Photon frequency conversion induced by gravitational radiation

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We consider the propagation of gravitational radiation in a magnetized multicomponent plasma. It is shown that large density perturbations can be generated, even for small deviations from flat space, provided the cyclotron frequency is much larger than the plasma frequency. Furthermore, the induced density gradients can generate frequency conversion of electromagnetic radiation, which may give rise to an indirect observational effect of the gravitational waves.

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I. INTRODUCTION

Recently there has been an increased interest in gravitational waves, mainly due to the possibility of direct detection by LIGO (Laser Interferometer Gravitational-Wave Observatory) [1]. Naturally the effects of gravitational waves on Earth are very small—which is illustrated by the large dimensions required for detection. Closer to the source the influence of the gravitational waves may be larger, but generally it is nontrivial to predict the possible influence of the emitted radiation—in particular the coupling to the electromagnetic (EM) field complicates the description. For a discussion of the interaction between electromagnetic fields and gravitational radiation in an astrophysical context, see for example Refs. [2–8], and references therein.

In the present paper we will study the propagation of gravitational perturbations in a magnetized plasma, with the direction of propagation perpendicular to the magnetic field. It turns out that large density gradients driven by the gravitational perturbation can be generated, even for small deviations from flat space, provided the cyclotron frequency is much larger than the plasma frequency. Furthermore, as is well known from laboratory plasmas (see, e.g., [9]), moving density gradients can increase (or decrease) the frequency of electromagnetic wave packets, the so-called photon acceleration. The density gradients in our case are propagating with exactly the speed of light, in contrast with the laboratory application [9]. In principle this means that a given photon may increase its energy by several orders of magnitude, *independent of its initial energy*. Applying our results to gravitational radiation generated by binary systems, it turns out that the regime of most interest is the infrared regime. In this case a frequency conversion by an order of magnitude is possible for a binary system close to merging.

II. PLASMA RESPONSE TO A GRAVITATIONAL WAVE PULSE

A. Basic equations

The metric of a linearized gravitational wave propagating in the z direction can be written as [10]

$$\mathrm{d}s^2=-\mathrm{d}t^2+[1+h(u)]\mathrm{d}x^2+[1-h(u)]\mathrm{d}y^2+\mathrm{d}z^2, \quad (1)$$

where we have assumed linear polarization and $u\equiv z-t$. For an observer comoving with the time coordinate, the natural frame for measurements is given by

$$\begin{aligned} e_0&=\partial_t, \\ e_1&=(1-\tfrac{1}{2}h)\partial_x, \\ e_2&=(1+\tfrac{1}{2}h)\partial_y, \\ e_3&=\partial_z. \end{aligned} \quad (2)$$

It can be shown [7] that in such a frame, Maxwell's equations can be written

$$\nabla\cdot\boldsymbol{E}=\rho/\epsilon_0, \quad (3a)$$

$$\nabla\cdot\boldsymbol{B}=0, \quad (3b)$$

$$\frac{\partial\boldsymbol{E}}{\partial t}-\nabla\times\boldsymbol{B}=-\boldsymbol{j}_E-\mu_0\boldsymbol{j}, \quad (3c)$$

$$\frac{\partial\boldsymbol{B}}{\partial t}+\nabla\times\boldsymbol{E}=-\boldsymbol{j}_B, \quad (3d)$$

where the effective gravitational current densities are defined as

$$j_E^1=j_B^2=\tfrac{1}{2}(E^1-B^2)\frac{\partial h}{\partial z}, \quad (4a)$$

$$j_E^2=-j_B^1=-\tfrac{1}{2}(E^2+B^1)\frac{\partial h}{\partial z}, \quad (4b)$$

and $\nabla\equiv(e_1,e_2,e_3)$.

To first order in h, the fluid equations become

$$\frac{\partial n}{\partial t}+\nabla\cdot(n\boldsymbol{v})=0, \tag{5a}$$

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\gamma\boldsymbol{v}=\frac{q}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}), \tag{5b}$$

where $\gamma\equiv(1-v_\parallel^2)^{-1/2}$, $v_\parallel\equiv v_3$, and $n=\gamma\tilde{n}$, where $\tilde{n}$ is the proper number density. These equations hold for each particle species. Note that, in general, terms proportional to $v_1 h$ and $v_2 h$ appear in the equations [3]. Throughout this paper, we will assume that $v_1, v_2 \ll 1$, and thus neglect terms of order $v_1 h$, $v_2 h$.

B. Electromagnetic fields driven by a gravitational perturbation

From now on we assume $\partial/\partial t \ll \omega_c$, where $\omega_c\equiv qB/m$ is the cyclotron frequency, for all particle species (since the gravitational perturbation is assumed to be the driver of all perturbations this scaling thereby holds for $\partial/\partial t$ acting on all fields). Furthermore, we assume the presence of an external magnetic field $\boldsymbol{B}_0=B_0\boldsymbol{e}_1$ [where the total field is $\boldsymbol{B}=(B_0+\delta B)\boldsymbol{e}_1$]. The electric field takes the form $\boldsymbol{E}=E_\perp \boldsymbol{e}_2$.

Looking for solutions driven by the gravitational perturbation and thus using $\partial/\partial t=-\partial/\partial z$, we first consider Faraday's law for $\delta B\ll B_0$, which gives

$$\delta B=-E_\perp+hB_0. \tag{6}$$

Next we note that if the excited fields $E_\perp$ and δB grow (invalidating $\delta B\ll B_0$), the quantity $E_\perp+B$ that appears in the effective current still becomes $E_\perp+\delta B=hB_0$ and thereby the above formula holds for arbitrary electromagnetic amplitude. Taking the time derivative of Ampere's law, using Eq. (6), we obtain

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}\right]E_\perp+\mu_0\sum_i\frac{\partial j_{\perp(i)}}{\partial t}=-2\frac{\partial^2 h}{\partial t^2}B_0, \tag{7}$$

where the sum is over particle species and $j_\perp\equiv j_2$. For $\partial/\partial t=-\partial/\partial z$, the term (explicitly) involving $E_\perp$ vanishes. The currents are determined by the equation of motion, noting that the condition $\partial/\partial t\ll\omega_c$ means that the current contributions from different particle species cancel to lowest order in an expansion in the operator $\omega_c^{-1}\partial/\partial t$. The equation of motion gives

$$v_\parallel=-\frac{E_\perp}{B_0+\delta B} \tag{8}$$

to lowest order. Note that, using Eq. (6), we can now approximate the denominator in Eq. (8) by $B_0-E_\perp$. The error this approximation introduces will not have any noticeable effects. This is because $v_\parallel$ can only be altered significantly by the omitted term if $\delta B\approx B_0$, but this regime is inaccessible since—from Eq. (8)—it corresponds to superluminal speeds. From the parallel component of Eq. (5b) we can calculate the first order correction to the induced velocity, which subsequently determines the current. We obtain

$$v_\perp=-\frac{m}{q}\frac{1-v_\parallel}{B_0-E_\perp}\frac{\partial(\gamma v_\parallel)}{\partial t}. \tag{9}$$

Furthermore, the continuity equation gives

$$\delta n=\frac{n_0 v_\parallel}{1-v_\parallel} \tag{10}$$

where we have divided the density into a perturbed and an unperturbed part, $n=n_0+\delta n$.

From (7) and the relations above we can thus determine the induced velocity and density in terms of the metric perturbation h. The result (for all particle species) is

$$v_\parallel=\frac{1-(1-\mathcal{H})^2}{1+(1-\mathcal{H})^2}, \tag{11a}$$

$$\delta n=\frac{n_0}{2}\left[\frac{1}{(1-\mathcal{H})^2}-1\right], \tag{11b}$$

where $\mathcal{H}\equiv 2h/\Sigma_i(\omega_{p(i)}^2/\omega_{c(i)}^2)$ and $\omega_{p(i)}\equiv(q_{(i)}^2 n_0/\epsilon_0 m_{(i)})^{1/2}$ is the plasma frequency for the unperturbed plasma species i. Thus it is clear that even a moderate or small value of the gravitational perturbation may cause significant density perturbation, provided the plasma is strongly magnetized in the sense that $\Sigma_i(\omega_{p(i)}^2/\omega_{c(i)}^2)\ll 1$. This is because the fast magnetosonic (or compressional Alfvén) wave fulfills approximately the same dispersion relation as the gravitational wave, with the mismatch being proportional to $\Sigma_i(\omega_{p(i)}^2/\omega_{c(i)}^2)$ [11]. The divergence that occurs for $\mathcal{H}\rightarrow 1$ is clearly unphysical and its removal will be discussed in the next subsection.

For future considerations it will also be useful to have the relation between the relative magnetic field perturbation and the relative density perturbation. When $|\delta B|\gg|hB_0|$, which is the case of most interest, the last term of Eq. (6) can be neglected and the desired relation can be derived by combining the resulting formula with Eqs. (8) and (10). The simple result is

$$\frac{\delta n}{n_0}=\frac{\delta B}{B_0}. \tag{12}$$

C. Removal of the divergence

The purpose in this subsection is to explain the reason for the occurrence of divergence when $\mathcal{H}$ approaches unity, and to discuss various modifications of the assumptions that lead to a more physical behavior. From Eq. (9) we note that for infinitesimal velocity perturbations, $v_\perp$ (and thereby $j_\perp$) depends linearly on $v_\parallel$, but for large parallel velocities; in particular when $v_\parallel\rightarrow 1$, $v_\perp$ remains finite due to the factor $1-v_\parallel$. From Eq. (7) it is thus clear that we *cannot* have a stationary solution where $E_\perp$ depends only on $z-t$ for large enough h, and from Eq. (11a) we see that this limit for the gravitational perturbation is reached when $\mathcal{H}$ becomes unity. Basically, the physical reason is the following: In a vacuum the electromagnetic and gravitational modes obey the same dispersion relation, and therefore—due to the mode coupling provided by the unperturbed magnetic field—the system evolves in a nonstationary way. In particular, gravitational wave energy may be continuously converted into electromagnetic wave energy, as will be examined in more detail below. In the presence of a plasma, however, the induced currents change the dispersion relation of the electromagnetic wave, and the resulting detuning of the modes saturates the conversion of energy between them, making a steady state solution (in a frame moving with the velocity of light) possible in principle. For a strongly magnetized plasma, on the other hand, the induced plasma currents cannot grow continuously with h, as we have seen above. For sufficiently high gravitational amplitude this means that the plasma currents are of little significance, practically the plasma appears as a vacuum for $\mathcal{H}\geqslant 1$ and, in particular, solutions depending on $z-t$ are impossible. This conclusion is *not* dependent on the absence of thermal effects in our calculations in Sec. II B. Generally the addition of thermal motion only modifies our expressions (11) by a factor of the order $1+(v_t/c)^2$, where v_t is the thermal velocity. In particular, the divergence of (11b) still occurs for a finite value of $\mathcal{H}$.

On the other hand, it is clear that our omission of the back reaction of the electromagnetic wave on the gravitational pulse in principle could change this picture, since obviously certain components of the energy momentum tensor also diverge when $\mathcal{H}\rightarrow 1$, implying that the gravitational wave amplitude could indeed be diminished due to the influence of the generated EM wave. The effects of the self-consistent gravitational field caused by the plasma perturbations are discussed in the Appendix, but will be omitted here since it turns out that the back reaction on the gravitational wave is negligible in the application to be discussed in this article.

Since it is clear that for $\mathcal{H}\geqslant 1$ the generated currents cannot stop the growth of the EM wave, we simplify the picture from now on by putting the density to zero and thus totally ignoring the plasma effects. The general solution to Eq. (7) for the electric field in the presence of a monochromatic gravitational wave $h=\tilde{h}\cos[k(z-t)]$ can then be written

$$\delta B=E_{\perp}=\tfrac{1}{2}k(C_z z+C_t t)B_0\tilde{h}\sin[k(z-t)]+E_{+}(z-t)+E_{-}(z-t), \tag{13}$$

where $C_z+C_t=1$ and E_+ and E_- are arbitrary functions. For an initial value problem where the plasma is unperturbed in the absence of the pulse $C_z=0$, $C_t=1$, and $E_+=E_-=0$, i.e., the electromagnetic amplitude grows linearly with time. For a boundary value problem, on the other hand, where the external magnetic field B_0 occupies a region $z\geqslant 0$ and there is a gravitational wave but no EM waves propagating into the magnetized region, clearly $C_z=1$, $C_t=0$, and $E_+=E_-=0$, i.e., we have a linear spatial growth instead. For the applications to be discussed later on we will be interested in a situation where B_0 is not necessarily static. We thus note that qualitatively the solution given by Eq. (13) still applies for a quasi-static situation, i.e., where the dependence of B_0 on time is slow enough such that the electric fields E associated with the time variations fulfill $E/B_0\ll 1$.

In principle, we can also achieve very large EM-wave amplitudes when we abandon the specific solutions depending on $z-t$. However, since the growth is only linear in t and/or z, apparently we need large times/distances of coherent interaction. For a boundary value problem we can roughly define the effective distance of interaction as $z_{\rm eff}$,

$$\delta B_{\max}\simeq z_{\rm eff}B_{0,{\rm char}}h'_{\rm char}, \tag{14}$$

where the index ‘‘char’’ denotes the characteristic values of the various quantities in the region of interest and the prime denotes differentiation with respect to the argument.

To summarize: Eq. (11b) has a class of physically sound solutions, but also unphysical ones with the property $\delta n\rightarrow\infty$ as $\mathcal{H}\rightarrow 1$. The singular behavior is caused by the insistence to look for solutions that move with a specific velocity, together with the omission of the self-consistent gravitational field from the plasma perturbations. The divergent solutions can be removed either by considering a boundary or an initial value problem, as discussed in this subsection, or by considering the back reaction of the plasma perturbations on the gravitational wave, as discussed in the Appendix. The alternative considered here is the most relevant one with regard to astrophysical applications. Real astrophysical systems have finite distances of interaction between gravitational waves and plasma waves that can be estimated on physical grounds. Thus when estimating the maximum magnetic field perturbation that can be produced by a gravitational wave in a given situation, we can in principle apply solutions (11b) together with (12) but we must note the upper bound for $\delta B_{\max}$ that exists for a given $z_{\rm eff}$ and is given by Eq. (14).

III. PHOTON FREQUENCY SHIFT

We now consider the effect of the gravitational wave perturbations on high frequency photons in a plasma. For simplicity we assume that the photons propagate parallel to the gravitational waves and let them be represented by the vector potential $\boldsymbol{A}=\tilde{\boldsymbol{A}}\exp(\mathrm{i}\,\theta)+\mathrm{c.c.}$, where c.c. stands for complex conjugate. Using the approach of geometrical optics [10], the wave number $k\equiv\partial_z\theta$ and frequency $\omega\equiv-\partial_t\theta$ satisfy some local dispersion relation $\omega=W(z,t,k)$. The amplitude of the vector potential is assumed small and by high frequency photons we mean $\omega\gg\omega_{\mathrm{p}(i)},\omega_{\mathrm{c}(i)}$.

Due to the gravitational waves the plasma has a background of possibly large fields δn, $v_{\parallel}$, δB, and $E_{\perp}$ all being functions of $z-t$ and varying on a time and length scale much longer than that of $\boldsymbol{A}$.

Since $\omega\gg\omega_{\mathrm{c}(i)}$ the high frequency pulse approximately behaves as if the plasma is unmagnetized. The equation of motion linearized in the high frequency (hf) variables reads

$$\left[\frac{\partial}{\partial t}+v_{\parallel}\frac{\partial}{\partial z}\right]\boldsymbol{v}_{(i)}^{\rm hf}=\frac{-q_{(i)}}{\gamma m_{(i)}}\left(\frac{\partial\boldsymbol{A}}{\partial t}+v_{\parallel}\frac{\partial\boldsymbol{A}}{\partial z}\right), \tag{15}$$

and thus $\boldsymbol{v}_{(i)}^{\rm hf} = -q_{(i)}\boldsymbol{A}/\gamma m_{(i)}$, where the large scale variations have been neglected. The induced high frequency current is therefore $\boldsymbol{j}^{\rm hf} = -\omega_{\rm p}^2\boldsymbol{A}/\mu_0$, where the plasma frequency is $\omega_{\rm p} \equiv (\Sigma_i q_{(i)}^2 n/\epsilon_0 m_{(i)}\gamma)^{1/2}$. Taking the time derivative of Ampere's law gives the following wave equation for the photons:

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + \omega_{\rm p}^2\right]\boldsymbol{A} = 0. \tag{16}$$

We recognize the dispersion relation as $\omega = [k^2 + \omega_{\rm p}^2(z-t)]^{1/2}$, where we assume that the variations in the plasma frequency are determined from Eqs. (11) together with $\gamma = (1 - v_\parallel^2)^{-1/2}$.

The change in the wave number and frequency as the wave propagates through the nonuniform and time-varying media with velocity $v_g = \partial\omega/\partial k$ is given by the ray equations

$$\frac{dk}{dt} = -\frac{\partial W}{\partial z}, \quad \frac{d\omega}{dt} = \frac{\partial W}{\partial t}. \tag{17}$$

We note that W is a function of $z-t$ and introduce coordinates $\xi = z - v_g t$, $\tau = t$ locally moving with the photons, i.e., it should be understood that $v_g = v_g(\tau = \tau_0)$ for some τ_0. Then, in a small neighborhood of τ_0 it holds that $d\omega/d\tau = -\partial W/\partial\xi$. Using $\partial_\xi = (1-v_g)^{-1}\partial_\tau$, this can be integrated from time 1 to 2 (which need not be a small interval), noting that $1 - v_g \approx \omega_{\rm p}^2/2\omega^2$. The result is

$$\frac{\omega_1}{\omega_2} = \frac{\omega_{\rm p1}^2}{\omega_{\rm p2}^2}, \tag{18}$$

where the indices 1 and 2 denote the values at τ_1 and τ_2, respectively. An interesting aspect of Eq. (18) is that the frequency conversion factor $N = \omega_1/\omega_2$ is independent of the frequency regime of the EM wave. Thus, in principle, x rays can be turned into gamma rays, just as well as infrared waves can be converted into the visible regime. This is in contrast to laser excited wake fields [9], where efficient frequency shifts can only take place provided the frequency of the converted pulse roughly lies in the same frequency regime as the exciting laser pulse. The reason for the difference is that the density gradients propagate with exactly the speed of light in our case, whereas, naturally, the corresponding velocity is slightly less than c in the laboratory experiments. The necessary distance of acceleration for a given conversion factor N is proportional to ω^2, however, and this puts certain limits for the applicability to the highest frequency regimes.

IV. EXAMPLE

We have found that large density perturbations traveling with the velocity of light can be induced by small gravitational wave perturbations, provided the cyclotron frequency is much larger than the plasma frequency, as described by Eqs. (11). Furthermore, photons propagating in a moving density gradient can undergo frequency up conversion (or down conversion), as described by Eq. (18). In principle the effects can be large, even for a moderate deviation from flat space-time. It is not yet clear that the predicted frequency conversion can be observed during reasonable conditions, however, and our aim in this section is to provide estimates to shed light on this question. In this section we reinstate the speed of light in all expressions.

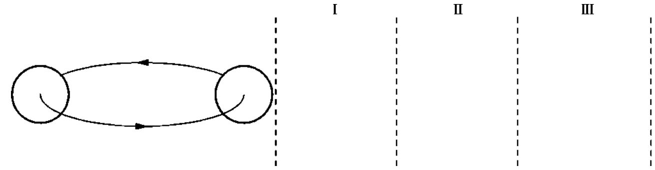

FIG. 1. The neighborhood of the binary system is divided into three regions: region I ($20R_S$–$30R_S$), region II ($30R_S$–$3500R_S$), and region III ($3500R_S$–10^6R_S). In regions I and II we are situated in the near zone of the magnetic field of the pulsar. Thus the plasma density is low, and we assume that the plasma particles do not interfere with the approximations made in the example. For this to be true, $\mathcal{H} \gg 1$ must hold in region I, which is satisfied even for very high densities. Furthermore $\Sigma_i(\omega_{{\rm p}(i)}^2/\omega_{{\rm c}(i)}^2) \gg 1$ should apply in region II, which is fulfilled for $n_0 \leq 10^6$ cm^{-3}. In region III, which is mainly outside the light cylinder of the pulsar, we assume the plasma density n_0 to be of the order of 10^{12} cm^{-3}.

As a source of gravitational radiation we consider a binary system. At least one of the objects should have a moderate to strong magnetic field [in order to make the parameter $\Sigma_i(\omega_{{\rm p}(i)}^2/\omega_{{\rm c}(i)}^2)$ small], and the objects should be compact (as to make the gravitational wave frequency and amplitude before merging large). Thus, for definiteness (and calculational simplicity due to symmetries) we assume that the system consists of two neutron stars of equal mass $M_\odot$ separated by a distance of $40R_S$, where $R_S = 2GM_\odot/c^2 \approx 3$ km. Furthermore, the surface magnetic field of each neutron star is assumed to be 10^6 T. For the unperturbed plasma density profile, see Fig. 1.

The area surrounding the binary system can loosely be divided into three regions (Fig. 1). The interval $20R_S$–$30R_S$ from the center of mass (c.m.) roughly constitutes region I, which is the region where most of the gravitational energy is gained by the EM wave. Using a Newtonian approximation, with $d = \alpha R_S$ and $r = \beta R_S$, it is straightforward to show that $|h| \sim (2\alpha\beta)^{-1}$, where d is the separation distance between the binary objects and r is the observation distance from the center of mass of the system. In order to obtain an estimate of the amplitude of the generated EM wave, we combine the above expression for the gravitational wave amplitude with Eq. (14) and the data given above. The result is

$$\frac{\delta B}{B_0} \sim 7 \times 10^{-5} \tag{19}$$

at the end of region I. In region II (approximate interval $30R_S$–$3500R_S$ from the c.m.) $\delta B/B_0$ is still small, and—as seen by Eq. (12)—the relative density perturbation is thereby small as well, which limits the frequency conversion effect in this region. However, the gravitationally induced EM wave suffers spherical attenuation, whereas the unperturbed magnetic field is that of a dipole, and consequently the relative density perturbation grows quadratically with distance. The end of region II is defined as the necessary distance to make $\delta B/B_0$ of the order unity due to this increase. (For pulsars with periods longer than 35 ms, regions I and II lie in the near zone, and thus the unperturbed magnetic field indeed decays cubically in the region of interest, although the unperturbed field becomes a radiation field outside the light cylinder of the pulsar.) In region III (approximate interval $3500R_S$–$10^6\ R_S$), the relative density perturbation is appreciable, and thus the main frequency conversion occurs here [12].

At the beginning of region III the relative density perturbation is $\delta n/n_0 \sim 1$, in agreement with Eq. (12). An EM wave with initial frequency $\omega \equiv \omega_{\mathrm{min}} = 10^{12}$ rad/s can move from a density minimum to a density maximum during a ''laboratory system distance''

$$L_{\mathrm{freq}} = cT_{\mathrm{freq}} = cL_{\mathrm{grad}}/(c - v_g) \sim \omega_{\mathrm{max}}^2 L_{\mathrm{grad}}/\omega_{\mathrm{p}}^2,$$

where L_{grad} is a typical density gradient scale length. For definiteness we assume that the pulsars have periods of the order of 350 ms, in which case $\delta B/B_0$ may increase to $\delta B/B_0 \sim 10$ for the most of region III. In our example the maximum frequency magnification N thus is

$$N = \frac{\omega_{\mathrm{max}}}{\omega_{\mathrm{min}}} = \frac{\omega_{\mathrm{p,max}}^2}{\omega_{\mathrm{p,min}}^2} \sim 10. \qquad (20)$$

Inserting $\omega_{\mathrm{max}} = 10^{13}$ rad/s, and letting $\omega_{\mathrm{p,max}}^2 = 10^{11}$ rad/s (corresponding to $n_0 \simeq 10^{12}\ \mathrm{cm}^{-3}$) we obtain $L_{\mathrm{freq}} \simeq 10^6 R_{\mathrm{S}}$, i.e., the acceleration can take place within region III. Strictly applying our one-dimensional calculations of Sec. III means that frequency up converted EM waves will be down converted and vice versa, if the gravitational source and the induced density perturbation are indeed periodic. In our example, on the other hand, the successive frequency conversion effects will decrease with the distance from the source, and thus for an earth-based observer the radiation generated in region III should show periodic up and down conversions. The frequency conversion ratio of Eq. (20) is, of course, a maximum value of our example that occurs for radiation generated at a density extremum, but all radiation generated in region III will be up or down converted with a factor in the interval 1–N, and consequently the effect should be observable provided the object is close enough for radiation generated in region III in the approximate frequency interval 10^{11} rad/s$\leqslant \omega \leqslant 10^{14}$ rad/s to be detected, where the upper limit is imposed by the fact that the system has a finite distance of interaction. If we try to increase the interaction efficiency by considering higher plasma densities the electromagnetic wave damping due to Thomson scattering becomes prohibitive [12].

V. SUMMARY AND DISCUSSION

We have considered the generation of traveling density perturbations in a magnetized plasma induced by gravitational radiation. Provided $\Sigma_i(\omega_{\mathrm{p}(i)}^2/\omega_{\mathrm{c}(i)}^2) \ll 1$, significant density perturbations, i.e., $\delta n/n_0 \sim 1$, can be induced even by a small gravitational wave with $h \ll 1$, provided $\mathcal{H} \sim 1$. Basically the large effect is possible because of the approximate agreement of the dispersion relations between the fast magnetosonic and gravitational modes in the regime $\Sigma_i(\omega_{\mathrm{p}(i)}^2/\omega_{\mathrm{c}(i)}^2) \ll 1$, which in turn allows for a long distance of coherent interaction.

In order to find a mechanism where the induced density perturbations may give rise to earth-based observational effects, we have studied frequency conversions of electromagnetic wave packets traveling in the moving density gradients. The formula (18), relating the frequency of the wave packet for two different positions in the moving density profile, is in conceptual agreement with the corresponding results of Ref. [9], which considered an analogous situation but where the density perturbation was due to plasma oscillations traveling with a phase velocity slightly less than the speed of light c. In our case the gradients move with exactly c, however, and thereby the maximum frequency conversion factor N does not decrease with the initial frequency (as for conventional photon acceleration), *in principle* allowing for up conversion even of γ rays.

The idealizations made in Secs. II and III are somewhat too strong for our results to be directly applicable to a situation of astrophysical relevance. In particular, we cannot consider the unperturbed plasma as homogeneous and the geometry as one-dimensional when making estimates. In our example with a binary system as a source of gravitational radiation, we have thus been forced to divide the neighborhood of the system into three regions: region I where most of the energy transfer into electromagnetic wave energy occurs, region II where the relative density perturbation grows, and region III where the frequency conversion takes place. In order to describe the physics in region I adequately we must abandon solutions that depend on $z - ct$ only, and the basis for this has been discussed in Sec. II C. By making estimates based on our analytical calculations, we conclude that the gravitational waves emitted by a system of binary pulsars close to merging may result in periodic frequency up and down conversions of electromagnetic radiation in the infrared part of the spectrum. The frequency of the up and down conversions coincides with the gravitational wave frequency, i.e., it is twice the orbital frequency.

APPENDIX

In this appendix we investigate the regime of validity for the multi-component test fluid approach. Normally we think that by continually decreasing the parameters proportional to the unperturbed energy density, at some point the fluid in an external gravitational field can be treated as a test fluid. In our case the situation is not quite that simple, since we can decrease the electromagnetic ($\propto B_0^2$) and the rest mass energy density ($\propto n_0$) at the same rate keeping $\Sigma_i(\omega_{\mathrm{p}(i)}^2/\omega_{\mathrm{c}(i)}^2)$ constant. Since our solution in Sec. II B has a diverging energy-momentum tensor whenever $\mathcal{H} \equiv 2h/\Sigma_i(\omega_{\mathrm{p}(i)}^2/\omega_{\mathrm{c}(i)}^2) \rightarrow 1$, clearly we cannot justify the test fluid approach simply by assuming a sufficiently low unperturbed energy density. To shed light on the physical effects due to the self-consistent gravitational field, we will first consider the linearized theory. This will provide a guide for making estimates of the regime of validity of our (nonlinear) test matter solution in Sec. II B and also makes it possible to justify the omission of self-consistent gravitational effects in Sec. IV.

We divide all quantities into an unperturbed part (i.e., the value in the absence of the gravitational perturbation) and a perturbed part. We note that the only variables that are nonzero in the unperturbed state are the density ($=n_0$), the magnetic field ($=B_0 e_1$), and the metric ($=\eta_{\mu\nu}$). It should be emphasized that in addition to the direct effect on the dispersion relation from the matter, which we will study below, there is also an indirect contribution (that will be omitted here) to the dispersion relation from the background curvature produced by the (unperturbed) matter. In the regime where the gravitational wave length is much shorter than the background curvature, however, the shortwave approximation can be applied, which implies that these two effects can be studied separately and their contribution to the dispersion relation of the gravitational wave can be added; see, e.g., Ref. [13]. In the above scenario (provided thermal effects are still neglected) the only effects from the gravitational wave on the plasma perturbations are from the effective currents in (4a),(4b), where, in the present case, we have $j_E^2 = -j_B^1 = -(1/2)B_0(\partial h/\partial z)$ and the other components are zero. Thus using Maxwell's equations and the set of fluid equations for each particle species and the same approximations as in Sec. II (but avoiding the ansatz $\partial/\partial t = \partial/\partial z$) we will obtain a wave equation for the fast magnetosonic wave, modified from the standard textbook form by allowing for an arbitrary value of $\Sigma_i(\omega_{\mathrm{p}(i)}^2/\omega_{\mathrm{c}(i)}^2)$ and with a gravitational ''source term'' due to the effective gravitational currents above. The result is

$$\left(\frac{\partial^2}{\partial t^2} + \frac{C_A^2}{1+C_A^2}\frac{\partial^2}{\partial z^2}\right)\delta B = 2\frac{\partial^2 h}{\partial t^2}B_0, \tag{A1}$$

where we have introduced the Alfvén velocity $C_A = (\Sigma_i(\omega_{\mathrm{p}(i)}^2/\omega_{\mathrm{c}(i)}^2))^{1/2}$. (Note that C_A may be larger than unity, but, as can be seen above, the actual magnetosonic wave velocity is smaller or equal to C_A.) The system is closed self-consistently by Einstein's field equations, which, after linearization reduces to (cf. Eq. 4.9 in Ref. [13])

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2}\right)h = 16\pi G[T_{11} - T_{22}]_{\mathrm{lin}} = \frac{16\pi G}{\mu_0}B_0\delta B, \tag{A2}$$

where lin stands for ''linear part of.'' It is simple to combine Eqs. (A1) and (A2) into a single wave equation for the coupled fast magnetosonic and gravitational mode. However, it is probably more illustrative to proceed by considering the corresponding dispersion relation. Making a plane wave ansatz, $\delta B = \widetilde{\delta B}\exp[i(kz-\omega t)]$ and $h = \tilde{h}\exp[i(kz-\omega t)]$, we directly find the dispersion relation:

$$\omega^2 - k^2 = \frac{32\pi G B_0^2}{\mu_0}\left(\frac{\omega^2}{\omega^2 - k^2 C_A^2/(1+C_A^2)}\right) \tag{A3}$$

from Eqs. (A1) and (A2). Thus the presence of matter causes a phase velocity $\omega/k > 1$ and a group velocity $d\omega/dk < 1$. A further consequence is that the gravitational wave also becomes dispersive. Apparently the relation between $\widetilde{\delta B}$ and $\tilde{h}$ is

$$\widetilde{\delta B} = B_0\tilde{h}\left(\frac{\omega^2}{\omega^2 - k^2 C_A^2/(1+C_A^2)}\right) \tag{A4}$$

where the omission of the self-consistent gravitational field is a valid approximation only if we can use the vacuum dispersion relation $\omega^2 - k^2 = 0$ as an approximation instead of Eq. (A3) when calculating $\widetilde{\delta B}$ from (A4). From now on we will focus on the regime $C_A \gg 1$, which makes the magnetosonic phase velocity close to unity. Since the (typically small) right-hand side of (A3) now must be compared to the small phase velocity difference of the (uncoupled) magnetosonic and gravitational waves, the condition for omitting the self-consistent gravitational field is significantly stronger if one should get an approximately correct *magnetic field*, and not just a small contribution from the right-hand side in the dispersion relation (A3). For $C_A \gg 1$ the condition for omitting the self-consistent gravitational field and still obtaining an approximate expression for $\widetilde{\delta B}$, becomes

$$\frac{32\pi G B_0^2}{\mu_0} \ll \frac{\omega^2}{C_A^4}. \tag{A5}$$

The above validity condition is obtained by comparing the magnetic field obtained from the full self-consistent dispersion relation and its vacuum approximation. A much simpler way to arrive at the same condition as in (A5) is to demand that the relative contribution from the energy momentum tensor terms in Einstein's equations should be much smaller than the relative velocity difference between the magnetosonic and gravitational waves. The advantage with this latter formulation of the validity condition is that it can be easily applied also when the relation between δB and h as well as the expression for the energy momentum tensor are nonlinear. Adopting this condition for omitting the self-consistent gravitational field when the plasma response to the metric perturbation is nonlinear we write

$$32\pi G\,\max(\delta T) \ll \frac{\omega_{\mathrm{char}}^2}{C_A^2}h_{\mathrm{char}}, \tag{A6}$$

where max(δT) denotes the maximum deviation from the unperturbed value of the perturbed energy momentum tensor for any of its components, and the index ''char'' denotes the characteristic value of the gravitational wave frequency and metric perturbation, respectively. For the regime when Eq. (A6) is violated, obviously our solution in Sec. II B must be modified to take the self-consistent gravitational field into account and this may result in new types of solutions describing, for example, nonlinear solitary gravitational pulses. This problem is outside the scope of our present article, however. We note that our example in Sec. IV fulfills the validity condition (A6) by a margin of several orders of magnitude.

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[11] Note that in the regime of interest, the linear dispersion relation of the fast magnetosonic wave is $\omega^2 = k^2c^2/[1 + \Sigma_i(\omega_{p(i)}^2/\omega_{c(i)}^2)]$.
[12] For the assumed electron number density $n_0 = 10^{12}/\mathrm{cm}^3$, the mean free path due to Thomson scattering is roughly $5\times 10^6 R_S$. Thus, the outer boundary imposed by Thomson scattering is larger than the size of region III.
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