

Super-Hubble nonlinear perturbations during inflation

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We show that the nonlinear evolution of long-wavelength perturbations may be important in a wide class of inflationary scenarios. We develop a solution for the evolution of such nonlinear perturbations which is exact to first order in a gradient expansion. As a first application, we demonstrate that in single field models of inflation there can be no parametric amplification of super-Hubble modes during reheating. We consider the implications of the solution for recent discussions of the back-reaction effect of long wavelength perturbations on the background geometry, give a new derivation of the equation of motion of stochastic inflation, and demonstrate that if the (generalized) slow-rolling condition is not satisfied, then inevitably long wavelength vector modes for gravitational fluctuations will be generated.

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I. INTRODUCTION

The hypothesis that causally unrelated regions of space-time evolve independently is one of the cornerstones in the development of relativity and one of the basic assumptions of all relativistic theories. As an implication of this hypothesis in cosmology we expect that perturbations with long wavelengths¹ evolve independently at different points of space, as if they were independent homogeneous patches of the Universe (see e.g. [1] and references therein). This expectation has been explored in the linear theory of cosmological perturbations. In this context, it is possible to derive integral constraints (see e.g. [2]) which imply that in the case of adiabatic fluctuations there can be no generation of fluctuation modes on super-Hubble scales. However, it is possible for entropy fluctuations to be generated on such scales, one well known example being the formation of cosmic defects during a phase transition which leads to super-Hubble structures (see e.g., [3]). Another example is the generation of axion fluctuations (see e.g. [4]). Recently, the super-Hubble range effects of entropy fluctuations have been widely studied in the context of inflationary reheating (see e.g. [5–7]).

The aim of this work is to investigate the process of structure formation in the non-linear regime, for inflationary models based on a single scalar field. In Sec. II, we show that the presence of nonlinearities is inevitable in a wide class of inflationary models. In Sec. III we find a solution (which is exact to leading order in the gradient expansion) for nonlinear modes in the case when (but not only when) perturbations satisfy a *generalized slow-rolling condition*. This condition is naturally satisfied during inflation. For our solution

we can define a generalized Bardeen parameter which is constant on scales larger than the Hubble radius. As a first application (Sec. IV), we demonstrate that there is no amplification of long wavelength gravitational fluctuations during reheating. In Sec. V we discuss some implications of our solution for the back reaction of perturbations on the background geometry. In Sec. VI, the Langevin equation for stochastic inflation [8] is re-derived in a way which includes the effects of gravitational (and not just matter) fluctuations, using the results of Sec. III which can be self-consistently applied in an inflating Universe with nonlinear super-Hubble modes. Finally in Sec. VII we demonstrate that in the case in which the generalized slow-rolling condition is violated, the generation of vector modes is inevitable.

II. GENERATION OF FLUCTUATIONS

Perturbations are generated in the de Sitter phase of an inflationary Universe as a consequence of quantum vacuum fluctuations (see e.g. [9] for a comprehensive review). The analysis is based on the consistent quantization of the linearized metric and matter perturbations in a classical expanding background space-time. In this framework, it can be shown that the Sasaki-Mukhanov parameter v [10,11], defined as²

$$v = a \left[\delta\varphi + \left(\frac{\dot{\phi}}{H} \right) \phi \right] = \frac{a\dot{\phi}}{H} \zeta, \quad (1)$$

in the linear regime obeys the equation of motion of a free scalar field. Here, ζ is the Bardeen parameter [12]

$$\zeta = \phi - \frac{H}{\dot{H}} (H\phi + \dot{\phi}), \quad (2)$$

¹Throughout this paper, we use the terms super-Hubble, infrared and long wavelength interchangeably. All refer to modes with wavelengths larger than the Hubble radius.

²The cosmological scale factor is denoted by $a(t)$, the Hubble expansion rate by H , and the scalar matter field by ϕ .

where ϕ is the variable which describes scalar metric perturbations in longitudinal gauge in a spatially flat universe:

$$ds^2 = (1 + 2\phi)dt^2 - a^2(t)(1 - 2\phi)\delta_{ij}dx^i dx^j. \quad (3)$$

The scalar sector is the dominant one for cosmological perturbations generated through slow-roll inflation (see e.g. [13] and references therein).

The action for linearized metric and matter perturbations about a classical expanding background becomes [9] the action of a free scalar field v with time-dependent mass (the time dependence is determined by the background geometry and is negligible on scales smaller than the Hubble radius). Hence, the perturbations can be canonically quantized, and expectation values of v (and hence also of ζ) can be easily calculated (see e.g. [9]) once the state of the system is specified. The state can be specified mode by mode in Fourier space. Our choice is the following: We fix the initial conditions for each Fourier mode when its wavelength is equal to the Planck length,³ and assume that we have vacuum expectation values at that time. This assumption is very similar to the usual choice of a vacuum state at the onset of inflation, since—at least for the usual dispersion relations [14,15]—the wave function is almost constant on sub-Hubble scales. The advantage of our choice compared to the usual one is that we do not extrapolate the physics above the Planck energy. However, for the usual dispersion relations for a free field, the difference is immaterial. Using the equation of motion for ζ in the inflationary era, we obtain the following expression for $\langle \zeta^2 \rangle$:

$$\langle \zeta^2 \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{H_p^4}{2k^3 \dot{\phi}_p^2} \left(1 + \frac{k^2}{a^2 H^2} \right). \quad (4)$$

Note that ζ contains the full information about the linearized metric (and matter) fluctuations (valid as long as the slow-roll approximation holds).

Now we turn our attention to the infrared part of the spectrum ($k/aH \ll 1$) for which

$$\phi \simeq -\frac{\dot{H}}{H^2} \zeta. \quad (5)$$

Thus

$$\langle \phi^2 \rangle_{IR} = \frac{\dot{H}^2}{H^4} \int_{IR} \frac{d^3k}{(2\pi)^3} \frac{H_p^4}{2k^3 \dot{\phi}_p^2}, \quad (6)$$

where the subscript *IR* indicates that we only include infrared modes.

Integrating the equation of motion for the background field and the Friedmann equations in the slow rolling regime gives

$$\int_{\phi_p}^{\phi} \frac{V d\phi}{V'} = m_p^2 \log\left(\frac{k}{am_p}\right), \quad (7)$$

where $m_p = (8\pi G)^{-1/2}$ is the Planck mass and $V(\phi)$ is the scalar field potential. Assuming a power law potential

$$V(\phi) = M^4 \left(\frac{\phi}{m_p}\right)^\alpha, \quad (8)$$

we get

$$\phi_p^2 = \phi^2 - 2m_p^2 \alpha \log\left(\frac{k}{am_p}\right). \quad (9)$$

Substituting in Eq. (6) and carrying out the integral gives

$$\langle \phi^2 \rangle_{IR} = \frac{\alpha}{48(4+\alpha)\pi^2} \left(\frac{M}{m_p}\right)^4 \left(\frac{\phi}{m_p}\right)^\alpha \left(\frac{\phi_i}{\phi}\right)^{4+\alpha}, \quad (10)$$

for $|\phi| \ll |\phi_i|$, where ϕ_i is the background field value when inflation starts.

We see that if inflation lasts long enough, i.e. if ϕ_i is large enough, then $\langle \phi^2 \rangle_{IR}$ can become larger than one and thus infrared perturbations can go nonlinear. As a matter of fact, although the amplitude of each mode may be small, due to the large phase space of infrared modes, the actual value of ϕ in real space may be greater than 1, and so we may not be allowed to use perturbation theory to expand the Einstein equations. The possible importance of nonlinear effects for the evolution of space-time was pointed out in [16,17], though with a different interpretation. We will return to this issue in Sec. V.

Let us now study under which conditions $\langle \phi^2 \rangle_{IR}$ is indeed nonlinear. As a first step, note that the value of M is constrained by the observational data. The amplitude A of a linear cosmological perturbation mode (measured in terms of ζ) when it enters the Hubble radius after inflation is the same as when it exits the Hubble radius during inflation.⁴ Consider, for example, the largest wavelength observable mode, the mode which is entering the Hubble radius today and is given by

$$k = a_0 H_0 \simeq \left(\frac{M}{T_{CMB}}\right) \left(\frac{H_0}{m_p}\right) (a_e m_p), \quad (11)$$

where the subscript 0 refers to the value of quantities today, T_{CMB} denotes the present temperature of the microwave background, and a_e is the value of the scale factor when inflation ends. Then, combining Eqs. (4) and (9) gives us the amplitude of the mode:

$$A = \frac{H_p^4}{\dot{\phi}_p^2} \simeq \left(\frac{M}{m_p}\right)^4 \left[\log\left(\frac{T_{CMB} m_p}{M H_0}\right) \right]^{1+\alpha/2}. \quad (12)$$

³The values of the variables at this time are indicated by a subscript p , and they are functions of k .

⁴Although this statement is conventionally taken to be true, there are exceptions as has recently been shown for a certain class of inflationary models (see Sec. IV and references quoted there).

Thus we end up with

$$\frac{M}{m_p} \simeq A^{1/4} \left[67 - \frac{1}{4} \log A \right]^{-(\alpha+2)/8}. \quad (13)$$

The value of A is constrained by various observations, in particular by the amplitude of the cosmic microwave anisotropies on large angular scales [18], which give $A \sim 10^{-10}$.

A theoretical requirement for a successful inflationary scenario is that all the observable modes must have exited the Planck length during inflation. The consequence is that φ_i is larger than φ_p for all the observable modes. Using Eqs. (9), (12) and (13), this requirement becomes

$$\varphi_i > m_p \left[\log \left(\frac{T_{CMB} m_p}{M H_0} \right) \right]^{1/2} \simeq m_p \left[67 - \frac{1}{4} \log A \right]^{1/2} \quad (14)$$

(for this order of magnitude estimate we can set $\alpha=1$). Inserting this constraint in Eq. (10) gives

$$\langle \phi^2 \rangle_{IR} > A \left[67 - \frac{1}{4} \log A \right]^{3+\alpha/2} \sim 10^{-4.41+0.93\alpha}, \quad (15)$$

at the end of inflation, for $A \sim 10^{-10}$. We see that even if inflation starts at the latest possible time which is allowed by theoretical or observational considerations (it is usually taken to start much earlier), $\langle \phi^2 \rangle_{IR}$ almost reaches the nonlinear regime for a massless theory (i.e. for $\alpha=4$) just before reheating starts. If inflation starts at values of ϕ for which the energy density is comparable to the Planck density, then the phase space of infrared modes is much larger and the nonlinear regime will be reached much earlier.

III. GRADIENT EXPANSION

We use the following ansatz for our metric:

$$ds^2 = e^{2\phi} dt^2 - e^{-2\psi} \delta_{ij} dx^i dx^j, \quad (16)$$

which is a generalization of a metric with linear scalar perturbations. This ansatz, though not a general solution, represents the most important sector of the metric since any such metric can be continuously connected to a metric with linear scalar perturbations which is the dominant sector in a Universe with linear perturbations generated during slow-roll inflation [13]. We also consider the simplest model of matter which has one scalar field (the inflaton).

The Lagrangian and energy-momentum tensor are given by

$$\mathcal{L} = \frac{1}{2} \partial^\alpha \varphi \partial_\alpha \varphi - V(\varphi), \quad (17)$$

$$T^\mu_\nu = \partial^\mu \varphi \partial_\nu \varphi - \mathcal{L} \delta^\mu_\nu. \quad (18)$$

In writing the Einstein equations we keep the leading terms in their gradient expansions, assuming that the wavelengths are large. Naturally, in the G_{00} and G_{ij} equations, all

the gradient terms drop out⁵ and we are left with the Friedmann equations for the above metric:

$$\psi'^2 = \frac{8\pi G}{3} \left(\frac{\varphi'^2}{2} + V(\varphi) \right), \quad (19)$$

$$-2\psi'' + 3\psi'^2 = 8\pi G \left(-\frac{\varphi'^2}{2} + V(\varphi) \right), \quad (20)$$

and of course the more interesting G_{0i} equation:

$$(\psi')_{,i} = 4\pi G \varphi' \varphi_{,i} \quad (21)$$

where $'$ denotes $e^{-\phi}(\partial/\partial t)$.

We immediately see that not every φ can satisfy the last equation since the right hand side must be a perfect gradient. The condition that φ must satisfy is

$$(\varphi')_{,i} \varphi_{,j} = (\varphi')_{,j} \varphi_{,i}, \quad (22)$$

or, equivalently,⁶

$$\nabla \varphi' \propto \nabla \varphi. \quad (23)$$

From Eq. (23), we can see that surfaces of constant φ' and constant φ must be tangential to each other and consequently are the same. This means that φ' must be a function of φ , at least in the finite regions of space in which the gradient of φ does not vanish:

$$\varphi' = \frac{\partial}{\partial \varphi} g(\varphi, t), \quad (24)$$

where $g(\varphi, t)$ can be any function of φ and t . Then the G_{0i} equation reduces to⁷

$$\psi' = 4\pi G g(\varphi, t). \quad (25)$$

Since there is no explicit time dependence in this formalism, we assume that g has also no explicit time dependence:

$$g(\varphi, t) = g(\varphi). \quad (26)$$

From now on, we call Eq. (24) (with no explicit time dependence) the generalized slow-rolling condition due to its resemblance to the field equation during the slow-rolling phase. Note that at this point, we have not shown that the absence of time dependence is required in Eq. (26), but simply that this ansatz gives a consistent solution. However, below we show that in the case of single field inflationary

⁵It is possible to show that the G_{ij} equations for $i \neq j$ can be satisfied only if we include the vector and tensor sectors of the metric perturbations. However their magnitudes go to zero in the long wavelength limit and hence, in this limit, they do not affect the equations for scalar perturbations.

⁶Equation (22) implies that the cross product of two gradient vectors vanishes, so the vectors must be in the same direction.

⁷The integration constant can be absorbed in $g(\varphi, t)$.

models which initially satisfy the slow rolling condition, Eq. (26) is indeed satisfied.

As one can easily see, in the linear regime Eqs. (25) and (26) lead to the vanishing of the non-adiabatic pressure defined as

$$\delta p_{nad} \equiv \dot{p} \Gamma \equiv \dot{p} \left(\frac{\delta p}{\dot{p}} - \frac{\delta \rho}{\dot{\rho}} \right). \quad (27)$$

This is the condition derived in [6,7] for the constancy of ζ for the perturbations.

Combining Eqs. (24) and (26) gives

$$\psi'' = 4\pi G \varphi'^2, \quad (28)$$

which can also be obtained by combining the Friedmann equations (19) and (20).

After neglecting the gradient terms and using Eqs. (24) and (26), both the G_{00} and G_{ij} equations reduce to

$$4\pi G g'^2 - 3(4\pi G g)^2 = -8\pi G V(\varphi). \quad (29)$$

Thus we see that, in the long wavelength limit, the G_{0i} equation is consistent with the Friedmann equations given that the field φ satisfies the generalized slow-rolling condition (24).

As a matter of fact, this is what happens during slow-roll inflation, when we have

$$\varphi' = -\frac{1}{3H} \frac{\partial V(\varphi)}{\partial \varphi} = \frac{1}{3\psi'} \frac{\partial V(\varphi)}{\partial \varphi}, \quad (30)$$

and hence Eq. (24) is satisfied.

Because of this property, even after the end of inflation, as long as the perturbation is in the super-Hubble regime, φ' remains a function of φ . The reason is that, using the field equation, we can track the evolution of φ and φ' into the reheating era, as functions of the initial φ during inflation. By eliminating the initial φ , it is possible to find φ' as a function of φ at any time. However it will not be a single-valued function when φ is oscillating. Figure 1 shows the behavior of $g(\varphi)$ (solid curve) during the reheating era, obtained by solving Eq. (29). The dashed curve is $\sqrt{V(\varphi)/(6\pi G)}$, the asymptotic limit of $g(\varphi)$ for $\varphi \gg m_p$.

Dividing Eq. (24) by Eq. (25) and integrating the result with respect to φ , we find that

$$\tilde{\zeta}(x) \equiv \psi - 4\pi G \int \left(\frac{\partial \log(g)}{\partial \varphi} \right)^{-1} d\varphi \quad (31)$$

is an integration constant and as such does not depend on time.

It can easily be verified that for linear perturbations $\delta \tilde{\zeta}(x)$ reduces to the usual Bardeen parameter ζ , which is known to remain constant for adiabatic super-Hubble modes⁸

⁸In linear perturbation theory, $\delta \phi$ is forced to be equal to $\delta \psi$ by the off-diagonal G_{ij} equations for all theories in which the off-diagonal components of T_{ij} vanish to linear order, which is in particular the case for scalar field matter.

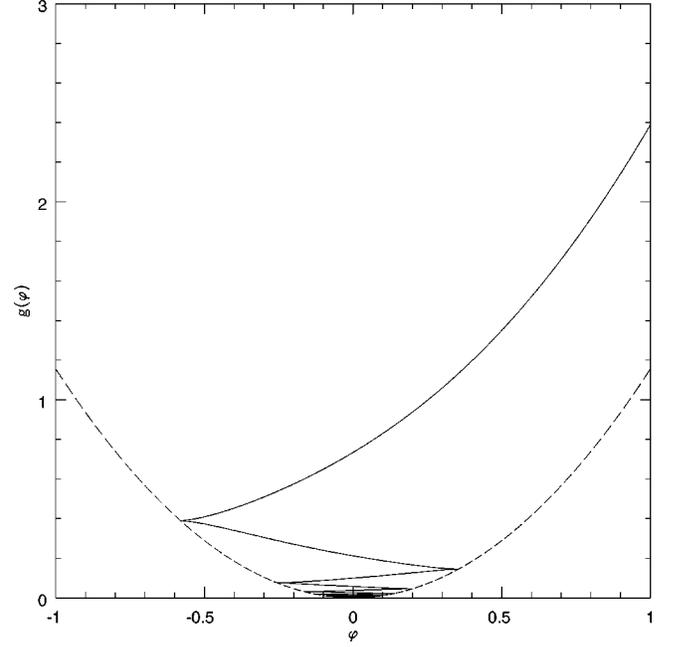


FIG. 1. The solid curve is a solution for $g(\varphi)$ during the reheating era and the dashed curve is $\sqrt{V(\varphi)/(6\pi G)}$. The horizontal axis is in units of Planck mass while the vertical axis has arbitrary units.

$$\begin{aligned} \delta \tilde{\zeta}(x) &= \delta \psi - 4\pi G \frac{g \delta \varphi}{g'} = \delta \psi - \frac{g \delta(e^{-\phi} \dot{\psi})}{g'^2} \\ &= \delta \psi - \frac{H}{\dot{H}} (H \delta \phi + \delta \dot{\psi}) = \zeta(x). \end{aligned} \quad (32)$$

Thus, we have been able to generalize the Bardeen parameter to the case of nonlinear long-wavelength fluctuations.

Let us now consider the case of multiple scalar fields. In this case, Eq. (21) changes to

$$(\psi')_{,i} = 4\pi G \varphi'_a \varphi_{a,i}, \quad (33)$$

where the subscript a denotes the field index and is summed over, if repeated. The consistency condition (22) takes the form

$$(\varphi'_a)_{,i} \varphi_{a,j} = (\varphi'_a)_{,j} \varphi_{a,i}. \quad (34)$$

We see that if the fields satisfy the generalized slow rolling condition

$$\varphi'_a = \frac{\partial}{\partial \varphi_a} g(\varphi, t), \quad (35)$$

then Eq. (34) is satisfied and we again arrive at Eq. (25). However, for the multiple field problem this result is not general. Note that Eq. (35) is only a sufficient condition.

During inflation, if the effective mass of a field component is larger than the Hubble constant, it is damped in a few e-foldings. Otherwise, it slow-rolls down the potential. So a few e-foldings after the onset of inflation, all the remaining field components undergo slow-rolling and satisfy Eq. (35).

We conclude that during multiple-field inflation, we can still use the above results. However, when the inflation ends, Eq. (35) breaks down and, unlike the single field case, we end up with a mixture of scalar and vector metric perturbations, as described in Sec. VII.

We are also able to define the generalized Bardeen parameter (31) for each slow-rolling path in field space, since the evolution is effectively of single field nature along each path. However, the parameter does not remain constant after inflation ends since Eq. (35) breaks down.

In conclusion, we have shown that causally disconnected regions of the Universe evolve independently in the sense that locally observable parameters satisfy the conservation law (31), which is a generalization of the conservation of the Bardeen parameter in linear perturbation theory. As mentioned, our result is only true in the case of (in general) non-linear scalar perturbations satisfying the slow-roll condition.

In the next two sections we consider some of the applications of this result.

IV. PARAMETRIC RESONANCE OF SUPER-HUBBLE MODES DURING REHEATING

It was recently suggested [19] that parametric resonance during the reheating phase of an inflationary Universe [20] may lead to an exponential amplification of super-Hubble scale gravitational perturbations. If true, this would affect the usual predictions of inflationary model for observables such as the matter power spectrum and the spectrum of cosmic microwave anisotropies.

In Ref. [21] it was shown that, although there is no causality constraint which prohibits the amplification of super-Hubble (but sub-horizon) modes during reheating, the effect does not occur in a simple massive scalar field model of chaotic inflation (i.e. with $\alpha=2$). This result was shown to be true even beyond the linear analysis, using numerical methods [22]. Recently, a general no-go theorem for resonance of long wavelength scalar gravitational fluctuations in the context of a single scalar matter field theory was suggested in [23]. In this last reference, the effect was also investigated numerically for a matter theory with both quadratic and quartic terms (see also [24]).

The problem with the analysis of [23] or with every other analytical approach which is based on considering the evolution of ζ (in the linear regime) through the turning point of φ is that ζ is ill-defined when $\dot{\varphi}$ vanishes. However, $\tilde{\zeta}$ is continuous at this point since the integrand of Eq. (30) diverges as $(\varphi - \varphi_0)^{-1/2}$,⁹ so the integral is continuous. Therefore, $\tilde{\zeta}$ is well defined through the turning. We now argue that $\tilde{\zeta}$ is in fact constant throughout. For the argument we refer back to Fig. 1 which shows a sketch of the time evolution of $g(\varphi)$ during the period of oscillation of φ . During each time interval between the turning points, $\tilde{\zeta}$ and hence ζ

are constant. At the turning points, $\tilde{\zeta}$ is continuous, and hence it is not possible that at these points $\tilde{\zeta}$ undergoes a jump. The fact that $\tilde{\zeta}$ is constant implies the absence of parametric amplification of super-Hubble modes.

In contrast to the single field case, as pointed out at the end of Sec. III, this formalism cannot be applied to the multiple-field case during reheating, and thus we may expect non-trivial behavior of the fields during this era. In fact, such behavior has been observed for a specific two-field potential [5,6,25,26], where parametric amplification of isocurvature modes can lead to non-linear amplitudes of infrared modes. We will return to this effect in Sec. VII.

V. BACK REACTION OF ADIABATIC INFRARED MODES

It has been claimed [16,17] that the growth of $\langle \phi^2 \rangle$ during slow-roll inflation can lead to significant corrections to the background Friedmann equations at second order in perturbation theory (see also [27,28] for similar discussions based on the back reaction of infrared gravitational waves). We claim that, as a result of the analysis of Sec. III, this effect, though formally valid, cannot be identified by local observers as an effect of inhomogeneities. The reason is that, as we showed, for perturbations generated during slow-roll inflation, the local Friedmann equations are always satisfied. Note that our analysis includes the effects of the leading back-reaction terms in the above-mentioned references. Obviously, back-reaction terms which are of higher order in gradients are not included in our analysis.

Due to the fact that inflation is followed by a radiation dominated era, modes which enter the Hubble radius after inflation are damped by the Hubble expansion. Consequently, the phase space of constant amplitude perturbations is shrinking after inflation. This effect can smooth out the density field so that it becomes of linear order again at some point in the radiation dominated era. This will happen in models in which inflation does not last much longer than the minimal number of e-foldings required to solve the problems of standard big bang cosmology. However, in many models of chaotic inflation, inflation lasts long enough, so that the remaining phase space at the time of equal matter and radiation is large enough to give nonlinearity. However, even in this case there has not been enough time for the nonlinearities to enter the observable region (sub-Hubble scales).

This does not mean, however, that the presence of nonlinearities has no effect at all. As we will see in the next section, the process of generation of perturbations during inflation is influenced by nonlinear effects and thus the spectrum is distorted compared to what would be predicted in linear theory.

VI. EQUATION OF MOTION FOR STOCHASTIC INFLATION

In this section we will use coarse graining on the scale of the Hubble radius to provide a new derivation of the equation of motion for stochastic inflation [8] which takes into account the gravitational fluctuations.

⁹This can be obtained by considering Eq. (21) close to the turning point.

Let us define the coarse-grained Bardeen parameter, ζ_c , in Fourier space as follows:

$$\zeta_c(k) = \tilde{\zeta}(k) W\left(\frac{k}{aH}\right) \quad (36)$$

where the window function $W(k)$ can be chosen to be¹⁰

$$W\left(\frac{k}{aH}\right) = \left(1 + \frac{k^2}{a^2 H^2}\right)^{-2} \quad (37)$$

which is smooth on the Hubble scale. In a patch larger than the Hubble radius but small enough so the non-linearity in ζ is negligible, we can use ζ (usual Bardeen parameter) instead of $\tilde{\delta\zeta}$. Making use of the fact that background $\tilde{\zeta}$ is constant in this patch we have

$$\tilde{\zeta} = \delta\tilde{\zeta} = \dot{\zeta}. \quad (38)$$

We can use our knowledge of the linear quantum generation of perturbations (Sec. II) to find the statistical properties of $\tilde{\zeta}'$, with the result

$$\langle \tilde{\zeta}'(\mathbf{x}_1, t_1) \tilde{\zeta}'(\mathbf{x}_2, t_2) \rangle = \frac{5H^6}{48\pi^2 \varphi'^2} F(Ha|\mathbf{x}_1 - \mathbf{x}_2|, H|t_1 - t_2|), \quad (39)$$

where $F(0,0) = 1$ and $F(\beta, \tau) \rightarrow 0$ as β or τ goes to infinity [more precisely, $F(\beta, \tau) \ll 1$ if $\beta \gg 1$ or $\tau \gg 1$]. The explicit form of F is complicated and we do not write it down here.

Taking the time derivative of the definition of $\tilde{\zeta}$ [see Eq. (31)], and using the fact that $\psi' = -H \simeq -\sqrt{(8\pi G/3)V(\varphi)}$ during inflation, we get

$$\tilde{\zeta}' = -H - 4\pi G \left(\frac{\partial \log(g)}{\partial \varphi}\right)^{-1} \varphi'. \quad (40)$$

We can neglect g' in Eq. (29) to find g as a function of $V(\varphi)$ in the slow-roll approximation. Plugging this into the above formula, we obtain the coarse-grained equation of motion:

$$\varphi' = -m_p^2 H \frac{\partial \log(V)}{\partial \varphi} - \sqrt{\frac{5}{48}} \frac{H^2}{\pi} \xi(x, t). \quad (41)$$

Here, all quantities are coarse-grained, and the variable ξ is a random Gaussian field whose two-point correlation function is given by

$$\langle \xi(\mathbf{x}_1, t_1) \xi(\mathbf{x}_2, t_2) \rangle = F(Ha|\mathbf{x}_1 - \mathbf{x}_2|, H|t_1 - t_2|). \quad (42)$$

¹⁰This is only one example of a cutoff function. In general, one can use any function which is close to one for $k < aH$ and tends to zero as k goes to infinity fast enough to eliminate the ultraviolet divergence.

Note that since ζ is a metric perturbation variable, the above equations correspond to metric back reaction yielding stochastic dynamics.

Let us compare this calculation with the original one in [8]. There, instead of the window function (37), a step function was used. More importantly, gravitational perturbations were neglected. As a result, the numerical factors are different. In addition, in the original analysis the random variable had a white noise spectrum whereas our noise has a finite correlation time. In spite of these differences, we expect that the qualitative behavior of the solutions will stay the same.

One major difference between our analysis and the usual analysis of stochastic inflation (see e.g. [29]) is that if inflation stops at some point, and consequently the generalized slow rolling condition breaks down (see the end of Sec. III), then, as elaborated in Sec. VII, the evolution takes a nonlocal form. In particular, the end of inflation in some region of space can affect the inflating regions in an acausal way.

Limiting the analysis to the linear regime leads to the standard results of the stochastic inflationary scenario. Namely, since in the large picture different points in space undergo independent random evolution, we end up with a scale invariant spectrum for ζ (or $\tilde{\zeta}$) with a logarithmic correction in Fourier space due to the finite correlation length of the random field ξ .

However, as argued in Sec. II, there is a large class of scenarios which allow infrared nonlinear perturbations. As a matter of fact, the main result of Sec. II is that the cumulative effect of the second term in Eq. (41) may become important even when its magnitude is negligible. Hence, even when the self-reproduction of the stochastic scenario does not take place, infrared nonlinearities may be important [16].

This nonlinearity affects the generation of perturbations by changing the background parameters which appear in the amplitudes as well as the Gaussianity of the inhomogeneities. We postpone the calculation of this correction to a future work.

VII. VECTOR MODES

In this section we investigate the case in which the generalized slow-rolling condition (24) or (35) is not satisfied. For the general metric,

$$ds^2 = N^2 dt^2 - \gamma_{ij} dx^i dx^j, \quad (43)$$

the $0i$ Einstein tensor elements are given by

$$G_i^0 = N^{-1} (K_{,i} - K_{i,j}^j), \quad (44)$$

where

$$K_j^i = -\frac{1}{2} N^{-1} \dot{\gamma}^{ik} \dot{\gamma}_{kj}, \quad (45)$$

$$K = K_i^i. \quad (46)$$

For the metric used in Sec. III,

$$N = e^\phi, \quad \gamma_{ij} = e^{-2\psi} \delta_{ij} \quad (47)$$

and thus

$$K_{ij} = \psi' \gamma_{ij}. \quad (48)$$

The G_i^0 equation is not satisfied in general. It is well known that beyond linear order in perturbation theory, tensor and scalar perturbations mix. However, in our case we cannot satisfy the equation by adding tensor perturbations. We thus include vector metric perturbations in order to satisfy the G_i^0 equation.

The presence of vector modes would modify K as follows:

$$K_{ij} = \psi' \gamma_{ij} + A_{i;j} + A_{j;i}, \quad (49)$$

where A_i is some vector field made up from the metric variables representing vector perturbations. Substituting this into Eq. (44) leads to the following G_i^0 equation:

$$2\nabla \psi' - 2\mathbf{R} \cdot \mathbf{A} + \nabla \times (\nabla \times \mathbf{A}) = 8\pi G \varphi' \nabla \varphi \quad (50)$$

where \mathbf{R} is the Ricci tensor. If \mathbf{R} vanishes, it is easy to find \mathbf{A} and ψ' in Fourier space:

$$\mathbf{A}_{\mathbf{k}} = -\frac{8\pi G}{k^4} \mathbf{k} \times (\mathbf{k} \times \mathbf{C}_{\mathbf{k}}), \quad (51)$$

$$\psi'_k = \frac{-4i\pi G}{k^2} (\mathbf{k} \cdot \mathbf{C}_{\mathbf{k}}), \quad (52)$$

where

$$\mathbf{C}_{\mathbf{k}} \equiv 8\pi G \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} \varphi' \nabla \varphi \quad (53)$$

is the Fourier transform of the right hand side of Eq. (50).

One can take the \mathbf{R} term to the right hand side of Eq. (50) and write a perturbative expansion in \mathbf{R} for \mathbf{A} and ψ' . We conclude that for general nonlinear scalar field perturbations the metric which satisfies the Einstein equations, in general, must include nonlinear vector perturbations. One known example of this phenomenon is the formation of vector pertur-

bations in the case of topological defect formation during a phase transition, where super-Hubble structures form in a short time [3]. Another example considered in [5,6,25] (see the end of Sec. IV) occurs in inflationary reheating, since during multiple field reheating the generalized slow-rolling condition (35) breaks down. Note that in the back-reaction calculation of [25] and in models in which the infrared modes lead to nonlinearities, it is not enough to include only scalar perturbations since vector perturbations are generated as well.

VIII. CONCLUSION

We have demonstrated that in many models of inflation the large phase space of infrared modes leads to nonlinearities, even when the amplitude of each Fourier mode is small. We were able to find a solution of the Einstein field equations for nonlinear fluctuations which is exact to leading order in the gradient expansion, and thus will be accurate to describe the infrared modes. For this solution, we were able to define a generalized Bardeen parameter which is conserved in time.

As a first application of this formalism, we were able to show that in models with a single matter field, there can be no parametric amplification of super-Hubble cosmological fluctuations during inflationary reheating. Applied to the problem of back reaction of infrared modes, our solution implies that the back-reaction effect is not locally identifiable as an effect due to inhomogeneities since the local Friedmann equations are satisfied. We were able to use our formalism to give a re-derivation of the equation of motion for stochastic inflation which takes into account the effects of gravitational fluctuations. Finally, we have shown that nonlinearities inevitably lead to the generation of vector perturbations if a *generalized slow-rolling condition* is not satisfied.

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