

Fast convolution on the sphere

Benjamin D. Wandelt

*Department of Physics, Princeton University, Princeton, New Jersey 08544
and Theoretical Astrophysics Center, Copenhagen, Denmark*

Krzysztof M. Górski

*European Southern Observatory, Garching bei München, Germany
and Warsaw University Observatory, Warsaw, Poland*

(Received 14 October 2000; published 15 May 2001)

We propose fast, exact and efficient algorithms for the convolution of two arbitrary functions on the sphere which speed up computations by a factor $\mathcal{O}(\sqrt{N})$ compared to present methods where N is the number of pixels. No simplifying assumptions are made other than band limitation. This reduces typical computation times for convolving the full sky with the asymmetric beam pattern of a megapixel cosmic microwave background (CMB) mission from months to minutes. Our methods enable realistic simulation and careful analysis of data from such missions, taking into account the effects of asymmetric “point spread functions” and far side lobes of the physical beam. While motivated by CMB studies, our methods are general and hence applicable to the convolution or filtering of any scalar field on the sphere with an arbitrary, asymmetric kernel. We show in an Appendix that the same ideas can be applied to the inverse problems of map-making and beam reconstruction by similarly accelerating the *transpose convolution* which is needed for the iterative solution of the normal equations.

DOI: 10.1103/PhysRevD.63.123002

PACS number(s): 95.75.Pq, 98.70.Vc, 98.80.-k

I. INTRODUCTION

A major near-term objective in the field of cosmology today is to gain a detailed measurement and statistical understanding of the anisotropies of the cosmic microwave background (CMB). While the theory of primary CMB anisotropy is well developed (see [1] for a review) and we are facing a veritable flood of data from a new generation of instruments and missions, perhaps the single most limiting factor for interpreting these data is the exorbitant computational cost involved in realistic mission simulation and careful analysis of the data products [2,3].

Important and computationally expensive tasks for both simulation and analysis of microwave data are to simulate and to correct for the systematic errors due to imperfections of realistic microwave telescopes, such as beam asymmetries and far side lobes. The effect of an asymmetric “point spread function” is to distort the shapes of the detected anisotropies. What makes far side lobes an important issue is the fact that the CMB anisotropy signal has an amplitude of one in 10^5 relative to the 2.7 K background. In regions of low galactic latitude, foregrounds from galactic synchrotron radiation and dust emission are expected to exceed this signal by many orders of magnitude over a wide range of frequencies [4,5]. Even though CMB experiments will obviously not target these regions to obtain measurements of the background anisotropy, the large amplitudes of these galactic sources may induce systematic errors even when “looking” in directions far away from the galactic plane if the instrument allows diffraction of stray light into the detectors. Solar system bodies, including the Earth, are other possible sources of stray light.

To assess these problems and formulate solutions we must be able to compute the detector response at every pointing of

the telescope. The inputs are a physical model of the “beam” over 4π steradian and a model of the “sky” containing both simulated signal as well as foreground sources possibly including ground emission. Note that in the general case not just the direction of the pointing is important but also the orientation of the beam about the pointing axis. The detector response is then the solution to a quadrature problem at each orientation.

Analysis methods of CMB data have neglected this difficulty by assuming azimuthal symmetry of the beam which greatly simplifies the calculation [6–8]. Simulation work which did include an asymmetric beam and far side lobes using pixel based methods [9–11] ran up against computational challenges for angular scales smaller than one degree, running for hundreds of hours even with optimized adaptive mesh algorithms. Such algorithms are clearly inadequate for modern high resolution experiments which achieve resolutions of a few minutes of arc.

In this paper we describe a numerical method which greatly accelerates the computations which are necessary to correctly account for realistic beam profiles in simulation and analysis of directional data on the sphere. This is achieved by rewriting the problem in such a way that we can take advantage of the Cooley-Tukey fast Fourier transform (FFT) algorithm.

The following section of this paper defines the general problem in terms of rotations of the beam with respect to the sky. We then introduce a geometrically motivated split of the rotation operator in Sec. III. This enables us, in Sec. IV, to derive the general solution for the detector response for *all* possible relative orientations of the beam and the sky within a given section on the sphere. Section V then discusses the solution and derives special cases from it, amongst others the well-known algorithm for convolution with azimuthally

symmetric kernels. We conclude in Sec. VI with a timing example. An Appendix applies the same ideas to accelerating the computation of the transverse convolution, an operation which becomes important in the inverse problem of map estimation.

While we are motivated by the goal of achieving and interpreting precision measurements of the anisotropies of the cosmic microwave background, the methods we present are general and apply to the convolution or filtering of any scalar field on the sphere with an arbitrary, asymmetric but constant kernel. We generalize our methods to tensor fields on the sphere in Ref. [12].

II. STATEMENT OF THE PROBLEM

Consider two band limited functions on the sphere $b(\vec{\gamma})$ and $s(\vec{\gamma})$. For definiteness and to aid the imagination we will refer to them in the following as the *beam* and the *sky*, respectively, but they could be completely general band limited functions—in particular neither of them is constrained to be positive definite or even real.

The task is to compute the scalar product of the beam and the sky at a set of beam orientations. To describe these orientations, we use the Euler angles Φ_1, Θ and Φ_2 .¹ The convolved signal for each beam orientation (Φ_1, Θ, Φ_2) can then be written as

$$T(\Phi_2, \Theta, \Phi_1) = \int d\Omega_{\vec{\gamma}} [\hat{D}(\Phi_2, \Theta, \Phi_1) b](\vec{\gamma}) * s(\vec{\gamma}). \quad (1)$$

Here the integration is over all solid angles, \hat{D} is the operator of finite rotations such that $\hat{D}b$ is the rotated beam, and the asterisk denotes complex conjugation.

If (Φ_1, Θ, Φ_2) can be written as a continuous function of a parameter $t \in [0, T]$, say, then we call the ordered set of tuples $(\Phi_1(t), \Theta(t), \Phi_2(t))$ a *scan path*. Note that Eq. (1) assumes that time varying signals in the sky vary either on time scales much longer than the duration of the scan or much smaller than the integration time per sample. In the context of CMB missions this is a good approximation with the exceptions of planets (for long duration missions), time varying point sources, and atmospheric foregrounds. Of these only atmospheric foregrounds present a problem for the convolution, because they are extended—convolution with a point source is a simple operation in position space (the pixel basis) and can be computed separately. Linearity allows us to then add the results of the 4π convolution of extended sources to the point source convolution.

In the most general case, the band limits [see Eq. (4) below for a definition] of the beam and the sky are L_b and L_s , respectively. Define $L \equiv \min(L_b, L_s)$. Note that we actu-

¹Our Euler angle convention refers to active right handed rotations of a physical body in a fixed coordinate system. The coordinate axes stay in place under all rotations and the object rotates around the x , y and z axes by Φ_1 , Θ and Φ_2 , respectively, according to the right handed screw rule.

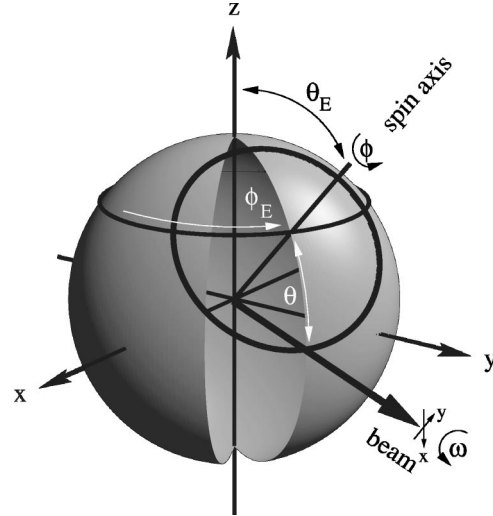


FIG. 1. Our coordinate system for efficient convolutions. The beam is shown at the position corresponding to $\theta=35^\circ$, $\theta_E=50^\circ$, $\phi_E=60^\circ$, $\phi=0^\circ$ and $\omega=0^\circ$. The cross-hairs on the beam mark its orientation, here shown for $\omega=0$. In the null position ($\theta=\theta_E=\phi_E=\phi=\omega=0$) the beam is aligned with the z axis, the vertical cross hair pointing along increasing x and the horizontal cross hair pointing along increasing y .

ally only need one of s , b to be band limited as long as the multipoles of the other are bounded as $l \rightarrow \infty$. Then the numerical evaluation of the integral in Eq. (1) takes $\mathcal{O}(L^2)$ operations for each tuple (Φ_1, Θ, Φ_2) . These integrals need to be evaluated for a grid of beam locations that has to contain at least $\mathcal{O}(L^3)$ grid points to allow subsequent interpolation at arbitrary locations. Therefore the total computational cost for evaluating the convolution using Eq. (1) scales as $\mathcal{O}(L^5)$.

III. FACTORIZING THE ROTATION

It is possible to simplify the evaluation of Eq. (1) significantly by factorizing the rotation into two auxiliary rotations such as

$$\hat{D}(\Phi_2, \Theta, \Phi_1) \equiv \hat{D}(\phi_E, \theta_E, 0) \hat{D}(\phi, \theta, \omega). \quad (2)$$

We will define the various angles and motivate this split in the following. Figure 1 is intended to illustrate this discussion.

To introduce these coordinates let us first consider *basic scan paths*. Imagine a scan path where the beam sweeps over the sky by scanning on rings of angular radius $\theta \in [0, \pi/2)$. The centers of these scanning circles lie on a ring of constant latitude, a polar angle $\theta_E \in [0, \pi)$ away from the north pole. The angle $\phi_E \in [0, 2\pi)$ selects a given scanning circle and is defined as the longitude of its center, while $\phi \in [0, 2\pi)$ measures the angle along each scanning ring defined as increasing in a right-handed way about the outward normal at the center, starting from zero at the southernmost point on the ring. Hence, for such a basic scan path we can write the

convolution as set of scalar products $T(\phi_E, \phi)$. The angles θ and θ_E are thought of as parameters which fixed to define the scanning geometry.

As a generalization of basic scan paths, we allow as a further degree of freedom an additional right handed rotation of the beam about its outward axis by an angle $\omega \in [0, \pi/2)$.

Now we can see geometrically that all beam orientations on generalized basic scan paths can be arrived at by successively applying the two rotations in Eq. (2). Define as the null position of the beam when it is oriented along the z axis $\theta = \theta_E = \phi_E = \phi = \omega = 0$. Starting from the null position, acting on it with $\hat{D}(\phi, \theta, \omega)$ rotates it about its axis by ω and moves it out onto a ring with opening angle θ at an azimuthal angle ϕ . Then acting with $\hat{D}(\phi_E, \theta_E, 0)$ moves the beam into position.

Using the factorization Eq. (2) we can re-write Eq. (1) as

$$T(\phi_E, \phi, \omega) = \int d\Omega_{\vec{\gamma}} [\hat{D}(\phi_E, \theta_E, 0) \hat{D}(\phi, \theta, \omega) b](\vec{\gamma})^* s(\vec{\gamma}) \quad (\theta, \theta_E \text{ fixed}). \quad (3)$$

The function $T(\phi_E, \phi, \omega)$ contains all possible integrals for a given scanning geometry. In fact, for the special case $\theta = \theta_E = \pi/2$, these angles parametrize all possible orientations of the beam on the sky, i.e. (ϕ_E, ϕ, ω) parametrize the group of rotations in three dimensions. It is well known that in this case these coordinates cover $SO(3)$ twice, but this can be easily remedied by restricting the range of one of the angles to half its range. We defer removing this redundancy until the end of our calculation.

IV. SOLUTION

To exploit the form of Eq. (3), it is expedient to represent the functions s and b as well as the rotation operators in the spherical harmonic basis. A band limited function $f(\vec{\gamma})$ can be expanded in spherical harmonics as

$$f(\vec{\gamma}) = \sum_{l=0}^{L_f} \sum_{m=-l}^{m=l} f_{lm} Y_{lm}(\vec{\gamma}), \quad (4)$$

where $\vec{\gamma}$ denotes a unit vector. For practical applications the band limit L_f is chosen such that higher terms contribute insignificantly. We use the notation where all quantities carrying both an l and an m index vanish for $m > l$. This saves having to write explicit limits for sums over azimuthal quantum numbers.

Invariance of the scalar product under a change of basis then allows us to re-write Eq. (3) as

$$\begin{aligned} T(\phi_E, \phi, \omega) &= \sum_{lm} s_{lm} [\hat{D}(\phi_E, \theta_E, 0) \hat{D}(\phi, \theta, \omega) b]_{lm}^* \\ &= \sum_{lmMM'} s_{lm} D_{mM}^{l*}(\phi_E, \theta_E, 0) D_{MM'}^{l*}(\phi, \theta, \omega) b_{lM'}^*. \end{aligned} \quad (5)$$

A simple explicit expression for the matrix elements $D_{mm'}^l(\phi_2, \theta, \phi_1)$ can be given. One can define a real function $d_{mm'}^l(\theta)$ such that

$$D_{mm'}^l(\phi_2, \theta, \phi_1) = e^{-im\phi_2} d_{mm'}^l(\theta) e^{-im'\phi_1}. \quad (6)$$

Thus the dependence of D on the Euler angles ϕ_1 and ϕ_2 is only in terms of complex exponentials. While explicit formulas for the d -functions exist [13], they are more conveniently computed numerically using their recursion properties [14].

Substituting into Eq. (5) and defining the three-dimensional Fourier transform of $T(\phi_E, \phi, \omega)$ as

$$\begin{aligned} T_{mm'm''} &= \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi_E d\phi d\omega T(\phi_E, \phi, \omega) e^{-im\phi_E - im'\phi - im''\omega}, \end{aligned} \quad (7)$$

we obtain

$$T_{mm'm''} = \sum_l s_{lm} d_{mm'}^l(\theta_E) d_{m'm''}^l(\theta) b_{lm}^*. \quad (8)$$

This equation is the main result of this paper, in effect generalizing fast 2D Fourier transform convolution from the plane to the sphere. Its properties and specializations will be discussed in the next section. Here we give a geometrical interpretation. We have arrived at Eq. (8) by writing convolution problems in such a way that the results are fields on complete 3-tori instead of subsets of the 3-sphere, which is the group manifold of rotations in 3 dimensions. Convolutions over azimuthally symmetric and connected sections of the 2-sphere (such as polar caps or annuli) can be parametrized by θ and θ_E and hence can be extended to 3-tori as shown. Since exponentials are a complete and orthonormal basis on the 3-torus and because we assumed that s and b are band-limited, the $T_{mm'm''}$ contain all information about the inverse transform, Eq. (3),

$$T(\phi_E, \phi, \omega) = \sum_{m, m', m''=-L}^L T_{mm'm''} e^{im\phi_E + im'\phi + im''\omega}. \quad (9)$$

Not all tuples (ϕ_E, ϕ, ω) correspond to distinct beam orientations but this redundancy is more than compensated for by the efficiency of the method.

V. DISCUSSION

Several remarks about Eq. (8) are in order.

1. Computational cost

Computing the $T_{mm'm''}$ in Eq. (8) costs $\mathcal{O}(L^4 |\sin \theta_E| 2\theta/\pi)$ operations. The factors θ and $|\sin \theta_E|$ come from the fact that the band limit L for the sky implies a band limit $\propto 2\theta L/\pi$ on a ring of radius θ and hence the ranges of m and m' can be reduced by factors of $|\sin \theta_E|$ and $2\theta/\pi$, respectively, if the

rings and the ring of ring centers are not great circles. Using the fast Fourier transform (FFT) algorithm, the inverse Fourier transform takes $\mathcal{O}(L^3 \log L |\sin \theta_E| 2\theta/\pi)$ operations. If the convolved sky is assumed to be real, we have

$$T_{mm'm''} = T_{-m-m'-m''}^*, \quad (10)$$

reducing memory and processor requirements by a factor 2.

2. Quadrature and interpolation

In pixel space each evaluation of $T(\phi_E, \phi, \omega)$ is an explicit quadrature problem and hence necessarily approximate. In our approach, all sums have a finite number of terms and the results are exact as long as s and b are band-limited. Quadrature issues only have to be dealt with if b or s are given in pixel space and we have to evaluate the beam and sky multipole coefficients b_{lm} and s_{lm} . The details of which pixelization to choose on the sphere and how to solve this generalized quadrature problem for the multipole coefficients are outside of the scope of this work but an efficient and practical approach to the quadrature problem is implemented in the HEALPix package [16] and will be discussed in a future publication [17].

An interesting property of Eq. (8) is that as long as L was chosen appropriately one is guaranteed to have the convolved sky sampled sufficiently densely for worry-free interpolation on either of the three indices.

A. Special cases

We will now discuss certain special cases of Eq. (8).

1. Total convolution

Let us obtain the convolved sky at all possible beam orientations ω on an equidistant coordinate grid in ϕ (corresponding to the polar angle) and ϕ_E (corresponding to the azimuthal angle). We will refer to this case as the *total convolution*. This can be achieved by evaluating Eq. (8) setting $\theta = \theta_E = \pi/2$. In this case we only need to know $d_{m'm}^l(\pi/2)$. This means we only have to evaluate a single recursion relation to evaluate the sum on l , which simplifies the computation. The inverse FFT gives the desired result.

A further simplification arises in this case from the fact that if $\theta_E = \pi/2$, not all components of $T_{mm'm''}$ are independent. The redundancy in the parametrization where the polar angle $\phi \in [0, 2\pi)$ leads to the symmetry

$$T(\phi_E, \phi, \omega) \equiv T(\pi + \phi_E, 2\pi - \phi, \pi + \omega). \quad (11)$$

This translates into the identity

$$T_{mm'm''} \equiv (-1)^{m+m''} T_{-m-m''m'}, \quad (12)$$

which cuts the required memory and computation time by a factor 2.

2. Exact or approximate azimuthal symmetry of the beam

In many practical situations the ‘‘beam’’ represents the response function of an optical system with only mild imper-

fections. If this is the case, the beam has only slowly varying azimuthal structure, implying a cutoff wave number M such that $b_{lm} \sim 0$ for $m \geq M$. In this case the computational cost for a total convolution scales as $\mathcal{O}(L^3 M \theta \sin \theta_E)$.

In the limit of an azimuthally symmetric beam, $M=0$, we obtain an $\mathcal{O}(L^3 \theta \sin \theta_E)$ method. However, it is known [15] that at least in principle there exist faster methods for convolution of a function on the full sphere ($\theta = \theta_E = \pi/2$) with an azimuthally symmetric beam which scale as $\mathcal{O}(L^2 [\log(L)]^2)$. We can show how this limit is obtained from Eq. (8) by using the facts that the $d_{mm'}^l(\pi/2)$ are the Fourier coefficients of the $d_{mm'}^l(\theta)$ and that $d_{m0}^l(\theta) = P_{lm}(\theta)$. Then Eq. (8) reduces to the form

$$T(\phi_E, \phi) = \sum_{lm} Y_{lm}(\pi - \phi, \phi_E + \pi/2) b_{l0} s_{lm}, \quad (13)$$

where the arguments of Y_{lm} are the polar angle and the azimuthal angle, respectively. The algorithm by [15] succeeds precisely in reducing the computational cost of evaluating this expression to $\mathcal{O}(L^2 [\log(L)]^2)$ under the proviso of the technical difficulties there outlined.

We note here for completeness, that by choosing a delta function beam (and hence $b_{lm} = \text{const}$), we recover the Fourier summation method for the spherical harmonic transform, described in Eqs. (5.2) to (5.4) in [14]. This computes Eq. (4) on an equidistant coordinate grid by doing Fourier transforms on latitudinal and longitudinal lines. The forward transform is obtained by simply working all steps in reverse (see Appendix).

3. Basic scan paths

Consider an application where the convolution is required only along a ‘‘basic scan path.’’ This is one of the proposed scan strategies for the Planck satellite mission. From our definition of basic scan paths in Sec. III we see that they correspond to setting $\omega=0$ in Eq. (8).

Computing the inverse Fourier transform of Eq. (8) with $\omega=0$ just amounts to summing over m'' . Then only the two-dimensional Fourier transform

$$T_{mm'}(\omega=0) = \sum_l s_{lm} d_{mm'}^l(\theta_E) X_{lm'} \quad (14)$$

remains to be evaluated. The quantity

$$X_{lm} \equiv \sum_M d_{mM}^l(\theta) b_{lM}^* \quad (15)$$

can be precomputed. All in all the computational time needed for evaluating these expressions is $\mathcal{O}(L^3 \theta \sin \theta_E)$. Storage requirements scale only as $\mathcal{O}(L^2)$.

Note that in this case azimuthal symmetry of the beam does not necessarily imply reduced computational cost. If the beam is concentrated at the north pole into a region of size σ then X_{lm} will have m modes populated up to $M \sim \theta/\sigma$.

Geometrically, the basic scan path corresponds to a 2-torus which is the section of the 3-sphere of rotations at

constant ω . Note that in this case there is no redundancy in the parametrization—every tuple $(\phi_E, \phi, 0)$ corresponds to a distinct beam orientation.

4. Perturbations about basic scan paths

A slight generalization of the previous case are scan paths which are close to basic but include a variation in θ_E from scanning circle to scanning circle. Such paths result for example from precessing or “wobbling” the spin axis of a scanning satellite.

Such scan paths can be composed by computing several convolutions along basic scan paths for different angles θ_E and then choosing scanning circles at will from among these basic ones. This method suggests itself if the precession angle is small and hence a small number of convolutions is sufficient to sample the variation in θ_E . Convolutions at points which do not coincide with sampling points can then be determined by interpolation.

Another approach to this type of problem and further generalizations are discussed in the next paragraphs.

5. Other special cases

Other potentially interesting special cases of Eq. (8) can be worked out by fixing any of the parameters to special values and evaluating the inverse transform, analogous to the calculation for basic scan paths. For example one obtains expressions for

(i) All possible beam orientations along a circle of constant latitude θ_E . In this case ϕ and ω have the same meaning; formally, $T_{mm'm''} = T_{mm'} \delta_{m'm''}$ and we obtain an $\mathcal{O}(L^2 M \sin \theta_E)$ method:

$$T_{mm'} = \sum_l s_{lm} d_{mm'}^l(\theta_E) b_{lm}^*. \quad (16)$$

(ii) Individual scanning rings of a basic scan path. Here, $\omega = 0$, and the only free parameter is ϕ .

The details of the calculations for this and similar cases are now easy exercises.

6. Generalizations

Further, it is clear from the derivation that more general types of paths can be constructed by factorizing the rotation operator more than twice, so as to generate for example a ring of ring of rings, etc. For particular applications some of these may be advantageous, for example if they simplify the interpolation problem on the output ring set. A specific example is the precessing scan path mentioned in a previous subsection. Inserting another rotation operator $\hat{D}(0, \theta_P, \phi_P)$ between the two operators in Eq. (2), and setting $\omega = 0$ produces a set of rings whose centers lie on circles of radius θ_P about θ_E . This may simplify the interpolation problem. The rotation corresponding to ϕ_P can be sampled sparsely if θ_P is small ($M_P \sim \sin \theta_P \sin \theta_E L$ with obvious notation) and the interpolation problem becomes simpler.

VI. CONCLUSIONS

This paper presents a general algorithm which greatly reduces the computational cost of convolving two band limited but otherwise arbitrary functions on the sphere. The speedup increases linearly with the smallest angular scale of the smoother of the two functions in the problem. The scalings of the necessary operation counts are discussed in detail in Sec. V.

We quote in the Appendix formulas showing how the ideas presented in this paper can be applied to the inverse problem of “deconvolution” by speeding up the iterative solution of the normal equation in an analogous way.

This paper focuses on the convolution of scalar valued functions on the sphere such as temperature, elevation, etc. In order to be able to deal with the polarization of the cosmic microwave background we extend the methods presented here to tensor valued functions on the sphere in Ref. [12].

The algorithms which are presented here are already being used as a core component of the prototype simulation pipeline of the Planck satellite and are being implemented for use by the MAP mission. To give an example for the timing gains one makes by applying this method, we computed the following case: both sky and beam were interpolated and pixelized very densely, with millions of pixels each to resolve the steep variations over many orders of magnitude. The band limit was somewhat generously chosen as $L = 1024$. Then the convolution of the sky with the beam of a single detector for a whole year of mission data, consisting of $(2049)^2 \sim 4 \times 10^6$ convolved samples along a basic scan path was generated in less than 15 minutes on a single Silicon Graphics R10000 processor. This compares with several days of computation on a severely coarsened sampling grid with several hundred times fewer samples on the same machine, using the adaptive mesh method [11]. For the same resolution which we achieved with our methods, the adaptive mesh code would have run for months.

Using our methods, future CMB missions can go beyond having to approximate the treatment of realistic beams. Our methods lend themselves to being used in conjunction with iterative map-making methods to remove from the data artifacts which are due to beam distortions and far side lobes (see Appendix).

Lastly, we feel that the geometric constructions, analogies to group properties and algebraic results we introduce in this article are useful more generally for CMB data analysis. We are currently exploring these issues, with promising first results [18].

ACKNOWLEDGMENTS

We would like to thank F. Hansen for coding help in the early stages of this project, F. Bouchet for suggesting consideration of the transpose operation and R. Caldwell for reading the manuscript before submission. Part of this work was supported by the Dansk Grundforskningsfond through its funding for TAC. B.D.W. is supported by the NASA MAP/MIDEX program.

APPENDIX: THE DECONVOLUTION PROBLEM

We sketch here how to set up the inverse problem of reconstructing the true sky from the convolved observations. If we start with a noise free set of convolutions, then the equation to be inverted in order to estimate the true underlying sky is, schematically,

$$\mathbf{A}\mathbf{s}=\mathbf{d}. \quad (\text{A1})$$

Here, \mathbf{s} is the true sky, \mathbf{d} is the vector containing the time-ordered data after observation. The convolution operator is represented by \mathbf{A} .

The least-square estimator for the true sky, $\hat{\mathbf{s}}$ then satisfies the *normal equation*

$$\mathbf{A}^T\mathbf{A}\hat{\mathbf{s}}=\mathbf{A}^T\mathbf{d}. \quad (\text{A2})$$

For a perfect observation with a delta function beam, $\mathbf{A}^T\mathbf{A}\equiv\mathbf{1}$. So it may be reasonable to expect that we can make

progress by considering a mildly imperfect optical system and consider iterative techniques for solving the normal equation. In this case the ability to solve for $\hat{\mathbf{s}}$ iteratively (e.g. using a conjugate gradient technique) relies on convergence, which is assured up to numerical effects because the normal matrix $\mathbf{A}^T\mathbf{A}$ is positive definite and being able to compute the matrix products in Eq. (A2) quickly. The application of \mathbf{A} can be computed efficiently using the formulas set out in Secs. IV and V. We now find an algorithm for the efficient application of \mathbf{A}^T , the *transpose convolution*.

The transpose convolution acts on the time-ordered data. One can think of it as running through all samples and superposing beam patterns on the sky, one for each observation, in the direction and orientation the beam had when that sample was taken. This gives rise to a smoothed sky map which we denote by $y(\vec{\gamma})$.

We can write down the expression for the transpose convolution \mathbf{A}^T in a similar way to Eq. (3)

$$y(\vec{\gamma})=\int d\phi_E d\phi d\omega[\hat{D}(\phi_E,\theta_E,0)\hat{D}(\phi,\theta,\omega)b](\vec{\gamma})^*T(\phi_E,\phi,\omega). \quad (\text{A3})$$

Now the derivation is analogous to the one preceding Eq. (8). Doing the convolution in spherical harmonic space yields the spherical harmonic coefficients y_{lm} in terms of the Wigner d functions as

$$y_{lm}=\sum_{m'm''}d_{mm'}^l(\theta_E)d_{m'm''}^l(\theta_E)b_{lm''}^*T_{mm'm''}. \quad (\text{A4})$$

This formula can be evaluated, generalized or applied to special cases just as we showed in Sec. V for Eq. (8).

-
- [1] W. Hu, N. Sugiyama, and J. Silk, *Nature (London)* **386**, 37 (1997).
 [2] J. Borrill, *Phys. Rev. D* **59**, 027302 (1999).
 [3] J. R. Bond, R. G. Crittenden, A. H. Jaffe, and L. Knox, *Comput. Sci. Eng.* **1**, 21 (1999).
 [4] C. G. T. Haslam, *Astron. Astrophys.* **47**, 1 (1982).
 [5] D. Schlegel, D. Finkbeiner, and M. Davis, *Astrophys. J.* **500**, 525 (1998).
 [6] M. White and M. Srednicki, *Astrophys. J.* **443**, 6 (1995).
 [7] L. Knox, *Phys. Rev. D* **60**, 103516 (1999).
 [8] J. H. P. Wu *et al.*, *Astrophys. J. Suppl.* **132**, 1 (2001).
 [9] P. de Maagt, A. M. Polegre, and G. Crone, ‘‘Straylight Evaluation of the Carrier Configuration,’’ ESA technical report (1998).
 [10] J. Delabrouille, Ph.D. thesis, University of Chicago, 1999.
 [11] C. Burigana *et al.*, *Astron. Astrophys.* (to be published), astro-ph/0010113.
 [12] A. Challinor *et al.*, *Phys. Rev. D* **62**, 123002 (2000).
 [13] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988); D. M. Brink and G. R. Satchler, *Angular Momentum* (Clarendon, Oxford, 1975).
 [14] T. Risbo, *J. Geodesy, Berlin* **70**, 383 (1996).
 [15] J. R. Driscoll and D. M. Healy, *Adv. Appl. Math.* **15**, 202 (1994).
 [16] The HEALPIX Homepage, <http://www.eso.org/~kgorski/healpix/>
 [17] B. D. Wandelt *et al.*, ‘‘Fast Multipole Quadrature on the Sphere with Applications to HEALPIX’’ (in preparation).
 [18] B. D. Wandelt, in *Mining the Sky*, Proceedings of MPA/ESO/MPE Joint Conference, edited by A. J. Bandy, Garching bei Muenchen, Germany, 2000, astro-ph/0012416.