Small, dense quark stars from perturbative QCD

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As a model for nonideal behavior in the equation of state of QCD at high density, we consider cold quark matter in perturbation theory. To second order in the strong coupling constant α_s, the results depend sensitively on the choice of the renormalization mass scale. Certain choices of this scale correspond to a strongly first order chiral transition, and generate quark stars with maximum masses and radii approximately half that of ordinary neutron stars. At the center of these stars, quarks are essentially massless.

Strongly interacting matter under extreme conditions can reveal new phenomena in quantum chromodynamics (QCD). Compact stars serve as an excellent observatory to probe QCD at large density, as their interior might be dense enough to allow for the presence of chirally symmetric quark matter, i.e., quark stars [1–13].

The usual model used for quark stars is a bag model, with at most a correction $\sim \alpha_s$ from perturbative QCD [6]. In the massless case, the first order correction cancels out in the equation of state, so that one ends up finally with a free gas of quarks modified only by a bag constant. If the bag constant is fit from hadronic phenomenology, then the gross features of quark stars are very similar to those expected for neutron stars: the maximum mass is $\approx 2.M_\odot$, with a radius ≈ 10 km.

In this Rapid Communication we consider quark stars, using the equation of state for cold, dense QCD in perturbation theory to $\sim \alpha_s^2$ [2,3]. These results are well known, and our only contribution is to use modern determinations of the running of the QCD coupling constant [14]. At the outset, we stress that we do *not* suggest that the perturbative equation of state is a good approximation for the densities of interest in quark stars. Rather, we use it merely as a model for the equation of state of QCD.

To $\sim \alpha_s^2$, there is significant sensitivity to the choice of the renormalization mass scale. Under our assumptions, we find that this choice is tightly constrained by the physics. We consider two illustrative values of this parameter. One choice corresponds to a weakly first order chiral transition (or no true phase transition), and gives maximum masses and radii very similar to that of neutron stars. The second choice corresponds to a strongly first order chiral transition [15], and generates two types of stars. One type has densities a few times that of nuclear matter, and looks like the stars of a weakly first order chiral transition. In addition, however, there is a new class of star [7,12], with densities much higher than that of nuclear matter. For this new class, the maximum mass is $\approx 1.M_\odot$, with a radius ≈ 5 km. Other models with nonideal behavior also generate small, dense quark stars [8–11].

Assume that the chiral phase transition occurs at a chemical potential μ_χ [16]. Our perturbative equation of state is applicable only in the chirally symmetric phase, when the quark chemical potential $\mu > \mu_\chi$. In this phase, the effects of a strange quark mass, $m_s \approx 100$ MeV [18], are small relative to the quark chemical potentials, $\mu > 300$ MeV. Thus we take three flavors of massless quarks with equal chemical potentials [1,19].

The thermodynamic potential of a plasma of massless quarks and gluons was calculated perturbatively to $\sim \alpha_s^2$ by Freedman and McLerran [2] and by Baluni [3], using the momentum-space subtraction (MOM) scheme. The MOM coupling is related to that in the modified mimimal substraction scheme ($\overline{\text{MS}}$) as [3,20–22]

$$\frac{\alpha_s^{\text{MOM}}}{\pi} = \frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\left[1 + \mathcal{A}\ \frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right]; \qquad (1)$$

$\alpha_s = g^2/(4\pi)$, with g the QCD coupling constant, and $\mathcal{A} = 151/48 - (5/18)N_f$, with N_f the number of massless flavors. (While we take $N_f = 3$, we give formulas for arbitrary N_f.) In the $\overline{\text{MS}}$ scheme, to $\sim \alpha_s^2$ the thermodynamic potential is then

$$\Omega(\mu) = -\frac{N_f \mu^4}{4\pi^2}\left\{1 - 2\left(\frac{\alpha_s}{\pi}\right)\right.$$
$$\left. -\left[G + N_f \ln\frac{\alpha_s}{\pi} + \left(11 - \frac{2}{3}N_f\right)\ln\frac{\bar{\Lambda}}{\mu}\right]\left(\frac{\alpha_s}{\pi}\right)^2\right\}, \quad (2)$$

where $G = G_0 - 0.536 N_f + N_f \ln N_f$, $G_0 = 10.374 \pm .13$ [23], and $\bar{\Lambda}$ is the renormalization subtraction point. In $\overline{\text{MS}}$ scheme, the thermodynamic potential is manifestly gauge invariant. We take the scale dependence of the strong coupling constant, $\alpha_s \equiv \alpha_s(\bar{\Lambda})$ as [14,24]

$$\alpha_s(\bar{\Lambda}) = \frac{4\pi}{\beta_0 u}\left[1 - \frac{2\beta_1}{\beta_0^2}\frac{\ln(u)}{u} + \frac{4\beta_1^2}{\beta_0^4 u^2}\left(\left(\ln(u) - \frac{1}{2}\right)^2\right.\right.$$
$$\left.\left. + \frac{\beta_2 \beta_0}{8\beta_1^2} - \frac{5}{4}\right)\right]; \qquad (3)$$

$u = \ln(\bar{\Lambda}^2/\Lambda_{\overline{MS}}^2)$, $\beta_0 = 11 - 2N_f/3$, $\beta_1 = 51 - 19N_f/3$, and $\beta_2 = 2857 - 5033N_f/9 + 325N_f^2/27$. The scale $\Lambda_{\overline{MS}}$ is fixed by requiring that $\alpha_s = 0.3089$ at $\bar{\Lambda} = 2$ GeV [14]; for $N_f = 3$, Eq. (3) gives $\Lambda_{\overline{MS}} = 365$ MeV.

All thermodynamic quantities follow consistently from $\Omega(\mu)$. The pressure is given by $p(\mu) = -\Omega(\mu)$, the quark number density by $n(\mu) = (\partial p/\partial \mu)$, and the energy density by $\epsilon = -p + \mu n$. Given our stated assumptions, the only freedom we have in the model is the choice of the ratio $\bar{\Lambda}/\mu$. To illustrate this, we take the values $\bar{\Lambda}/\mu = 1, 2, 3$.

For reasons which will become clear later, we find the choice $\bar{\Lambda} = 2\mu$ especially interesting. Start with a very large chemical potential, such as $\mu = 100$ GeV, for which $\alpha_s \sim .095$ (for the purposes of discussion, assume $N_f = 3$ at this scale). The first order term decreases the ideal gas pressure by $\sim 6\%$; the sum of the first and second order terms decreases the pressure by $\sim 7\%$ of the ideal gas value. Because the strong coupling constant runs relatively slowly with μ at large μ, even at $\mu = 1$ GeV, where $\alpha_s \sim .31$, the first order term decreases the ideal gas pressure only by $\sim 20\%$; the first and second order terms, by $\sim 30\%$.

As can be seen from Eq. (2), the perturbative expansion of the thermodynamic potential is an expansion in a power series not just in α_s, but in $\alpha_s \log(\alpha_s)$. The logarithm of α_s arises from the plasmon effect, where the Debye mass squared $m_D^2 \sim \alpha_s \mu^2$. Because gluons at $T = 0$ have four-dimensional phase space in loop integrals, however, the plasmon effect is relatively innocuous, and only produces logarithms, $\log(m_D/\mu) \sim \log(\alpha_s)$.

This is in stark contrast to the perturbative expansion of the free energy at nonzero temperature, $T \neq 0$. While there is again a plasmon effect, $m_D^2 \sim \alpha_s T^2$, because in loop integrals static gluons at $T \neq 0$ have a three dimensional phase space, the perturbative expansion is not in α_s, but in $\sqrt{\alpha_s}$. The series in $\sqrt{\alpha_s}$ is much worse behaved than that at $\mu \neq 0$, $T = 0$, and does not converge until very high temperatures [25]. The convergence appears to improve after resummation [26,21,22], or by using Padé methods [27].

Consequently, the perturbative series for the thermodynamic potential may be much better behaved at $\mu \neq 0$ (and $T = 0$) than at $T \neq 0$ [28]. This does not imply that a given value of α_s, which is adequate to compute the thermodynamic potential, works equally well for all other quantities. In particular, the gaps for color superconductivity are non-perturbative, $\phi \sim \exp(-1/\sqrt{\alpha_s})$ [29,17], and much smaller values of α_s appear to be required to reliably compute them [30]. In QCD, effective models find that even when $\mu \sim 400$ MeV, these gaps are at most ~ 100 MeV [17]. As the relative change in the thermodynamic potential is only $\sim (\phi/\mu)^2$, then, for the equation of state in QCD, color superconductivity is never a large effect.

To truly know how well perturbation theory converges at $\mu \neq 0$, it is imperative to compute the thermodynamic potential to $\sim \alpha_s^3$. Unlike the case of $T \neq 0$, which is sensitive to nonperturbative effects from static magnetic gluons from $\sim \alpha_s^3$ on, at $\mu \neq 0$ (and $T = 0$), the entire power series in α_s is well defined [29].
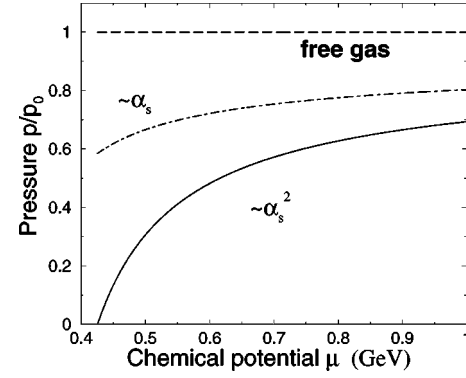

FIG. 1. The total pressure, relative to the pressure of an ideal gas, p_0; including terms to order $\sim \alpha_s$ and to order $\sim \alpha_s^2$, as a function of μ; $\bar{\Lambda} = 2\mu$.

We now use the perturbative calculation of the thermodynamic potential for $\mu < 1$ GeV. Since both terms $\sim \alpha_s$ and $\sim \alpha_s^2$ have negative coefficients, as $\alpha_s(\mu)$ increases with decreasing μ, eventually the pressure vanishes. While it is clearly invalid using perturbation theory when $p = 0$, it at least provides a well defined model of dense QCD. In Fig. 1 we show the pressure for $\bar{\Lambda} = 2\mu$; it vanishes at $\mu_c = 425$ MeV, where $\alpha_s \sim .65$. This corresponds to a quark density $\sim 4.35\rho_0$, where ρ_0 is the density of quarks in nuclear matter, $\sim 3 \times .16/\text{fm}^3$.

A weakness in our model is how to match the equation of state for massless quarks, Eq. (2), onto that for massive quarks and hadrons. The quark chemical potential must be larger than one third of the nucleon rest mass, minus one third the binding energy of nuclear matter, $\mu > \mu_{min} \approx 313 - 5$ MeV. While the pressure vanishes at μ_{min}, hadronic (or quark) matter certainly exists, with nonzero pressure, for all $\mu > \mu_{min}$. Thus we imagine that a very "soft" equation of state for massive quarks (and hadrons) matches onto the equation of state in Eq. (2) at some $\mu > \mu_c$ (see also below). Consequently, μ_c cannot be much higher than μ_{min}.

It is this which limits the choice of $\bar{\Lambda}/\mu$ in our model. For $\bar{\Lambda} = \mu$, $\mu_c = 767$ MeV. It is absurd to think that the pressure of massive quarks could be small to densities $\sim 33\rho_0$. Thus we do not consider this case further.

For $\bar{\Lambda} = 3\mu$, $\mu_c = 300$ MeV when $\alpha_s \sim .6$. By the Hugenholtz–van Hove theorem, when the pressure vanishes, the ratio of the total energy to the baryon number is $E/A = 3\mu_c$. For iron, $E/A = 930$ MeV. Thus for $\bar{\Lambda} = 3\mu$, $E/A = 3\mu_c = 900$ MeV, and, as suggested by Bodmer and Witten [4,5], strange quark matter is absolutely stable relative to hadronic matter.

While possible, we prefer an alternate view. Our perturbative equation of state is valid only in the chirally symmetric phase, for $\mu > \mu_\chi$. Perhaps when $\mu < \mu_\chi$, the true equation of state is close to our perturbative model, but vanishes smoothly as $\mu \to \mu_{min}$. As discussed later, this is a model for a weakly first order (or no) chiral phase transition.

The structure of a quark star is determined by the solution to the Tolman-Oppenheimer-Volkov (TOV) equations [1]. For the TOV equations, all that matters is the relationship

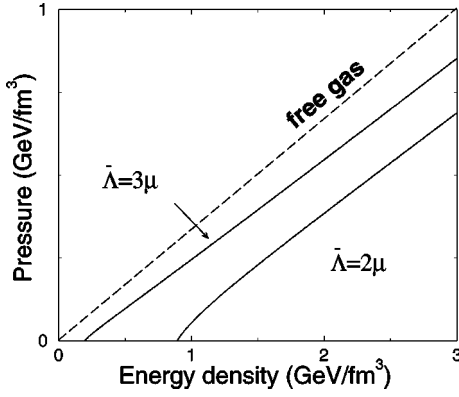FIG. 2. Equation of state for cold quark matter, for $\bar{\Lambda}/\mu=2,3$.

between pressure and energy density. This is shown in Fig. 2 for $\bar{\Lambda}/\mu=2,3$. The numerical solution of the TOV equations gives the mass radius relationships of Fig. 3. For any solution to the TOV equation, the chemical potential reaches its maximum value at the center of the star; as one goes out in radius, the chemical potential decreases, and equals μ_c (where the pressure vanishes) at the edge of the star. For $\bar{\Lambda}=3\mu$, the maximum mass is $M_{\max}\approx2.14M_{\odot}$. At this mass, the radius is $R_{\max}\approx12$ km; the chemical potential at the center of the star is $\mu\approx456$ MeV, which corresponds to a quark density of $\rho_{\max}\approx5.1\rho_0$. When $\bar{\Lambda}=2\mu$, $M_{\max}\approx1.05M_{\odot}$. At this mass, the radius is $R_{\max}\approx5.81$ km; the chemical potential at the center is $\mu\approx649$ MeV, corresponding to a quark density of $\rho_{\max}\approx15\rho_0$.

To help understand these results, it is useful to compare to the equation of state of a nonideal bag model:

$$\Omega(\mu)=-\frac{N_f}{4\pi^2}a_{\text{eff}}\mu^4+B_{\text{eff}};\qquad(4)$$

B_{eff} is an effective bag constant, and the parameter a_{eff} measures deviations from ideality. A common choice is to take a_{eff} from the thermodynamic potential to one loop order, with a fixed value of the coupling constant: $a_{\text{eff}}=1-2\alpha_s/\pi$ [6]. In a bag model, the relationship between pressure and energy density is linear, $p=(\epsilon-4B)/3$, *irrespective* of the value of a_{eff}. Thus we can uniformly scale p, ϵ, and B
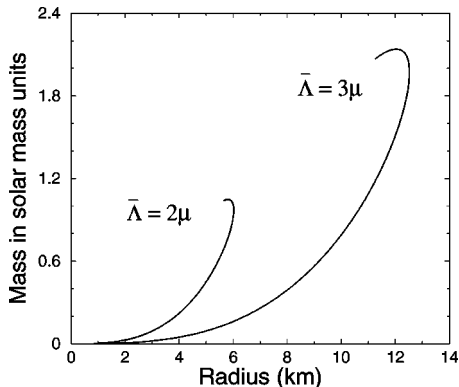

FIG. 3. Mass-radius relation of the quark star for $\bar{\Lambda}/\mu=2,3$.

together, so the maximum mass and radius satisfy a simple scaling relation, $M_{\max}\sim R_{\max}\sim1/B^{1/2}$ [5].

For the chemical potentials of relevance to a quark star, somewhat surprisingly we find numerically that the pressure in Eq. (2) can be well approximated by the effective bag model, Eq. (4). This can be seen from Fig. 2, where the relationship between pressure and energy is very nearly linear. When $\bar{\Lambda}=3\mu$, the pressure agrees with a bag model with $B_{\text{eff}}^{1/4}=140$ MeV and $a_{\text{eff}}=.626$ to within 2% for $\mu:300\rightarrow470$ MeV. This is close to the usual value in the MIT bag model, $B^{1/4}=145$ MeV [31]. When $\bar{\Lambda}=2\mu$, the pressure agrees with a bag model with $B_{\text{eff}}^{1/4}=199$ MeV and $a_{\text{eff}}=.628$ to within 4% for $\mu:425\rightarrow650$ MeV.

Consequently, the mass radius relationships for our quark stars agree well with a bag equation of state. To $\sim5\%$, the maximum masses and radii scale according to $\sim1/B_{\text{eff}}^{1/2}$. The shape of the mass radius curve is also the same as for a bag model. Notably, light quark stars have small radii. This is because for light stars, $M\ll M_{\odot}$, the chemical potential at the center of the star is near μ_c, and the equation of state is controlled by that of massless fields, minus a bag constant.

Our results for $\bar{\Lambda}=2\mu$ can be compared to other equations of state for dense QCD [8–12]. All of these can be viewed as models in which there is nonideality at a scale significantly higher than nuclear matter densities. Reference [8] uses the results of Ref. [2], but finds standard results, $M_{\max}\approx2.0M_{\odot}$. Reference [9] finds dense stars, with $M_{\max}\approx1.4M_{\odot}$ and $R_{\max}\approx7.0$ km; however, they do not suggest that there could be two types of compact, hadronic stars, as we find for $\bar{\Lambda}=2\mu$. Reference [10] uses a Schwinger-Dyson model, and finds $M_{\max}\approx0.7M_{\odot}$ and $R_{\max}\approx9.0$ km. References [11] and [12] use models with massive quasiparticles, so that the masses act as a type of nonideality. Reference [11] finds $M_{\max}\approx0.8M_{\odot}$ and $R_{\max}\approx4.0$ km; Ref. [12] finds $M_{\max}\approx1.35M_{\odot}$ and $R_{\max}\approx10.0$ km. We note that in Nambu–Jona-Lasino models, stars with a quark core do not arise, even at the maximum mass [13].

What about the manifestly nonperturbative phase in which chiral symmetry is spontaneously broken, $\mu<\mu_\chi$? To understand this, consider an expansion in a large number of colors [32]. The usual large N_c limit is to let $N_c\rightarrow\infty$ at fixed N_f. Since quark loops are suppressed in this limit, however, gluons are only affected by quarks when $\mu\sim N_c^{1/4}$ or larger. This is in contrast with the transition at a nonzero temperature, which occurs at a temperature $\sim N_c^0$ [33]. We then consider a generalized large N_c limit, in which $N_f\rightarrow\infty$ at fixed N_f/N_c [34–36]. The quark thermodynamic potential is $\Omega\sim-N_fN_c\mu^4$, and the quark number density $n\sim N_fN_c\mu^3$. In this limit, $\mu_\chi\sim N_c^0$, as at nonzero temperature.

For the purposes of our discussion, all that matters is that baryon masses $m_B\sim N_c$. The baryon chemical potential is related to the quark chemical potential as $\mu_B=N_c\mu\sim N_c$. When the baryons are nonrelativistic, so their Fermi momenta $k_f\sim1$, the baryon number density is $n_B\sim d_Bk_f^3$. The degeneracy of baryons is at least $\sim N_f$, and could easily be larger, $\sim N_f^2$. The baryon thermodynamic potential is (naively) $\Omega_B\sim-d_Bk_f^5/m_B$. For $k_f\sim1$, however, everything is

fine: the baryon density is $\sim N_c$ that of quarks, and $\Omega_B \lesssim (1/N_c)\Omega$. In terms of the quark chemical potential, however, $\mu = \mu_B/N_c$; with $m_q = m_B/N_c$, $\mu \approx m_q + k_f^2/(2m_B N_c) + \dots$. That is, for $k_f \sim 1$, the region in μ over which hadrons are a reasonable description is *small*, $\sim 1/N_c^2$. Also, the binding energy of nuclear matter is automatically $\sim 1/N_c^2$.

From this large N_c argument, we conclude that a hadronic description is applicable *only* in a very narrow region of μ; for larger μ, still of order one, a quark description is appropriate. This need not be a true phase transition; rather, simply that the thermodynamic potential may be very difficult to compute in terms of hadrons, but relatively simple in terms of quarks. For example, when $k_f \sim N_c^{1/2}$, so $\mu - m_q \sim 1/N_c$, naively $\Omega_B \sim -d_B N_c^{3/2} \sim N_c^{1/2}\Omega$. This cannot be right — the thermodynamic potential of baryons cannot dominate that of quarks. The only resolution is that there are cancellations — analogous to those which occur for baryon-meson couplings [37] — which greatly reduce the baryon thermodynamic potential, so that it is comparable to that of quarks. In other words, the hadronic thermodynamic potential *must* ''soften'' whenever $\mu - m_q > 1/N_c^2$.

Once one is away from this (narrow) hadronic window in μ, the appropriate equation of state for $\mu < \mu_\chi$ is that for massive quarks. There are then two possibilities.

The first is that the thermodynamic potential for massless quarks matches, more or less smoothly, onto that of massive quarks. This requires either a weakly first order chiral transition, or perhaps just crossover. Below μ_χ, as $\mu \to \mu_{\min}$ the quark thermodynamic potential vanishes in a fashion typical of massive particles; within $\sim 1/N_c^2$ of $\mu_{\min}$, a hadronic description is applicable.

This is illustrated by the choice $\bar{\Lambda} = 3\mu$. For a star at its maximum mass, at the center $\mu \approx 456$ MeV; as the radius increases, μ decreases, until $\mu = \mu_{\min}$ at the surface. Thus, if $\mu_\chi < 456$ MeV, there is chirally symmetric quark matter at the center of the star, but for larger radii, one enters first a phase with massive quarks, and finally, a hadronic shell.

Consequently, for both the choices $\bar{\Lambda} = 3\mu$ and 2μ, all of our quark stars are, properly speaking, hybrid stars, with shells of massive quarks, and then hadrons. For stars near the maximum mass, we assume this mantle is thin, and does not greatly alter its properties. As the mass of the star decreases, however, so does the central density; eventually, the entire star is composed entirely of massive quarks and hadrons. At this point, the relationship between the star's mass and radius is no longer like that of Fig. 3. Instead, it looks like that of nonrelativistic matter, for which the radius increases as the mass decreases. It is necessary for the chiral transition to be weakly first order, or a smooth crossover, for the mass-radius curve to be continuous.

The second possibility is that the equation of state for massless quarks does not match smoothly onto that for massive quarks, with a strongly first order chiral transition [15]. As the thermodynamic potential approaches the ideal gas limit at large μ, and vanishes at $\mu_{\min}$, this requires that the pressure is small at a value of $\mu_\chi \gg \mu_{\min}$. Below μ_χ, by construction the pressure of massive quarks is small, with a ''soft'' equation of state [38].

This occurs if $\bar{\Lambda} = 2\mu$. At the maximum mass, $\mu \sim 649$ MeV at the center; if $\mu_\chi \lesssim 649$ MeV, most of the star is composed of massless quarks. As the mass of the star decreases, so does the amount in the chirally symmetric phase. If the chiral phase transition is strongly first order, eventually one jumps to a second branch, in which the chemical potentials are always $< \mu_\chi$ throughout the star. Stars on this second branch are composed only of massive quarks and hadrons, with a maximum mass and radius like that of ''ordinary'' neutron stars. A strong first order chiral transition is necessary to ensure that there are two, distinct branches. Using toy models for the thermodynamic potential, numerically we obtained solutions to the TOV equations which display two branches: we patched a thermodynamic potential for massive quarks, for $\mu < \mu_c \approx \mu_\chi$, onto that for massless quarks, for $\mu > \mu_c$. In this case, our stars of massless quarks constitute a third class of compact stars, after white dwarfs and ''ordinary'' neutron stars [7,12].

Most pulsars have masses $\sim 1.5 M_\odot$ [1]; experiments will soon provide an estimate of the radii [40]. The massive compact halo object (MACHO) project has also reported microlensing events for the Large Magellanic Cloud with masses $M = 0.15$–$0.9 M_\odot$ [39]. For a weakly first order chiral phase transition, if MACHO events are hadronic stars, they must have large radii. For a strongly first order chiral phase transition [15], MACHO events could be quark stars, with small radii, and pulsars might represent the second branch. We stress that our numbers for the maximum mass and radius are meant only to be suggestive. Even so, we believe that our conclusions are qualitatively correct; a weak (or no) chiral phase transition leads to one type of compact objects; a strongly first order chiral transition, to two.

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