Class of kinks in $SU(N) \times Z_2$

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In a classical, quartic field theory with $SU(N) \times Z_2$ symmetry, a class of kink solutions can be found analytically for one special choice of parameters. We construct these solutions and determine their energies. In the limit $N \to \infty$, the energy of the kink is equal to that of a kink in a Z_2 model with the same mass parameter and quartic coupling [coefficient of $\mathrm{Tr}(\Phi^4)$]. We prove the stability of the solutions to small perturbations but global stability remains unproven. We then argue that the continuum of choices for the boundary conditions leads to a whole space of kink solutions. The kinks in this space occur in classes that are determined by the chosen boundary conditions. Each class is described by the coset space H/I where H is the unbroken symmetry group and I is the symmetry group that leaves the kink solution invariant.

I. INTRODUCTION

Classical solutions in field theories have been of long-standing interest and have played an important role in our understanding of a variety of phenomena. The existence of solutions can be predicted on the basis of topology though the actual construction of the solutions is usually quite difficult and requires a certain amount of guesswork. Even after a solution is found, one needs to check its perturbative stability. And even after a solution is proved to be perturbatively stable, there is no guarantee that it is the globally stable solution.

In this paper, we shall construct a class of kink solutions in purely scalar, $SU(N) \times Z_2$ field theories. The analytic construction is made possible by a special choice of parameters. We then prove the perturbative stability of the kink solutions. As there is no known extension of the Bogomol'nyi's method [1] to this case, global stability will remain unproven. We will then address the question of whether there might be other kink solutions in the model. We discuss the possibility that various boundary conditions imposed at spatial infinity, while topologically equivalent, lead to different kink solutions. Each kink solution is continuously degenerate with a degeneracy determined by the invariance group of the solution. We describe the problem of mapping the space of kink solutions.

The following few sections deal with the construction and perturbative stability of the kink solutions that have been found. The space of kinks is described in Sec. VII and this can be read with minimal reference to the prior sections.

II. MODEL

We will consider the (1+1)-dimensional, classical field theory[1]

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[1]The addition of gauge fields can easily be accommodated in what follows as discussed in Sec. VI.

$$L = \mathrm{Tr}[(\partial_\mu \Phi)^2] - V(\Phi) \tag{1}$$

$$V(\Phi) = -m^2 \mathrm{Tr}[\Phi^2] + h(\mathrm{Tr}[\Phi^2])^2$$
$$+ \lambda \mathrm{Tr}[\Phi^4] + V_0 \tag{2}$$

where Φ transforms in the adjoint representation of $SU(N)$ ($N = 2n+1$) and hence is an $N \times N$ Hermitian matrix. Φ can also be written in terms of components:

$$\Phi = \sum_{a=1}^{N^2-1} \Phi^a T_a \tag{3}$$

where T_a are the (traceless) generators of $SU(N)$ and will be taken to be in the Gell-Mann representation [2] and normalized so that

$$\mathrm{Tr}[T_a T_b] = \frac{1}{2} \delta_{ab}. \tag{4}$$

The field components Φ^a are real. The constant V_0 is chosen so that the minimum of the potential is at $V = 0$. Note the absence of a cubic term in the model.

The center of $SU(N)$ is Z_N and the group elements of the center transform Φ by multiplication by factors of $\exp(i2\pi/N)$. For even N, the transformation $\Phi \to -\Phi$ is included in the center of $SU(N)$. However, for odd N it is not, and the full symmetry of the model is $SU(N) \times Z_2$. We will only be interested in odd values of N since it is in this case that the model admits topological kink solutions.

We will consider parameters h and λ such that the symmetry breaking is

$$SU(N) \times Z_2 \to \frac{SU(n+1) \times SU(n) \times U(1)}{Z_{n+1} \times Z_n} \tag{5}$$

where $N = 2n+1$ (n is a positive integer). This symmetry breaking pattern is achieved if [3]

$$\frac{h}{\lambda} > -\frac{N^2+3}{N(N^2-1)}. \tag{6}$$

In fact, in the next section we will choose

$$\frac{h}{\lambda} = -\frac{3}{N(N-1)}. \tag{7}$$

This choice is consistent with Eq. (6) as long as $N>3$.

To fix the constant V_0, let the minimum of $V(\Phi)$ occur at

$$\Phi = \Phi_0 \equiv R \, \mathrm{diag}(p_1, p_2, \ldots, p_N) \tag{8}$$

where R is an overall normalization factor and the matrix is normalized so that

$$\sum_{i=1}^{N} p_i^2 = \frac{1}{2}. \tag{9}$$

Then the potential is extremized for

$$R = \frac{m}{\sqrt{\lambda'}} \tag{10}$$

where

$$\lambda' = h + 4\lambda \sum_{i=1}^{N} p_i^4 = h + \frac{N^2+3}{N(N^2-1)}\lambda. \tag{11}$$

Then, $V(\Phi_0)=0$ gives

$$V_0 = -\frac{m^4}{4\lambda'}. \tag{12}$$

The condition that $\lambda'>0$ is precisely the condition in Eq. (6). We shall assume the parameters as constrained by Eq. (6), in which case the unbroken symmetry is given in Eq. (5) and the vacuum expectation value of Φ is

$$\Phi_0 = R\sqrt{\frac{2}{N(N^2-1)}} \begin{pmatrix} n\mathbf{1}_{n+1} & \mathbf{0} \\ \mathbf{0} & -(n+1)\mathbf{1}_n \end{pmatrix} \tag{13}$$

where $\mathbf{1}_n$ denotes the $n \times n$ unit matrix, $N=2n+1$, and R is defined in Eq. (10).

III. KINK SOLUTION

The model contains the breaking of a discrete Z_2 symmetry whenever N is odd. Hence kink solutions exist. Across these kink solutions, the vacuum expectation value of Φ must be related by an element of the Z_2 group. Hence, if Φ_k denotes the kink solution,

$$\Phi_k(x=-\infty) = -U\Phi_k(x=+\infty)U^{-1}$$

where U is an element of $SU(N)$.

Based on the experience with kinks in $SU(5)$ [4], we conjecture the following form for the kink solution:

$$\Phi_k(x) = f(x)\mathbf{M} + g(x)\mathbf{P} \tag{14}$$

where $f(x)$ and $g(x)$ are unspecified functions as yet, and $\mathbf{M}$ and $\mathbf{P}$ are $SU(N)$ generators such that

$$\mathrm{Tr}(\mathbf{M}^2) = \frac{1}{2} = \mathrm{Tr}(\mathbf{P}^2) \tag{15}$$

$$\mathrm{Tr}(\mathbf{MP}) = 0 = \mathrm{Tr}(\mathbf{M}^3\mathbf{P}) = \mathrm{Tr}(\mathbf{MP}^3). \tag{16}$$

Insertion of Φ_k in the potential gives

$$V(\Phi_k) = -\frac{m^2}{2}(f^2+g^2) + \frac{h}{4}(f^4+g^4) + \lambda[f^4\mathrm{Tr}(\mathbf{M}^4) \\ + g^4\mathrm{Tr}(\mathbf{P}^4)] + \left[\frac{h}{2} + 6\lambda\,\mathrm{Tr}(\mathbf{M}^2\mathbf{P}^2)\right]f^2g^2 - V_0. \tag{17}$$

The important realization that permits an analytic solution to be found is that the cross-term containing both f and g disappears if we choose

$$h = -12\lambda\,\mathrm{Tr}(\mathbf{M}^2\mathbf{P}^2). \tag{18}$$

In this case, the energy of the kink will separate into two pieces, one depending only on f and the other depending only on g. So the energy will be a sum of two single field energies, each of which can be treated directly or by Bogomol'nyi's method.

Let us further choose the matrices $\mathbf{M}$ and $\mathbf{P}$ in such a way that the kink boundary conditions imply

$$f(+\infty) = -f(-\infty), \quad g(+\infty) = +g(-\infty). \tag{19}$$

Then, we find that f and g must satisfy

$$f'' + m^2 f + 4\lambda[3\mathrm{Tr}(\mathbf{M}^2\mathbf{P}^2) - \mathrm{Tr}(\mathbf{M}^4)]f^3 = 0 \tag{20}$$

$$g^2 = -\frac{m^2}{4\lambda}\frac{1}{3\mathrm{Tr}(\mathbf{M}^2\mathbf{P}^2) - \mathrm{Tr}(\mathbf{P}^4)}. \tag{21}$$

The equation for f will have the solution

$$f(x) = f_0\tanh\left(\frac{m}{\sqrt{2}}x\right) \tag{22}$$

where

$$f_0^2 = \frac{m^2}{4\lambda}\frac{1}{\mathrm{Tr}(\mathbf{M}^4) - 3\mathrm{Tr}(\mathbf{M}^2\mathbf{P}^2)}. \tag{23}$$

The solution for f is valid provided

$$\mathrm{Tr}(\mathbf{M}^4) > 3\mathrm{Tr}(\mathbf{M}^2\mathbf{P}^2) \tag{24}$$

and the (constant) solution for g is valid provided

$$\mathrm{Tr}(\mathbf{P}^4) > 3\mathrm{Tr}(\mathbf{M}^2\mathbf{P}^2). \tag{25}$$

In addition to the properties in Eqs. (15) and (16), the conditions in Eqs. (24) and (25), $\mathbf{M}$ and $\mathbf{P}$ must yield the

desired vacuum expectation values for Φ at spatial infinity. A choice of **M** and **P** that satisfies all these conditions is

$$\mathbf{M}=\beta\begin{pmatrix}\mathbf{1}_{N-1} & 0 \\ 0 & -(N-1)\end{pmatrix},$$

$$\mathbf{P}=\sqrt{N}\beta\begin{pmatrix}\mathbf{1}_n & 0 & 0 \\ 0 & -\mathbf{1}_n & 0 \\ 0 & 0 & 0\end{pmatrix} \quad (26)$$

where

$$\beta=\frac{1}{\sqrt{2N(N-1)}}. \quad (27)$$

Straightforward calculation shows that all the conditions on **M** and **P** are met provided that $N>3$.

With this choice of **M** and **P**, the kink solution can now be written as

$$\Phi_k(x)=\frac{m}{\sqrt{\lambda}}\sqrt{\frac{N-1}{N-3}}\left[\tanh\left(\frac{m}{\sqrt{2}}x\right)\mathbf{M}+\sqrt{N}\mathbf{P}\right]. \quad (28)$$

An alternative, more transparent, form of this solution is

$$\Phi_k(x)=\left(\frac{1-F(x)}{2}\right)\Phi_-+\left(\frac{1+F(x)}{2}\right)\Phi_+ \quad (29)$$

where

$$F(x)=\tanh\left(\frac{m}{\sqrt{2}}x\right), \quad (30)$$

and $\Phi_\pm\equiv\Phi(x=\pm\infty)$. Note that the alternate form does not work for any chosen boundary condition. For example, if $\Phi_+=-\Phi_-$, the form leads to the embedded kink solution which is known to be unstable [4].

Now that we have shown that Φ_k is a solution within the restricted ansatz in Eq. (14), we also need to show that it is a solution of the full theory. This is most simply done by writing

$$\Phi=\Phi_k+\Psi=\Phi_k+\sum\psi_a T^a \quad (31)$$

and then checking that the energy density does not contain any terms that are linear in Ψ. In the quadratic terms in the energy density, this will clearly by the case since Φ_k satisfies the equations of motion (20) and (21) and the generators satisfy the orthogonality condition in Eq. (4). The only terms that can potentially lead to a term linear in the ψ_a come from the quartic term, $\text{Tr}(\Phi^4)$, in the potential and are of the form

$$\text{Tr}(\Phi_k^3 T^a)\psi^a.$$

For off-diagonal T^a, this vanishes since Φ_k^3 is diagonal and the product of a diagonal and an off-diagonal matrix is off diagonal. For diagonal T^a, it is better to choose a represen-

tation of the generators other than the Gell-Mann representation. The choice of diagonal generators other than **M** and **P** are

$$T^i=\begin{pmatrix}\tau_i & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0}_n & 0 \\ 0 & 0 & 0\end{pmatrix} \quad (i=1,\ldots,n-1) \quad (32)$$

$$T^i=\begin{pmatrix}\mathbf{0}_n & \mathbf{0} & 0 \\ \mathbf{0} & \tau_{i-n+1} & 0 \\ 0 & 0 & 0\end{pmatrix} \quad (i=n,\ldots,2n-2) \quad (33)$$

where τ_i $(i=1,\ldots,n-1)$ are the normalized, diagonal $SU(n)$ generators in the Gell-Mann representation. The off-diagonal generators are the same as in the Gell-Mann basis. It is easy to check that this basis is complete and the generators satisfy the normalization in Eq. (4).

Now we are interested in checking if there are terms in the potential that are linear in the components of Ψ expanded in the new basis. It is easy to check that

$$\text{Tr}(\mathbf{M}^3 T^i)=0=\text{Tr}(\mathbf{P}^3 T^i).$$

Hence

$$\text{Tr}(\Phi_k^3 T^i)=0$$

and there are no linear terms in the components of Ψ occurring in the energy density. The terms linear in the perturbations along the generators **M** and **P** will vanish because these have already been chosen to satisfy the equations of motion.

Therefore there are no linear terms in Ψ in the energy density and Φ_k in Eq. (28) is indeed a solution.

IV. KINK ENERGY

The energy of the kink is

$$E_k=\int dx[\text{Tr}(\Phi_k')^2+V(\Phi_k)]. \quad (34)$$

Insertion of Φ_k from Eq. (28) and evaluation yields

$$E_k=2\sqrt{\frac{2}{3}}\frac{m^3}{\lambda}\left(\frac{N-1}{N-3}\right). \quad (35)$$

In the limit that $N\to\infty$, this gives

$$E_k\to2\sqrt{\frac{2}{3}}\frac{m^3}{\lambda} \quad (36)$$

which is precisely the energy of the kink in the Z_2 model:

$$L=\frac{1}{2}(\partial_\mu\phi)^2-\frac{m^2}{2}\phi^2+\frac{\lambda}{4}\phi^4+\frac{m^4}{4\lambda}. \quad (37)$$

In the large N limit, the kink solution (28) goes to

$$\Phi_k \rightarrow \frac{m}{\sqrt{2\lambda}} \begin{pmatrix} \mathbf{1}_n & \mathbf{0} & 0 \\ \mathbf{0} & -\mathbf{1}_n & 0 \\ 0 & 0 & -\tanh(mx/\sqrt{2}) \end{pmatrix}. \qquad (38)$$

Note that this is not traceless because we have discarded a large number (N) of small (order $1/N$) terms.

V. PERTURBATIVE STABILITY

The procedure for proving perturbative stability is straightforward though tedious. We consider small deviations Ψ from the kink solution as in Eq. (31) and then find the change in the energy due to the perturbations:

$$\delta E[\Psi] = \mathrm{Tr} \int dx \Psi \left[-\frac{1}{2}\frac{d^2}{dx^2} + \mathbf{V}_2(\Phi_k) \right] \Psi \qquad (39)$$

where the matrix $\mathbf{V}_2$ is obtained by expanding the potential V up to quadratic order in Ψ. If the Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + \mathbf{V}_2(\Phi_k) \right] \Psi = \omega \Psi \qquad (40)$$

does not have any negative eigenvalues—i.e. regular solutions where $\Psi \rightarrow 0$ at spatial infinity only exist for $\omega \geq 0$—then the solution Φ_k is perturbatively stable.

The tedious part of this calculation is the evaluation of $\mathbf{V}_2(\Phi_k)$. Note that $\mathbf{V}_2$ is an $(N^2-1) \times (N^2-1)$ matrix since there are N^2-1 components of Ψ, one for each generator of $SU(N)$. Let us write

$$\Psi = \sum_{a=1}^{N^2-1} \psi^a T^a \qquad (41)$$

and the elements of $\mathbf{V}_2$ as V_{2ab}.

Let us discuss off-diagonal perturbations first. These fall into 5 types. The first type is when

$$T_1^a \propto \begin{pmatrix} \rho_n & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (42)$$

where ρ_n denotes a non-trivial, off-diagonal $n \times n$ matrix and $\mathbf{0}$ the trivial $n \times n$ matrix. The second to fifth types of perturbations are

$$T_2^a \propto \begin{pmatrix} \mathbf{0} & \rho_n & 0 \\ \rho_n^\dagger & \mathbf{0} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (43)$$

$$T_3^a \propto \begin{pmatrix} \mathbf{0} & \mathbf{0} & c \\ \mathbf{0} & \mathbf{0} & 0 \\ c^\dagger & 0 & 0 \end{pmatrix} \qquad (44)$$

$$T_4^a \propto \begin{pmatrix} \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \rho_n & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (45)$$

$$T_5^a \propto \begin{pmatrix} \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & c \\ 0 & c^\dagger & 0 \end{pmatrix} \qquad (46)$$

where c is a non-vanishing complex n-dimensional multiplet. Then the elements of $\mathbf{V}_2$ for type 1 and type 4 perturbations are

$$V_{2aa} = m^2 \frac{N \pm 3F}{N-3} \qquad (47)$$

and for type 3 (minus signs) and 5 (plus signs) are

$$V_{2aa} = \pm \frac{m^2}{2} F(1 \pm F) \qquad (48)$$

where the function $F(x)$ is defined in Eq. (30). The elements of $\mathbf{V}_2$ vanish for type 2 perturbations and there is no mixing between different off-diagonal perturbations either.

The potential V_{2aa} for type 1 and 4 perturbations is everywhere non-negative for $N > 3$ and hence the Schrödinger equation will not have any bound states. The situation for type 3 and 5 perturbations is less clear since $\pm F(1 \pm F)$ can have either sign. However, the Schrödinger equation has the $\omega = 0$ solution $\psi = 1 \mp F$ and this solution has no nodes, i.e. $\psi \neq 0$ except at infinity.[2] The zero mode solution does not go to zero at either $x = -\infty$ or at $x = +\infty$. Then, in order for a solution to go to zero at both spatial infinities, ω has to be chosen to be greater than zero. The reason is that the $\omega = 0$ solution needs to have smaller curvature (i.e. the curvature needs to be more negative) so that it can vanish at infinity. The curvature is proportional to $-\omega$ in the region where the potential is small. Therefore ω has to be larger. Hence, once again, there are no bound states to the Schrödinger equation. This shows that the solution is stable to off-diagonal perturbations.

Next we consider diagonal perturbations. These can be classified in three types. If we write

$$T_a = \begin{pmatrix} \mathbf{R}_a & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{S}_a & 0 \\ 0 & 0 & Q_a \end{pmatrix} \qquad (49)$$

where $\mathbf{R}_a$ and $\mathbf{S}_a$ are $n \times n$, diagonal matrices, then the three types of T_a are (1) $\mathbf{R}_a \neq 0$, $\mathbf{S}_a = 0$, $Q_a = 0$, (2) $\mathbf{R}_a \neq 0$, $\mathbf{S}_a \neq 0$, $Q_a = 0$, and (3) $\mathbf{R}_a \neq 0$, $\mathbf{S}_a \neq 0$, $Q_a \neq 0$. Then there are six cases to be treated in finding the elements of $\mathbf{V}_2$. If both T_a and T_b are type (1), we find

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[2]This zero mode will be of special interest in Sec. VII.

$$V_{2ab} = m^2 \frac{N+3F}{N-3} \delta_{ab} > 0 \qquad (50)$$

where we note that we are considering $N > 3$. Since V_{2ab} is everywhere non-negative, the Schrödinger equation has no bound states and the solution is stable to these perturbations.

If both T_a and T_b are type (3)—there is only one generator of type (3)—we find

$$V_{2aa} = \frac{m^2}{2}(3F^2 - 1). \qquad (51)$$

Now the Schrödinger equation is precisely that obtained when considering fluctuations about a Z_2 kink [5]. The complete eigenspectrum of this equation is known [6] and the lowest eigenvalue is $\omega = 0$ corresponding to the zero mode which describes translations of the kink. So there are no bound states and no instability to these perturbations.

The only other non-trivial perturbations are when T_a and T_b are both type (2). For these perturbations, we have

$$\mathbf{R}_a = \alpha_a \mathbf{1}_n \qquad (52)$$

where α_a is determined from the normalization of T_a. After some algebra, we find

$$\psi_a V_{2ab} \psi_b = \frac{m^2}{N-3}[(N-3F)\psi_\perp^2 + \{(N-3F) \\ - 6\sigma^2(N-1)(1-F)\}\psi_\parallel^2] \qquad (53)$$

where we have defined the unit vector $\hat{\alpha}_b$, decomposed ψ_a parallel ($\psi_\parallel = \hat{\alpha}_a \psi_a$) and perpendicular ($\psi_\perp$) to the vector α_a, and written

$$\sigma^2 \equiv \sum_a \alpha_a \alpha_a.$$

The coefficient of $\psi_\perp^2$ is positive since $N > 3$ and hence this perturbation cannot cause an instability. The coefficient of $\psi_\parallel^2$ needs to be checked.

Using the normalization of T_a, we obtain

$$\sigma^2 = \frac{1}{2(N-1)}. \qquad (54)$$

Inserting this into Eq. (53), we find that the coefficient of $\psi_\parallel^2$ is simply $+m^2$. Hence there is no instability to diagonal type (2) perturbations.

This explicit analysis shows that the solution is stable to all perturbations but this does not imply that the solution minimizes the energy globally.

VI. GAUGE FIELDS

The inclusion of gauge fields, $A_\mu = A_\mu^a T^a$, has no effect on the existence of the static solution in Eq. (28). This can be seen by noting that the terms in the energy that involve the gauge fields are $\mathrm{Tr}(\mathbf{E}^2 + \mathbf{B}^2)$, $-\mathrm{Tr}([A_x, \Phi]^2)$ and $i\mathrm{Tr}A_x[\Phi, \partial_x \Phi]$. The first two terms are quadratic in the gauge fields while the last one vanishes because $[\Phi, \partial_x \Phi] = 0$ for the solution. Hence $A_\mu = 0$ is a solution of the equations of motion.

The presence of gauge fields does not have any effect on the perturbative stability of the solution. To see this one can check that both the quadratic order terms in the gauge field are non-negative. For the first term this is explicit while for the second term one uses the fact $[A_x, \Phi] = -[A_x, \Phi]^\dagger$ since A_x and Φ are Hermitian. Hence the kink solution with $A_\mu = 0$ is stable to perturbations in the gauge fields.

VII. SPACE OF KINKS

Different boundary conditions will, in general, lead to different kink solutions. Therefore kinks with different boundary conditions fall into different classes—kinks belonging to different classes cannot be transformed into one another by global $SU(N)$ rotations. Here we would like to find the degeneracy of a kink solution, i.e. the space of boundary conditions that lead to degenerate kink solutions. There is clearly an $SU(N)$ global degeneracy but this is not very interesting since it applies to any field configuration in the theory. It is of greater interest to only consider those global $SU(N)$ transformations that leave $\Phi_- \equiv \Phi(x = -\infty)$ unchanged but act non-trivially on $\Phi_+ \equiv \Phi(x = +\infty)$. If we denote the unbroken symmetry groups at $x = \pm\infty$ by $H_\pm$, such transformations belong to H_-. But the transformations that belong to $K \equiv H_+ \cap H_-$ will act trivially on both Φ_- and on Φ_+. Therefore the space of boundary conditions at $x = +\infty$ leading to degenerate kinks is given by H_-/K.

In addition to the degeneracy due to different boundary conditions, any kink solution will have an "internal" symmetry group, denoted by I. This group will contain all those transformations that leave unchanged the whole kink solution (including the boundary conditions). So we have $I \subseteq K$ and the space of degenerate kinks is H_-/I.

The kink classification problem can be described in more detail as follows. Suppose we fix the boundary condition at $x = -\infty$ to be Φ_-; then the only constraint on the boundary condition at $x = +\infty$ is that it should be in the distinct topological sector. There is a full vacuum manifold (mod Z_2) worth of choices for $\Phi(x = +\infty) \equiv \Phi_+$. For certain choices of Φ_+ we can solve the equations of motion and obtain a set (described by the space K/I) of kink solutions that extremize the energy. Let the value of this energy be $U[\Phi_+; \Phi_-]$ where we have explicitly indicated that different choices of boundary conditions can lead to kink solutions of different energy. We are interested in the space of minima of the "potential" $U[\Phi_+; \Phi_-]$ with respect to Φ_+. The global minima of this potential will describe the lowest energy kink solutions in the model and may be termed the "kink vacuum manifold." Other local minima will describe kink solutions that are separated from the lightest kink by an energy barrier. In this way it might be possible to obtain "generations" of stable kinks having the same topological charge but differing in their energies.

The existence of the kink solution found in the previous sections does not tell anything about whether it is a mini-

mum of $U[\Phi_+ ; \Phi_-]$. To examine if the solution is a local minimum we can do a perturbative analysis where the perturbations are required to vanish at $x = -\infty$—so as to hold Φ_- fixed—but are not required to vanish at $x = +\infty$–since we want to find changes in the energy when the boundary conditions are varied. Going back to Sec. V we find that all but one of the non-trivial Schrödinger potentials are positive (for $N > 3$) at both spatial infinities. This means that a perturbation that does not vanish at both spatial infinities gives a divergent contribution to the energy. Hence the kink solution is stable under these perturbations of the boundary conditions. The only exceptional case is the off-diagonal perturbation of the type in Eq. (44). The potential [Eq. (48)] goes to a positive value at $x = -\infty$ but vanishes from below at $x = +\infty$. In fact, as described below Eq. (48), there is a zero mode for this perturbation that vanishes at $x = -\infty$ but goes to a non-vanishing constant at $x = +\infty$. So this mode is a ''dangerous'' one and needs to be examined further.

A closer inspection of this mode shows that it corresponds to gauge rotations of the field Φ which leave Φ_- invariant but rotate Φ_+. In other words, the zero mode rotates the kink within its own class described by the space H/I. Hence, the perturbation under consideration is not an instability but a gauge rotation. Therefore the class of kinks that has been found describes a set of local minima of $U[\Phi_+ ; \Phi_-]$.

There is a subtlety in the discussion above which we have glossed over. To obtain the Schrödinger equation in Eq. (39) we have to perform an integration by parts and assume that the boundary contributions vanish. However, here we are considering perturbations that do not vanish at infinity. This fact means that there is an extra contribution to δE_∞ in Eq. (39) given by

$$\delta E_\infty = \frac{1}{2} \text{Tr} \left[\Psi \frac{d\Psi}{dx} \right]_{-\infty}^{+\infty}. \tag{55}$$

The contribution at $x = -\infty$ vanishes because we are choosing $\Psi(-\infty) = 0$ but the contribution at $x = +\infty$ does not obviously vanish and depends on the derivative of the perturbation at infinity. However, since we are only interested in finite energy field configurations and the energy density contains the term $\text{Tr}(\partial_x \Psi)^2$, we must require that the derivative of the perturbations vanish at spatial infinity. So it is rigorous to take $\delta E_\infty = 0$ even if the perturbations do not vanish at infinity.

A more extensive discussion of the kink classification problem is left for future work.

VIII. SUMMARY

We have considered $SU(N) \times Z_2$ models. On the basis of topology, the model will contain topological kink solutions. A general technique for constructing the solutions is not known. Here, by using some guesses and by choosing a special relation between parameters [Eq. (7)], we have analytically constructed a class of kink solutions [see Eq. (28)]. The energy of the solutions is

$$E = \frac{2\sqrt{2}}{3} \left(\frac{N-1}{N-3} \right) \frac{m^3}{\lambda}. \tag{56}$$

The limiting value for large N is equal to the energy of a Z_2 kink with mass parameter m and coupling constant λ. We have explicitly checked that these kink solutions are perturbatively stable. It is not known if the solutions are globally stable and this remains an interesting open problem.

We have then described the space of kinks as partitioning into distinct classes. All members have the same topology, yet elements of different classes are not expected to have the same energy. The solutions constructed above describe only one of the (unknown number of) classes of kinks and might lie on the ''kink vacuum manifold''—the manifold consisting of the least energetic kinks in the model.

We hope that the solutions found here can be used as a guide to the construction of other topological defect solutions in complicated field theories.

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