{aligned} & \frac{p(p+1)}{2} (g'(z)^2 - c_0^2) \\ & = -\kappa_d^2 \left(\Lambda_B + \frac{(p+1)\alpha}{L^{p+2}} [c_0(t-t_0) + g(z)]^{p+2} \right). \end{aligned} \quad (2.10)$$

The right-hand side (RHS) of the above equation becomes time-independent, like the LHS, only if $\alpha=0$ or $c_0=0$. Let us consider first the case $\alpha=0$, i.e., disregard the Casimir stress. Upon integration, Eq. (2.10) then provides the solution for the function $g(z)$, and the warp factor takes the final form

$$a(t, z) = \frac{1}{c_0(t-t_0) + c_1 z},$$

where $c_1 = \pm \sqrt{c_0^2 - \frac{2\kappa_d^2}{p(p+1)} \Lambda_B}$. (2.11)

Note that the above bulk solution exists for every value, positive, negative or zero, of the bulk cosmological constant. The singular distribution of energy at the location of the two branes generates the so-called *jump* conditions [27], which, in conformal coordinates, take the form

$$\left. \frac{[a']}{a^2} \right|_{z_i} = -\frac{\kappa_d^2}{p} \Lambda_i \quad (i=1,2). \quad (2.12)$$

Substituting the solution (2.11) in the above conditions, we arrive at the constraints

$$c_1 = \frac{\kappa_d^2}{2p} \Lambda_1 = -\frac{\kappa_d^2}{2p} \Lambda_2, \quad (2.13)$$

which reveals the extreme fine tuning between the positive and negative self-energies of the two branes necessary for the stabilization of the inter-brane distance in the absence of a stabilizing potential, even in the time-dependent case. The fine tuning which is now relaxed, due to the coefficient c_0 , is the one that relates the brane self-energies to the bulk cosmological constant. As a result, the effective 4D cosmological constant does not vanish, thus, inducing the assumed evolution in time. One can easily check that, in the limit $c_0 \rightarrow 0$, the extreme fine-tuning between all of the parameters of the theory reappears, i.e.,

$$\Lambda_1^2 = \Lambda_2^2 = \frac{8p}{(p+1)} \frac{|\Lambda_B|}{\kappa_d^2}. \quad (2.14)$$

In this limit, note that the solution exists only for negative bulk cosmological constant and the warp factor assumes the form $a(z) = l/z$, where $l = \sqrt{p(p+1)}/(2\kappa_d^2 \Lambda_B)$ is the d -dimensional AdS radius. For the specific case of a 3-brane embedded in a 5-dimensional spacetime we easily recover the solution of Ref. [3] as presented in Ref. [19] in conformal coordinates.

If, instead of $\alpha=0$, we choose alternatively $c_0=0$, we re-introduce the Casimir stress in our analysis but, at the same time, we loose the time dependence of our solution. In the presence of the Casimir stress, time-dependent solutions most probably require a departure from conformal geometry which immensely complicates the whole problem. However, it is possible to construct an effective theory exploiting the large separation of the energy scales: Casimir stress and Λ_B

are presumably determined in units of a large fundamental d -dimensional gravitational scale. On the contrary, the characteristic frequencies of time-dependent perturbations are very small as compared to this scale for any cosmological epoch starting from reheating. Therefore, it seems reasonable to find static solution(s) in d dimensions to perform a dimensional reduction about. Studying small and ‘‘soft’’ perturbations around these solutions, one can perform the stability analysis for the extra dimension in the resulting $(d-1)$ -dimensional, effective field theory.

B. Static solutions

Assuming the presence of the Casimir stress in the bulk, and of a bulk cosmological constant, we seek static solutions for the warp factor $a = a(z)$. The system of Eqs. (2.5)–(2.8) then reduces to the following set of independent equations:

$$p \frac{a''}{a} + \frac{p}{2}(p-3) \frac{a'^2}{a^2} = \kappa_d^2 a^2 \left[-\Lambda_B + \frac{\alpha}{(aL)^{p+2}} \right], \quad (2.15)$$

$$\frac{p}{2}(p+1) \frac{a'^2}{a^2} = \kappa_d^2 a^2 \left[-\Lambda_B - \frac{(p+1)\alpha}{(aL)^{p+2}} \right]. \quad (2.16)$$

As we will see these equations can be integrated explicitly. Multiplying Eq. (2.15) by $(p+1)$ and adding Eq. (2.16), one obtains

$$p(p+1) \frac{a''}{a} + \frac{p}{2}(p+1)(p-2) \frac{a'^2}{a^2} = -(p+2) \kappa_d^2 a^2 \Lambda_B. \quad (2.17)$$

In order to solve Eq. (2.17), we perform a coordinate transformation $dy = a(z)dz$ which corresponds to the following parametrization of the line element:

$$ds^2 = a^2(y) (-dt^2 + dx_1^2 + dx_2^2 + \dots + dx_p^2) + dy^2. \quad (2.18)$$

In these coordinates one gets

$$p(p+1) \left(\frac{a''}{a} + \frac{p}{2} \frac{a'^2}{a^2} \right) = -(p+2) \kappa_d^2 \Lambda_B, \quad (2.19)$$

while Eq. (2.16) becomes

$$\frac{p}{2}(p+1) \left(\frac{a'}{a} \right)^2 = \kappa_d^2 \left[-\Lambda_B - \frac{(p+1)\alpha}{(aL)^{p+2}} \right], \quad (2.20)$$

where now the prime stands for differentiation with respect to y . For $\Lambda_B < 0$ the general solution to Eq. (2.19) is

$$a^{(p+2)/2}(y) = A_1 \cosh(\omega y) + A_2 \sinh(\omega y), \quad (2.21)$$

where

$$\omega^2 = \frac{(p+2)^2}{2p(p+1)} \kappa_d^2 |\Lambda_B|, \quad (2.22)$$

and $A_{1,2}$ are integration constants. For $\Lambda_B > 0$ the solution is given by Eq. (2.21) with \cosh and \sinh replaced by \cos and \sin , respectively. Finally, for $\Lambda_B = 0$, the solution reads²

$$a^{(p+2)/2}(y) = A_1 y + A_2. \quad (2.23)$$

Let us concentrate first on the case $\Lambda_B < 0$. The solution (2.21) can be rewritten in a more convenient form:

$$a^{(p+2)/2}(y) = a_0^{(p+2)/2} \begin{cases} \cosh[\omega(y-y_0)] & \text{if } A_1 > A_2, \\ \sinh[\omega|y-y_0|] & \text{if } A_1 < A_2. \end{cases} \quad (2.24)$$

Inserting these solutions into Eq. (2.20), we obtain the following consistency conditions:

$$\tanh^2[\omega(y-y_0)] = 1 - \frac{(p+1)\alpha}{|\Lambda_B| (a_0 L)^d} \cosh^{-2}[\omega(y-y_0)] \quad (2.25)$$

$$\coth^2[\omega(y-y_0)] = 1 - \frac{(p+1)\alpha}{|\Lambda_B| (a_0 L)^d} \sinh^{-2}[\omega(y-y_0)], \quad (2.26)$$

respectively. It follows from Eqs. (2.25) and (2.26) that

$$|\Lambda_B| L^d = (\pm) \frac{(p+1)\alpha}{a_0^{p+2}}. \quad (2.27)$$

Equation (2.27) demands α to be positive for the \cosh -type and negative for the \sinh -type solution. For the 5-dimensional case this was also realized in Ref. [21]. Note that the parameter $L = z_2 - z_1$, appearing in the above constraints, may be written in terms of the coordinates y_1 and y_2 in the non-conformal frame as

$$L = \int_{y_1}^{y_2} \frac{dy}{a(y)} = \frac{1}{a_0 \omega} I(\omega y_1, \omega y_2), \quad (2.28)$$

where the integral I is defined as follows:

$$I(\omega y_1, \omega y_2) \equiv \int_{\omega y_1}^{\omega y_2} \frac{d\zeta}{\cosh^{2/(p+2)}(\zeta - \zeta_0)}, \quad (2.29)$$

where $\zeta_0 = \omega y_0$, for the \cosh -type solution. A similar expression (with \cosh replaced by \sinh) holds for the \sinh -type solution.

Finally, the *jump* conditions (2.12) for the derivative of the scale factor on the branes, expressed in non-conformal coordinates, lead to the constraints

²In the conformal frame an exact form of the solution for $\Lambda_B = 0$ was presented in Ref. [19].

$$\frac{4\omega}{p+2} \tanh[\omega(y_k - y_0)] = (-1)^k \frac{\kappa_d^2}{p} \Lambda_k \quad (k=1,2) \quad (2.30)$$

for the cosh-type solution and

$$\frac{4\omega}{p+2} \coth[\omega(y_k - y_0)] = (-1)^k \frac{\kappa_d^2}{p} \Lambda_k \quad (k=1,2) \quad (2.31)$$

for the sinh-type. The position of each brane with respect to the point $y=y_0$ determines the sign of its tension. Note that the cosh-type solution is characterized by the existence of a minimum at $y=y_0$ while, at the same point, the sinh-type solution has a singularity. We always assume that the first brane is located to the left of y_0 ($y_0 - y_1 > 0$), and Λ_1 is positive. In order to avoid the singularity in the case of the sinh-type solution we place the second brane on the same side of y_0 . According to Eq. (2.31), Λ_2 then should be negative. On the other hand, in the case of the cosh-type solution the second brane can be placed to the right of the minimum thus allowing for a configuration with two positive-tension branes (such a solution was first studied in Ref. [11]). An alternative way of compactifying the extra dimension arises in this case: discarding the second brane and identifying the two minima at $y = \pm y_0$, single-brane models with compact extra dimension may be constructed. The construction and investigation of the cosmological properties of these models were conducted in Refs. [7,9].

A remark about the fine-tuning of parameters in our model is in order at this point. By fixing the position of one of the two branes and imposing that $a(y_*) = 1$, for some $y_* \neq y_0$, we find y_0 as a function of a_0 . Invoking the consistency condition (2.27), a_0 is also fixed in terms of Λ_B and the parameter L , which is related to the location of the second brane, and, thus, to the inter-brane distance, through Eq. (2.28). Before considering the jump conditions, these two quantities together with the two brane tensions, are free parameters of the model. The two jump conditions, evaluated at the locations of the two branes, will fix two of these parameters leaving the other two free. Here, we choose to fix the bulk cosmological constant and the inter-brane distance and have as input parameters the two brane tensions. In Sec. III, we will demonstrate³ that the fine-tuning of Λ_B and of the location of the second brane guarantees the existence of static solutions with vanishing $(d-1)$ -dimensional effective cosmological constant and a constant radion field.

Let us investigate the reason that made the construction of such a geometry possible in the case of the physically more interesting cosh-like solution. In Ref. [11] a simplified ap-

proach was pursued: An unknown mechanism created an effective potential for the scale factor of the extra dimension. Varying this contribution with respect to the metric, a y -dependent (55)-component of the energy-momentum tensor was found in addition to the contribution due to the bulk cosmological constant. Since the stabilization mechanism was unknown, it was assumed in Ref. [11] that the components T_{00} and T_{ii} were not altered. In the present work we observe that the Casimir stress qualitatively leads to the behavior of the scale factor as in Ref. [11] where the existence of a minimum allowed for a configuration with two positive-tension branes. Moreover, the y -dependent contribution to the total T_{55} given as

$$\begin{aligned} T_{55} &= |\Lambda_B| - \frac{(p+1)\alpha}{L^d a_0^{p+2} \cosh^2[\omega(y-y_0)]} \\ &= |\Lambda_B| - \frac{|\Lambda_B|}{\cosh^2[\omega(y-y_0)]} \end{aligned} \quad (2.32)$$

has the same form which the analysis in Ref. [11] demanded. In addition, the known behavior of the solution in the bulk allowed us to derive its contribution to the remaining components of the energy-momentum tensor and thus to include its effect on the spacetime structure. Therefore, the non-isotropic contributions to T_{00} and T_{ii} are not zero. However, they are different from that to T_{55} indicating the breaking of the 5-dimensional Poincaré covariance. As a result, the Casimir stress generated by a conformally coupled scalar field fulfills all the conditions necessary for the existence of a configuration with two positive-tension branes. The stability of this configuration, and of the one with a pair of positive-negative tension branes, under adiabatic perturbations, needs to be studied.

Concluding this section, let us make a few comments on the solutions in the case of positive and zero bulk cosmological constant. For $\Lambda_B > 0$, Eq. (2.20) leads to a constraint similar to Eq. (2.27) with (\pm) replaced by $(-)$, for both cos and sin-type solutions. Thus, the solution exists only in the case where $\alpha < 0$. Depending on the location of the branes, the system can accommodate positive-negative, negative-negative, or even positive-positive tension branes due to the oscillating behavior of the solution. The branes of the last configuration are geodesically disconnected by a pair of singularities. In any case, the hierarchy problem cannot be solved since the warp factor is bounded by a small value. For $\Lambda_B = 0$, the solution (2.23), when substituted in Eq. (2.20), leads to the result

$$A_1^2 = -\frac{2\kappa_d^2}{p} \frac{\alpha}{L^d}, \quad (2.33)$$

which again requires α to be negative. The *jump* conditions then yield

$$\left[\frac{a(y_2)}{a(y_1)} \right]^{(p+2)/2} = -\frac{\Lambda_1}{\Lambda_2}, \quad (2.34)$$

³In Sec. III, we will not impose the condition $a(y_*) = 1$. In return, the coordinates y_1 and y_0 will conveniently be fixed, in order to simplify our analysis. The result is the same with both methods: we are left with one integration constant, a_0 (or alternatively a_+ , see Sec. III), whose equilibrium value will be fixed by the consistency condition (2.27). The two jump conditions will fix, in turn, the bulk cosmological constant and the location of the second brane.

meaning that one of the two branes has negative tension. This result could also be inferred from the monotonic behavior of the solution (2.23). The resolution of the hierarchy seems possible at first sight. However, the fine-tuning of the ratio of the two brane tensions introduces a new hierarchy into the model.

III. PROPERTIES OF THE RADION FIELD

In this section we perform the derivation of the 4-dimensional effective theory following from the higher-dimensional theory (2.1) upon dimensional reduction. Our main goal is the determination of a kinetic term and an effective potential for the fluctuations of proper size of the extra dimension which are related to those of the canonically normalized radion field. These two quantities decide whether and how stable the static configurations of the previous section are. We will consider “+” single-brane models as well as “++” and “+-” two-brane models (“+” and “-” refer to the sign of the brane tensions). In what follows we specialize to the case $d=5$.

In order to study the size fluctuations, we restore the scale factor b of the extra dimension and consider it as a four-dimensional field

$$dy = b(x_{\perp}) d\xi. \quad (3.1)$$

Thereby, ξ is a dimensionless coordinate and $x_{\perp} \equiv (t, x^1, x^2, x^3)$. The static solution corresponds to the equilibrium value $\langle b \rangle = b_0$ which satisfies the constraints listed in the previous section. In terms of the coordinate ξ we choose the position of the first brane to be fixed at the point $\xi_1 = -1$ while the point y_0 corresponds to $\xi_0 = 0$. The second brane is located at $\xi = \lambda$ with $\lambda > 0$ corresponding to $\Lambda_2 > 0$ for the cosh-type solution. For the sinh-type solution positive values of λ are excluded due to the singularity at $\xi = 0$. The radion field is proportional to the deviation of b from its equilibrium value b_0 ,

$$b(x_{\perp}) - b_0 \equiv Z^{-1/2} r(x_{\perp}), \quad (3.2)$$

where the factor Z as well as the factor in front of $[b(x_{\perp}) - b_0]^2$ can be obtained by an explicit integration of the action (2.1) over the transverse coordinate ξ . Fluctuations of $b(x_{\perp})$ about b_0 entail x_{\perp} -dependent fluctuations of a [14]. Considering only terms up to quadratic order in the fluctuations, our final result will contain four dimensional gravity, a kinetic and a mass term for the radion, and the coupling of the radion to gravity. In addition, there will be a massless scalar $\phi_{(4)}$, the remnant of the five-dimensional conformally coupled scalar field ϕ . The four-dimensional, effective action takes the following form:

$$S_{\text{eff}} = - \int d^4x \sqrt{-g_{(4)}} \left\{ - \frac{1}{2\kappa_4^2} R_{(4)} + \frac{1}{2} \partial_{\mu} r \partial^{\mu} r + \frac{1}{2} \partial_{\mu} \phi_{(4)} \partial^{\mu} \phi_{(4)} + \frac{1}{2} m_r^2 r^2 + \gamma_{(4)} \phi_{(4)}^2 R_{(4)} + \mathcal{L}(\phi_{(4)}, \partial_{\mu} r \partial^{\mu} r) \right\}. \quad (3.3)$$

$\mathcal{L}(\phi_{(4)}, \partial_{\mu} r \partial^{\mu} r)$ denotes possible interaction terms of the radion with the $\phi_{(4)}$ field which are of no interest in this paper. For the stability analysis it is only necessary to investigate the free sector of the effective theory for the radion field. We treat cosh- and sinh-type solutions separately in the remainder of this section.

A. Sinh-type solution

In order to determine the effective, four-dimensional theory for the radion field, we need to start from the five-dimensional action and perform the ξ -integration explicitly. For this purpose we rewrite the dependence of the scale factor a in terms of its value on the first brane, in the case of sinh-type solutions, as follows:

$$a(x_{\perp}, \xi) = a_+(x_{\perp}) \frac{\sinh^{2/5}(\omega b(x_{\perp}) |\xi|)}{\sinh^{2/5}(\omega b(x_{\perp}))}, \quad (3.4)$$

with

$$a_+(x_{\perp}) = a_0 \sinh^{2/5}(\omega b(x_{\perp})). \quad (3.5)$$

Inserting Eq. (3.4) into Eq. (2.1) under consideration of Eq. (3.1), we obtain the following decomposition of the radion action:

$$S_r^{\text{eff}} = S_r^{\text{kinetic}} + S_r^{\text{interaction}} + S_r^{\text{potential}}. \quad (3.6)$$

Thereby, the terms S_r^{kinetic} and $S_r^{\text{interaction}}$ may be computed together. They include kinetic terms for b as well as an interaction term with $R_{(4)}$ and follow from the terms of the 5-dimensional scalar curvature $R_{(5)}$ which involve derivatives of the metric with respect to the 4-dimensional coordinates. We have

$$S_r^{\text{kinetic}} + S_r^{\text{interaction}} = \int d^4x \sqrt{-g_{(4)}} \{ A(\omega b) R_{(4)} + B(\omega b) \partial_{\mu} b \partial^{\mu} b \}, \quad (3.7)$$

where

$$A(\omega b) = \frac{b}{\kappa_5^2} \int_{|\lambda|}^1 d\xi \frac{\sinh^{4/5}(\omega b |\xi|)}{\sinh^{4/5}(\omega b)}, \quad (3.8)$$

$$B(\omega b) = \frac{12\omega}{5\kappa_5^2} \int_{|\lambda|}^1 d\xi \frac{\sinh^{4/5}(\omega b|\xi|)}{\sinh^{4/5}(\omega b)} \times \left\{ -[\coth(\omega b) - \xi \coth(\omega b\xi)] + \frac{2\omega b}{5} [\coth(\omega b) - \xi \coth(\omega b\xi)]^2 \right\}. \quad (3.9)$$

Equation (3.8), upon integration will lead to the definition of the four-dimensional Newton's constant in terms of the 5-dimensional one and the size of the extra dimension.

The potential part of Eq. (3.6) includes several contributions due to the brane tensions, the bulk cosmological constant, the Casimir energy, the terms from the five-dimensional curvature $R_{(5)}$ involving ξ -derivatives of the metric, and the delta-functional contributions from the discontinuities of a' at the boundaries $\xi = -1, \lambda$. After integrating over ξ , the expression for $S_r^{\text{potential}}$ reads

$$\begin{aligned} S_r^{\text{potential}} &= - \int d^4x \sqrt{-g_{(4)}} V^{\text{eff}} \\ &= - \int d^4x \sqrt{-g_{(4)}} \\ &\quad \times \left[\Lambda_1 - \frac{5}{2} \frac{|\Lambda_B|}{\omega} \coth(\omega b) + \frac{\sinh^{8/5}(\omega b|\lambda|)}{\sinh^{8/5}(\omega b)} \right. \\ &\quad \times \left(\Lambda_2 + \frac{5}{2} \frac{|\Lambda_B|}{\omega} \coth(\omega b|\lambda|) \right) \\ &\quad \left. - \frac{1}{\sinh^{8/5}(\omega b)} \left(\frac{|\Lambda_B|}{2\omega} I(\omega b) - \frac{2\alpha\omega^4}{I^4(\omega b)} \right) \right], \end{aligned} \quad (3.10)$$

where $\sqrt{-g_{(4)}} = a_+^4$. Note that the result of integration over ξ can be expressed in terms of the integral I , defined in Eq. (2.29), which now takes the form

$$I(\omega b) = \int_{\omega b|\lambda|}^{\omega b} \frac{d\xi}{\sinh^{2/5}\xi}. \quad (3.11)$$

We may now expand the effective potential up to second order in $b - b_0$, i.e.

$$\begin{aligned} V^{\text{eff}}(b) &= V^{\text{eff}}(b_0) + \frac{\delta V^{\text{eff}}}{\delta b} \Big|_{b_0} (b - b_0) + \frac{1}{2} \frac{\delta^2 V^{\text{eff}}}{\delta b^2} \Big|_{b_0} (b - b_0)^2 \\ &+ \dots \end{aligned} \quad (3.12)$$

With the help of the *jump* conditions (2.31), which, when expressed in terms of ξ , can be written as

$$\Lambda_1 = \frac{5}{2} \frac{|\Lambda_B|}{\omega} \coth(\omega b), \quad \Lambda_2 = -\frac{5}{2} \frac{|\Lambda_B|}{\omega} \coth(\omega b|\lambda|), \quad (3.13)$$

and the additional constraint (2.27), it is not difficult to show that both $V^{\text{eff}}|_{b_0}$ and $\delta V^{\text{eff}}/\delta b|_{b_0}$ vanish. Hence, we conclude that the constraint (2.27) in conjunction with the *jump* conditions causes the effective four-dimensional cosmological constant to be zero and, at the same time, extremizes the radion effective potential. Calculating the second derivative of V^{eff} with respect to b and inserting the result in the above expansion, we obtain the following expression for the effective potential of the radion:

$$\begin{aligned} V^{\text{eff}} &= (b - b_0)^2 \frac{24}{5} \frac{\omega^3}{\kappa_5^2} \sinh^{-8/5}(\omega b_0) \\ &\quad \times \left[\frac{1}{I(\omega b_0)} \left(\frac{1}{\sinh^{2/5}(\omega b_0)} - \frac{|\lambda|}{\sinh^{2/5}(\omega b_0|\lambda|)} \right)^2 \right. \\ &\quad \left. + \frac{2}{5} \left(\frac{\coth(\omega b_0)}{\sinh^{2/5}(\omega b_0)} - \frac{|\lambda|^2 \coth(\omega b_0|\lambda|)}{\sinh^{2/5}(\omega b_0|\lambda|)} \right) \right]. \end{aligned} \quad (3.14)$$

Varying λ between -1 and 0 , we realize that the radion mass squared is not sign definite. In order to draw some definite conclusions about the sign of the radion mass and, thus, the stability of the sinh-type solutions, we will study separately the cases of nearly flat ($\omega b \ll 1$) and highly warped ($\omega b \gg 1$) extra dimension.

(A) *Nearly flat extra dimension.* In the limit $\omega b \ll 1$, the hyperbolic functions in Eqs. (3.8),(3.9) can be expanded in powers of ωb . Keeping only the dominant terms and performing the integration, the kinetic part (3.7) takes the simplified form

$$\begin{aligned} S_r^{\text{kinetic}} + S_r^{\text{interaction}} &= \int d^4x \sqrt{-g_{(4)}} \left\{ \frac{R_{(4)}}{2\kappa_4^2} \left(\frac{b}{b_0} \right) \right. \\ &\quad \left. - \frac{8b_0}{171} (\omega b_0) [10 - |\lambda|^{9/5}] \right. \\ &\quad \left. \times (19 - 9|\lambda|^2) \partial_\mu b \partial^\mu b \right\}, \end{aligned} \quad (3.15)$$

where

$$\kappa_4^2 = \frac{9\kappa_5^2}{10b_0} \frac{1}{(1 - |\lambda|^{9/5})}. \quad (3.16)$$

In order to recover Einstein's gravity, we perform the conformal transformation $g_{\mu\nu}^{(4)} = \Omega(b) \hat{g}_{\mu\nu}^{(4)} = (b_0/b) \hat{g}_{\mu\nu}^{(4)}$. The interaction term between b and the 4-dimensional scalar curvature $R_{(4)}$ will then disappear and an extra contribution to the kinetic term of b will arise. The final form of the kinetic part, then, is given by

$$S_r^{\text{kinetic}} + S_r^{\text{interaction}} = \int d^4x \sqrt{-\hat{g}_{(4)}} \times \left\{ \frac{\hat{R}_{(4)}}{2\kappa_4^2} - \frac{3}{4\kappa_4^2 b_0^2} \partial_\mu b \partial^\mu b \right\}, \quad (3.17)$$

where the subdominant contribution to the kinetic term for b in Eq. (3.15), proportional to (ωb_0) , has been dropped. Note that, after the conformal transformation and the derivation of the kinetic term of b , we may safely set $b=b_0$ in all places other than $\partial_\mu b$. From the above expression, we may easily read the definition of the canonically normalized radion field

$$r(x_\perp) = \sqrt{\frac{3}{2\kappa_4^2 b_0^2}} (b - b_0). \quad (3.18)$$

We now turn to the form of the effective potential (3.14). In the same limit, $\omega b_0 \ll 1$, it assumes the form

$$V^{\text{eff}} = (b - b_0)^2 \frac{24}{5} \frac{(1 - |\lambda|^{3/5})}{\kappa_5^2 b_0^3}. \quad (3.19)$$

Under the conformal transformation, the effective potential changes as: $V^{\text{eff}} \rightarrow \hat{V}^{\text{eff}} = \Omega^2(b) V^{\text{eff}}$. However, this does not affect either the existence of the extremum or the sign of the second derivative, since

$$\left. \frac{\delta \hat{V}^{\text{eff}}}{\delta b} \right|_{b_0} = \Omega^2(b) \left. \frac{\delta V^{\text{eff}}}{\delta b} \right|_{b_0} + 2\Omega \left. \frac{\delta \Omega(b)}{\delta b} \right|_{b_0} V^{\text{eff}}|_{b_0} = 0, \quad (3.20)$$

$$\begin{aligned} \left. \frac{\delta^2 \hat{V}^{\text{eff}}}{\delta b^2} \right|_{b_0} &= \Omega^2(b) \left. \frac{\delta^2 V^{\text{eff}}}{\delta b^2} \right|_{b_0} + 4\Omega \left. \frac{\delta \Omega(b)}{\delta b} \right|_{b_0} \left. \frac{\delta V^{\text{eff}}}{\delta b} \right|_{b_0} \\ &\quad + 2 \left[\left(\left. \frac{\delta \Omega(b)}{\delta b} \right|_{b_0} \right)^2 + \Omega \left. \frac{\delta^2 \Omega(b)}{\delta b^2} \right|_{b_0} \right] V^{\text{eff}}|_{b_0} \\ &= \Omega^2(b) \left. \frac{\delta^2 V^{\text{eff}}}{\delta b^2} \right|_{b_0}. \end{aligned} \quad (3.21)$$

Using the definition of the radion field (3.18) and of the 4-dimensional Newton's constant (3.16), we find the radion mass to be

$$m_r^2 = \frac{144}{25b_0^2} \frac{(1 - |\lambda|^{3/5})}{(1 - |\lambda|^{9/5})}. \quad (3.22)$$

From the above we may conclude that the radion mass remains always positive in the limit of nearly flat extra dimension, thus, ensuring the stability of the sinh-type solution, independently of the position of the negative brane. Moreover, it remains well defined for the whole range of values

$-1 < \lambda < 0$, increasing when $|\lambda| \rightarrow 0$, as the size of the extra dimension increases, and decreasing in the opposite limit $|\lambda| \rightarrow 1$.

(B) *Highly warped extra dimension.* In this case we assume that $\omega b \gg 1$ and write the hyperbolic functions, appearing in the kinetic and potential parts of the effective action, in terms of exponentials. Neglecting the exponentially suppressed contributions, we may then easily perform the corresponding integration over the extra dimension. Note, however, that our approximation breaks down when $|\xi| \sim (\omega b)^{-1}$, i.e. when the negative tension brane is located close to the singularity. We may easily see, by simple numerical analysis that, as we increase ωb_0 , the positivity of the effective potential (3.14) demands the negative tension brane to be located further and further away from the point $\xi=0$. For example, for $\omega b_0=20$, the radion mass squared will be negative definite if the negative tension brane is located in the interval $-0.07 < \lambda < 0$, while for $\omega b_0=86$, relevant to the Randall-Sundrum setup, if $-0.09 < \lambda < 0$. Therefore, in order to ensure the stability of our sinh-type solutions in the case of large warping, we need to place the second brane away from the singularity. In this case, the area around $\xi \approx 0$ is excluded and our assumption that $\omega b|\xi| \gg 1$ will always hold for the stable solutions.

The kinetic and interaction terms in this case take the form

$$S_r^{\text{kinetic}} + S_r^{\text{interaction}} = \int d^4x \sqrt{-g_{(4)}} \left\{ \frac{R_{(4)}}{2\kappa_4^2} (1 - e^{4\omega b(|\lambda|-1)/5}) - \frac{6\omega}{5\kappa_5^2} (|\lambda|-1)^2 e^{4\omega b(|\lambda|-1)/5} \partial_\mu b \partial^\mu b \right\}, \quad (3.23)$$

where

$$\kappa_4^2 = \frac{2\omega}{5} \kappa_5^2. \quad (3.24)$$

Once again, we need to perform a conformal transformation, of the form $g_{\mu\nu}^{(4)} = (1 - e^{4\omega b(|\lambda|-1)/5})^{-1} \hat{g}_{\mu\nu}^{(4)}$, in order to recover Einstein's gravity. Then, we arrive at the result

$$S_r^{\text{kinetic}} + S_r^{\text{interaction}} = \int d^4x \sqrt{-\hat{g}_{(4)}} \left\{ \frac{\hat{R}_{(4)}}{2\kappa_4^2} - \frac{6\omega}{5\kappa_5^2} (|\lambda|-1)^2 e^{4\omega b_0(|\lambda|-1)/5} \times \partial_\mu (b - b_0) \partial^\mu (b - b_0) \right\}, \quad (3.25)$$

where the subdominant contributions to the kinetic term, after the conformal transformation and the expansion around b_0 , have been ignored. In this case, the canonically normalized radion field is defined as

$$r(x_{\perp}) = \sqrt{\frac{12\omega}{5\kappa_5^2} (|\lambda| - 1)^2 e^{2\omega b_0(|\lambda| - 1)/5} (b - b_0)}. \quad (3.26)$$

The effective potential (3.14), in the limit $\omega b_0 \gg 1$, reads

$$\begin{aligned} \hat{V}^{\text{eff}} &= (b - b_0)^2 \\ &\times \frac{192 \omega^3}{25 \kappa_5^2} \frac{(1 - |\lambda|)^2}{(1 - e^{-4\omega b_0(1 - |\lambda|)/5})^2} \frac{e^{-2\omega b_0}}{(1 - e^{-2\omega b_0(1 - |\lambda|)/5})}, \end{aligned} \quad (3.27)$$

which, when expressed in terms of the radion field, it yields the corresponding radion mass

$$m_r^2 = \frac{32}{5} \omega^2 \left(\frac{M_W}{M_P} \right)^3 e^{-2\omega b_0 |\lambda|}. \quad (3.28)$$

In the above expression, we have used the geometrical explanation of the Planck scale–weak scale hierarchy which in the limit of large warping is equivalent to

$$\frac{a_-}{a_+} \simeq e^{-2\omega b_0(1 - |\lambda|)/5} = \frac{M_W}{M_P} \quad (3.29)$$

and ignored the exponentially suppressed terms in the denominator of Eq. (3.27). As expected, since we restricted ourselves to the study of the stable sinh-type solutions, the radion mass squared turned out to be positive and proportional to the ratio M_W^3/M_P , since $\omega^2 \sim M_P^2$. However, there is an extra suppression factor that decreases very rapidly as $\omega b_0 \rightarrow \infty$ leading to an additional strong suppression of the radion mass. This factor depends on the location of the negative tension brane, leading to a larger suppression when the second brane is located close to the first one and to a smaller suppression when the second brane is located closer to the singularity. The relative change however is very small since $|\lambda| = \mathcal{O}(1)$.

B. Cosh-type solutions

For positive α (cosh-type solution) we may write the scale factor as follows:

$$a(x_{\perp}, \xi) = a_+(x_{\perp}) \frac{\cosh^{2/5}(\omega b(x_{\perp}) \xi)}{\cosh^{2/5}(\omega b(x_{\perp}))}, \quad (3.30)$$

where, similarly to the previous subsection, we have defined

$$a_+(x_{\perp}) = a_0 \cosh^{2/5}(\omega b(x_{\perp})). \quad (3.31)$$

We first focus on the kinetic and interaction terms. Substituting the above ansatz in the 5-dimensional action and decomposing $R_{(5)}$, we obtain

$$\begin{aligned} S_r^{\text{kinetic}} + S_r^{\text{interaction}} &= \int d^4x \sqrt{-g_{(4)}} \{ \tilde{A}(\omega b) R_{(4)} \\ &+ \tilde{B}(\omega b) \partial_{\mu} b \partial^{\mu} b \}, \end{aligned} \quad (3.32)$$

where now

$$\tilde{A}(\omega b) = \frac{b}{\kappa_5^2} \int_{-1}^{\lambda} d\xi \frac{\cosh^{4/5}(\omega b \xi)}{\cosh^{4/5}(\omega b)}, \quad (3.33)$$

$$\begin{aligned} \tilde{B}(\omega b) &= \frac{12\omega}{5\kappa_5^2} \int_{-1}^{\lambda} d\xi \frac{\cosh^{4/5}(\omega b \xi)}{\cosh^{4/5}(\omega b)} \\ &\times \left\{ -[\tanh(\omega b) - \xi \tanh(\omega b \xi)] \right. \\ &\left. + \frac{2\omega b}{5} [\tanh(\omega b) - \xi \tanh(\omega b \xi)]^2 \right\}. \end{aligned} \quad (3.34)$$

In analogy to the previous subsection we obtain for the potential part the expression

$$\begin{aligned} S_r^{\text{potential}} &= - \int d^4x \sqrt{-g_{(4)}} V^{\text{eff}} \\ &= - \int d^4x \sqrt{-g_{(4)}} \left[\Lambda_1 - \frac{5}{2} \frac{|\Lambda_B|}{\omega} \tanh(\omega b) \right. \\ &+ \frac{\cosh^{8/5}(\omega b \lambda)}{\cosh^{8/5}(\omega b)} \\ &\times \left(\Lambda_2 - \frac{5}{2} \frac{|\Lambda_B|}{\omega} \tanh(\omega b \lambda) \right) \\ &\left. + \frac{1}{\cosh^{8/5}(\omega b)} \left(\frac{|\Lambda_B|}{2\omega} I(\omega b) - \frac{2\alpha\omega^4}{I^4(\omega b)} \right) \right], \end{aligned} \quad (3.35)$$

where

$$I(\omega b) = \int_{-\omega b}^{\omega b \lambda} \frac{d\xi}{\cosh^{2/5}\xi}. \quad (3.36)$$

Using the *jump* conditions (2.30), which can be written in the simple form

$$\Lambda_1 = \frac{5}{2} \frac{|\Lambda_B|}{\omega} \tanh(\omega b), \quad \Lambda_2 = \frac{5}{2} \frac{|\Lambda_B|}{\omega} \tanh(\omega b \lambda), \quad (3.37)$$

we once again see that both V^{eff} and $\delta V^{\text{eff}}/\delta b$ vanish when evaluated at the static solution. Performing the expansion around b_0 , we finally get the following expression for the effective potential:

$$V^{\text{eff}} = -(b-b_0)^2 \frac{24}{5} \frac{\omega^3}{\kappa_5^2} \cosh^{-8/5}(\omega b_0) \times \left[\frac{1}{I(\omega b_0)} \left(\frac{1}{\cosh^{2/5}(\omega b_0)} + \frac{\lambda}{\cosh^{2/5}(\omega b_0 \lambda)} \right)^2 + \frac{2}{5} \left(\frac{\tanh(\omega b_0)}{\cosh^{2/5}(\omega b_0)} + \frac{\lambda^2 \tanh(\omega b_0 \lambda)}{\cosh^{2/5}(\omega b_0 \lambda)} \right) \right]. \quad (3.38)$$

This is negative definite for any value of $b \neq b_0$. As a result, we conclude that a positive α leads to a negative m_r^2 and thus to a tachyonic potential for the radion field. Although the static configuration with two positive tension branes, presented in the previous section, corresponds to an extremum of the effective potential this extremum is a maximum. Therefore, any small perturbation of this static solution induces an instability causing the radion field to run away from its equilibrium value. Hence, there is no stabilization of the extra dimension.

For completeness, we now briefly give the results for the expression of the radion field, in terms of $(b-b_0)$, and of the radion mass in the two cases of nearly flat ($\omega b \ll 1$) and highly warped ($\omega b \gg 1$) extra dimension.

(A) *Nearly flat extra dimension.* In the limit $\omega b \ll 1$, the integration in Eq. (3.33) gives rise to an interaction term between b and $R_{(4)}$, as in the case of the sinh-type solution. The same conformal transformation, $g_{\mu\nu}^{(4)} = (b_0/b) \hat{g}_{\mu\nu}^{(4)}$, removes this term and gives the dominant contribution to the kinetic term of b . Then, the kinetic part of the effective action reduces again to Eq. (3.17), leading to the same definition (3.18) for the radion field, while

$$\kappa_4^2 = \frac{\kappa_5^2}{2b_0(\lambda+1)}. \quad (3.39)$$

In the same limit, the effective potential (3.38) assumes the form

$$\hat{V}^{\text{eff}} = -(b-b_0)^2 \frac{24\omega^2}{5\kappa_5^2 b_0} (\lambda+1) = \frac{1}{2} \left(-\frac{16\omega^2}{5} \right) r^2(x_\perp), \quad (3.40)$$

from which we may easily read the expression of the radion mass squared which is indeed negative. However, in the limit $\omega \rightarrow 0$, this mass goes to zero leaving behind a static cosh-type solution which is a saddle point of the effective action instead of a maximum. Nevertheless, this semi-stable solution is plagued by the existence of a massless radion field.

(B) *Highly warped extra dimension.* In this case, the analysis is similar to the one of the previous subsection, in the limit $\omega b \gg 1$. Upon integration of Eqs. (3.33), (3.34), we find the result

$$S_r^{\text{kinetic}} + S_r^{\text{interaction}} = \int d^4x \sqrt{-g_{(4)}} \left\{ \frac{R_{(4)}}{2\kappa_4^2} (e^{4\omega b(\lambda-1)/5} - e^{-8\omega b/5}) + \frac{6\omega}{5\kappa_5^2} [(\lambda-1)^2 e^{4\omega b(\lambda-1)/5} - 4e^{-8\omega b/5}] \partial_\mu b \partial^\mu b \right\}, \quad (3.41)$$

where κ_4^2 is defined as in Eq. (3.24). The conformal transformation $g_{\mu\nu}^{(4)} = (e^{4\omega b(\lambda-1)/5} - e^{-8\omega b/5})^{-1} \hat{g}_{\mu\nu}^{(4)}$ restores Einstein's gravity and leads to a canonically normalized kinetic term for the radion field, which is given by Eq. (3.26) with $(|\lambda|-1)$ replaced by $(\lambda+1)$. The effective potential (3.38), in the limit $\omega b_0 \gg 1$, reads

$$\hat{V}^{\text{eff}} = -(b-b_0)^2 \frac{192\omega^3}{25\kappa_5^2} e^{-2\omega b_0} \frac{e^{-4\omega b_0/5} (1 + \lambda e^{-2\omega b_0(\lambda-1)/5})^2 + (1 + \lambda^2 e^{-2\omega b_0(\lambda-1)/5})}{(e^{4\omega b(\lambda-1)/5} - e^{-8\omega b/5})^2}, \quad (3.42)$$

which yields the radion mass squared

$$m_r^2 = -\frac{32}{5} \frac{\omega^2}{(\lambda+1)^2} e^{-2\omega b_0 \lambda/5} \begin{cases} \lambda^2 \left(\frac{a_1}{a_2} \right)^2 & (\lambda \leq 1), \\ \left(\frac{a_1}{a_2} \right) & (\lambda > 1), \end{cases} \quad (3.43)$$

where $(a_1/a_2) \simeq e^{2\omega b_0(1-\lambda)/5}$ is the ratio of the warp factors of the two branes located at $\xi = -1, \lambda$, respectively. The radion mass squared is always negative definite, as expected. Its value strongly depends on the position of the second positive brane and may vary significantly for various values of λ .

As mentioned in the previous section, the cosh-type solution allows for a single positive-tension brane configuration with compact extra dimension. The above analysis applies also in that case. Setting $\Lambda_2 = 0$ and taking the limit $\lambda \rightarrow 0$, i.e. moving the ‘‘empty’’ brane to the location of the minimum in all the above expressions, one obtains the corresponding results for the single-brane configuration. It is obvious that the effective potential (3.38) maintains its negative sign which also indicates the instability of the ‘‘+’’ brane configuration.

IV. CONCLUSIONS

It is often thought that staticity of gravity in a spacetime with a compact, extra dimension *requires* the introduction of

two branes possessing tensions of opposite sign. This statement is usually illustrated using an analogue in electrostatics. The presence of one positive-tension brane creates a gravitational field with the arrows of the field strength lines directed away from the brane. Since the dimension is compact these lines should meet at a point associated with a negative-tension brane. This argument is flawed since it is possible that at one point the gravitational field strength simply becomes zero, thus allowing to complete the circle *without* introducing a negative tension object. This idea was advocated in [7,9], and the concrete realization of such a compactification due to the Casimir stress was discussed in [21] and in this paper.

Interestingly enough, a classical force created by a scalar field in the bulk does not yield a static situation without introducing a metric singularity or a compensating negative-tension brane as it was shown in [11,17]. The presence of the quantum effects (Casimir stress), however, does allow for single and two-brane configurations with only positive tensions as we showed in Sec. II. The deeper reason for this is the violation of the classical energy condition by the Casimir stress [21] since the pressure in the extra dimension exceeds the energy density in the bulk. Positive-tension brane configurations arise in the case of an attractive Casimir force, whereas a repulsive Casimir force implies static pairs of positive-negative tension branes.

In Sec. II we found exact static solutions to the Einstein's equations, in a $d=p+1+1$ -dimensional spacetime, in the presence of a bulk cosmological constant being positive, negative or zero and of the Casimir stress created by a conformally coupled scalar field. It was shown that time-dependent cosmological solutions in a d -dimensional, conformally-flat spacetime are excluded if the Casimir stress is implemented in the theory. Ignoring the Casimir stress, non-static solutions, which are generalizations of the RS solution in de Sitter and anti de Sitter spacetime, were derived. These solutions allowed only for pairs of positive-negative tension branes with extremely fine-tuned brane tensions. Restoring the Casimir stress and considering a negative bulk cosmological constant, we found two different types of static solutions for the scale factor in the bulk, namely a cosh-type solution for an attractive Casimir force ($\alpha>0$) and a sinh-type solution for a repulsive Casimir force ($\alpha<0$). The monotonic behavior of the sinh-type solution was consistent only with the presence of a positive-negative tension pair of branes. On the other hand, the existence of a minimum in the behavior of the cosh-type solution allowed for the introduction of two positive tension branes or even for the compactification of the extra spatial dimension with a single positive-tension brane. These types of configurations were studied before in Refs. [7,9,11,12], by assuming the existence of an unknown mechanism to create an isotropy breaking contribution to the (55) -component of the energy-momentum tensor.

In the above context we addressed the issue of the stabilization of the extra dimension. Up to quadratic fluctuations about the static solutions and for all possible brane configurations, we derived the dimensionally reduced effective theory for the radion field. We calculated the mass squared of the radion field in each case. According to our results, there is indeed stabilization of the positive-negative pair of branes arising for a repulsive Casimir force and for the branes situated in the exponential regime of the sinh-solution. This implies that the corresponding static solution corresponds to a minimum of the radion effective potential. Interestingly, the situation becomes unstable if the negative-tension brane is placed close to the singularity. However, in the case of stabilization the hierarchy between the scales on the two branes produces a radion mass in the KeV range or smaller which makes this possibility phenomenologically unacceptable. Similar results demonstrating the stability of a positive-negative brane configuration in the case of a repulsive Casimir force, which was also plagued by an unnaturally small radion mass, were presented by Garriga *et al.* [19]. The possibility of stabilizing a similar brane configuration by taking into account quantum fluctuations of bulk fields was also studied by Goldberger and Rothstein [20] and the same inconsistency between the resolution of the hierarchy problem and the natural stabilization of the inter-brane distance was again pointed out. The role of the singularity in the stability of the system, when the second brane is placed close to it, became apparent only in the framework of our analysis, where exact solutions for the spacetime background, in the presence of the Casimir force, were derived. This feature was not revealed in any of the above works, where the quantum effects were considered as small perturbations in a fixed background. On the other hand, the static cosh-type solution arising for an attractive Casimir force, turned out to be unstable independently of the position of the two positive tension branes. This is in agreement with the results by Nam [20] where a purely attractive Casimir force due to a scalar field fails to stabilize the radion field. The single positive-tension brane configuration, that also arises in this case, shares the same kind of instability. As a result, the Casimir force fails to stabilize any configuration involving solely positive-tension branes, thus, leaving this question open for future study.

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