Dynamics of effective gluons

Stanisław D. Głazek
Institute of Theoretical Physics, Warsaw University, ul. Hoża 69, 00-681 Warsaw, Poland
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Renormalized Hamiltonians for gluons are constructed using a perturbative boost-invariant renormalization group procedure for effective particles in light-front QCD, including terms up to third order. The effective gluons and their Hamiltonians depend on the renormalization group parameter λ, which defines the width of momentum-space form factors that appear in the renormalized Hamiltonian vertices. Third-order corrections to the three-gluon vertex exhibit asymptotic freedom, but the rate of change of the vertex with λ depends in a finite way on regularization of small-x singularities. This dependence is shown in some examples, and a class of regularizations with two distinct scales in x is found to lead to the Hamiltonian running coupling constant whose dependence on λ matches the known perturbative result from Lagrangian calculus for the dependence of gluon three-point Green's function on the running momentum scale at large scales. In the Fock-space basis of effective gluons with small λ, the vertex form factors suppress interactions with large kinetic energy changes and thus remove direct couplings of low-energy constituents to high-energy components in the effective bound-state dynamics. This structure is reminiscent of parton and constituent models of hadrons.

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I. INTRODUCTION

Current studies of hadronic structure are guided by three physical pictures. The first picture is based on the constituent quark model, which serves as a classification of hadrons in particle data tables [1]. Quantum numbers of a hadron in this model correspond to a simple Hamiltonian with only kinetic energy of two or three quarks and interquark potentials in the hadron rest frame, with no gluons. The second picture is provided by the parton model for hadrons in the infinite-momentum frame [2]. Modern versions of the model introduce a slew of quarks and gluons with distribution functions in variable x—a fraction of the hadron momentum that is carried by a parton. About half of the hadron momentum is carried by gluons, with mostly small values of x, so that many partons can share the hadron momentum. Binding of partons is not described by the parton model. In the third picture, hadrons are considered to be excitations of a complicated ground state (vacuum) that contains condensates of quarks and gluons. Understanding of hadronic structure in the third way relies on the assumed ground-state properties [3]. Despite recent progress in experimental and theoretical studies of hadronic structure, including the lattice approach [4,5], the three basic pictures are not yet unified in a single quantitative formulation of QCD. To connect constituent quarks and partons with QCD degrees of freedom, one needs a relativistic description of effective particles in quantum field theory.

This paper describes a perturbative third-order calculation of renormalized Hamiltonians for effective gluons in the light-front Fock space. The effective gluons are derived in a boost-invariant renormalization group procedure for particles [6], which originates in the similarity approach to renormalization of Hamiltonians [7] and the notion of vertex form factors for extended strongly interacting particles [8]. The renormalization procedure provides a connection between the canonical quantum field theory and the concepts of bound-state constituents in the rest and infinite-momentum frames. For simplicity, this paper is limited to gluons. Quark effects in the gluon dynamics are mentioned only in passing. Gluons alone are worth a discussion since their interactions are responsible for asymptotic freedom. This feature requires understanding in Hamiltonian approach independently of the quark dynamics. Also, asymptotically free effective gluon interactions display specific sensitivity to the regularization of small-x singularities.

Section II presents the initial regularized Hamiltonian for gluons. The Hamiltonian includes ultraviolet counterterms that are calculable order by order in the procedure described in Sec. III. The procedure introduces vertex form factors in the effective gluon interactions. The form factor width parameter λ is reduced from infinity down to the scale of hadronic masses through a solution of a differential equation, which eliminates large momentum transfers from the bound-state eigenvalue problem for effective gluons with small λ. The coupling strength of the three-gluon vertex as a function of λ is calculated in Sec. IV and analyzed in Sec. V. These two sections show how asymptotic freedom of effective gluons emerges in the light-front Fock space Hamiltonians for QCD. Section VI provides a short summary and a brief discussion of how the effective particle calculus can be applied to electron-hadron scattering in a simplest approximation.

II. INITIAL HAMILTONIAN

The canonical light-front QCD Hamiltonian requires regularization and counterterms [9]. To regulate the Hamiltonian, momenta $p^+=p^0+p^3$ and $p^\perp=(p^1,p^2)$ are parametrized using the + momentum ratios x and relative transverse momenta $\kappa^\perp$ that will be described below. Regularization is imposed through factors that exclude large $|\kappa^\perp|$ and small x, preserving all kinematical symmetries of the light-front dynamics (i.e., the Poincaré symmetries of the surface $x^+=x^0+x^3=0$ in space-time), and processes of creation of particles from the bare vacuum are absent. Power counting and the renormalization strategy for the absolute coordinates, $x^- = x^0 - x^3$ and $x^\perp = (x^1, x^2)$ or p^+ and $p^\perp$ [9], are modified when one goes over to the variables x and $\kappa^\perp$. However, key features remain similar and perturbative results described in the next sections agree with the expectation that the ultraviolet renormalization of light-front Hamiltonians involves functions of x.

A. Canonical terms

The classical Lagrangian density for gluon fields is

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\, F^{\mu\nu}F_{\mu\nu}, \tag{2.1}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu]$ and $A^\mu = A^{a\mu}t^a$, with $[t^a, t^b] = if^{abc}t^c$. The Lagrangian implies equations of motion, $\partial_\mu F^{\mu\nu} = ig[F^{\mu\nu}, A_\mu]$, and for fields satisfying these equations the canonical energy-momentum density tensor is $T^{\mu\nu} = -F^{a\mu\alpha}\partial^\nu A^a_\alpha + g^{\mu\nu}F^{a\alpha\beta}F^a_{\alpha\beta}/4$.

In the gauge $A^{a+} = 0$, the Lagrange equations constrain A^- to $\tilde{A}^- = (1/\partial^+)2\partial^\perp A^\perp - (2/\partial^{+2})ig[\partial^+ A^\perp, A^\perp]$ and the independent field degrees of freedom are $A^\perp$. The first term in $\tilde{A}^-$ is independent of the coupling constant g and can by definition be included in a new constrained field $A^\mu = [A^+ = 0,\, A^- = (1/\partial^+)2\partial^\perp A^\perp,\, A^\perp]$. The second term can be kept explicitly as part of the interactions. Using this convention and freely integrating by parts, one obtains an expression for the light-front energy of the constrained gluon field:

$$P^- = \frac{1}{2}\int dx^- d^2x^\perp \mathcal{H}|_{x^+=0}, \tag{2.2}$$

where $\mathcal{H} = T^{+-}$ and

$$\frac{1}{2}T^{+-} = \mathcal{H}_{A^2} + \mathcal{H}_{A^3} + \mathcal{H}_{A^4} + \mathcal{H}_{[\partial AA]^2}, \tag{2.3}$$

with

$$\mathcal{H}_{A^2} = -\frac{1}{2}A^\perp(\partial^\perp)^2 A^\perp, \tag{2.3a}$$

$$\mathcal{H}_{A^3} = gi\partial_\alpha A^a_\beta [A^\alpha, A^\beta]^a, \tag{2.3b}$$

$$\mathcal{H}_{A^4} = -\frac{1}{4}g^2[A_\alpha, A_\beta]^a[A^\alpha, A^\beta]^a, \tag{2.3c}$$

$$\mathcal{H}_{[\partial AA]^2} = \frac{1}{2}g^2[i\partial^+ A^\perp, A^\perp]^a \frac{1}{(i\partial^+)^2}[i\partial^+ A^\perp, A^\perp]^a. \tag{2.3d}$$

This expression is a candidate for further consideration in analogy to QED [10–13]. A heuristic expression for the quantum gluon energy operator is obtained by substitution:

$$A^\mu = \sum_{\sigma c}\int [k][t^c \varepsilon^\mu_{k\sigma} a_{k\sigma c} e^{-ikx} + t^c \varepsilon^{\mu *}_{k\sigma} a^\dagger_{k\sigma c} e^{ikx}]_{x^+=0}, \tag{2.4}$$

where $[k] = \theta(k^+)k^+ d^2k^\perp/(16\pi^3 k^+)$ and $\varepsilon^\mu_{k\sigma} = (\varepsilon^+_{k\sigma} = 0, \varepsilon^-_{k\sigma} = 2k^\perp \varepsilon^\perp_\sigma/k^+, \varepsilon^\perp_\sigma)$. Here σ numbers gluon spin polarization states and c is a color index. The creation and annihilation operators satisfy the commutation relations

$$[a_{k\sigma c}, a^\dagger_{k'\sigma' c'}] = k^+\tilde{\delta}(k-k')\,\delta^{\sigma\sigma'}\delta^{cc'}, \tag{2.4a}$$

where $\tilde{\delta}(p) = 16\pi^3\delta(p^+)\delta(p^1)\delta(p^2)$ and commutators among all a's and among all $a^\dagger$'s vanish. For all momenta, spins, and colors, $a_{k\sigma c}|0\rangle \equiv 0$ and $a^\dagger_{k\sigma c}$ creates bare gluons from the state $|0\rangle$.

The plain insertion of Eq. (2.4) into P^- produces terms with creation and annihilation operators appearing in all possible orders. All terms are then ordered so that creation operators stand to the left of annihilation operators. The process of ordering produces commutators of creation and annihilation operators, which lead to diverging integrals. All such terms can be dropped at this stage entirely since they will be either removed by regularization, in the case of modes with $k^+ = 0$, or, after regularization, they will be replaced by well-defined mass counterterms that result from a renormalization group procedure and contain free additive finite parts.

The ordered operator, denoted $P^-_{\rm quantum}$, is highly divergent. For example, a correction of order g^2 to the free energy $k^- = k^{\perp 2}/k^+$ of a single bare gluon state $|k\sigma c\rangle = a^\dagger_{k\sigma c}|0\rangle$ diverges due to integration over an infinite range of transverse momenta of virtual gluons that appear in the intermediate states of second-order perturbation theory. The energy correction diverges also due to small-x singularities. Namely, the gluon momentum k^+ can be shared by two intermediate gluons carrying fractions x and $1-x$. The sum over intermediate states involves an integral over x from 0 to 1, while the polarization vectors of intermediate gluons provide x and $1-x$ in denominator of the integrand, cf. $\varepsilon^\mu_{k\sigma}$ in Eq. (2.4). As another example, the product $P^-_{\rm quantum}P^-_{\rm quantum}$ is even more divergent than the energy correction because it does not contain the energy denominator that reduces the contribution of intermediate states with large momenta in perturbation theory. Consequently, $\exp(-iP^-_{\rm quantum}x^+/2)$ as a candidate for a unitary evolution operator in time x^+, is not defined before one regulates $P^-_{\rm quantum}$ by limiting the range of momentum that the bare gluons may have.

B. Regularization

The first step in the regularization procedure is made by limiting the range of momentum integration in Eq. (2.4); cf. Ref. [9]. Let $|k^\perp| < \Omega$ and $k^+ > \epsilon^+$, with the understanding that $\Omega \to \infty$ and $\epsilon^+ \to 0$ when the regularization is being removed.

The lower bound on k^+ implies that the regulated expression for $P^-_{\rm quantum}$, denoted by $P^-_{\Omega\epsilon^+}$, does not contain any terms with exclusively creation or annihilation operators. Such terms would be forced by a translationally invariant integral over x^- to preserve momentum k^+, while the momentum they would have to create or destroy is at least $n\epsilon^+$, where n denotes the number of creation or annihilation operators in such terms, respectively. These two conditions are incompatible. Hence $P^-_{\Omega\epsilon^+}$ does not contain terms that could alter the bare vacuum state $|0\rangle$. The coupling constant g is assumed to be sufficiently small for stability of the regularized theory built on top of $|0\rangle$; cf. [14].

The limits on absolute momenta in $P^-_{\Omega\epsilon^+}$ violate the boost invariance of light-front dynamics, and in the term (2.3d) with inverse powers of ∂^+, one may still have 0 in the denominator. To eliminate the violation of boost invariance and regulate the $1/\partial^+$ singularities, $P^-_{\Omega\epsilon^+}$ is further curbed through the following step.

Interaction terms in $P^-_{\Omega\epsilon^+}$ are modified so that *changes* of the particle momenta are limited. In this work, the transverse momentum changes are limited by a parameter Δ and changes of x by a parameter δ. As an example, it is useful to consider the three-gluon vertex

$$H_{A^3\Omega\epsilon^+}=\sum_{123}\int[123]\tilde{\delta}(k_1+k_2-k_3)[C_{\Omega\epsilon^+}(123)a_1^\dagger a_2^\dagger a_3 + C^*_{\Omega\epsilon^+}(123)a_3^\dagger a_2 a_1]. \tag{2.5}$$

Momentum conservation implies that one can write $k_1^+ = x_1 k_3^+$, $k_1^\perp = x_1 k_3^\perp + \kappa_{12}^\perp$, $k_2^+ = x_2 k_3^+$, $k_2^\perp = x_2 k_3^\perp - \kappa_{12}^\perp$, with $x_1+x_2=1$ and $\kappa_{12}^\perp = x_2 k_1^\perp - x_1 k_2^\perp$. Here x_1, x_2, and $\kappa_{12}^\perp$ are invariant under seven kinematical transformations of light-front dynamics. It is helpful to call the momentum carried together by all annihilated or created particles in a single vertex a *parent* momentum in the vertex. In the vertex (2.5), k_3 is a parent momentum. Also, a slash in a subscript is used below to indicate that the momentum before the slash sign is considered to be a daughter of the parent momentum after the slash sign. For example, $k_1^\perp = x_{1/3}k_3^\perp + \kappa_{1/3}^\perp$, where $x_{1/3} = x_1/x_3$ and $\kappa_{1/3}^\perp = \kappa_{12}^\perp$.

The momentum changes in the vertex (2.5) are limited by inserting a factor $r_{\Delta\delta}(\kappa_{i/3}^{\perp 2}, x_{i/3})$ for each creation and annihilation operator. Since k_3 is a parent momentum, $x_{3/3}=1$ and $\kappa_{3/3}^\perp = 0$, and the regularization factor for the term (2.5) equals $r_{\Delta\delta}(\kappa^{\perp 2},x) r_{\Delta\delta}(\kappa^{\perp 2},1-x)$, where $x=x_{1/3}$ and $\kappa^\perp = \kappa_{12}^\perp$. Factors $r_{\Delta\delta}$ are chosen to have the form

$$r_{\Delta\delta}(\kappa^{\perp 2},x) = r_\Delta(\kappa^{\perp 2}) r_\delta(x)\,\theta(x), \tag{2.6}$$

where

$$r_\Delta(z) = \exp(-z/\Delta^2) \tag{2.7}$$

and $r_\delta(x)$ suppresses the region of x smaller than δ. That $r_\Delta(z)$ falls off exponentially is a guarantee for ultraviolet convergence of all transverse momentum integrals that appear in perturbation theory. Integrals that behave as $\ln\Delta$ or Δ^n with positive n for $\Delta\to\infty$ will be called ultraviolet divergent. The small-x regulating function $r_\delta(x)$ must vanish sufficiently quickly for $x\to 0$ to regulate all small-x singularities. In a sense to become clear in Sec. V, the factors r_δ considered in this work lie in the vicinity of two cases:

$$r_\delta(x) = x/(x+\delta) \tag{2.8}$$

and

$$r_\delta(x) = \theta(x-\delta). \tag{2.9}$$

Integrals that behave as $\ln\delta$ or δ^n with negative n will be called small-x divergent. Mixing of the ultraviolet and small-x regularizations through expressions of the type $\exp[-(\kappa^{\perp 2}/x)/\Delta^2]$ is discussed in Appendix E.

Every creation and annihilation operator in all vertices in the operator $P^-_{\Omega\epsilon^+}$ is supplied with a factor $r_{\Delta\delta}$, k_3 being replaced by the corresponding parent momentum. Terms that contain four operators are recast as contracted products of terms with only three operators, which are already regulated. This step produces regularization factors $\tilde{r}$ that are given in full detail in Appendix A. The instantaneous terms containing inverse powers of ∂^+ are regulated in the same way, by interpreting the momentum that is transferred along the inverted ∂^+ as a momentum carried by a virtual particle that connects two vertices [11,12]. The fully regulated operator $P^-_{\Omega\epsilon^+}$ is denoted by $[P^-_{\Omega\epsilon^+}]_{\Delta\delta}$.

In the last step of defining the initial Hamiltonian $H_{\Delta\delta}$, the limits of $\Omega\to\infty$ and $\epsilon^+\to 0$ are taken with Δ and δ kept constant:

$$H_{\Delta\delta} = \lim_{\Omega\to\infty}\lim_{\epsilon^+\to 0}[P^-_{\Omega\epsilon^+}]_{\Delta\delta} + X_{\Delta\delta}. \tag{2.10}$$

$X_{\Delta\delta}$ denotes counterterms, which need to be found. They cannot depend on Ω and ϵ^+ in all orders of perturbation theory in the limits $\Omega\to\infty$ and $\epsilon^+\to 0$, because all changes of finite momenta are now bounded by the parameters Δ and δ, and a finite momentum cannot be connected in a finite number of fixed-size steps to the region of absolute cutoffs Ω and ϵ^+ when they are removed. Thus the regularization parameters Δ and δ define a theory whose ultraviolet structure can be analyzed using perturbative renormalization group strategy independently of Ω and ϵ^+ when these cutoff parameters are sent to their respective limits. The renormalization group procedure that provides means for finding the counterterms $X_{\Delta\delta}$ order by order in perturbation theory is described in the next sections. The initial Hamiltonian has then the form [cf. Eq. (2.3)].

$$H_{\Delta\delta} = H_{A^2} + H_{A^3} + H_{A^4} + H_{|\partial AA|^2} + X_{\Delta\delta}, \tag{2.11}$$

where, for example,

$$H_{A^2} = \sum_{\sigma c}\int[k]\frac{k^{\perp 2}}{k^+}a^\dagger_{k\sigma c}a_{k\sigma c} \tag{2.12a}$$

and

$$H_{A^3} = \sum_{123}\int[123]\tilde{\delta}(p^\dagger - p)\tilde{r}_{\Delta\delta}(3,1)[gY_{123}a_1^\dagger a_2^\dagger a_3 + gY^*_{123}a_3^\dagger a_2 a_1]. \tag{2.12b}$$

These two terms are quoted from Appendix A, where all regulated canonical terms are listed and the notation is explained. $X_{\Delta\delta}$ will be discussed below and in the next sections. Note that the free Hamiltonian (2.12a) contains no regularization. This is necessary to preserve kinematical light-front symmetries. Differences between the bare three-gluon vertex (2.12b) and effective gluon vertex from Sec. V are described there.

III. RENORMALIZATION GROUP PROCEDURE FOR PARTICLES

In Eq. (2.11), $H_{\Delta\delta}$ is expressed in terms of the creation and annihilation operators for bare gluons in regulated local theory. This section describes the renormalization group procedure [6] that is used in Sec. IV to rewrite the initial Hamiltonian $H_{\Delta\delta}$ in terms of operators that create or annihilate effective gluons, instead of bare ones. The effective gluon operators are obtained by applying a unitary transformation $\mathcal{U}_\lambda$ to the initial bare operators. The effective operators depend on the parameter λ that labels $\mathcal{U}_\lambda$.

The parameter λ has dimension of mass and distinguishes different kinds of effective gluons according to the following rule. *Effective gluons of type* λ *can change their relative motion kinetic energy through a single effective interaction by no more than about* λ. The transformation $\mathcal{U}_\lambda$ is mathematically designed in perturbation theory so that resulting interaction terms contain vertex form factors and the latter limit the kinetic energy changes by their width parameter, which equals λ. All Hamiltonians with different λ's are equal, and the rewriting does not introduce any change in the theory, although the same Hamiltonian appears differently when expressed in terms of different gluons. For brevity, the effective gluons corresponding to some value of λ are referred to as gluons of width λ.

A. General features

The form factor width parameter λ greatly differs from the regularization cutoffs, because it may be kept finite, even small, while the cutoffs have to be made extremely large to approximate the initial theory. Even if the expansion in terms of bare particles is hopelessly complicated, a hadron may still have a well-defined, convergent expansion in the basis of effective constituents with small width λ.

Thanks to the vertex form factors, in the Fock-space basis built from gluons of width λ, the effective Hamiltonian matrix elements quickly tend to zero when the effective gluons change kinetic energy across the matrix element by more than λ. Therefore, the effective Hamiltonian matrix is narrow. This is important for applications to bound-state physics because eigenstates of narrow matrices may have a small number of dominant components [15]. In the case of hadrons, the constituent model suggests that the intricate complexity of QCD is buried in the structure of constituents and their interactions, while the number of effective constituents is small. The success of perturbative QCD in reproducing changes of deep inelastic structure functions with momentum transfer down to fairly small values suggests that the structure of effective constituents can be approximated using perturbation theory. The idea of effective particles is by no means new [17]. The new element is the renormalization group procedure for effective particles in QCD.

The renormalization group procedure provides the means to find counterterms $X_{\Delta\delta}$ that have to be included in the initial Hamiltonian to compensate for the spurious effects of ultraviolet regularization. One takes advantage of the form factors in the Hamiltonian vertices in analogy to Ref. [7]. The narrow dynamics can smear states of effective particles with finite energy, only by less than $n\lambda$ towards high energies in nth-order perturbation theory. For to raise the free energy of a state by $n\lambda$, the effective interaction must act on the state about n times when the form factors die out exponentially for energy changing by more than λ. The highest-order n that is still independent of Δ approaches ∞ when $\Delta \to\infty$. Therefore, to obtain the ultraviolet regularization-independent results, at least in perturbation theory to all orders, it is sufficient to demand that the Hamiltonian coefficients in front of creation and annihilation operators with finite λ be independent of the ultraviolet regularization. Hence one can read the diverging structure of ultraviolet counterterms from the coefficients: see Eq. (3.11). However, the effects of small-x regularization are not under control of the renormalization group procedure and the coefficients with finite λ's may depend on $r_\delta(x)$. The dependence on $\delta \to 0$ sometimes drops out, but finite effects may remain, as will be shown with examples in Sec. V.

In the perturbative renormalization group procedure for deriving effective particles and their interactions, one never encounters genuine infrared singularities associated with small energy denominators. This is explained below Eq. (3.10), where the differences of invariant masses are taken care of in analogy to differences of energies in the similarity renormalization group procedure for Hamiltonians [7]. The perturbative denominators are effectively limited from below by λ. The nonperturbative part of the dynamics with relative motion kinetic energy changes smaller than λ is first tackled when one proceeds to solve the effective Schrödinger equation. Since the form factors keep the effective dynamics in a well-defined range of energies, numerical methods may apply in finding approximate solutions to the full theory [15].

B. Construction of $\mathcal{H}_\lambda$

Let a commonly denote the bare operators $a_{k\sigma c}$ or $a^\dagger_{k\sigma c}$. Operators a are transformed by the unitary operator $\mathcal{U}_\lambda$ into operators a_λ that create or annihilate effective particles of width λ, with identical quantum numbers:

$$a_\lambda = \mathcal{U}_\lambda a \mathcal{U}_\lambda^\dagger . \tag{3.1}$$

The initial Hamiltonian $H_{\Delta\delta}$ is rewritten in terms of a_λ, $H_{\Delta\delta} = H_\lambda(a_\lambda)$. If quarks were included, $H_{\Delta\delta}$ would correspond to the QCD Hamiltonian written in terms of canonical quarks and gluons, associated with bare partons or bare currents. H_λ for λ comparable with masses would represent the same Hamiltonian written in terms of constituent quarks and gluons. Applying $\mathcal{U}_\lambda$, one obtains

$$\mathcal{H}_\lambda \equiv H_\lambda(a) = \mathcal{U}_\lambda^\dagger H_{\Delta\delta} \mathcal{U}_\lambda . \tag{3.2}$$

$\mathcal{H}_\lambda$ has the same coefficient functions in front of products of a's as the effective H_λ has in front of the unitarily equivalent products of a_λ's. Differentiating $\mathcal{H}_\lambda$ with respect to λ, one obtains

$$\mathcal{H}'_\lambda = -[\mathcal{T}_\lambda, \mathcal{H}_\lambda], \tag{3.3}$$

where $\mathcal{T}_\lambda = \mathcal{U}^\dagger_\lambda \mathcal{U}'_\lambda$. Here $\mathcal{T}_\lambda$ is constructed below using the notion of vertex form factors. For example, if an operator without form factors has the structure

$$\hat{O}_\lambda = \int [k_1 k_2 k_3] V_\lambda(1,2,3) a^\dagger_{\lambda k_1} a^\dagger_{\lambda k_2} a_{\lambda k_3}, \tag{3.4}$$

the operator with form factors is written as $f_\lambda \hat{O}_\lambda$ and has the structure

$$f_\lambda \hat{O}_\lambda = \int [k_1 k_2 k_3] f_\lambda(\mathcal{M}_{12}, \mathcal{M}_3) V_\lambda(1,2,3) a^\dagger_{\lambda k_1} a^\dagger_{\lambda k_2} a_{\lambda k_3}, \tag{3.5}$$

where

$$f_\lambda(\mathcal{M}_{12}, \mathcal{M}_3) = \exp[-(\mathcal{M}^2_{12} - \mathcal{M}^2_3)^2/\lambda^4]. \tag{3.6}$$

For any operator $\hat{O}$ expressible as a linear combination of products of creation and annihilation operators, $f\hat{O}$ contains a form factor $f_\lambda(\mathcal{M}_c, \mathcal{M}_a)$ in front of each product, where $\mathcal{M}_c$ and $\mathcal{M}_a$ stand for the total free invariant masses of particles created (c) and annihilated (a) through the product, respectively. For gluons in Eq. (3.6), $\mathcal{M}_3 = 0$.

The relative motion kinetic energy changes in interaction vertices of effective particles are limited by demanding that $H_\lambda = f_\lambda G_\lambda$, with some unknown G_λ. Then one derives equations for G_λ that result from the choice for f_λ and some definition of $\mathcal{T}_\lambda$. In practice, one first uses $\mathcal{U}_\lambda$ to transform the Hamiltonian to $\mathcal{H}_\lambda = f_\lambda \mathcal{G}_\lambda$ and then one calculates coefficients of a's in $\mathcal{G}_\lambda$, which are the same as the coefficients of a_λ in G_λ. This calculation includes the construction of $\mathcal{T}_\lambda$ and proceeds as follows (the subscript λ is dropped for simplicity of notation):

$$\mathcal{H}' = f'\mathcal{G} + f\mathcal{G}' = -[\mathcal{T}, \mathcal{G}_0] - [\mathcal{T}, \mathcal{G}_I]. \tag{3.7}$$

$\mathcal{G}$ is split into two parts. The free part $\mathcal{G}_0$ is bilinear in a's and independent of the coupling constant g. The remaining interaction-dependent part is denoted by $\mathcal{G}_I$. In the present work, $\mathcal{G}_0$ is taken to be independent of λ. The definitions of f and $\mathcal{G}_0$ imply then that $f\mathcal{G}_0 = \mathcal{G}_0$ and $f'\mathcal{G}_0 = 0$. Equation (3.7) contains two unknowns $\mathcal{T}$ and $\mathcal{G}_I$. Without loss of generality, one assumes that $\mathcal{T} \to 0$ when $\mathcal{G}_I \to 0$, and one expands operators in powers of $\mathcal{G}_I$ with the goal of enabling the procedure to work order by order. The expansion of $\mathcal{T}$ starts from the term of order $\mathcal{G}_I$. Changes of $\mathcal{G}_I$ with λ should start from second power. If

$$f\mathcal{G}'_I = -f[\mathcal{T}, \mathcal{G}_I], \tag{3.8}$$

then

$$[\mathcal{T}, \mathcal{G}_0] = [(1-f)\mathcal{G}_I]' \tag{3.9}$$

and

$$\mathcal{G}'_I = [f\mathcal{G}_I, \{(1-f)\mathcal{G}_I\}'_{\mathcal{G}_0}], \tag{3.10}$$

where the curly brackets with subscript $\mathcal{G}_0$ indicate the solution for $\mathcal{T}$ that follows from Eq. (3.9). The choice of f made above implies that perturbation theory for $\mathcal{T}$ and $\mathcal{G}$ does not lead to small energy differences in the denominators, since $1-f$ vanishes quadratically with the energy difference. $\mathcal{G}_I$ contains only connected interactions because Eq. (3.10) has a commutator on the right-hand side. The initial condition for Eq. (3.10) is provided by $H_{\Delta\delta}$, so that Eq. (3.10) in integral form reads

$$\mathcal{G}_\lambda = H_{\Delta\delta} + \int_\infty^\lambda ds [f_s \mathcal{G}_{Is}, \{(1-f_s)\mathcal{G}_{Is}\}_{\mathcal{G}_0}], \tag{3.11}$$

which allows one to find the counterterms $X_{\Delta\delta}$ using the condition that they remove the dependence on regularization from the second term for finite λ and relative momenta of interacting particles. The counterterms contain free finite parts that need to be determined using experimental data, including symmetries such as Poincaré symmetry of observables or current conservation. Finally, $\mathcal{H}_\lambda = f_\lambda \mathcal{G}_\lambda$.

C. Perturbation theory for $\mathcal{G}_{I\lambda}$

This section contains formulas used in Sec. IV for solving Eq. (3.11) in perturbation theory up to third order. Since the perturbative expansion has formally the same structure as in scalar theory [18], only the main steps are listed. In the first step, $\mathcal{G}_{I\lambda}$ is expanded into a series of terms $\tau_n \sim g^n$:

$$\mathcal{G}_I = \sum_{n=1}^\infty \tau_n. \tag{3.12}$$

It immediately follows from Eq. (3.11) that τ_1 is independent of λ and equal to the second term in the initial Hamiltonian from Eq. (2.11), i.e., Eq. (2.12b). One has $\tau_1 = \alpha_{21} + \alpha_{12}$, where α_{21} denotes the first and α_{12} the second term on the right-hand side of Eq. (2.12b). The left subscript denotes the number of creation and the right subscript the number of annihilation operators. The corresponding Hamiltonian interaction term is obtained by multiplying the integrand in Eq. (2.12b) by $f_\lambda = \exp[-(k_1+k_2)^4/\lambda^4]$ and transforming a's into a_λ's.

For $\tau_2 = \beta_{11} + \beta_{31} + \beta_{13} + \beta_{22}$, Eq. (3.10) implies

$$\tau'_2 = [\{f'\tau_1\}, f\tau_1] \equiv f_2[\tau_1 \tau_1], \tag{3.13}$$

where $f_2 = \{f'\}f - f\{f'\}$, with the understanding that the first factor f in all terms of f_2 is for the first τ in the square brackets and the second factor f in all terms of f_2 is for the second τ in the brackets. The square brackets denote all connected terms that result from contractions replacing products $a_i a^\dagger_j$ by commutators $[a_i, a^\dagger_j]$. The solution for τ_2 is then

$$\tau_{2\lambda} = \mathcal{F}_{2\lambda}[\tau_1 \tau_1] + \tau_{2\infty}, \tag{3.14}$$

where $\mathcal{F}_{2\lambda} = \int_\infty^\lambda f_2$ depends on incoming and outgoing momenta in the two vertices formed by the operators in the square brackets. In the sequence $a\tau_{ab} b \tau_{bc} c$, the three successive configurations of particle momenta are labeled by a, b, and c. To write down compact expressions for $\mathcal{F}_{2\lambda}$, the symbol $uv=\mathcal{M}_{uv}^2-\mathcal{M}_{vu}^2$ is defined, where $\mathcal{M}_{uv}^2$ denotes the free invariant mass of a set of particles from configuration u that are connected to the particles in configuration v by an interaction τ_{uv} in the sequence $u\tau_{uv}v$. Spectators of the interaction τ_{uv} do not count. In this notation,

$$f_\lambda(\mathcal{M}_{ab},\mathcal{M}_{ba})=\exp[-(ab^2/\lambda^4)]\equiv f_{ab}. \quad (3.15)$$

The parent momentum for the vertex connecting two configurations u and v is denoted by P_{uv}, and in the following equations, p_{uv} is written in place of P_{uv}^+. In all expressions, the minus component of the momentum of every gluon is given by the eigenvalue of $\mathcal{G}_0=H_{A^2}$ from Eq. (2.12a), i.e., $k^-=k^{\perp 2}/k^+$. Thus, with the Gaussian vertex form factors,

$$\mathcal{F}_2(a,b,c)=\frac{p_{ba}ba+p_{bc}bc}{ba^2+bc^2}[f_{ab}f_{bc}-1]. \quad (3.16)$$

In Eq. (3.14),

$$\tau_{2\infty}=H_{A^4}+H_{[\partial AA]^2}+X_{\Delta\delta 2}, \quad (3.17)$$

where $X_{\Delta\delta 2}$ denotes all ultraviolet counterterms proportional to g^2.

For third-order terms $\tau_3=\gamma_{21}+\gamma_{12}+\gamma_{41}+\gamma_{14}+\gamma_{32}+\gamma_{23}$, Eq. (3.10) gives

$$\tau_3'=[f\tau_1,\{(1-f)\tau_2\}']+[\{f'\tau_1\},f\tau_2]. \quad (3.18)$$

After integration,

$$\tau_{3\lambda}=\mathcal{F}_{31\lambda}[\tau_1[\tau_1\tau_1]]-\mathcal{F}_{32\lambda}[[\tau_1\tau_1]\tau_1] + \mathcal{F}_{2\lambda}[\tau_{2\infty}\tau_1+\tau_1\tau_{2\infty}]+\tau_{3\infty}, \quad (3.19)$$

where, for any sequence $a\tau_{ab}b\tau_{bc}c\tau_{cd}d$,

$$\mathcal{F}_{32}(a,b,c,d)=-\mathcal{F}_{31}(d,c,b,a), \quad (3.20)$$

$$\mathcal{F}_{31}(a,b,c,d)=\frac{p_{cb}cb+p_{cd}cd}{cd^2+cd^2}\Bigg[(p_{bd}bd+p_{ba}ba)\times\left(\frac{f_{ab}f_{bc}f_{cd}f_{bd}-1}{ab^2+bc^2+cd^2+bd^2}-\frac{f_{ab}f_{bd}-1}{ab^2+bd^2}\right)+p_{bd}\frac{bc^2+cd^2}{db}\left(\frac{f_{ab}f_{bc}f_{cd}-1}{ab^2+bc^2+cd^2}-\frac{f_{ab}f_{bc}f_{cd}f_{bd}-1}{ab^2+bc^2+cd^2+bd^2}\right)\Bigg]. \quad (3.21)$$

The last term in Eq. (3.19) denotes counterterms proportional to g^3, i.e., $\tau_{3\infty}=X_{\Delta\delta 3}$. The next section describes the calculation of γ_{21}. The calculation requires knowledge of β_{11}, β_{22}, and β_{31}. For all terms, $\pi_{ij}=\pi_{ji}^\dagger$.

IV. INTERACTIONS OF EFFECTIVE GLUONS

This section describes the derivation of effective gluon dynamics in third-order perturbation theory in the coupling constant g_λ that measures the strength of the effective three-gluon vertex in the Hamiltonian $H_\lambda(a_\lambda)$. Since $g_\lambda=g+o(g^3)$, the Hamiltonian terms of order g_λ and g_λ^2 can be calculated using an expansion in powers of g before one proceeds to the third-order terms that define g_λ. All calculations are carried out in the framework described in Sec. III.

In the rewritten Hamiltonian $H_\lambda(a_\lambda)$, the terms that have coefficients of order 1 are

$$H_{(0)}=\sum_{\sigma c}\int[k]\frac{k^{\perp 2}}{k^+}a_{\lambda k\sigma c}^\dagger a_{\lambda k\sigma c}. \quad (4.1)$$

The subscript λ indicates that $a_{\lambda k\sigma c}$ annihilate and $a_{\lambda k\sigma c}^\dagger$ create effective gluons of width λ. The effective gluons can also be interpreted as having a spatial transverse width on the order of $1/\lambda$ for moderate values of x. This interpretation is explained below Eq. (4.3).

The terms in $H_\lambda(a_\lambda)$ that have coefficients order g are (see Appendix A for details of the notation)

$$H_{(1)}=\sum_{123}\int[123]\tilde{\delta}(p^\dagger-p)f_\lambda(\mathcal{M}_{12},0)\tilde{r}_\delta(x_1)\times[gY_{123}a_{\lambda 1}^\dagger a_{\lambda 2}^\dagger a_{\lambda 3}+gY_{123}^*a_{\lambda 3}^\dagger a_{\lambda 2}a_{\lambda 1}], \quad (4.2)$$

where

$$\tilde{r}_\delta(x)=r_\delta(x)r_\delta(1-x), \quad (4.3)$$

and the form factor $f_\lambda(\mathcal{M}_{12},0)=\exp[-\kappa_{12}^{\perp 4}/(x_1x_2\lambda^2)^2]$ falls off as a function of the relative transverse momentum at a rate that depends on x carried by gluons. For moderate values around 1/2, the transverse momentum width is on the order of $\lambda/2$, but for x approaching 0, the transverse momentum width of the vertex becomes very small, leading to a spread of the interaction strength in the transverse spatial directions. Thus the coupling of effective gluons to the wee region is quite different from the canonical coupling in Eq. (2.12a).

Terms with coefficients of order g^2 are derived by changing a to a_λ in $\tau_{\lambda 2}=\beta_{\lambda 11}+\beta_{\lambda 31}+\beta_{\lambda 13}+\beta_{\lambda 22}$ and by inserting form factors as described in Sec. III B. The contribution of $\tau_{2\infty}$ includes a counterterm induced by the ultraviolet regularization r_Δ. Namely, to evaluate the terms $\beta_{\lambda 11}$ one needs to know the counterterm $\beta_{\infty 11}$. It follows from Eq. (3.14) that

$$\beta_{\lambda 11}=\sum_{\sigma c}\int[k]\frac{\mu_\lambda^2}{k^+}a_{k\sigma c}^\dagger a_{k\sigma c}, \quad (4.4)$$

where

$$\mu_\lambda^2=\frac{g^2}{16\pi^3}\int_0^1\frac{dx}{x(1-x)}\int d^2\kappa^\perp\frac{1}{k^+}\mathcal{F}_{2\lambda}(k,K,k)\times 2\sum_{12}|Y_{12k}|^2\tilde{r}_{k,1}^2+\mu_\infty^2, \quad (4.5)$$

and the last term μ_∞^2 is contributed by the counterterm $\beta_{\infty 11}$. The structure of $\beta_{\infty 11}$ is known here from hindsight: i.e., the regularization dependence of the integral in Eq. (4.5) results in a number that depends on the function $r_\Delta(\kappa^2)$, but does not depend on the gluon quantum numbers. One has $k^- = k^{\perp 2}/k^+$, $K^- = (\mathcal{M}^2 + k^{\perp 2})/k^+$, $\mathcal{M}^2 = \kappa^{\perp 2}/[x(1-x)]$, and $\tilde r_{k,1}$ is given at the end of Appendix A. The sum over quantum numbers of intermediate two-gluon states is $\Sigma_{12}|Y_{12k}|^2 = N_c \kappa^2[1 + 1/x^2 + 1/(1-x)^2] = \kappa^2 P(x)/[2x(1-x)]$, where $P(x)$ is the Altarelli-Parisi gluon splitting function $P_{GG}(x)$ [19]. Here $N_c = 3$ denotes the number of colors.

Assuming that for some $\lambda = \lambda_0$ the effective gluon mass squared should have the value $\mu^2_{0\delta}$, one can calculate the counterterm mass μ^2_∞ from Eq. (4.5). The resulting effective mass takes the form

$$\mu^2_\lambda = \mu^2_{0\delta} + \frac{g^2}{16\pi^3}\int_0^1 \frac{dx\, r_{\delta\mu}(x)}{x(1-x)} \int d^2\kappa^\perp \frac{1}{k^+} \times [\mathcal{F}_{2\lambda} - \mathcal{F}_{2\lambda_0}]\kappa^2 P(x)/[x(1-x)], \tag{4.6}$$

which is independent of the ultraviolet regularization:

$$r_{\delta\mu}(x) = [\tilde r_\delta(x)]^2 = r^2_\delta(x) r^2_\delta(1-x). \tag{4.7}$$

To find a more convenient notation for μ^2_λ, the arbitrary, possibly δ-dependent number $\mu^2_{0\delta}$ can be replaced by an expression that follows from a second-order perturbative result for a single effective gluon mass correction obtained from the eigenvalue equation for $H_\lambda(a_\lambda)$. This expression introduces a new parameter μ^2_δ that would equal the second-order result for physical gluon mass squared, if such gluons existed. The present paper does not discuss effects that may be induced by different choices of μ_δ in effective gluon Hamiltonians calculated to orders higher than g^3 or in solving the effective Schrödinger equation with $H_\lambda(a_\lambda)$. With the form factor f of Eq. (3.15), one obtains

$$\mu^2_\lambda = \mu^2_\delta + \frac{g^2}{(4\pi)^2}\int_0^1 dx\, r_{\delta\mu}(x) P(x) \int_0^\infty dz\, \exp[-2z^2/\lambda^4], \tag{4.8}$$

while the counterterm mass is

$$\mu^2_\infty = \mu^2_\delta + \frac{g^2}{(4\pi)^2}\int_0^1 dx\, r_{\delta_\mu}(x) P(x) \times \int_0^\infty dz\, \exp[-4zx(1-x)/\Delta^2]. \tag{4.9}$$

Inclusion of n_f flavors of quarks with bare masses m_f produces an additional term in the integrand in Eq. (4.8) equal to $\Sigma_{f=1}^{n_f} Q_f(x)\exp[-2z_f^2/\lambda^4]$, with $Q_f(x) = 1 - 2x(1-x) + 2m_f^2/z_f$ resembling the gluon splitting function into a pair of massive quarks of flavor f and $z_f = z + m_f^2/[x(1-x)]$. Quarks would also add $\Sigma_{f=1}^{n_f} Q_f(x)$ to $P(x)$ in Eq. (4.9). Note that for convergence of the integral over x in the effective mass term (4.8), it is sufficient that the regulating function behaves as $r_\delta(x) \sim x^\epsilon$ with $\epsilon > 0$. In the counterterm (4.9), $r_\delta(x)$ has to vanish at small x at least as fast as $x^{1/2+\epsilon}$.

Having the result for $\beta_{\lambda 11}$, one replaces the bare a with effective a_λ. The second-order effective gluon mass term is

$$H_{(2)11} = \sum_{\sigma c} \int [k] \frac{\mu^2_\lambda}{k^+} a^\dagger_{\lambda k \sigma c} a_{\lambda k \sigma c}. \tag{4.10}$$

Other terms of order g^2 are derived following the same path, but they do not require ultraviolet counterterms for regularization adopted in Sec. II.

Of main interest to this work is the term $\gamma_{\lambda 21}$ whose calculation involves τ_1, β_{11}, β_{22}, and β_{31}. Details of the calculation are described in Appendixes B and C. Equation (B1) for $\gamma_{\lambda 21}$ has the following structure [cf. Eq. (3.11)]:

$$\gamma_{\lambda 21} = \gamma_{\infty 21} + \int dx\, d^2\kappa^\perp [F_\lambda C + D] R_\Delta. \tag{4.11}$$

F_λ denotes form factors that multiply terms C, which means that momentum integrals in terms C cannot produce a dependence on the ultraviolet regularization factors R_Δ in the limit $\Delta \to \infty$. The first step in calculating $\gamma_{\lambda 21}$ is to evaluate the terms $\int D R_\Delta$ and find their dependence on R_Δ, to construct the counterterms $\gamma_{\infty 21}$. This step is carried out in Appendix C, including n_f flavors of quarks, with the result [see Eqs. (B9) and (C18)] that

$$\gamma_{\infty 21} = \sum_{123} \int [123] \tilde\delta(k_1 + k_2 - k_3) \frac{g^3}{16\pi^3} \gamma_\infty a_1^\dagger a_2^\dagger a_3, \tag{4.12}$$

where

$$\gamma_\infty = Y_{123} \frac{-\pi}{3} \ln\frac{\Delta}{\mu} \{N_c[11 + h(x_1)] - 2n_f\} + \gamma_{\text{finite}} \tag{4.13}$$

and

$$h(x_1) = 6\int_{x_1}^1 dx\, r_{\delta t}(x)\left[\frac{2}{1-x} + \frac{1}{x-x_1} + \frac{1}{x}\right] - 9\tilde r_\delta(x_1)\int_0^1 dx\, r_{\delta\mu}(x)\left[\frac{1}{x} + \frac{1}{1-x}\right] + (1 \leftrightarrow 2). \tag{4.13a}$$

$\tilde r_\delta(x_1)$ is given by Eq. (4.3), $r_{\delta\mu}(x)$ by Eq. (4.7), and $r_{\delta t}(x)$ by Eq. (B4a). The counterterm part γ_{finite} removes finite effects of ultraviolet regularization. It contains unknown functions of x_1 that multiply the structures Y_{12}, Y_{13}, Y_{23} in Y_{123}, and the additional structure Y_4: see Eqs. (C10) and (C15). Here γ_{finite} differs from canonical terms, but it does not influence the third-order running of the Hamiltonian coupling constant.

Having found the counterterm, one obtains a finite answer for $\gamma_{\lambda 21}$ in the limit $\Delta \to \infty$:

$$\gamma_{\lambda 21}=\sum_{123}\int[123]\tilde{\delta}(p^{\dagger}-p)[W_{\lambda 12}Y_{12}+W_{\lambda 13}Y_{13}+W_{\lambda 23}Y_{23}+W_{\lambda 4}Y_4]a_1^{\dagger}a_2^{\dagger}a_3, \tag{4.14}$$

where the four coefficients W_λ are functions of x_1 and $\kappa_{12}^{\perp 2}$. The complete expression for $\gamma_{\lambda 21}$ is given in Appendix B. The effective three-gluon interaction term of order g^3 in $H_\lambda(a_\lambda)$ is $H_{(3)21}+H_{(3)12}$, where $H_{(3)12}=H_{(3)21}^{\dagger}$ and

$$H_{(3)21}=f_\lambda \mathcal{U}_\lambda \gamma_{\lambda 21}\mathcal{U}_\lambda^{\dagger}. \tag{4.15}$$

The operator $\mathcal{U}_\lambda$ is unitary with accuracy to terms of order higher than third, and its action is equivalent here to changing a's to a_λ's as in lower-order terms.

V. THREE-GLUON VERTEX

This section describes the running coupling constant g_λ that appears in Hamiltonian interaction vertices written in terms of gluons of width λ. The word "running" means changing with λ. The coupling constant is defined through the strength of the three-gluon vertex for some value of $x_1=x_0$ when $\kappa_{12}^{\perp}\to 0$, in direct analogy to the coupling constant in ϕ^3 theory in six dimensions [18] or electric charge in the Thomson limit in QED. The effective three-gluon vertex is a sum of $H_{(1+3)21}$ and $H_{(1+3)21}^{\dagger}$, where $H_{(1+3)}=H_{(1)}+H_{(3)}$ and

$$H_{(1+3)21}=\sum_{123}\int[123]\tilde{\delta}(p^{\dagger}-p)f_\lambda[V_{\lambda 12}Y_{12}+V_{\lambda 13}Y_{13}+V_{\lambda 23}Y_{23}+W_{\lambda 4}Y_4]a_{\lambda 1}^{\dagger}a_{\lambda 2}^{\dagger}a_{\lambda 3}, \tag{5.1}$$

with $f_\lambda=\exp\{-[\kappa_{12}^{\perp 2}/(x_1x_2\lambda^2)]^2\}$. The effective vertex contains creation and annihilation operators for effective gluons of width λ, instead of the bare ones in Eqs. (2.12b) and (4.14). This change is reflected by the presence of the form factor f_λ, which strongly suppresses the emission of effective gluons with transverse momentum larger than $x_1x_2\lambda$. The vertex functions $V_{\lambda 12}(x_1,\kappa_{12}^{\perp 2})$, $V_{\lambda 13}(x_1,\kappa_{12}^{\perp 2})$, and $V_{\lambda 23}(x_1,\kappa_{12}^{\perp 2})$ are used below to evaluate the running coupling constant. The fourth function $W_{\lambda 4}(x_1,\kappa_{12}^{\perp 2})$ multiplies Y_4, which is distinct from canonical structures. $W_{\lambda 4}$ is independent of λ in the limit $\kappa_{12}^{\perp}\to 0$ and does not contribute to the running coupling constant.

When $\kappa_{12}^{\perp}\to 0$, all three vertex functions $V_{\lambda ij}$, for $ij=12,13,23$, vary with λ equally. Using results from Appendix D without quarks, one can introduce a single function

$$W_\lambda(x)=V_{\lambda ij}(x,0^{\perp})-V_{\lambda_0 ij}(x,0^{\perp})=-\frac{g^3}{48\pi^2}N_c[11+h(x)]\ln\frac{\lambda}{\lambda_0}, \tag{5.2}$$

to describe the behavior of all three functions $V_{\lambda ij}(x,0^{\perp})=g\tilde{r}_\delta(x)+W_\lambda(x)$ for finite x and λ in the limit $\delta\to 0$.

$W_\lambda(x)$ depends on x through $h(x)$. The case $h(x)=0$ corresponds to the standard asymptotic freedom result [16] in the following sense. At a certain value of $\lambda=\lambda_0$, the three potentially different numbers $g_{\lambda_0 ij}\equiv V_{\lambda_0 ij}(x_0,0^{\perp})$ can be set equal to a common value $g_0=g_{\lambda_0}$ by adjusting finite parts of the counterterm (4.13) to match the gauge symmetry result from the bare canonical Hamiltonian of QCD. For $h(x)=0$, the choice of x_0 does not matter as long as δ is negligible in comparison to x_0 and $1-x_0$. Then, for $\lambda\neq\lambda_0$, the three coupling constants remain equal, $g_{\lambda ij}=V_{\lambda ij}(x_0,0^{\perp})=g_\lambda=g_0+W_\lambda(x_0)+o(g_0^5)$, with $g=g_0$ in $W_\lambda(x_0)$, or

$$g_\lambda=g_0-\frac{g_0^3}{48\pi^2}11N_c\ln\frac{\lambda}{\lambda_0}. \tag{5.3}$$

Expressing g_0 by g_λ in the differential Eq. (3.10) or in its integrated forms, Eq. (3.11) or Eq. (5.3), in the perturbative expansion up to third order in g_λ, one has

$$\lambda\frac{d}{d\lambda}g_\lambda=\beta_0 g_\lambda^3, \tag{5.4}$$

with

$$\beta_0=-\frac{11N_c}{48\pi^2}, \tag{5.5}$$

which is equal to the β-function coefficient in Feynman calculus in QCD. Thus, if one identifies the effective Hamiltonian form factor width parameter λ with the running momentum scale in Feynman diagrams, the standard result from off-shell S-matrix calculus would be recovered in effective Hamiltonians in the case $h(x)=0$.

The function $h(x)$ is determined by the initial Hamiltonian and small-x regularization factor r_δ in it through Eq. (4.13a). In the limit $\delta\to 0$, for r_δ of Eq. (2.8),

$$h(x)=12\left[3+\frac{1-x-x^2}{(1-x)(1-2x)}\ln x+\frac{(1-x)^2-x}{x(1-2x)}\ln(1-x)\right] \tag{5.6}$$

and, in the case of r_δ from Eq. (2.9),

$$h(x)=12\ln\min(x,1-x). \tag{5.7}$$

These two cases are shown in Fig. 1 along with case (c), which corresponds to $h(x)=0$. The vertex function $V(x)$ in Fig. 1 is defined by

$$g_0V(x)=g_0+W_\lambda(x) \tag{5.8}$$

and plotted using $\alpha_0=g_0^2/(4\pi)=0.1$ for $\lambda_0=100$ GeV and $N_c=3$. In part (d) of Fig. 1, the value of $V(1/2)$ is plotted against λ. The sharp cutoff case (b) of $\theta(x-\delta)$ is visibly different from (c), which corresponds to $h(x_1)=0$, but the continuous case (a) of $x/(x+\delta)$ differs from (c) only by 8%. By extrapolation, the results (a) and (b) suggest that there may exist a regularization factor $r_\delta(x)$ that matches the case (c). The mathematical structure of Eq. (4.13a) and the Euclidean integrals in Feynman diagrams with dimensional regularization [20] both hint at the power-law functions $r_\delta(x)=x^\delta$. Inspection shows that, in the limit $\delta\to 0$,

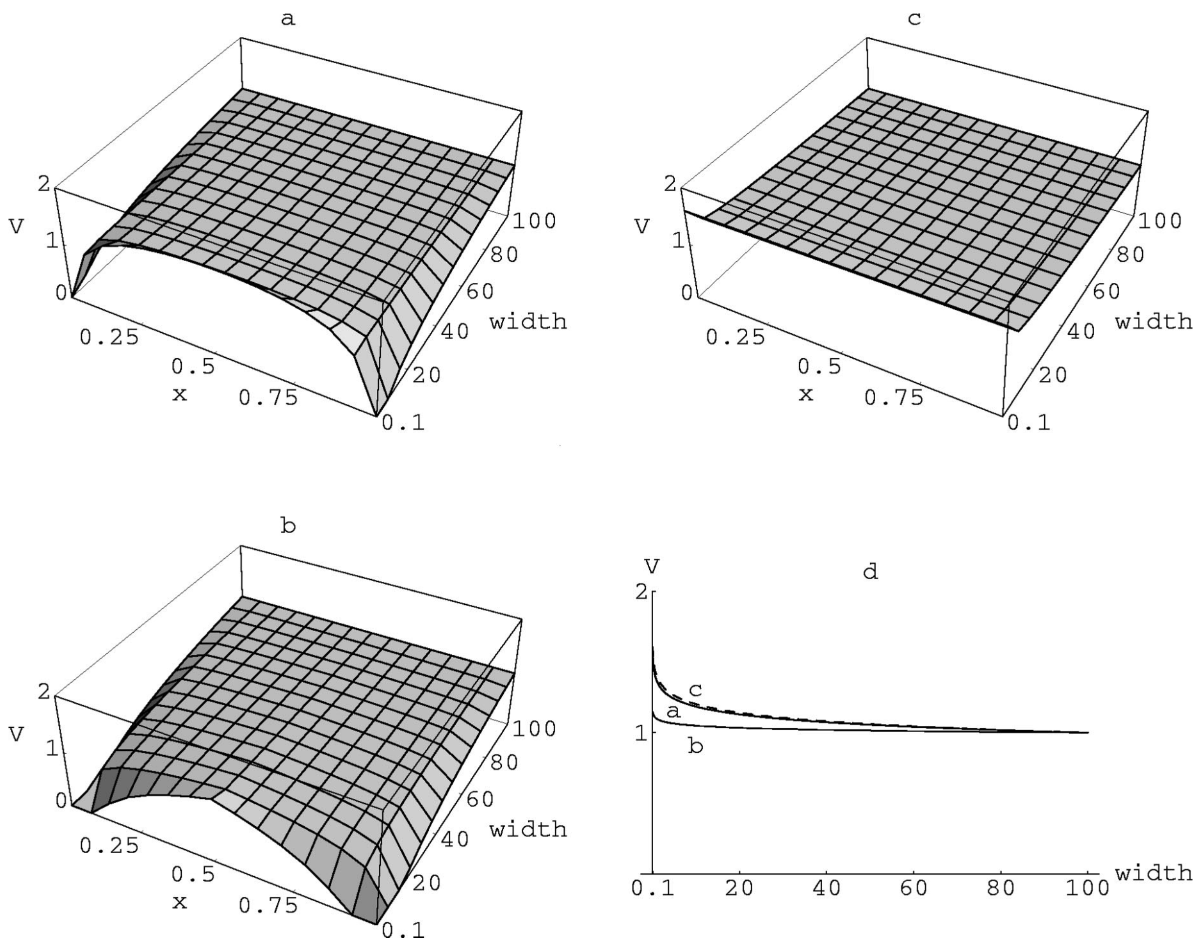

FIG. 1. Change of the effective gluon vertex function $V(x)$ from Eq. (5.8) with the width λ varying from $\lambda_0 = 100$ GeV down to 100 MeV, for three different small-x regularization functions: (a) $r_\delta = x/(x+\delta)$, (b) $r_\delta = \theta(x-\delta)$, and (c) $r_\delta = x^\delta \theta(x-\epsilon)$. Part (d) shows the dependence of $V(1/2)$ on λ for the three cases, correspondingly. The case (c), dashed line, matches the QCD running coupling constant result obtained from Feynman diagrams. Note the dynamical suppression of the effective gluon coupling for extreme values of x in cases (a) and (b). See the text for details.

$$r_\delta(x) = x^\delta \theta(x-\epsilon) \tag{5.9}$$

leads to $h(x_1) = 0$ for all fixed values of x_1 between 0 and 1, as long as $\epsilon/\delta \to 0$. This result sets a path for developing a connection between the Hamiltonian dynamics of effective gluons in the light-front Fock space and Feynman diagrams for Green's functions in the Lagrangian calculus [16].

The second scale ϵ in Eq. (5.9) is required to regulate linear small-x divergences such as in the mass counterterms in Eq. (4.9). Details of $r_\delta(x)$ at $x \sim \epsilon$ are not important for pointwise convergence of $h(x_1)$ to 0, and a whole class of regularizations with a second scale $\epsilon \ll \delta$ gives the same result. Ultraviolet regularizations using invariant mass soften small-x divergences in the mass counterterms so that the second scale ϵ appears unnecessary, but such regularizations mix small-x effects with large $\kappa^\perp$ divergences: see Appendix E. In evaluation of $h(x_1)$, factorization of the ultraviolet renormalization group flow in λ from small-x effects requires $\delta \to 0$. The small-x behavior of other terms in this limit has to be compared with the size of g to discuss the validity of the perturbative analysis. The warning is warranted by the fact that in Fig. 1, where the value of $\alpha \sim 0.1$ for λ on the order of 100 GeV is taken from phenomenology based on Feynman diagrams, $g_0 \sim 1.1 > 1$.

Figure 1 shows that small-x regularizations of Eqs. (2.8) and (2.9) lead to a suppression of effective gluon interactions when x_1 moves away from 1/2 for $\kappa_{12}^\perp \to 0$, where $f_\lambda \to 1$. They also slow down the rate of growth of the coupling constant with lowering λ. Similar results follow for f_λ depending on p^- instead of $\mathcal{M}^2$. Here λ^2 is replaced by $k_3^+ \lambda$ and one loses boost invariance, but in the ratio $(\lambda_1 k_3^+)/(\lambda_2 k_3^+)$ the dependence on k_3^+ cancels out.

Thorn [21] calculated the diverging part of a four-point gluon Green's function using the $A^+ = 0$ gauge and integrating Feynman diagrams first over k^-. Using sharp cutoffs $k^+ > \epsilon^+$ and $|k^\perp| \leq \Lambda$ in the remaining integrals, Thorn found that to understand asymptotic freedom from the cutoff dependence of Green's functions, one has to include direct and crossed box diagrams that cancel skew p^+-dependent terms from the three-point function whose own cutoff dependence had opposite sign to the asymptotic freedom. Lepage and Brodsky [13] developed a whole formalism for hard exclusive processes involving hadrons. Perry [22] used their rules in old fashioned perturbation theory to obtain the ultraviolet diverging part of a set of third-order terms in the quark-gluon-quark off-energy-shell vertex function, with similar cutoffs to Thorn's. In the limit of ultraviolet cutoffs being sent to infinity, Perry found asymptotic freedom in the cutoff dependence of the quark-gluon vertex function. Irrelevant parts diverged as functions of the small-p^+ cutoff and depended on the off-shell energy parameter, but Perry reported that they could be made small by using an invariant mass cutoff. In calculations of the Green's functions, small energy denominators require an extra step of setting a lower bound on the transverse momenta and factorization of the contributions from below. Such contributions complicate issues that arise in the introduction of the renormalization scale at small scales, and gluon coupling must be brought under control [23]. The dependence on the energy-shell parameter involves Hamiltonian eigenvalues, which depend on a solution of the hadronic binding problem.

Effective Hamiltonians are different and are calculated differently from Green's functions. Asymptotic freedom appears in the dependence of the Hamiltonian vertices on the width λ. The width may be finite and adjusted to the scale of processes one is interested in, independently of the diverging ultraviolet cutoffs. Thus, for example, Thorn's results for Green's functions calculated in the scattering theory for gluons in asymptotic states do not contradict the results of this paper, which concern Hamiltonian vertices for gluons of width λ in a boost-invariant approach, independently of any non-boost-invariant regularization cutoffs on individual bare gluon momenta, such as those studied in Thorn's work. Note also that $\mathcal{U}_\lambda$ is unitary and the procedure involves no wave function renormalization, in distinction from the standard renormalization group concept, and no Ward identity is used. λ effectively limits energy denominators from below, so that issues of binding are clearly separated from logarithmic evolution of effective Hamiltonians. No off-shell energy parameter is introduced. In summary, no need arises to rely on the arbitrary ultraviolet regularization, no extra lower bound on momenta is needed, and no off-shell energy parameter is introduced. The notion of effective constituents is then a natural candidate for the phenomenology of hadronic wave functions [24] to be put on the firm ground of Hamiltonian quantum mechanics, with an open path to make connection with diagrammatic techniques [25] for scattering amplitudes.

Although the renormalized Hamiltonian vertices display a tendency to decouple dynamics of effective gluons from the small-x region, many x-dependent terms completely drop out from the third-order running of the coupling constant. These terms require careful investigation since vacuum effects may enter through the small-x region [14]. Once these terms are calculated in detail using the perturbative procedure, they can be subsequently studied in the Schrödinger equations with effective Hamiltonians beyond perturbation theory. In particular, variational studies could aim at understanding gluon condensation and spontaneous chiral symmetry breaking, including the cases where the number of flavors approaches the critical value at which asymptotic freedom goes away [26]. The small-x regularization clearly interfaces in this transition: see Eq. (4.13).

VI. CONCLUSION

Coefficients of products of creation and annihilation operators for effective gluons of width λ in the renormalized light-front QCD Hamiltonian contain vertex form factors f_λ and vary with λ in a perturbatively calculable way. Obtained in third-order perturbation theory, the effective three-gluon vertex with vanishing transverse momentum is a linear function of $\ln\lambda$. The coefficient of $\ln\lambda$ matches the coefficient of $\ln Q$ that appears in the running coupling constant dependence on the running scale Q in the Lagrangian calculus for Green's functions. This happens for small-x regularizations of the type $r_\delta(x)=x^\delta\theta(x-\epsilon)$ in the limit $\delta\to 0$ for $\epsilon/\delta \to 0$. For other regularizations, such as $r_\delta(x)=x/(x+\delta)$ or $r_\delta(x)=\theta(x-\delta)$, the coefficient of $\ln\lambda$ contains an additive term $h(x)$, defined in Eq. (4.13a) and described in Sec. V. Here $h(x)$ suppresses interactions at small x in addition to the suppression implied by the vertex form factors f_λ.

In the effective particle light-front Fock-space basis in gauge $A^+=0$, asymptotic freedom is a consequence of a single effective gluon containing a pair of bare gluons. This component amplifies the strength with which effective gluons can split into effective gluons when λ gets smaller. The mechanism is the same as in the case of scalar particles in six dimensions [18], although the source of $\ln\lambda$ is different. Namely, in scalar theory it is the integration over additional transverse momentum dimensions, while in QCD it is the transverse momentum factors in gluon polarization vectors. In a perturbative description of processes characterized by a physical momentum scale Q, using $H_\lambda(a_\lambda)$, there will appear powers of g_λ and $\ln Q/\lambda$. For $Q/\lambda=1$, $\ln Q/\lambda=0$, so that the theoretical predictions will have the form of a series in powers of the asymptotically free running coupling constant g_Q.

The dynamics of effective particles includes binding, described through the Schrödinger equation with $H_\lambda(a_\lambda)$. Initial studies of some simplified bound-state eigenvalue problems for light-front QCD Hamiltonian matrices have recently been carried out for heavy quarkonia [27] and gluonium states [28], reporting reasonable results. Although the effective particle approach described here is different, the bound-state equations resulting from the second-order perturbation theory may be similar. So the initial studies suggest that the effective particle approach should be tested in application to bound states of constituent quarks and gluons.

Besides the bound-state dynamics of low-energy constituents in the hadronic rest frame, the boost-invariant effective particle approach provides a theoretical tool for studies of layers of hadronic structure in other frames of reference, including the infinite-momentum frame. The theory is simplest, and in lowest order of the same type as for quarks and gluons, in the case of electron-hadron interactions through one-photon exchange. In the case of massive vector bosons, there exist three instead of only two polarization states and the choice of gauge $A^+=0$ is not directly available. A scheme to attempt the more complicated theory was formulated by Soper [29] in massive QED, but the required renormalization procedure for Hamiltonians with massive gauge bosons is completely undeveloped in comparison to the current status of renormalization in Lagrangian calculus [30].

Physical processes that involve one-photon exchange have amplitudes proportional to e^2. To obtain an amplitude for a simplest scattering event with large momentum transfer that involves strong interactions, it is sufficient to calculate the strong subprocess only to order g^2 and study cross sections to order $e^4 g^2$. The complete initial Hamiltonian contains four terms,

$$H = H_{\mathrm{QED}} + H_{\mathrm{eq}\gamma} + H_{\mathrm{QCD}} + X, \tag{6.1}$$

where the second term includes couplings of quarks to photons and instantaneous electromagnetic coupling of quarks and electrons. The transformation $\mathcal{U}_\lambda$ of Eq. (3.1) can be calculated from the trajectory of $H_{\lambda \mathrm{QCD}}$ alone, together with the required counterterms in X. Then the Hamiltonians $H_{\mathrm{eq}\gamma}$ can be rewritten in terms of quarks and gluons of width λ. This step produces a dependence on regularization, which should be removed by additional counterterms in X. Here QED degrees of freedom are not changed. The resulting effective Hamiltonian dynamics can be then tested for covariance using perturbative expansion for the amplitude $e^+e^- \to$hadrons including terms of order e^2, $e^2 g$, and $e^2 g^2$, which contribute to the cross section through orders e^4 and $e^4 g^2$.

Such tests are of interest since the renormalization group for particles produces light-front Hamiltonians without reference to the vacuum structure and scattering amplitudes, and wave function renormalization and Ward identities do not enter the Hamiltonian. In a much simplified model without gauge symmetry, an adjustment of finite parts of counterterms led to covariant scattering with proper threshold behavior [31]. It should be verified in the Hamiltonian approach based on Eq. (6.1) if QCD binding effects known in order g^2 interfere with obtaining covariant results. Genuinely perturbative fourth-order test calculations in QED should verify if in the limit $\delta \to 0$ the perturbative Hamiltonian approach is able to produce fully covariant results for observables through adjustment of finite parts of the ultraviolet counterterms that contain otherwise undetermined functions of x.

ACKNOWLEDGMENT

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APPENDIX A: DETAILS OF THE INITIAL HAMILTONIAN

The initial QCD Hamiltonian for gluons in Eq. (2.11) contains the following terms:

$$H_{A^2} = \sum_{\sigma c} \int [k] \frac{k^{\perp 2}}{k^+} a^\dagger_{k\sigma c} a_{k\sigma c}, \tag{A1}$$

$$H_{A^3} = \sum_{123} \int [123] \tilde{\delta}(p^\dagger - p) \tilde{r}_{\Delta\delta}(3,1) [g Y_{123} a_1^\dagger a_2^\dagger a_3 + g Y^*_{123} a_3^\dagger a_2 a_1], \tag{A2}$$

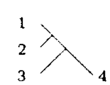
where

$$Y_{123} = i f^{c_1 c_2 c_3} \left[\varepsilon_1^* \varepsilon_2^* \cdot \varepsilon_3 \kappa - \varepsilon_1^* \varepsilon_3 \cdot \varepsilon_2^* \kappa \frac{1}{x_{2/3}} - \varepsilon_2^* \varepsilon_3 \cdot \varepsilon_1^* \kappa \frac{1}{x_{1/3}} \right], \tag{A2a}$$

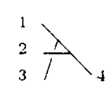
with $\varepsilon \equiv \varepsilon^\perp$ and $\kappa \equiv \kappa^\perp_{1/3}$,

$$H_{A^4} = \sum_{1234} \int [1234] \tilde{\delta}(p^\dagger - p) \frac{g^2}{4} [\Xi_{A^4 1234} a_1^\dagger a_2^\dagger a_3^\dagger a_4 + X_{A^4 1234} a_1^\dagger a_2^\dagger a_3 a_4 + \Xi^*_{A^4 1234} a_4^\dagger a_3 a_2 a_1], \tag{A3}$$

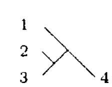
$$\Xi_{A^4\,1234} = \tfrac{2}{3}\big[\, \tilde{r}_{1+2,1}\tilde{r}_{4,3}\,(\varepsilon_1^*\varepsilon_3^* \cdot \varepsilon_2^*\varepsilon_4 - \varepsilon_1^*\varepsilon_4 \cdot \varepsilon_2^*\varepsilon_3^*)\, f^{ac_1c_2} f^{ac_3c_4} +$$



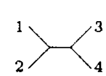
$$\tilde{r}_{1+3,1}\tilde{r}_{4,2}\,(\varepsilon_1^*\varepsilon_2^* \cdot \varepsilon_3^*\varepsilon_4 - \varepsilon_1^*\varepsilon_4 \cdot \varepsilon_2^*\varepsilon_3^*)\, f^{ac_1c_3} f^{ac_2c_4} +$$



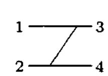
$$\tilde{r}_{3+2,3}\tilde{r}_{4,1}\,(\varepsilon_1^*\varepsilon_3^* \cdot \varepsilon_2^*\varepsilon_4 - \varepsilon_3^*\varepsilon_4 \cdot \varepsilon_2^*\varepsilon_1^*)\, f^{ac_3c_2} f^{ac_1c_4}\big].$$

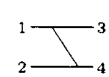


$$X_{A^4\,1234} = \tilde{r}_{1+2,1}\tilde{r}_{3+4,3}\,(\varepsilon_1^*\varepsilon_3 \cdot \varepsilon_2^*\varepsilon_4 - \varepsilon_1^*\varepsilon_4 \cdot \varepsilon_2^*\varepsilon_3)\, f^{ac_1c_2} f^{ac_3c_4} +$$

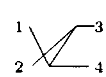


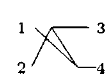
$$[\tilde{r}_{3,1}\tilde{r}_{2,4} + \tilde{r}_{1,3}\tilde{r}_{4,2}]\,(\varepsilon_1^*\varepsilon_2^* \cdot \varepsilon_3\varepsilon_4 - \varepsilon_1^*\varepsilon_4 \cdot \varepsilon_2^*\varepsilon_3)\, f^{ac_1c_3} f^{ac_2c_4} +$$





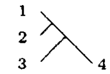
$$[\tilde{r}_{3,2}\tilde{r}_{1,4} + \tilde{r}_{2,3}\tilde{r}_{4,1}]\,(\varepsilon_1^*\varepsilon_2^* \cdot \varepsilon_3^*\varepsilon_4 - \varepsilon_1^*\varepsilon_3 \cdot \varepsilon_2^*\varepsilon_4)\, f^{ac_1c_4} f^{ac_2c_3}.$$



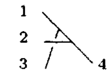


$$H_{[\partial AA]^2}=\sum_{1234}\int[1234]\tilde{\delta}(p^\dagger-p)g^2[(\Xi_{[\partial AA]^2 1234}a_1^\dagger a_2^\dagger a_3^\dagger a_4+\text{H.c.})+X_{[\partial AA]^2 1234}a_1^\dagger a_2^\dagger a_3 a_4], \tag{A4}$$

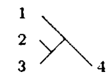
$$\Xi_{[\partial AA]^2\,1234}=-\tfrac{1}{6}\Big[\tilde{r}_{1+2,1}\tilde{r}_{4,3}\,\varepsilon_1^*\varepsilon_2^*\cdot\varepsilon_3^*\varepsilon_4\,\frac{(x_1-x_2)(x_3+x_4)}{(x_1+x_2)^2}\,f^{ac_1c_2}f^{ac_3c_4}+$$



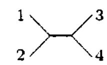
$$\tilde{r}_{1+3,1}\tilde{r}_{4,2}\,\varepsilon_1^*\varepsilon_3^*\cdot\varepsilon_2^*\varepsilon_4\,\frac{(x_1-x_3)(x_2+x_4)}{(x_1+x_3)^2}\,f^{ac_1c_3}f^{ac_2c_4}+$$



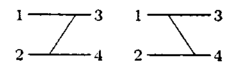
$$\tilde{r}_{3+2,3}\tilde{r}_{4,1}\,\varepsilon_3^*\varepsilon_2^*\cdot\varepsilon_1^*\varepsilon_4\,\frac{(x_3-x_2)(x_1+x_4)}{(x_3+x_2)^2}\,f^{ac_3c_2}f^{ac_1c_4}\Big].$$



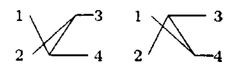
$$X_{[\partial AA]^2\,1234}=\tfrac{1}{4}\Big[\tilde{r}_{1+2,1}\tilde{r}_{3+4,3}\,\varepsilon_1^*\varepsilon_2^*\cdot\varepsilon_3\varepsilon_4\,\frac{(x_1-x_2)(x_3-x_4)}{(x_1+x_2)^2}\,f^{ac_1c_2}f^{ac_3c_4}-$$



$$[\tilde{r}_{3,1}\tilde{r}_{2,4}+\tilde{r}_{1,3}\tilde{r}_{4,2}]\,\varepsilon_1^*\varepsilon_3\cdot\varepsilon_2^*\varepsilon_4\,\frac{(x_1+x_3)(x_2+x_4)}{(x_2-x_4)^2}\,f^{ac_1c_3}f^{ac_2c_4}-$$



$$[\tilde{r}_{3,2}\tilde{r}_{1,4}+\tilde{r}_{2,3}\tilde{r}_{4,1}]\,\varepsilon_1^*\varepsilon_4\cdot\varepsilon_2^*\varepsilon_4\,\frac{(x_2+x_3)(x_1+x_4)}{(x_1-x_4)^2}\,f^{ac_1c_4}f^{ac_2c_3}\Big].$$



$p^\dagger$ and p are the total momenta of created and annihilated particles, respectively. For the vertex with parent momentum p and daughter momenta d and $p-d$, the regulating function in Eq. (A2) is $\tilde{r}_{\Delta\delta}(p,d)=r_{\Delta\delta}(p,d)r_{\Delta\delta}(p,p-d)$, where $r_{\Delta\delta}(p,d)=r_\Delta(\kappa_{d/p}^{\perp 2})r_\delta(x_{d/p})\theta(x_{d/p})$, with $x_{d/p}=d^+/p^+\equiv x_d/x_p$ and $\kappa_{d/p}^\perp=d^\perp-x_{d/p}p^\perp$. Equations (2.6) to (2.9) complete the definitions of $\tilde{r}_{\Delta\delta}(p,d)$. Also, $\tilde{r}_{p,d}\equiv\tilde{r}_{\Delta\delta}(p,d)$. The momentum integration measures contain the same factors as in Eq. (2.4) for each indicated particle.

APPENDIX B: EXPRESSION FOR $\gamma_{\lambda 21}$

Equation (3.19) gives

$$\begin{aligned}\gamma_{\lambda 21}=&\mathcal{F}_{3\lambda}8[\alpha_{12}\alpha_{21}\alpha_{21}]_{21}+\mathcal{F}_{2\lambda}2[\beta_{\infty 22}\alpha_{21}]_{21}\\&+\mathcal{F}_{2\lambda}2[\alpha_{12}\beta_{\infty 31}]_{21}+\mathcal{F}_{3\lambda}4[[\alpha_{12}\alpha_{21}]_{11}\alpha_{21}]_{21}\\&+\mathcal{F}_{2\lambda}[\alpha_{12}\beta_{\infty 31}]_{21}+\mathcal{F}_{2\lambda}2[\beta_{\infty 11}\alpha_{21}]_{21}\\&+\mathcal{F}_{3\lambda}2[\alpha_{21}[\alpha_{12}\alpha_{21}]_{11}]_{21}+\mathcal{F}_{2\lambda}\frac{1}{2}[\beta_{\infty 22}\alpha_{21}]_{21}\\&+\mathcal{F}_{2\lambda}[\alpha_{21}\beta_{\infty 11}]_{21}+\gamma_{\infty 21}.\end{aligned} \tag{B1}$$

The terms are grouped and ordered in one-to-one correspondence to Fig. 2, so that using the labels from Fig. 2, Eq. (B1) reads

$$\begin{aligned}\gamma_{21}=&\mathcal{F}_3 8(a)+\mathcal{F}_2 2(b)+\mathcal{F}_2 2(c)+\mathcal{F}_3 4(d)+\mathcal{F}_2(e)\\&+\mathcal{F}_2 2(f)+\mathcal{F}_3 2(g)+\mathcal{F}_2\frac{1}{2}(h)+\mathcal{F}_2(i)+(j).\end{aligned} \tag{B2}$$

In all terms, the external gluon quantum numbers are labeled in the same way: 1 and 2 refer to the created gluons and 3 to the annihilated one. The intermediate momenta are assigned the numbers 6, 7, and 8. For example, using the notation from Appendix A, we have $k_1^\perp=x_{1/7}k_7^\perp+\kappa_{16}^\perp$, where $\kappa_{16}^\perp\equiv\kappa_{1/7}^\perp$, $x_{1/7}\equiv x_1/x$. Also, $\kappa_{68}^\perp=\kappa^\perp-(1-x)\kappa_{12}^\perp/x_2$, $\kappa_{16}^\perp=-x_1\kappa^\perp/x+\kappa_{12}^\perp$, and $\kappa_{78}^\perp=\kappa^\perp$. All coefficients of creation operators ought to be symmetrized with respect to quantum numbers of gluons 1 and 2, and only one way of contracting the intermediate gluon annihilation and creation operators is included, so that the displayed numerical weights represent the result of all possible contractions. Thus, in Fig. 2(b), only one ordering of vertices from $\beta_{\infty 22}$ is included and the factor of 2 is put in front. The black dots in Fig. 2 indicate counterterms, $\beta_{\infty 11}$ in cases (f) and (i), and $\gamma_{\infty 21}$ in case (j). Thick

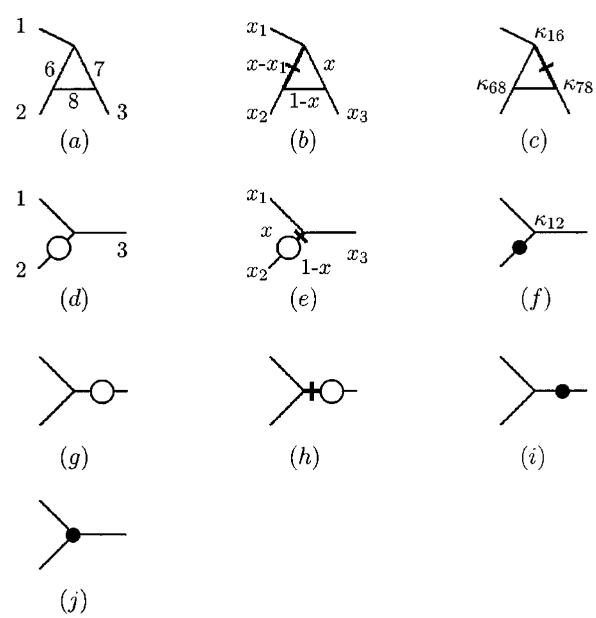


FIG. 2. Graphical illustration of Eq. (B1).

lines with transversal bars denote the combined contributions of the terms H_{A^4} and $H_{[\partial AA]^2}$ from Eqs. (A3) and (A4), $\beta_{\infty 22}=X_{A^4}+X_{[\partial AA]^2}$, and $\beta_{\infty 31}=\Xi_{A^4}+\Xi_{[\partial AA]^2}$. Since these terms are independent of the transverse momenta and the three-gluon vertices α_{12} and α_{21} are odd in the transverse momentum, both terms e and h in Eq. (B2), represented by diagrams (e) and (h) in Fig. 2, are equal zero.

The nonzero terms are listed in the order of their appearance in Eq. (B2), using the convention $\gamma_{21}=\Sigma_n \gamma_{21(n)}$, with n ranging from a to j, and

$$\gamma_{21(n)}=\sum_{123}\int[123]\tilde{\delta}(k_1+k_2-k_3)\frac{g^3}{16\pi^3}\frac{1}{2}\gamma_{(n)}a_1^\dagger a_2^\dagger a_3, \tag{B3}$$

$$\gamma_{(a)}=8\frac{N_c}{2}if^{c_1c_2c_3}\int_{x_1}^1\frac{dx\, r_{\delta t}(x)}{x(1-x)(x-x_1)}\int d^2\kappa^\perp r_{\Delta t}(\kappa^\perp)\frac{\mathcal{F}_{3\lambda(a)}}{k_3^{+2}}\kappa_{68}^i\kappa_{16}^j\kappa^k\varepsilon_{(a)}^{ijk}+(1\leftrightarrow 2), \tag{B4}$$

where

$$r_{\delta t}(x)=r_\delta(x)r_\delta(1-x)r_\delta(x_1/x)r_\delta[(x-x_1)/x]r_\delta[(x-x_1)/x_2]r_\delta[(1-x)/x_2], \tag{B4a}$$

$$r_{\Delta t}(\kappa^\perp)=\exp[-2(\kappa_{68}^{\perp 2}+\kappa_{16}^{\perp 2}+\kappa^{\perp 2})/\Delta^2], \tag{B4b}$$

$$\begin{aligned}\frac{\mathcal{F}_{3\lambda(a)}}{k_3^{+2}}=&\frac{-x\mathcal{M}_{16}^2+\mathcal{M}^2}{\mathcal{M}_{16}^4+\mathcal{M}^4}\left\{(\mathcal{M}_{bd}^2+x_2\mathcal{M}_{68}^2)\left[\frac{f_{68}f_{bd}f_{16}f-1}{\mathcal{M}_{68}^4+\mathcal{M}_{bd}^4+\mathcal{M}_{16}^4+\mathcal{M}^4}-\frac{f_{68}f_{bd}-1}{\mathcal{M}_{68}^4+\mathcal{M}_{bd}^4}\right]-\frac{\mathcal{M}_{16}^4+\mathcal{M}^4}{\mathcal{M}_{bd}^4}\left[\frac{f_{68}f_{16}f-1}{\mathcal{M}_{68}^4+\mathcal{M}_{16}^4+\mathcal{M}^4}\right.\right.\\&\left.\left.-\frac{f_{68}f_{bd}f_{16}f-1}{\mathcal{M}_{68}^4+\mathcal{M}_{bd}^4+\mathcal{M}_{16}^4+\mathcal{M}^4}\right]\right\}+\frac{x\mathcal{M}_{16}^2+x_2\mathcal{M}_{68}^2}{\mathcal{M}_{16}^4+\mathcal{M}_{68}^4}\left\{(2\mathcal{M}^2-\mathcal{M}_{12}^2)\left[\frac{ff_{ca}f_{16}f_{68}-1}{\mathcal{M}^4+(\mathcal{M}^2-\mathcal{M}_{12}^2)^2+\mathcal{M}_{16}^4+\mathcal{M}_{68}^4}\right.\right.\\&\left.\left.-\frac{ff_{ca}-1}{\mathcal{M}^4+(\mathcal{M}^2-\mathcal{M}_{12}^2)^2}\right]-\frac{\mathcal{M}_{16}^4+\mathcal{M}_{68}^4}{\mathcal{M}^2-\mathcal{M}_{12}^2}\left[\frac{ff_{16}f_{68}-1}{\mathcal{M}^4+\mathcal{M}_{16}^4+\mathcal{M}_{68}^4}-\frac{ff_{ca}f_{16}f_{68}-1}{\mathcal{M}^4+(\mathcal{M}^2-\mathcal{M}_{12}^2)^2+\mathcal{M}_{16}^4+\mathcal{M}_{68}^4}\right]\right\},\end{aligned} \tag{B4c}$$

$$\mathcal{M}^2=\frac{\kappa^{\perp 2}}{x(1-x)}, \tag{B4d}$$

$$\mathcal{M}_{68}^2=\frac{x_2^2\kappa_{68}^{\perp 2}}{(x-x_1)(1-x)}, \tag{B4e}$$

$$\mathcal{M}_{16}^2=\frac{x^2\kappa_{16}^{\perp 2}}{x_1(x-x_1)}, \tag{B4f}$$

with $f_u=\exp[-u^2/\lambda^4]$, $bd=\mathcal{M}_{bd}^2=\mathcal{M}_{68}^2/x_2+\mathcal{M}_{12}^2$, $ca=\mathcal{M}^2-\mathcal{M}_{12}^2$, $f\equiv f_{cd}$, $cd=\mathcal{M}^2$, and

$$\begin{aligned}\varepsilon_a^{ijk}=&\varepsilon_1^{*j}\varepsilon_2^{*i}\varepsilon_3^k\left[1-\frac{x}{x-x_1}+\frac{1}{x_1}-\frac{2x}{x_1}+\frac{x}{(1-x)x_1}+\frac{xx_2}{(1-x)x_1}+\frac{xx_2}{(x-x_1)x_1}\right]+\varepsilon_1^{*k}\varepsilon_2^{*i}\varepsilon_3^j\left[\frac{1}{x-x_1}-\frac{1}{1-x}\right]\\&+\varepsilon_1^{*k}\varepsilon_2^{*j}\varepsilon_3^i\left[\frac{-x_2}{(1-x)(x-x_1)}\right]+\varepsilon_1^{*i}\varepsilon_2^{*k}\varepsilon_3^j\left[\frac{x_2}{(1-x)(x-x_1)}\right]+\varepsilon_1^{*i}\varepsilon_2^{*j}\varepsilon_3^k\left[\frac{-x_2}{x-x_1}+\frac{xx_2}{(1-x)(x-x_1)}\right]\\&+\varepsilon_1^{*j}\varepsilon_2^{*k}\varepsilon_3^i\left[\frac{-x_2}{(1-x)x_1}-\frac{xx_2}{(1-x)(x-x_1)x_1}\right]+\varepsilon_1^*\varepsilon_2^*\left[\delta^{ik}\varepsilon_3^j\frac{x_2}{(1-x)^2}+\delta^{jk}\varepsilon_3^i\frac{x_2}{(1-x)x}+\delta^{ij}\varepsilon_k^3\left(\frac{xx_2}{(x-x_1)^2}-\frac{x_2}{1-x}\right)\right]\\&+\varepsilon_1^*\varepsilon_3\left[\delta^{jk}\varepsilon_2^{*i}\left(\frac{x}{(1-x)(x-x_1)}-\frac{1}{x}\right)-\delta^{ij}\varepsilon_2^{*k}\frac{xx_2}{(1-x)(x-x_1)^2}-\delta^{ik}\varepsilon_2^{*j}\frac{xx_2}{(1-x)^2(x-x_1)}\right]\\&+\varepsilon_2^*\varepsilon_3\left[\delta^{ij}\varepsilon_1^{*k}\frac{-x_2}{(x-x_1)^2}+\delta^{jk}\varepsilon_1^{*i}\frac{x_2}{x(x-x_1)}-\delta^{ik}\varepsilon_1^{*j}\left(\frac{xx_2}{(1-x)^2x_1}-\frac{x_2}{(x-x_1)x_1}\right)\right].\end{aligned} \tag{B4g}$$

$$\gamma_{(b)}=2\frac{N_c}{2}if^{c_1c_2c_3}\int_{x_1}^1\frac{dx\, r_{\delta t}(x)}{x(1-x)}\int d^2\kappa^\perp r_{\Delta t}(\kappa^\perp)\frac{\mathcal{F}_{2\lambda(b)}}{k_3^+}\varepsilon_{(b)}+(1\leftrightarrow 2), \tag{B5}$$

where

$$\frac{\mathcal{F}^{2\lambda(b)}}{k_3^{+2}}=\frac{2\mathcal{M}^2-\mathcal{M}_{12}^2}{(\mathcal{M}^2-\mathcal{M}_{12}^2)^2+\mathcal{M}^4}(f_{ac}f-1), \tag{B5a}$$

$$\varepsilon_{(b)}\equiv\varepsilon_b^{\perp}\kappa^{\perp}=\varepsilon_1^*\varepsilon_2^*\varepsilon_3\kappa\left(1-s_{(b)}-\frac{1}{x}-\frac{1}{1-x}\right)+\varepsilon_1^*\varepsilon_3\varepsilon_2^*\kappa\left(\frac{1}{x}+\frac{s_{(b)}}{1-x}\right)+\varepsilon_2^*\varepsilon_3\varepsilon_1^*\kappa\left(\frac{1}{1-x}+\frac{s_{(b)}}{x}\right), \tag{B5b}$$

and $s_{(b)}=(x_1+x)(x_2+1-x)/(x-x_1)^2$,

$$\gamma_{(c)}=2\,\frac{-N_c}{2}\,if^{c_1c_2c_3}\int_{x_1}^1\frac{dx\,r_{\delta t}(x)}{(x-x_1)(1-x)}\int d^2\kappa^{\perp}r_{\Delta t}(\kappa^{\perp})\,\frac{\mathcal{F}_{2\lambda(b)}}{k_3^+}\,\varepsilon_{(c)}+(1\leftrightarrow 2), \tag{B6}$$

where

$$\frac{\mathcal{F}_{2\lambda(c)}}{k_3^+}=\frac{x_2\mathcal{M}_{68}^2+\mathcal{M}_{bd}^2}{\mathcal{M}_{68}^4+\mathcal{M}_{bd}^4}(f_{68}f_{bd}-1), \tag{B6a}$$

$$\varepsilon_{(c)}\equiv\varepsilon_c^{\perp}\kappa_{68}^{\perp}=\varepsilon_1^*\varepsilon_2^*\varepsilon_3\kappa_{68}\left(\frac{-x_2}{x-x_1}+\frac{x_2 s_{(c)}}{1-x}\right)+\varepsilon_1^*\varepsilon_3\varepsilon_2^*\kappa_{68}\left(-1-s_{(c)}+\frac{x_2}{1-x}+\frac{x_2}{x-x_1}\right)+\varepsilon_2^*\varepsilon_3\varepsilon_1^*\kappa_{68}\left(\frac{-x_2}{1-x}+\frac{x_2 s_{(c)}}{x-x_1}\right), \tag{B6b}$$

and $s_{(c)}=(x_1-x+x_1)(1-x+1)/x^2$.

$$\gamma_{(d)}+\gamma_{(f)}=4N_cY_{123}\int_0^1\frac{dx\,r_{\delta\mu}(x)}{x(1-x)}\int d^2\kappa^{\perp}r_{\Delta\mu}(\kappa^{\perp})\kappa^{\perp 2}\left[1+\frac{1}{x^2}+\frac{1}{(1-x)^2}\right]\left[\frac{\mathcal{F}_{3\lambda(d)}}{x_2^2k_3^{+2}}+\frac{\mathcal{F}_{2\lambda(f)}}{x_2k_3^+\mathcal{M}^2}\right]+4Y_{123}\frac{\mathcal{F}_{2\lambda(f)}}{x_2k_3^+}\tilde{\mu}_\delta^2+(1\leftrightarrow 2), \tag{B7}$$

where

$$\frac{\mathcal{F}_{3\lambda(d)}}{x_2^2k_3^{+2}}+\frac{\mathcal{F}_{2\lambda(f)}}{x_2k_3^+\mathcal{M}^2}=\frac{1}{x_2^2}\,\frac{\mathcal{M}_{12}^2-x_2\mathcal{M}^2}{\mathcal{M}_{12}^4+\mathcal{M}^2}\left\{\left[\mathcal{M}^2\left(\frac{1}{x_2}+x_2\right)+\mathcal{M}_{12}^2\right]\left[\frac{f^2f_{bd}f_{12}-1}{2\mathcal{M}^4+(\mathcal{M}^2/x_2+\mathcal{M}_{12}^2)^2+\mathcal{M}_{12}^4}-\frac{ff_{bd}-1}{\mathcal{M}^4+(\mathcal{M}^2/x_2+\mathcal{M}_{12}^2)^2}\right]\right.$$
$$\left.-\frac{\mathcal{M}^4+\mathcal{M}_{12}^4}{\mathcal{M}^2/x_2+\mathcal{M}_{12}^2}\left[\frac{f^2f_{12}-1}{2\mathcal{M}^4+\mathcal{M}_{12}^4}-\frac{f^2f_{bd}f_{12}-1}{2\mathcal{M}^4+(\mathcal{M}^2/x_2+\mathcal{M}_{12}^2)^2+\mathcal{M}_{12}^4}\right]\right\}+\frac{\mathcal{M}_{12}^2}{x_2\mathcal{M}^2}\,\frac{f_{12}f^2-1}{\mathcal{M}_{12}^4+2\mathcal{M}^4}, \tag{B7a}$$

$\mathcal{M}_{bd}^2=\mathcal{M}^2/x_2+\mathcal{M}_{12}^2$, since $\mathcal{M}_{68}^2\equiv\mathcal{M}^2$ in terms d and f (here f is a subscript, not a form factor), $f_{12}\equiv f_{ad}$, $ad=\mathcal{M}_{12}^2$,

$$\frac{\mathcal{F}_{2\lambda(f)}}{x_2k_3^+}=\frac{f_{12}-1}{x_2\mathcal{M}_{12}^2}, \tag{B7b}$$

$r_{\delta\mu}(x)$ is given in Eq. (4.7), $r_{\Delta\mu}(\kappa^{\perp})=r_\Delta^4(\kappa^{\perp 2})$, and $2g^2\tilde{\mu}_\delta^2=16\pi^3\mu_\delta^2$.

$$\gamma_{(g)}+\gamma_{(i)}=2N_cY_{123}\int_0^1\frac{dx\,r_{\delta\mu}(x)}{x(1-x)}\int d^2\kappa^{\perp}r_{\Delta\mu}(\kappa^{\perp})\kappa^{\perp 2}\left[1+\frac{1}{x^2}+\frac{1}{(1-x)^2}\right]\left[\frac{\mathcal{F}_{3\lambda(g)}}{k_3^{+2}}+\frac{\mathcal{F}_{2\lambda(i)}}{k_3^+\mathcal{M}^2}\right]+2Y_{123}\frac{\mathcal{F}_{2\lambda(i)}}{k_3^+}\tilde{\mu}_\delta^2+(1\leftrightarrow 2), \tag{B8}$$

where

$$\frac{\mathcal{F}_{3\lambda(g)}}{k_3^{+2}}+\frac{\mathcal{F}_{2\lambda(i)}}{k_3^+\mathcal{M}^2}=\frac{-\mathcal{M}_{12}^2}{\mathcal{M}^2}\frac{f_{12}f^2-1}{\mathcal{M}_{12}^4+2\mathcal{M}^4}-\frac{\mathcal{M}^2+\mathcal{M}_{12}^2}{\mathcal{M}^4+\mathcal{M}_{12}^4}\left\{(2\mathcal{M}^2-\mathcal{M}_{12}^2)\left[\frac{f^2f_{ca}f_{12}-1}{2\mathcal{M}^4+(\mathcal{M}^2-\mathcal{M}_{12}^2)^2+\mathcal{M}_{12}^4}-\frac{ff_{ca}-1}{\mathcal{M}^4+(\mathcal{M}^2-\mathcal{M}_{12}^2)^2}\right]\right.$$
$$\left.+\frac{\mathcal{M}^4+\mathcal{M}_{12}^4}{\mathcal{M}_{12}^2-\mathcal{M}^2}\left[\frac{f^2f_{12}-1}{2\mathcal{M}^4+\mathcal{M}_{12}^4}-\frac{f^2f_{ca}f_{12}-1}{2\mathcal{M}^4+(\mathcal{M}^2-\mathcal{M}_{12}^2)^2+\mathcal{M}_{12}^4}\right]\right\}, \tag{B8a}$$

with $ca=\mathcal{M}^2-\mathcal{M}_{12}^2$, and

$$\frac{\mathcal{F}_{2\lambda(i)}}{k_3^+}=-\frac{f_{12}-1}{\mathcal{M}_{12}^2}, \tag{B8b}$$

$\gamma_{(j)}=\gamma_\infty+(1\leftrightarrow2)$, where γ_∞ denotes the counterterm coefficient in

$$\gamma_{\infty 21}=\sum_{123}\int[123]\tilde{\delta}(k_1+k_2-k_3)\frac{g^3}{16\pi^3}\gamma_\infty a_1^\dagger a_2^\dagger a_3, \tag{B9}$$

whose dependence on the quantum numbers 1, 2, and 3 needs to be found.

APPENDIX C: ULTRAVIOLET DIVERGENT TERMS IN $\gamma_{\lambda 21}$

The three-gluon vertex is given by Eqs. (B4)–(B9). The ultraviolet diverging parts of terms b and c vanish. This can be seen by changing the variables so that the sum of squares of relative transverse momenta in three subsequent vertices in Eq. (B4b) is written as

$$\eta_{68}\kappa_{68}^{\perp 2}+\eta_{16}\kappa_{16}^{\perp 2}+\eta\kappa^{\perp 2}=\zeta\rho^{\perp 2}+\chi\kappa_{12}^{\perp 2}, \tag{C1}$$

where $\zeta=\eta_{68}+\eta_{16}x_1^2/x^2+\eta$, $\rho^\perp=\kappa^\perp-\xi\kappa_{12}^\perp$, $\xi=[\eta_{68}(1-x)/x_2+\eta_{16}x_1/x]/\zeta$, and $\chi=[\eta_{68}\eta_{16}(x-x_1)^2/(xx_2)^2+\eta_{68}\eta(1-x)^2/x_2^2+\eta_{16}\eta]/\zeta$. The coefficients η are introduced for identification of the finite parts of the counterterms that may contain functions of x_1. All η's are equal to 2 in Eq. (B4b). Using the variable $\rho^\perp$, the potentially diverging relative transverse momentum integrals in terms b and c can be written as

$$\gamma_{(b)\mathrm{div}}=-2\frac{N_c}{2}if^{c_1c_2c_3}\int_{x_1}^1dx\,r_{\delta t}(x)\times\int d^2\rho^\perp e^{-\zeta\rho^{\perp 2}/\Delta^2}\frac{\kappa^\perp}{\kappa^{\perp 2}}\varepsilon_{(b)}^\perp+(1\leftrightarrow2), \tag{C2a}$$

$$\gamma_{(c)\mathrm{div}}=2\frac{N_c}{2}if^{c_1c_2c_3}\int_{x_1}^1dx\,r_{\delta t}(x)\times\int d^2\rho^\perp e^{-\zeta\rho^{\perp 2}/\Delta^2}\frac{\kappa_{68}^\perp}{\kappa_{68}^{\perp 2}x_2}\varepsilon_{(c)}^\perp+(1\leftrightarrow2). \tag{C2b}$$

Both terms contain an integral of the same structure,

$$I^\perp=\int d^2\kappa^\perp\frac{\kappa^\perp}{\kappa^2}\exp[-\zeta(\kappa^\perp-v^\perp)^2/\Delta^2], \tag{C2c}$$

and $\lim_{\Delta\to\infty}I^\perp=\pi v^\perp$, which is a finite ultraviolet regularization effect with no divergence. Thanks to the regularization factors $r_{\delta t}$, these terms can be integrated over x and the resulting function of x_1 depends on the choice of the coefficients η. If they were chosen to depend on x, an arbitrarily complex, ultraviolet-regularization-dependent finite function of x_1 can be obtained, but $\gamma_{(b)\mathrm{div}}=\gamma_{(c)\mathrm{div}}=0$. Next,

$$\gamma_{(g+i)\mathrm{div}}=-N_cY_{123}\tilde{r}_\delta(x_1)\int_0^1dx\,r_{\delta\mu}(x)\int_{\mu^2}^\infty\frac{\pi\,d\kappa^2}{\kappa^2}e^{-4\kappa^2/\Delta^2}\times x(1-x)\left[1+\frac{1}{x^2}+\frac{1}{(1-x)^2}\right]+(1\leftrightarrow2). \tag{C3}$$

μ^2 is an arbitrary parameter inserted here to simplify the notation for the well-defined integral over κ^2 at small κ when the terms with form factors are written explicitly and the small denominator effect is absent. One also obtains

$$\gamma_{(d+f)\mathrm{div}}=2\gamma_{(g+i)\mathrm{div}}. \tag{C4}$$

The logarithmically diverging term (C3) appears then with a factor of -3 in the counterterm coefficient γ_∞ in Eq. (B9). Its finite contribution to the counterterm is a finite number times Y_{123}.

The remaining term $\gamma_{(a)}$ of Eq. (B4) contains the tensor $\kappa^{ijk}=\kappa_{68}^i\kappa_{16}^j\kappa^k$ contracted with $\varepsilon_{(a)}^{ijk}$ and the latter does not depend on the transverse momenta. The ultraviolet regularization dependence comes only from

$$\left[\frac{\mathcal{F}_{3\lambda(a)}}{k_3^{+2}}\right]_\Delta=\frac{x_2}{\mathcal{M}^2\mathcal{M}_{68}^2}, \tag{C5}$$

and using Eq. (C1) one obtains

$$\gamma_{(a)\mathrm{div}}=8\,\frac{N_c}{2}\,if^{c_1c_2c_3}\int_{x_1}^{1}dx\,r_{\delta t}(x)\,\frac{1-x}{x_2}\,I^{ijk}\varepsilon^{ijk}_{(a)}+(1\leftrightarrow 2),\tag{C6}$$

where

$$I^{ijk}=\int d^2\rho^{\perp}e^{-\zeta\rho^{\perp 2}/\Delta^2}\frac{\kappa^{i}_{68}\kappa^{j}_{16}\kappa^{k}}{\kappa^{\perp 2}\kappa^{\perp 2}_{68}}.\tag{C7}$$

The ultraviolet-regularization-dependent part of I^{ijk} can be evaluated using Feynman's trick to combine denominators and making similar steps as in Eq. (C2):

$$[I^{ijk}]_{\Delta}=\frac{\pi}{2}\,\frac{x_1}{x}\,\frac{1-x}{x_2}\left[\kappa^{i}_{12}\delta^{jk}-\kappa^{k}_{12}\delta^{ij}+\frac{x-x_1+xx_2}{x_1(1-x)}\,\kappa^{j}_{12}\delta^{ik}\right]\ln\frac{\Delta}{|\kappa^{\perp}_{12}|}.\tag{C8}$$

A number of finite terms are obtained that depend on the choice of coefficients η in Eq. (C1). The terms include also a new tensor structure $\kappa^{i}_{12}\kappa^{j}_{12}\kappa^{k}_{12}/\kappa^{2}_{12}$. Inserting $[I^{ijk}]_{\Delta}$ in place of I^{ijk} in Eq. (C6), one obtains

$$\gamma_{(a)\mathrm{div}}=8\,\frac{N_c}{2}\,if^{c_1c_2c_3}\int_{x_1}^{1}dx\,r_{\delta t}(x)\,\frac{1-x}{x_2}\,[I^{ijk}]_{\Delta}\varepsilon^{ijk}_{(a)}+(1\leftrightarrow 2),\tag{C9}$$

where

$$if^{c_1c_2c_3}\frac{1-x}{x_2}[I^{ijk}]_{\Delta}\varepsilon^{ijk}_{(a)}=\pi\ln\frac{\Delta}{|\kappa^{\perp}_{12}|}[c_{12}Y_{12}+c_{13}Y_{13}+c_{23}Y_{23}],\tag{C9a}$$

and

$$c_{12}=\frac{2}{1-x}+\frac{1}{x-x_1}+\frac{1}{x}+\frac{(1-x)^2}{x_2^2}-\frac{2}{x_2},\tag{C9b}$$

$$c_{13}=\frac{2}{1-x}+\frac{1}{x-x_1}+\frac{1}{x}+\frac{(1-x)^2}{x_2}-2,\tag{C9c}$$

$$c_{23}=\frac{2}{1-x}+\frac{1}{x-x_1}+\frac{1}{x}-\frac{(1-x)^2}{x_2^2}-\frac{1+x^2}{x_2}-2,\tag{C9d}$$

with

$$Y_{12}=if^{c_1c_2c_3}\varepsilon^{*}_{1}\varepsilon^{*}_{2}\cdot\varepsilon_3\kappa_{12},\tag{C10a}$$

$$Y_{13}=-if^{c_1c_2c_3}\varepsilon^{*}_{1}\varepsilon_3\cdot\varepsilon^{*}_{2}\kappa_{12}\frac{1}{x_{2/3}},\tag{C10b}$$

$$Y_{23}=-if^{c_1c_2c_3}\varepsilon^{*}_{2}\varepsilon_3\cdot\varepsilon^{*}_{1}\kappa_{12}\frac{1}{x_{1/3}}.\tag{C10c}$$

The counterterm should cancel the divergence, so that

$$0=\gamma_{(a)\mathrm{div}}+\gamma_{(d+f)\mathrm{div}}+\gamma_{(g+i)\mathrm{div}}+\gamma_{\infty\mathrm{div}}+(1\rightarrow 2)$$
$$=2N_c\pi\ln\frac{\Delta}{\mu}\left\{2\int_{x_1}^{1}dx\,r_{\delta t}(x)[c_{12}Y_{12}+c_{13}Y_{13}+c_{23}Y_{23}]-3[Y_{12}+Y_{13}+Y_{23}]\tilde{r}_{\delta}(x_1)\int_0^1dx\,r_{\delta\mu}(x)x(1-x)\left[1+\frac{1}{x^2}+\frac{1}{(1-x)^2}\right]\right\}+\gamma_{\infty\mathrm{div}}+(1\leftrightarrow 2)\tag{C11}$$

and

$$\gamma_{(j)\mathrm{div}}=-Y_{123}\frac{2N_c\pi}{3}\ln\frac{\Delta}{\mu}[11+h(x_1)],\tag{C12}$$

where

$$h(x_1)=6\int_{x_1}^{1}dx\,r_{\delta t}(x)\left[\frac{2}{1-x}+\frac{1}{x-x_1}+\frac{1}{x}\right]-9\tilde{r}_{\delta}(x_1)\int_0^1dx\,r_{\delta\mu}(x)\left[\frac{1}{x}+\frac{1}{1-x}\right]+(1\leftrightarrow 2).\tag{C13}$$

The ultraviolet counterterm $\gamma_{\infty 21}$ has the form (B9), where

$$\gamma_{\infty}=Y_{123}\frac{-N_c\pi}{3}\ln\frac{\Delta}{\mu}[11+h(x_1)]+\gamma_{\mathrm{finite}}.\tag{C14}$$

Different choices of the coefficients η in Eq. (C1) lead to different finite terms γ_{finite}, which demonstrates a finite dependence on the ultraviolet regularization. To restore the covariance of observables, the finite terms must be then allowed to contain unknown functions of x_1 that multiply three structures Y_{12}, Y_{13}, and Y_{23} from Eq. (C10) and the fourth structure

$$Y_4=if^{c_1c_2c_3}\varepsilon^{*}_{1}\kappa_{12}\cdot\varepsilon^{*}_{2}\kappa_{12}\cdot\varepsilon_3\kappa_{12}/\kappa^{2}_{12}.\tag{C15}$$

Inclusion of n_f flavors of quarks produces in $\gamma_{(j)\mathrm{div}}$ additional diverging terms of the form

$$-2\pi n_f\ln\frac{\Delta}{\mu}[c_{f12}Y_{12}+cf_{13}Y_{13}+c_{f23}Y_{23}+c_{fm}Y_{123}],\tag{C16}$$

where

$$c_{f12}=-\int_{x_1}^{1}dx\,\frac{1-4x+2x^2+x_1}{2x_2^2}+(1\leftrightarrow 2),\tag{C17a}$$

$$c_{f13}=-\int_{x_1}^{1}dx\,\frac{1-4x+2x^2+x_1}{2x_2}+(1\leftrightarrow 2),\tag{C17b}$$

$$c_{f23}=-\int_{x_1}^{1}dx\,\frac{-2+4x-2x^2(1+x_2)+x_1x_2}{2x_2^2}+(1\leftrightarrow 2),\tag{C17c}$$

$$c_{fm}=\int_0^1 dx\,\frac{-3}{8}\,2[1-2x(1-x)]+(1\leftrightarrow 2). \tag{C17d}$$

After simplifications, integration, and symmetrization, the fermion contribution turns out to be not sensitive to small-x regularization and changes Eq. (C14) to

$$\gamma_{(j)\mathrm{div}}=-Y_{123}\frac{2\pi}{3}\ln\frac{\Delta}{\mu}\{N_c[11+h(x_1)]-2n_f\}. \tag{C18}$$

APPENDIX D: $W_\lambda(x)$ IN Eq. (5.2)

$W_{\lambda 12}(x_1,\kappa_{12}^{\perp 2})$, $W_{\lambda 13}(x_1,\kappa_{12}^{\perp 2})$, and $W_{\lambda 23}(x_1,\kappa_{12}^{\perp 2})$ in Eq. (4.14) all become equal to $W_\lambda(x_1)$ when $\kappa_{12}^{\perp}=0$. The calculation of $W_\lambda(x_1)$ is based on the extraction of the coefficients of terms linear in $\kappa_{12}^{\perp}$, which form Y_{ij} for $ij=12$, 13, 23. The procedure employs the following facts. The self-interaction terms d and g, and mass counterterms f and i from Fig. 2, contribute through

$$\frac{g^3}{16\pi^3}N_c\int_0^1 dx\int\frac{d^2\kappa^\perp}{\kappa^{\perp 2}}x(1-x)\left[1+\frac{1}{x^2}+\frac{1}{(1-x)^2}\right]\times[5f^2+4ff_{bd}-4f^2f_{bd}-2f^3-3\theta(\mu^2-\kappa^2)]. \tag{D1}$$

μ is introduced only for simplification and is canceled in the full formula for the dependence of $W_\lambda(x_1)$ on λ. In the term $\gamma_{(a)}$, the renormalization group factor of Eq. (B4c) contributes to the dependence of $W_\lambda(x_1)$ on λ only through a small A expansion of

$$\left[\frac{\mathcal{F}_{3\lambda(a)}}{k_3^{+2}}\right]_\lambda=-\frac{x_2}{[\mathcal{M}_{680}^2-2x_2A]\mathcal{M}^2}ff_{16}f_{68}, \tag{D2}$$

where $A=\kappa^\perp\kappa_{12}^\perp/(x-x_1)$ and the added subscript 0 indicates that $\kappa_{12}^\perp$ is set equal 0. Other parts do not contribute because when the form factors f are expanded, no dependence on λ is generated due to dimensional reasons and the remaining factors cannot contribute since they are multiplied by the differences of the form factors and there is an identity

$$\int_0^\infty\frac{dz}{z}[e^{-az^2/\lambda^4}-e^{-bz^2/\lambda^4}]=\frac{1}{2}\ln\frac{b}{a}, \tag{D3}$$

which shows that λ drops out. The expansion in A in Eq. (D2) produces the same tensor structure as in Eq. (C8), which leads then to Eq. (5.2).

APPENDIX E: REGULARIZATION MIXING FOR x AND $\kappa^\perp$

Regularization with invariant masses implies that the coefficients η in Eq. (C1) become equal to $\eta=1/x+1/(1-x)$, $\eta_{68}=x_2/(x-x_1)+x_2/(1-x)$, and $\eta_{16}=x/x_1+x/(x-x_1)$. These coefficients diverge when $x\to x_1$ or $x\to 1$. In mass counterterms, the coefficient η is the same. When integrating over transverse momenta in mass counterterms with $r_\Delta(2\eta\kappa^{\perp 2})$, one obtains in Eq. (4.9) only a logarithmically divergent integral over x and a regularization factor of the form $r_\delta(x)=x^\delta$ is sufficient to regulate it. In other words, in place of the second scale ϵ, in r_δ from Eq. (5.9), one can consider regularizations of transverse divergences that provide additional damping of the small-x region. The question is what are the consequences of the mixing of large κ and small x in $\gamma_{\lambda 21}$. Most representative is Eq. (B4). Although all three η's can grow to ∞, χ is limited and does not exceed $1/(x_1x_2)$, reaching this value at the ends of the integration range, while $\xi=1$ when $x=x_1$ and drops down to 0 at $x=1$. Therefore, $\chi\kappa_{12}^{\perp 2}/\Delta^2$ in the exponent is always a small number and vanishes when $\Delta\to\infty$ without contributing to the regularization dependence. On the other hand, the coefficient ζ grows to infinity at the ends of the integration region in x. This way the ultraviolet regularization factor depending on invariant masses changes the small-x singularities. The same phenomenon occurs in a simpler form in the instantaneous terms involving $1/\partial^{+2}$. The instantaneous terms themselves do not contain an integration over $\kappa^\perp$, but the integration occurs when these terms are included in the dynamics. The logarithmically divergent integrals over $\kappa^\perp$ produce logarithms of ζ as Δ-independent remnants of the ultraviolet regularization and one cannot exclude arbitrary functions of x in finite parts of the ultraviolet counterterms, including integrals that strengthen small-x logarithmic divergences. Invariant mass regularizations, including a small gluon mass μ [6], provide additional damping for $x\sim\mu^2/\Delta^2$.

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