Znajek-Damour horizon boundary conditions with Born-Infeld electrodynamics

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In this work, the interaction of electromagnetic fields with a rotating (Kerr) black hole is explored in the context of the Born-Infeld (BI) theory of electromagnetism instead of standard Maxwell theory and particularly BI theory versions of the four horizon boundary conditions of Znajek and Damour are derived. Naturally, an issue to be addressed is then whether they would change from the ones given in the Maxwell theory context and if they do, how. Interestingly enough, as long as one employs the same local null tetrad frame as the one adopted in the works of Damour and of Znajek to read out physical values of electromagnetic fields and a fictitious surface charge and currents on the horizon, it turns out that one ends up with exactly the same four horizon boundary conditions despite the shift of the electrodynamics theory from a linear Maxwell one to a highly nonlinear BI one. Close inspection reveals that this curious and unexpected result can be attributed to the fact that the concrete structure of BI equations happens to be such that it is indistinguishable *at the horizon* to a local observer, say, in Damour's local tetrad frame from that of standard Maxwell theory.

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I. INTRODUCTION

The idea of using rotating black holes as energy sources has a long history. To our knowledge, Salpeter [1] and Zel'dovich [1] were the first to point out that gigantic black holes might serve as power engines for quasars or radio galaxies. Realistic theoretical models to realize this type of energy extraction from rotating black holes also appeared afterwards and they are due to Penrose [2], Press and Teukolsky [2], Ruffini and Wilson [3], Damour [3], and Blandford and Znajek [4]. Among these models, that of Blandford and Znajek is particularly interesting in its formulation and looks quite plausible in its operational mechanism. At first, puzzling over the possible explanation for the observed twin jets pointing oppositely out of a black hole-accretion disk system, Blandford and Znajek conceived of a particular process in which the power going into the jets comes from the hole's enormous rotational energy. Schematically, their mechanism works as follows: suppose that the rotating hole is threaded by magnetic field lines. As the hole spins, it drags the field lines around, causing them to fling surrounding plasma upward and downward to form two jets. Then the jets shoot out along the hole's spin axis and their direction is firmly fixed to the hole's axis of rotation. The magnetic field lines, of course, come from the accretion disk around the hole. Namely, it is the magnetic fields that extract the rotational energy of a black hole and then act to power the jets. According to their careful analysis, on the other hand, as the energy is extracted, electric currents flow into the horizon near the hole's poles (in the form of positively charged particles falling inward), and currents flow out of the horizon near the equator (in the form of negatively-charged particles falling inward). It was as though the hole were a voltage generator of an electric circuit driving current out of the horizon's equator, then up magnetic field lines to a large distance, then through ''plasma load'' to other field lines near the hole's spin axis, then down those field lines and into the horizon. Namely, the magnetic field were the wires of the electric circuits, the plasma was the load that exerts power from the circuit. And the two pictures, one schematic and the other analytic, are just two different ways of describing the same phenomenon. This electric circuit description was totally unexpected and thus curious enough although it was resulted from a careful general relativistic treatment of the problem. Right after the post of this new mechanism, Znajek [5] and, independently, Damour [6] succeeded in translating the careful general relativistic formulation into a surprisingly simple nonrelativistic, flat spacetime electrodynamics language, the celebrated four horizon boundary conditions. And the assumption of central importance in this new picture is to endow the horizon with some fictitious surface charge and current as those previously imagined by Hanni and Ruffini [7]. It is really amusing that one now has an option to view the situation in terms of flat spacetime electrodynamics alone at least for rough understanding.

Speaking of the theory that governs the electromagnetism, however, it is interesting to note that historically, there has been another classical theory that can be thought of as a larger class of theory involving the standard Maxwell theory just as its limiting case. It is the theory proposed in the 1930s by Born and Infeld [9]. In spite of its long history, the Born-Infeld (BI) theory of electrodynamics has remained almost unnoticed and hence nearly uncovered in full detail. This theory may be thought of as a highly nonlinear generalization of or a nontrivial alternative to the standard Maxwell theory of electromagnetism. It is known that Born and Infeld had been led, when they first constructed this theory, by the considerations such as finiteness of the energy in electrodynamics, natural recovery of the usual Maxwell theory as a linear approximation and the hope to find solitonlike solutions representing pointlike charged particles. In the present work we would like to explore the interaction of electromag-

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netic fields with a rotating (Kerr) black hole but in the context of BI theory of electromagnetism instead of Maxwell theory. And our particular concern is to derive BI theory versions of the four horizon boundary conditions to see how they would change from the ones derived originally by Znajek and by Damour in the context of Maxwell theory. Now the motivation for shifting the theory of electromagnetism from that of standard Maxwell to that of BI to study the physics of interaction between ''test'' electromagnetic field and ''background'' rotating black hole geometry can be stated as follows. The BI theory, although appeared as a ''classical'' theory long before the advent of quantum electrodynamics (QED) theory, may be viewed as some kind of an effective low-energy theory of QED in that its highly nonlinear structure plays the role of eliminating the short-distance divergences. Normally, the strong magnetic field, believed to be anchored in the central black holes of typical gamma-ray bursters, is regarded as being originated, say, from that of neutron stars that has collapsed to form the black hole. A number of various observations indicate that in young neutron stars, the surface magnetic field strengths are of order 10^{11}–10^{13} (G) and in some extreme cases such as magnetars, magnetic field strengths are estimated to be as large as $\geqslant 5\times 10^{14}$ (G) [8]. Then the magnetic field of this ultra strength, in turn, stimulates our curiosity and leads us to ask questions such as what would happen if we choose to employ the BI theory that, as stated, can be thought of as an effective theory of QED, instead of linear Maxwell theory, to study the physics in the vicinity of rotating hole's horizon? And in doing so, we anticipate that perhaps the highly nonlinear nature of the BI theory may serve to uncover some hidden interplay between the strong electromagnetic field and ultra strong gravity near the hole's horizon. Since our main concern is the derivation of the four horizon boundary conditions in BI theory, we now recall some of the basic ingredients of these boundary conditions obtained in the conventional Maxwell theory.

The four ''horizon boundary conditions'' first derived in the works of Znajek [5] and of Damour [6] and reformulated later in the literature can be briefly described as follows. They may be called radiative ingoing boundary condition, Ohm's law, Gauss' law and Ampere's law, respectively. And in order to represent each boundary condition properly, we need to introduce in advance some quantities that will be derived carefully in the text shortly. They are electric and magnetic fields at the horizon $(\vec{E}_H,\vec{B}_H)$ as seen by a local observer in a null tetrad frame which has been made to be well behaved at the horizon by the amount of boost that becomes suitably infinite at the horizon and the *fictitious* charge and current densities $(\sigma,\vec{\kappa})$ that have been assigned at the horizon in such a way that the sum of real current 4-vector outside the horizon and this fictitious current 4-vector on the horizon together is conserved. Firstly, then, the radiative ingoing boundary condition first derived by Znajek [5] takes the form $\vec{B}_H=\vec{E}_H\times\hat{n}$ with $\hat{n}$ being the outer unit normal to the horizon. Evidently, it states that the electric and magnetic fields tangential to the horizon are equal in magnitude and perpendicular in direction and hence their Poynting energy flux is *into* the hole. Secondly, the Ohm's law reads $\vec{E}_H=4\pi\vec{\kappa}$. It has been derived rigorously in the work by Damour [6] and pointed out in the work by Znajek [5]. Clearly, this relation takes on the form of a nonrelativistic Ohm's law for a conductor and hence implies that if we endow the horizon with some charge and current densities (which are to be determined by the surrounding external electromagnetic field $F_{\mu\nu}$ as we shall see in the text), then the horizon behaves as if it is a conductor with finite surface resistivity of $\rho=4\pi\simeq 377(\text{ohms})$. Actually these two relations are the ones that have been explicitly derived in the works by Znajek and by Damour and play the central role in justifying that the introduction of fictitious charge and current densities on the horizon indeed provides a self-consistent picture. That is, one might wonder what would happen to the Joule heat generated when those surface currents work against the surface resistance and how it would be related to the electromagnetic energy going down the hole through the horizon. In their works, Znajek and Damour provided a simple and natural answer to this question. Namely, they showed in an elegant manner that the total electromagnetic energy flux (i.e., the Poynting flux) into the rotating Kerr hole through the horizon is indeed precisely the same as the amount of Joule heat (Ohmic dissipation) produced by the surface currents when they work against the surface resistivity of 4π. As a result, one may think of the rotating hole as a conducting sphere that absorbs the incident electromagnetic energy flux as a form of Joule heat that the surface current (driven by the electromagnetic fields) generates when it interferes with the surface resistivity. This is indeed an interesting and quite convincing alternative picture of viewing the interaction of external electromagnetic fields with a rotating black hole. Damour [6] also remarked that this result provides a clear confirmation of Carter's assertion [10] that a black hole is analogous to an ordinary object having finite viscosity and electrical conductivity. Thirdly, if one follows the formulation of Damour but in a slightly different way in taking the local tetrad frame and projecting the Maxwell field tensor and the surface current 4-vector onto that chosen tetrad frame, one also gets the relation $E_{\hat{r}}=4\pi\sigma$ which may be identified with the surface version of Gauss' law. It says that the fictitious surface charge density we assumed on the horizon plays the role of terminating the normal components of all electric fields that pierce the horizon. Lastly, if we combine the radiative ingoing boundary condition at the horizon that we obtained earlier, $\vec{B}_H=\vec{E}_H\times\hat{n}$ with the Ohm's law $\vec{E}_H=4\pi\vec{\kappa}$, we end up with the fourth relation $\vec{B}_H=4\pi(\vec{\kappa}\times\hat{n})$ which can be viewed as the surface version of Ampere's law. Again, consistently with our motivation for introducing fictitious current density on the horizon, this relation indicates that the current density we assumed plays the role of terminating any tangential components of all magnetic fields penetrating the horizon. And actually these four horizon boundary conditions later on provided a strong motivation for the proposal of so-called ''membrane paradigm [11]'' of black holes by Thorne and his collaborators. As we already mentioned, in the present work we would like to particularly derive BI theory versions of these four horizon boundary conditions to see if they would change from the ones given above and if they would, how. Interestingly enough, as far as we employ the same local null tetrad frame as the one adopted in the works by Damour and by Znajek, it turns out that we end up with exactly the same four horizon boundary conditions despite the shift of the electrodynamics theory from a linear Maxwell one to a highly non-linear BI one. As we shall see shortly in the text, this curious and unexpected result can be attributed to the fact that the nature of the BI theory or more precisely, the concrete structure of BI equations happens to be such that it is indistinguishable *at the horizon* to a local observer, say, in Damour's local tetrad frame from that of standard Maxwell theory. We find this point indeed quite amusing on theoretical side.

II. CHOICE OF COORDINATE SYSTEM AND TETRAD FRAME

As we stated in the Introduction above, we would like to derive Znajek-Damour-type boundary conditions at the horizon of Kerr black hole in the context of BI theory of electromagnetism. Generally speaking, all that is required of the ''correct'' boundary conditions for electric and magnetic fields at the horizon can be stated as follows. The physical field's components in the neighborhood of an event horizon should have ''nonspecial'' values. Or put another way, a physically well-behaved observer at the horizon should see the fields as having finite and nonzero values. And indeed, in order to discuss electrodynamics in terms of this *physical* field values, one should make relevant choice of coordinate system and proper choice and treatment of the associated tetrad frame [12] for the background Kerr black hole spacetime. It is well known that this can be achieved only when one takes the ingoing Kerr coordinates, the advanced null coordinates representing a reference frame of ''freely-falling'' photons, and employs an associated null tetrad frame such as that of Damour [6] which is well-behaved on the event horizon. Unlike the familiar Kinnersley's null tetrad [15] or the well-known Hawking-Hartle tetrad [13,14], however, the Damour's choice of tetrad is rather nonstandard and hence may not be so familiar. Even in the original work of Damour [6], the connection of his tetrad choice to these standard null tetrads is not discussed. Thus here in this section, we would like to briefly exhibit the derivation of Damour's tetrad so as to clarify this issue. We now start with some basics. Among the choices of coordinates for Kerr spacetime, the best-known Boyer-Lindquist coordinates $x^{\mu}=(t,r,\theta,\tilde{\phi})$ [12] can be viewed as the generalization of Schwarzschild coordinates to the stationary, axisymmetric case and the (ingoing) Kerr coordinates $x'^{\mu}=(v,r,\theta,\phi)$ [12] can be thought of as the axisymmetric generalization of Eddington-Finkelstein (advanced) null coordinates. They are related by the coordinate transformation given by [12]

$$dv=dt+\frac{(r^2+a^2)}{\Delta}dr,\quad d\phi=d\tilde{\phi}+\frac{a}{\Delta}dr, \tag{1}$$

where $\Delta=r^2+a^2-2Mr$ with M and a being the Arnowitt-Deser-Misner (ADM) mass and the angular momentum per unit mass of the hole respectively. Turning to the choice of tetrad frame, there are largely two types of tetrad frames; orthonormal tetrad $e_A=\{e_0=u,e_1,e_2,e_3\}$ and null tetrad $Z_A=\{l,n,m,\overline{m}\}$ which are related to each other by [12]

$$e_0=\frac{1}{\sqrt{2}}(l+n),\quad e_1=\frac{1}{\sqrt{2}}(l-n),$$
$$e_2=\frac{1}{\sqrt{2}}(m+\overline{m}),\quad e_3=\frac{1}{\sqrt{2}i}(m-\overline{m}), \tag{2}$$

with the null tetrad satisfying the orthogonality relation

$$-l^{\mu}n_{\mu}=1=m^{\mu}\overline{m}_{\mu}, \tag{3}$$

with all other contractions being zero. As is well known, the Hawking-Hartle tetrad is well-behaved at the horizon and can be constructed by performing an appropriate null rotation [13,14] on the Kinnersley's null tetrad given in ingoing Kerr coordinates. It is given by [13,14]

$$l^{\mu}_{HH}=\left(1,\ \frac{\Delta}{2(r^2+a^2)},\ 0,\ \frac{a}{(r^2+a^2)}\right),$$
$$n^{\mu}_{HH}=\left(0,\ \frac{-(r^2+a^2)}{\Sigma},\ 0,\ 0\right), \tag{4}$$
$$m^{\mu}_{HH}=\frac{1}{\sqrt{2}\Sigma^{1/2}}\left(ia\sin\theta,\ 0,\ 1,\ \frac{i}{\sin\theta}\right),$$

where $\Sigma=r^2+a^2\cos^2\theta$. Now, it is interesting to note that generally one can ''mix'' half of the null tetrad Z_A and half of the orthonormal tetrad e_A to form a ''quasiorthonormal'' or ''mixed'' tetrad

$$\{l^{\mu},-n^{\mu},e^{\mu}_2,e^{\mu}_3\},\quad \{-n_{\mu},l_{\mu},e^2_{\mu},e^3_{\mu}\}. \tag{5}$$

And if we construct this half-null, half-orthonormal, mixed tetrad from the above Hawking-Hartle null tetrad, it becomes Damour's quasiorthonormal tetrad as we can see shortly. Before we proceed, let us elaborate on the general construction of this mixed tetrad. Using the relations between the orthonormal tetrad e_A and null tetrad Z_A given in Eq. (2),

$$\begin{aligned}ds^2&=g_{\mu\nu}dx^{\mu}dx^{\nu}=\eta_{AB}e^A_{\mu}e^B_{\nu}dx^{\mu}dx^{\nu}\\&=(-l_{\mu}n_{\nu}-n_{\mu}l_{\nu}+m_{\mu}\overline{m}_{\nu}+\overline{m}_{\mu}m_{\nu})dx^{\mu}dx^{\nu}\end{aligned} \tag{6}$$

and hence $g_{\mu\nu}=-l_{\mu}n_{\nu}-n_{\mu}l_{\nu}+m_{\mu}\overline{m}_{\nu}+\overline{m}_{\mu}m_{\nu}$. However, since the pair (e_0,e_1) is related only to (l,n) while the pair (e_2,e_3) is related only to $(m,\overline{m})$, one can write, using $m_{\mu}\overline{m}_{\nu}+\overline{m}_{\mu}m_{\nu}=e^2_{\mu}e^2_{\nu}+e^3_{\mu}e^3_{\nu}$,

$$g_{\mu\nu}=-l_{\mu}n_{\nu}-n_{\mu}l_{\nu}+e^2_{\mu}e^2_{\nu}+e^3_{\mu}e^3_{\nu}. \tag{7}$$

This obviously implies that one may mix half of null tetrad and half of orthonormal tetrad to form a mixed tetrad as given in Eq. (5). Therefore, we now construct this mixed tetrad from the previous Hawking-Hartle tetrad as

$$\begin{aligned} e_0^\mu &\equiv l_{HH}^\mu = \left(1,\ \frac{\Delta}{2(r^2+a^2)},\ 0,\ \frac{a}{(r^2+a^2)}\right),\\ e_1^\mu &\equiv -n_{HH}^\mu = \left(0,\ \frac{(r^2+a^2)}{\Sigma},\ 0,\ 0\right),\\ e_2^\mu &= \frac{1}{\sqrt{2}}(m_{HH}^\mu + \bar{m}_{HH}^\mu) = \left(0,\ 0,\ \frac{1}{\Sigma^{1/2}},\ 0\right),\\ e_3^\mu &= \frac{1}{\sqrt{2}i}(m_{HH}^\mu - \bar{m}_{HH}^\mu)\\ &= \left(\frac{a\sin\theta}{\Sigma^{1/2}},\ 0,\ 0,\ \frac{1}{\Sigma^{1/2}\sin\theta}\right), \end{aligned} \tag{8}$$

and its dual is

$$\begin{aligned} e_\mu^0 &\equiv -n_\mu^{HH} = \left(\frac{(r^2+a^2)}{\Sigma},\ 0,\ 0,\ \frac{-(r^2+a^2)}{\Sigma}a\sin\theta\right),\\ e_\mu^1 &\equiv l_\mu^{HH}\\ &= \left(\frac{-\Delta}{2(r^2+a^2)},\ \frac{\Sigma}{(r^2+a^2)},\ 0,\ \frac{\Delta}{2(r^2+a^2)}a\sin^2\theta\right),\\ e_\mu^2 &= \frac{1}{\sqrt{2}}(m_\mu^{HH} + \bar{m}_\mu^{HH}) = (0,\ 0,\ \Sigma^{1/2},\ 0),\\ e_\mu^3 &= \frac{1}{\sqrt{2}i}(m_\mu^{HH} - \bar{m}_\mu^{HH})\\ &= \left(\frac{-a\sin\theta}{\Sigma^{1/2}},\ 0,\ 0,\ \frac{(r^2+a^2)}{\Sigma^{1/2}}\sin\theta\right), \end{aligned} \tag{9}$$

which we renamed as

$$l^\mu \to e_0^\mu,\quad n^\mu \to -e_1^\mu,$$

$$l_\mu \to e_\mu^1,\quad n_\mu \to -e_\mu^0,$$

to go from the null tetrad's orthogonality relations $-l^\mu n_\mu = 1 = m^\mu \bar{m}_\mu$ to the usual orthonormality condition $e_A^\mu e_\mu^B = \delta_A^B$, $e_A^\mu e_\nu^A = \delta_\nu^\mu$. Note that this mixed tetrad precisely coincides with Damour's choice of quasiorthonormal tetrad [6]. And the tetrad metric $\epsilon_{AB} = \epsilon^{AB}$ in

$$ds^2 = \epsilon_{AB} e^A e^B$$

can be identified with

$$\epsilon_{AB} = \epsilon^{AB} = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

Note that in all the calculations involved in this work to read off physical components of tensors such as Maxwell field tensor and current 4-vector, we shall strictly use this quasi-orthonormal tetrad given in Eqs. (8) and (9) and nothing else. In this sense, our choice of local tetrad frame is slightly different from that in the original work of Damour [6] in which he introduced, particularly on the 2-dimensional, $v = \text{const}$ section of the event horizon, some other orthonormal basis (slightly different from $\{e_2^\mu, e_3^\mu\}$ given above) specially adapted to the ''intrinsic geometry'' of the $v = \text{const}$ section of the horizon and used them to project out physical components of tensors.

Before closing our discussion, perhaps it might be worth mentioning the relevance of the choice of this Damour's tetrad over that of the usual zero angular momentum observer (ZAMO) tetrad in studying the electrodynamics in the vicinity of Kerr hole's horizon. Among other things, note that the 4-velocity of a local observer in this Damour's quasi-orthonormal frame, e_0^μ (in $e_0 = e_0^\mu \partial_\mu$) becomes, at the horizon where $\Delta = 0$, the usual Killing vector normal to the horizon, $\chi^\mu = (\partial/\partial v)^\mu + \Omega_H(\partial/\partial\phi)^\mu$ which has no pathological behavior whatsoever there. Thus we do not need any *ad hoc* regularization prescription to begin with. Certainly, this is in contrast to the corresponding quantity (i.e., the 4-velocity of a local observer) in ZAMO which becomes ill-defined as the horizon is approached and hence requires a cumbersome regularization treatment [11]. Thus in the present work, we choose to work with Damour's quasiorthonormal tetrad in ingoing Kerr coordinates and try to read out physical components of all tensors involved by projecting them onto Damour's tetrad frame.

III. IDENTIFICATION OF ELECTRIC AND MAGNETIC FIELDS ON THE HORIZON

In a sense, the BI electrodynamics can be thought of as a nonlinear generalization of the standard Maxwell theory as the BI field equation is a nonlinear differential equation that reduces to the Maxwell field equation in an appropriate limit. As we shall see in a moment, however, the highly nonlinear BI equations can be made to take on a seemingly linear structure similar to that of Maxwell equations. And to this end, we need to introduce two species of field strength tensors; the new one $G_{\mu\nu}$ for the inhomogeneous BI field equations and the usual one $F_{\mu\nu}$ for the homogeneous Bianchi identity. Despite this added technical complexity, however, the basic field quantities, namely the physical (finite and nonzero) components of electric field and magnetic induction can still be extracted from the standard field strength $F_{\mu\nu}$. And generally the typical procedure by which one can read off physical components of electric field and magnetic induction from $F_{\mu\nu}$ involves taking the projection of components of $F_{\mu\nu}$ onto the orthonormal tetrad frame chosen, $F_{AB} = F_{\mu\nu}(e^{\mu}_{A}e^{\nu}_{B})$. Since A,B are now tangent space indices in this locally-flat tetrad frame, the physical electric and magnetic field components then can be read off in a standard manner as

$$F_{AB}=\{F_{i0},F_{ij}\},$$

where

$$E_i=F_{i0},$$
$$B_i=\frac{1}{2}\epsilon_{ijk}F^{jk}=\frac{1}{2}\epsilon_{0ijk}F^{jk}=\tilde{F}_{0i}=-\tilde{F}_{i0}. \tag{11}$$

A. Brief review of BI electrodynamics in curved spacetimes

Eventually for the exploration of boundary conditions for BI electromagnetic fields at the horizon of Kerr hole, we now briefly describe general formulation of BI theory in a given curved spacetime. The BI theory of electromagnetism is, despite its long history and physically interesting motivations behind it, not well known and hence might be rather unfamiliar to relativists and workers in theoretical astrophysics community. Readers can find in the literature [16] some other works which discuss interesting aspects of this BI theory of electrodynamics from a modern perspective. In our discussion below, we are implicitly aimed at adapting the theory to the formulation of electrodynamics in a rotating uncharged black hole spacetime. Also at this point in seems worthy of mention that throughout, we will be assuming the ''weak field limit.'' To be a little more concrete, we consider the dynamics of electromagnetic field governed by the BI theory in the background of uncharged Kerr black hole spacetime. And we assume that the strength of this external electromagnetic field is small enough not to have any sizable backreaction to the background geometry. Then this means we are not considering phenomena described by solutions in coupled full Einstein-BI theory but an environment where the test electromagnetic field possesses dynamics governed by the BI theory rather than by the Maxwell theory. Also note that this assumption can be further justified as long as we confine our concern to the electrodynamics around the ''uncharged'' Kerr black hole. If, instead, one is interested in the same physics but in charged rotating black holes (which, however, is rather uninteresting since it is less likely to happen in realistic astrophysical environments where the black hole charge, if any, gets quickly neutralized by the surrounding plasma), one would have to deal with the full Einstein-BI theory in which, unfortunately, the charged rotating black hole solution is not available.

Thus we consider here the action of (4-dimensional) BI theory in a fixed background spacetime with metric $g_{\mu\nu}$. And to do so, some explanatory comments might be relevant. Coupling the BI gauge theory to gravity is not so familiar and hence we start first with the BI theory action in 4-dimensional flat spacetime:

$$S=\int d^4x\frac{1}{4\pi}\beta^2\left[1-\sqrt{-\det\left(\eta_{\mu\nu}+\frac{1}{\beta}F_{\mu\nu}\right)}\right]$$
$$=\int d^4x\frac{1}{4\pi}\beta^2\left[1-\sqrt{1+\frac{1}{2\beta^2}F_{\mu\nu}F^{\mu\nu}-\frac{1}{16\beta^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2}\right]$$

and then elevate it to its curved spacetime version by employing the minimal coupling scheme. This is really the conventional procedure and the result is

$$S=\int d^4x\sqrt{g}\left\{\frac{1}{4\pi}\beta^2\left[1-\sqrt{1+\frac{1}{2\beta^2}(g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta})-\frac{1}{16\beta^4}(g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}\tilde{F}_{\alpha\beta})^2}\right]+J^{\mu}A_{\mu}\right\}, \tag{12}$$

where $J^{\mu}=\rho_e u^{\mu}+j^{\mu}_e$ is the electric source current for the vector potential A_{μ}. Here, the generic parameter of the theory ''β'' having the canonical dimension $\dim[\beta]=\dim[F_{\mu\nu}]=+2$, probes the degree of deviation of BI theory from the standard Maxwell theory as the limit $\beta\to\infty$ obviously corresponds to the Maxwell theory action. Now extremizing this action with respect to A_{μ} yields the dynamical BI field equation

$$\nabla_{\nu}\left[\frac{F^{\mu\nu}-\frac{1}{4\beta^2}(F_{\alpha\beta}\tilde{F}^{\alpha\beta})\tilde{F}^{\mu\nu}}{\sqrt{1+\frac{1}{2\beta^2}(F_{\alpha\beta}F^{\alpha\beta})-\frac{1}{16\beta^4}(F_{\alpha\beta}\tilde{F}^{\alpha\beta})^2}}\right]=4\pi J^{\mu}, \tag{13}$$

while the Bianchi identity, which is a supplementary equation to this field equation is given by

$$\nabla_\nu \tilde{F}^{\mu\nu} = \frac{1}{\sqrt{g}} \partial_\nu [\sqrt{g} \tilde{F}^{\mu\nu}] = 0, \tag{14}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the Hodge dual of $F_{\mu\nu}$. Note that this Bianchi identity is just a geometrical equation independent of the detailed nature of a gauge theory action. Thus it remains the same as that in Maxwell theory. For later use, we also provide the energy-momentum tensor of this BI theory,

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} \\ &= \frac{1}{4\pi}\Bigg\{ \beta^2(1-R)g_{\mu\nu} \\ &\quad + \frac{1}{R}\left[F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4\beta^2}(F_{\alpha\beta}\tilde{F}^{\alpha\beta})F_{\mu\alpha}\tilde{F}^{\alpha}_{\nu}\right]\Bigg\}, \end{aligned} \tag{15}$$

where $R \equiv [1 + (1/2\beta^2)(F_{\alpha\beta}F^{\alpha\beta}) - (1/16\beta^4)(F_{\alpha\beta}\tilde{F}^{\alpha\beta})^2]^{1/2}$. Now the first thing that one can readily notice in this rather unfamiliar BI theory of electrodynamics might be the fact that even in the absence of the source current, the dynamical BI field equation and the geometrical Bianchi identity clearly are not dual to each other under $F_{\mu\nu} \to \tilde{F}_{\mu\nu}$ and $\tilde{F}_{\mu\nu} \to -F_{\mu\nu}$. Obviously, this is in contrast to what happens in the standard Maxwell theory and can be attributed to the fact that when passing from the Maxwell to this highly nonlinear BI theory, only the dynamical field equation undergoes nontrivial change (''nonlinearization'') and the geometrical Bianchi identity, as pointed out above, remains unchanged. Therefore in order to deal with this added complexity properly and formulate the BI theory in curved background spacetime in a manner parallel to that for the standard Maxwell theory, we find it relevant to introduce another field strength $G_{\mu\nu}$ which, however, is made up of $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$. To be more precise, consider introducing, for the inhomogeneous BI field equation,

$$G_{\mu\nu} = \frac{1}{R}\left[F_{\mu\nu} - \frac{1}{4\beta^2}(F_{\alpha\beta}\tilde{F}^{\alpha\beta})\tilde{F}_{\mu\nu}\right] \tag{16}$$

and defining the associated fields on each spacelike hypersurfaces, (D^{α}, H^{α}) as

$$\begin{aligned} D^{\alpha} &= G^{\alpha\beta}u_{\beta}, \\ H^{\alpha} &= -\frac{1}{2}\epsilon^{\alpha\beta\lambda\sigma}u_{\beta}G_{\lambda\sigma} = -\tilde{G}^{\alpha\beta}u_{\beta}, \end{aligned} \tag{17}$$

which also implies their purely spatial nature

$$u_{\alpha}D^{\alpha} = 0 = u_{\alpha}H^{\alpha}. \tag{18}$$

Here, u^{μ} is the 4-velocity of fiducial observer (FIDO) (or more precisely ZAMO for rotating Kerr geometry) having a timelike geodesic orthogonal to spacelike hypersurfaces. Then the inhomogeneous BI field equation now takes the form

$$\nabla_\nu G^{\mu\nu} = 4\pi J^{\mu}, \tag{19}$$

which relates the fields (D^{μ}, H^{μ}) as defined above to ''free'' charge and current $J^{\mu} = \rho_e u^{\mu} + j^{\mu}_e$. Despite this extra elaboration, the fundamental field quantities, namely the electric field and the magnetic induction still can be identified with

$$\begin{aligned} E^{\alpha} &= F^{\alpha\beta}u_{\beta}, \\ B^{\alpha} &= -\frac{1}{2}\epsilon^{\alpha\beta\lambda\sigma}u_{\beta}F_{\lambda\sigma} = -\tilde{F}^{\alpha\beta}u_{\beta}, \end{aligned} \tag{20}$$

which again implies $u_{\alpha}E^{\alpha} = 0 = u_{\alpha}B^{\alpha}$. Thus the homogeneous Bianchi identity equation

$$\nabla_\nu \tilde{F}^{\mu\nu} = 0 \tag{21}$$

is expressible in terms of usual (E^{μ}, B^{μ}) fields. Then in this new representation of a set of BI equations, we now imagine their space-plus-time decomposition. Obviously, the dynamical BI field equation would split up into two inhomogeneous equations involving (D^{μ}, H^{μ}) and the ''free'' source charge and current $J^{\mu} = \rho_e u^{\mu} + j^{\mu}_e$ whereas the geometrical Bianchi identity equation decomposes into two homogeneous equations involving (E^{μ}, B^{μ}). Incidentally, one can then realize that this indeed is reminiscent of Maxwell equations in a ''medium.'' Namely, in this new representation, the BI theory of electrodynamics can be thought of as taking on the structure of ordinary Maxwell electrodynamics in a medium with nontrivial electric susceptibility and magnetic permeability. In this interpretation of the new representation of the BI theory, then, it is evident that the system is of course not linear in that (D^{μ}, H^{μ}) and (E^{μ}, B^{μ}) are related by

$$\begin{aligned} D^{\mu} &= \frac{1}{R}\left[E^{\mu} + \frac{1}{\beta^2}(E_{\alpha}B^{\alpha})B^{\mu}\right], \\ H^{\mu} &= \frac{1}{R}\left[B^{\mu} - \frac{1}{\beta^2}(E_{\alpha}B^{\alpha})E^{\mu}\right], \end{aligned} \tag{22}$$

$$\text{with } R = \left[1 - \frac{1}{\beta^2}(E_{\alpha}E^{\alpha} - B_{\alpha}B^{\alpha}) - \frac{1}{\beta^4}(E_{\alpha}B^{\alpha})^2\right]^{1/2}$$

or inversely

$$\begin{aligned} E^{\mu} &= \frac{1}{R}\left[D^{\mu} - \frac{1}{\beta^2}(D_{\alpha}H^{\alpha})H^{\mu}\right], \\ B^{\mu} &= \frac{1}{R}\left[H^{\mu} + \frac{1}{\beta^2}(D_{\alpha}H^{\alpha})D^{\mu}\right], \end{aligned} \tag{23}$$

$$\text{with } R = \left[1 - \frac{1}{\beta^2}(H_{\alpha}H^{\alpha} - D_{\alpha}D^{\alpha}) - \frac{1}{\beta^4}(D_{\alpha}H^{\alpha})^2\right]^{1/2},$$

where we used Eqs. (16), (17) and (20) and $u^{\alpha}u_{\alpha}=-1$, $F_{\alpha\beta}F^{\alpha\beta}=-2(E_{\alpha}E^{\alpha}-B_{\alpha}B^{\alpha})$ and $F_{\alpha\beta}\tilde{F}^{\alpha\beta}=4E_{\alpha}B^{\alpha}$. It is also noteworthy from the above expressions that

$$E_{\alpha}B^{\alpha}=D_{\alpha}H^{\alpha}. \tag{24}$$

Thus, from now on, we may call $D^{\mu}=(0,D^{i})$ as the ''electric displacement'' 4-vector and $H^{\mu}=(0,H^{i})$ as the ''magnetic field strength'' 4-vector.

B. Electric field and magnetic induction on the horizon

In our treatment of electrodynamics in the context of BI theory given in the previous subsection, we introduced a set of fields $D^{\mu}=(0,D^{i}=D_{i})$, $H^{\mu}=(0,H^{i}=H_{i})$ in addition to $E^{\mu}=(0,E^{i}=E_{i})$, $B^{\mu}=(0,B^{i}=B_{i})$. Thus we would have to first evaluate (D_i,H_i) on the horizon and then from them identify (E_i,B_i) afterwards. With respect to Damour's quasiorthonormal tetrad, then, the physical components of electric displacement D_i and magnetic field strength H_i can be read off as

$$G_{AB}=G_{\mu\nu}(e^{\mu}_{A}e^{\nu}_{B}) \quad \text{and}$$

$$D_{i}=G_{i0}, \quad H_{i}=\frac{1}{2}\epsilon_{ijk}G^{jk}=-\tilde{G}_{i0}. \tag{25}$$

More concretely, since we are working in ingoing Kerr coordinates (v,r,θ,ϕ), the components of electric displacement *on the horizon* can be read off as

$$\begin{aligned}
D_{\hat{r}}&=D_1=G_{10}=G_{\mu\nu}(e^{\mu}_{1}e^{\nu}_{0})|_{r_+}\\
&=\frac{(r_+^2+a^2)}{\Sigma_+}\left[G_{rv}+\frac{a}{(r_+^2+a^2)}G_{r\phi}\right],\\
D_{\hat{\theta}}&=D_2=G_{20}=G_{\mu\nu}(e^{\mu}_{2}e^{\nu}_{0})|_{r_+}\\
&=\frac{1}{\Sigma_+^{1/2}}\left[G_{\theta v}+\frac{a}{(r_+^2+a^2)}G_{\theta\phi}\right],\\
D_{\hat{\phi}}&=D_3=G_{30}=G_{\mu\nu}(e^{\mu}_{3}e^{\nu}_{0})|_{r_+}\\
&=\frac{\Sigma_+^{1/2}}{(r_+^2+a^2)\sin\theta}G_{\phi v},
\end{aligned} \tag{26}$$

where $\Sigma_+\equiv(r_+^2+a^2\cos^2\theta)$. Next, the components of magnetic field strength again on the horizon can be read off as

$$\begin{aligned}
H_{\hat{r}}&=H_1=-\tilde{G}_{10}=G^{23}=G_{23}\\
&=G_{\mu\nu}(e^{\mu}_{2}e^{\nu}_{3})|_{r_+}=\frac{1}{\Sigma_+\sin\theta}[a\sin^2\theta G_{\theta v}+G_{\theta\phi}],\\
H_{\hat{\theta}}&=H_2=-\tilde{G}_{20}=G^{31}=G_{30}\\
&=D_{\hat{\phi}}=\frac{\Sigma_+^{1/2}}{(r_+^2+a^2)\sin\theta}G_{\phi v},\\
H_{\hat{\phi}}&=H_3=-\tilde{G}_{30}=G^{12}=G_{02}\\
&=-D_{\hat{\theta}}=-\frac{1}{\Sigma_+^{1/2}}\left[G_{\theta v}+\frac{a}{(r_+^2+a^2)}G_{\theta\phi}\right],
\end{aligned} \tag{27}$$

where we used the Damour's quasi-orthonormal tetrad metric

$$ds^2=2e^0e^1+e^2e^2+e^3e^3=\epsilon_{AB}e^Ae^B$$

to deduce

$$\begin{aligned}
G^{23}&=\epsilon^{2A}\epsilon^{3B}G_{AB}=G_{23},\\
G^{31}&=\epsilon^{3A}\epsilon^{1B}G_{AB}=G_{30},\\
G^{12}&=\epsilon^{1A}\epsilon^{2B}G_{AB}=G_{02}.
\end{aligned}$$

Thus it is interesting to note that on the horizon $H_{\hat{\theta}}=D_{\hat{\phi}}$ and $H_{\hat{\phi}}=-D_{\hat{\theta}}$ or in a vector notation in a tangent space to the horizon,

$$\vec{H}_H=\vec{D}_H\times\hat{n}, \tag{28}$$

where $\hat{n}=\hat{r}$ is the vector (outer) normal to the horizon. This relation indicates that $\{\vec{H}_H,\vec{D}_H,\hat{n}\}$ form a ''triad'' on the horizon and hence constitutes the so-called ''radiative ingoing (or, inward Poynting flux)'' boundary condition at horizon as seen by a local observer at rest in the quasiorthonormal tetrad frame. Here, however, it seems worthy of note that although this relation is one of the horizon boundary conditions eventually we are after, it has *not* been obtained essentially from the horizon specifics. As a matter of fact, it holds for any $r=$const sections and indeed its emergence can be attributed to the ''half-null'' $(e^{\mu}_0=l^{\mu},e^{\mu}_1=-n^{\mu})$ structure of Damour's quasiorthonormal tetrad. Given the observation that the same type of relation as this ''radiative ingoing boundary condition'' actually holds for any null surface, one might wonder what then would be the distinctive nature of the event horizon (among null surfaces) that actually endows this boundary condition with real physical meaning. Znajek [5] provided one possible answer to this question and it is the following: the special feature of the event horizon over all other null surfaces is that it is a ''stationary'' null surface and there is a natural class of time coordinates associated with the frame at infinity in which the black hole is at rest. And the physical components of electric and magnetic fields should be evaluated, in a unique way, in a frame at rest on the horizon. At this point, we remark on another crucial thing happening at the horizon. Namely we note that at the horizon,

$$\begin{aligned} D_\alpha H^\alpha &= g_{\alpha\beta} D^\alpha H^\beta = (\epsilon_{AB} e^A_\alpha e^B_\beta) D^\alpha H^\beta = \epsilon_{AB} D^A H^B \\ &= D^0 H^1 + D^1 H^0 + D^2 H^2 + D^3 H^3 \\ &= D^2 D^3 - D^3 D^2 = 0, \\ D_\alpha D^\alpha &= g_{\alpha\beta} D^\alpha D^\beta = (\epsilon_{AB} e^A_\alpha e^B_\beta) D^\alpha D^\beta = \epsilon_{AB} D^A D^B \\ &= D^0 D^1 + D^1 D^0 + D^2 D^2 + D^3 D^3 \\ &= D^2 D^2 + D^3 D^3 \\ &= H^3 H^3 + H^2 H^2 = \epsilon_{AB} H^A H^B \\ &= g_{\alpha\beta} H^\alpha H^\beta = H_\alpha H^\alpha. \end{aligned} \tag{29}$$

These relations also holds *not* only at the horizon but on any $r = \text{const}$ sections and again can be attributed to the half-null nature of Damour's quasiorthonormal tetrad. One immediate consequence of these relations $D_\alpha H^\alpha = 0$ and $D_\alpha D^\alpha = H_\alpha H^\alpha$ everywhere is that practically $E^\mu = D^\mu$ and $B^\mu = H^\mu$ everywhere [due to Eqs. (22) and (23)] as seen by a local observer at rest in the quasiorthonormal tetrad frame. In fact, the interpretation of this is straightforward. Since Damour's quasiorthonormal tetrad is half-null in $(v-r)$ sector, an observer in this tetrad frame is actually a null observer who, as a result of his motion, would see the electromagnetic field around him as a ''radiation field'' all the way which, in turn, turns the BI theory of electrodynamics effectively into the Maxwell theory. What is particularly remarkable concerning this study of electrodynamics in the background of Kerr black hole in the context of BI theory is that the nature of the theory or the concrete structure of BI equations happens to be such that it is indistinguishable to a local observer in Damour's quasiorthonormal tetrad frame (indeed to any null observers) from that of standard Maxwell theory. This point is indeed quite amusing on theoretical side. From now on, then, whenever we deal with quantities involving physical components of fields as seen by this observer in Damour's tetrad frame, we can freely replace $[D^\mu(E^\mu), H^\mu(B^\mu)]$ by $[E^\mu(D^\mu), B^\mu(H^\mu)]$. Thus the radiative ingoing boundary condition at the horizon obtained above can be given in terms of electric field and magnetic induction as

$$\vec{B}_H = \vec{E}_H \times \hat{n}. \tag{30}$$

As pointed out earlier, this relation states that the electric and magnetic fields tangential to the horizon are equal in magnitude and perpendicular in direction and hence their Poynting energy flux is *into* the hole. This boundary condition as seen by a local observer again in a null tetrad frame (which has been made to be well-behaved at the horizon by the amount of boost that becomes suitably infinite at the horizon) has been derived first by Znajek [5] in the context of standard Maxwell theory and here we just witnessed that precisely the same radiative ingoing boundary condition holds in the BI theory context as well.

IV. (FICTITIOUS) CHARGE AND CURRENT ON THE HORIZON

It is well appreciated that in any attempt to have an intuitive picture of Blandford-Znajek process for the rotational energy extraction from rotating black holes, the introduction of surface charge and current density on the (stretched) horizon proves to be quite convenient. For instance, the circuit analysis in the membrane paradigm [11] cannot do without the notion of the horizon surface charge and current density. If one follows the original argument of Damour [6], one can justify their introduction as follows. Suppose the existence of a 4-current $J^\mu(v,r,\theta,\phi)$ which is defined and conserved all over the spacetime. Let $r=r_+$ be the location of an event horizon, then obviously some charge and current can plunge into the hole and disappear from the region $r>r_+$. Nevertheless, imagine that we do not want to consider what happens inside the black hole $(r<r_+)$ and just wish to keep the charge and current conserved in the region $r>r_+$. Then we would have to endow the surface $r=r_+$ with charge and current densities in such a way that the real current outside the horizon and this fictitious current on the horizon together can complete the circuit. Then the task of constructing the horizon surface current can be described as a mathematical problem as follows: ''Given the bulk current $J^\mu(v,r,\theta,\phi)$ such that $\nabla_\mu J^\mu = 0$, find a complementary boundary (surface) current j^μ on the surface $r=r_+$ such that $I^\mu \equiv [J^\mu Y(r-r_+) + j^\mu]$ [where $Y(r)$ is the Heaviside function defined by $dY(r) = \delta(r)dr$] is conserved.'' And in this problem, a crucial point to be noted is that the conservation of the bulk current J^μ is ensured by the field equation. Obviously then, what changes from the ordinary Maxwell theory case to the present BI theory case is that now the conservation of J^μ is secured by the inhomogeneous BI field equation instead of the Maxwell equation, i.e.,

$$\nabla_\nu G^{\mu\nu} = 4\pi J^\mu \tag{31}$$

implies $\nabla_\mu J^\mu = \nabla_\mu \nabla_\nu G^{\mu\nu}/4\pi = 0$ outside the horizon. Then the condition for the conservation of the *total* current I^μ reads

$$\begin{aligned} 0 = \nabla_\mu I^\mu &= \nabla_\mu [J^\mu Y(r-r_+) + j^\mu] \\ &= \frac{1}{4\pi}(\nabla_\nu G^{\mu\nu})\left(\frac{x_\mu}{r}\right)\delta(r-r_+) + \nabla_\mu j^\mu, \end{aligned} \tag{32}$$

where we used $\nabla_\nu G^{\mu\nu} = 4\pi J^\mu$, $\nabla_\mu J^\mu = 0$ and $\partial_\mu Y = (x_\mu/r)\delta(r-r_+)$. Obviously, this equation is solved by the complementary surface current given as

$$j^\mu = \frac{1}{4\pi} G^{\mu\nu}(\partial_\nu r)\delta(r-r_+) \equiv \frac{1}{4\pi} G^{\mu r}\delta(r-r_+). \tag{33}$$

Further, it is convenient to introduce a ''Dirac distribution'' δ_H on the horizon normalized with respect to the time at infinity v and the local proper area dA such that

$$\int d^4x\sqrt{g}\;\delta_H\delta(v-v_0)f(v,r,\theta,\phi)=\int_H dA\, f(v_0,r_+,\theta,\phi) \tag{34}$$

which, then, yields

$$\delta_H=\frac{(r_+^2+a^2)}{\Sigma_+}\delta(r-r_+), \tag{35}$$

where we used $\sqrt{g}=\Sigma\sin\theta$ and $dA=(r_+^2+a^2)\sin\theta d\theta d\phi$. Finally, then, the complementary surface current 4-vector on the horizon can be written as $j^\mu=\kappa^\mu\delta_H$ with

$$\kappa^\mu=\frac{1}{4\pi}\frac{\Sigma_+}{(r_+^2+a^2)}G_+^{\mu r}. \tag{36}$$

As usual, what matters is the identification of ''physical'' (i.e., finite and nonzero) components of this current 4-vector (i.e., the horizon charge and current density) as seen by an observer in our quasiorthonormal tetrad frame. And they can be computed, using the dual of Damour's mixed tetrad given in Eq. (9), in a straightforward manner as

$$\begin{aligned}
\sigma=\kappa^0=\kappa^\mu e_\mu^0|_{r_+}&=\frac{1}{4\pi}[G_+^{vr}-a\sin^2\theta G_+^{\phi r}]\\
&=\frac{1}{4\pi}\left[\frac{(r_+^2+a^2)}{\Sigma_+}G_{rv}+\frac{a}{\Sigma_+}G_{r\phi}\right]=\frac{1}{4\pi}D_{\hat r},\\
\kappa^{\hat r}=\kappa^1=\kappa^\mu e_\mu^1|_{r_+}&=0,\\
\kappa^{\hat\theta}=\kappa^2=\kappa^\mu e_\mu^2|_{r_+}&=\frac{1}{4\pi}\frac{\Sigma_+^{3/2}}{(r_+^2+a^2)}G_+^{\theta r}\\
&=\frac{1}{4\pi}\left[\frac{1}{\Sigma_+^{1/2}}G_{\theta v}+\frac{a}{\Sigma_+^{1/2}(r_+^2+a^2)}G_{\theta\phi}\right]=\frac{1}{4\pi}D_{\hat\theta},\\
\kappa^{\hat\phi}=\kappa^3=\kappa^\mu e_\mu^3|_{r_+}&\\
&=\frac{1}{4\pi}\frac{\Sigma_+^{1/2}}{(r_+^2+a^2)}\\
&\quad\times[-a\sin\theta G_+^{vr}+(r_+^2+a^2)\sin\theta G_+^{\phi r}]\\
&=\frac{1}{4\pi}\frac{\Sigma_+^{1/2}}{(r_+^2+a^2)\sin\theta}G_{\phi v}=\frac{1}{4\pi}D_{\hat\phi},
\end{aligned} \tag{37}$$

where the subscript ''+'' denotes the value at the horizon $r=r_+$ and we compared these equations with Eq. (26) to relate the surface charge and current densities to the components of electric displacement on the horizon.

V. OHM'S LAW, GAUSS' LAW, AND AMPERE'S LAW

We now are in the position to demonstrate that, as results of central significance, a set of three relations, at the horizon, between the fields $(D_i=E_i, H_i=B_i)$ and the surface charge and current densities $(\sigma=\kappa^0,\kappa^i)$ that can be thought of as Ohm's law, Gauss' law and Ampere's law valid at the horizon of a rotating Kerr black hole. First, notice that

$$D_{\hat\theta}=4\pi\kappa^{\hat\theta},\quad D_{\hat\phi}=4\pi\kappa^{\hat\phi}. \tag{38}$$

These relations can be rewritten in a vector notation in a tangent space to the horizon as

$$\vec D_H=4\pi\vec\kappa\quad\text{or}\quad\vec E_H=4\pi\vec\kappa \tag{39}$$

and hence can be interpreted as the ''Ohm's law.'' Namely, this relation precisely takes on the form of a nonrelativistic Ohm's law for a conductor and hence implies that if we endow the horizon with some charge and current densities which are to be determined by the surrounding external electromagnetic field $F_{\mu\nu}$, then the horizon behaves as if it is a conductor with finite surface resistivity of

$$\rho=4\pi\simeq377(\text{ohms}). \tag{40}$$

The derivation of Ohm's law and this value of surface resistivity has been performed first by Damour [6] and by Znajek [5] independently in the context of standard Maxwell theory. Thus what is indeed remarkable here is that the Ohm's law above and the value of horizon's surface resistivity (4π) remain unchanged even when we replace the Maxwell theory by the BI theory of electrodynamics. This result cannot be naturally anticipated but close inspection reveals that it can be attributed to the peculiar structure of highly nonlinear inhomogeneous BI field equation given in Eqs. (19) and (16) which, at the horizon, shows some magic such that there the $(\vec D,\vec H)$ fields become exactly the same as $(\vec E,\vec B)$ as seen by a local observer in Damour's tetrad frame respectively as can be checked from Eqs. (23) and (27) [or Eq. (29)]. As Damour [6] pointed out, this result constitutes a clear confirmation of Carter's assertion [10] that a black hole is analogous to an ordinary object having finite viscosity and electrical conductivity. Next, we also notice that

$$D_{\hat r}=4\pi\sigma\quad\text{or}\quad E_{\hat r}=4\pi\sigma, \tag{41}$$

which evidently can be identified with the surface version of Gauss' law. It says that the fictitious surface charge density we assumed on the horizon plays the role of terminating the normal components of all electric fields that pierce the horizon just as we want it to. Lastly, if we combine the radiative ingoing boundary condition at the horizon that we obtained earlier, $\vec H_H=\vec D_H\times\hat n$ (or $\vec B_H=\vec E_H\times\hat n$) and the Ohm's law above, $\vec D_H=4\pi\vec\kappa$ (or $\vec E_H=4\pi\vec\kappa$), we end up with the third relation

$$\vec H_H=4\pi(\vec\kappa\times\hat n)\quad\text{or}\quad\vec B_H=4\pi(\vec\kappa\times\hat n), \tag{42}$$

which may be viewed as the surface version of Ampere's law. Again, consistently with our motivation for introducing fictitious current density on the horizon, this relation indicates that the current density we assumed plays the role of terminating any tangential components of all magnetic fields penetrating the horizon. To summarize, for the reason stated earlier, even the highly nonlinear BI theory of electrodynamics leads to the same horizon boundary conditions Eqs. (30), (39), (41), and (42) as those in the standard Maxwell theory and indeed this set of four curious boundary conditions on the horizon actually have provided the motivation for the proposal of membrane paradigm [11] of black holes later on.

VI. JOULE'S LAW OR OHMIC DISSIPATION AT THE HORIZON

Perhaps one of the most intriguing consequences of assuming the existence of fictitious charge and current densities on the horizon would be that if we choose to do so, the horizon behaves as if it is a conductor with finite conductivity as we stressed in the previous section. Since it is the surrounding external electromagnetic field that drives the surface currents on the horizon, one might naturally wonder what would happen to the Joule heat generated when those currents work against the surface resistance and how it would be related to the electromagnetic energy going down the hole through the horizon. Znajek and Damour also provided a simple and natural answer to this question. Namely, they showed in a consistent and elegant manner that the total electromagnetic energy flux (i.e., the Poynting flux) into the rotating Kerr hole through the horizon is indeed precisely the same as the amount of Joule heat produced by the surface currents when they work against the surface resistivity of 4π. In the following, we shall demonstrate again along the same line of formulation as Damour that indeed the same is true even in the context of BI theory of electrodynamics. It is well-known that for a stationary, axisymmetric black hole spacetime with the horizon-orthogonal Killing field

$$\chi^{\mu}=(\partial/\partial v)^{\mu}+\Omega_H(\partial/\partial\phi)^{\mu}\equiv\xi^{\mu}+\Omega_H\psi^{\mu} \tag{43}$$

the mass energy and the angular momentum flux into the hole through the horizon are given respectively by

$$\begin{aligned}\frac{dM}{dv}&=\int_H dA\, T^{\mu}_{\nu}\xi^{\nu}\chi_{\mu}=\int_H dA\, T^{\mu}_{v}\chi_{\mu},\\ \frac{dJ_z}{dv}&=-\int_H dA\, T^{\mu}_{\nu}\psi^{\nu}\chi_{\mu}=-\int_H dA\, T^{\mu}_{\phi}\chi_{\mu},\end{aligned} \tag{44}$$

where $dA=(r_+^2+a^2)\sin\theta d\theta d\phi$ is again the area element on the horizon and T^{μ}_{ν} is the matter energy-momentum tensor at the horizon. Now, combining these with the first law of black hole thermodynamics [17]

$$dQ=\frac{1}{8\pi}\hat{\kappa}_H dA=dM-\Omega_H dJ_z, \tag{45}$$

where dQ denotes the heat dissipated in the hole *not* charge (recall that we only consider here *uncharge* Kerr black hole) with $\hat{\kappa}_H$ being the surface gravity [17] of the hole, one gets

$$\begin{aligned}\frac{dQ}{dv}&=\frac{dM}{dv}-\Omega_H\frac{dJ_z}{dv}=\int_H dA\, T^{\mu}_{\nu}(\xi^{\nu}+\Omega_H\psi^{\nu})\chi_{\mu}\\ &=\int_H dA\, T^{\mu}_{\nu}\chi^{\nu}\chi_{\mu}.\end{aligned} \tag{46}$$

Perhaps a word of caution might be relevant here. As we mentioned earlier, we are only interested in the ''test'' electromagnetic field whose dynamics is governed particularly by the BI theory in the ''background'' of uncharged Kerr black hole spacetime which is a solution to the vacuum Einstein equation. Therefore, as long as we confine our concern to the case with *uncharged* Kerr black hole physics, the first law of black hole thermodynamics given above still remains to be valid. If, instead, one is interested in the case with charged, rotating black hole physics, one would have to deal with the full, coupled Einstein-BI theory context and then there the associated first law should get modified to the extended version like the one given by Rasheed [18] recently. Now, since the ''matter'' for the case at hand is the BI electromagnetic field, we have at the horizon

$$\begin{aligned}T_{\mu\nu}\chi^{\mu}\chi^{\nu}|_{r_+}&=\frac{1}{4\pi}\Bigg\{\beta^2(1-R)\chi^{\alpha}\chi_{\alpha}\\ &\quad+\frac{1}{R}\Bigg[F_{\mu\alpha}F^{\alpha}_{\nu}-\frac{1}{4\beta^2}(F_{\alpha\beta}\tilde{F}^{\alpha\beta})F_{\mu\alpha}\tilde{F}^{\alpha}_{\nu}\chi^{\mu}\chi^{\nu}\Bigg]\Bigg\}\Bigg|_{r_+}\\ &=\frac{1}{4\pi}(F_{\mu\alpha}F^{\alpha}_{\nu})\chi^{\mu}\chi^{\nu}\Bigg|_{r_+},\end{aligned} \tag{47}$$

where R is as defined earlier and in the second line we used that at the horizon where $g_{\alpha\beta}\chi^{\alpha}\chi^{\beta}=\chi^{\alpha}\chi_{\alpha}=0$,

$$F_{\alpha\beta}F^{\alpha\beta}=-2(E_{\alpha}E^{\alpha}-B_{\alpha}B^{\alpha})=-2(D_{\alpha}D^{\alpha}-H_{\alpha}H^{\alpha})=0,$$

$$F_{\alpha\beta}\tilde{F}^{\alpha\beta}=4E_{\alpha}B^{\alpha}=4D_{\alpha}H^{\alpha}=0, \quad \text{and hence}$$

$$R=\left[1+\frac{1}{2\beta^2}(F_{\alpha\beta}F^{\alpha\beta})-\frac{1}{16\beta^4}(F_{\alpha\beta}\tilde{F}^{\alpha\beta})^2\right]^{1/2}=1,$$

which also yields, at the horizon, $G_{\mu\nu}=F_{\mu\nu}$. Recall that in the standard Maxwell theory,

$$T_{\mu\nu}=\frac{1}{4\pi}\left[F_{\mu\alpha}F^{\alpha}_{\nu}-\frac{1}{4}g_{\mu\nu}(F_{\alpha\beta}F^{\alpha\beta})\right] \tag{48}$$

and thus at the horizon,

$$T_{\mu\nu}\chi^{\mu}\chi^{\nu}|_{r_+}=(1/4\pi)(F_{\mu\alpha}F^{\alpha}_{\nu})\chi^{\mu}\chi^{\nu}|_{r_+},$$

which is the same as its counterpart in BI theory obtained above. This means that, at the horizon, the amount of total electromagnetic energy flux into the hole turns out to be the same and hence indistinguishable between Maxwell theory and BI theory. Further,

$$
\begin{aligned}
T_{\mu\nu}\chi^{\mu}\chi^{\nu}|_{r_+} &= \frac{1}{4\pi}(F_{\mu\alpha}F^{\alpha}_{\nu})\chi^{\mu}\chi^{\nu}|_{r_+} \\
&= \frac{1}{4\pi}\left\{\left[\frac{\Sigma_+^{1/2}}{(r_+^2+a^2)\sin\theta}F_{\phi v}\right]^2 \right. \\
&\quad \left. + \left[\frac{1}{\Sigma_+^{1/2}}G_{\theta v} + \frac{a}{\Sigma_+^{1/2}(r_+^2+a^2)}G_{\theta\phi}\right]^2\right\} \\
&= 4\pi[(\kappa^{\hat{\phi}})^2 + (\kappa^{\hat{\theta}})^2] = 4\pi(\vec{\kappa})^2, \qquad (49)
\end{aligned}
$$

where we used $G_{\mu\nu}=F_{\mu\nu}$ and $\kappa^{\hat{r}}=0$ at the horizon. Thus, finally we end up with

$$
\frac{dQ}{dv} = \int_H dA\, T_{\mu\nu}\chi^{\mu}\chi^{\nu} = 4\pi\int_H dA(\vec{\kappa})^2 = \int_H dA(\vec{E}_H\cdot\vec{\kappa}), \qquad (50)
$$

where we used the Ohm's law $\vec{E}_H = 4\pi\vec{\kappa}$, we obtained earlier. As we promised to demonstrate, clearly this is the Joule's law which is again precisely the same as its Maxwell theory counterpart originally obtained by Znajek [5] and by Damour [6] and implies that the absorption of electromagnetic energy by Kerr holes through the horizon can be translated into an equivalent picture in which the holes gain energy by absorbing Joule heat (or Ohmic dissipation) generated when the surface current $\vec{\kappa}$ driven by the electric field $\vec{E}_H$ works against the surface resistivity of 4π. And as before, what is remarkable is the fact that even if we replace the Maxwell theory by the highly nonlinear BI electrodynamics, the physics of the horizon such as this horizon thermodynamics as well as the horizon boundary conditions remain unchanged. And as we pointed out earlier, this has much to do with the nature of Damour's quasiorthonormal tetrad frame (i.e., its half-null structure) in ingoing Kerr coordinates.

VII. CONCLUDING REMARKS

In the present work, we have explored the interaction of electromagnetic fields with a rotating (Kerr) black hole in the context of Born-Infeld (BI) theory of electromagnetism and particularly we have derived BI theory versions of the four horizon boundary conditions of Znajek and Damour. Interestingly enough, as far as we employ the same local null tetrad frame as the one adopted in the works by Damour and by Znajek, we ended up with exactly the same four horizon boundary conditions despite the shift of the electrodynamics theory from a linear Maxwell one to a highly nonlinear BI one. As we have seen in the text, this curious and unexpected result could be attributed to the fact that the concrete structure of BI equations happens to be such that it is indistinguishable *at the horizon* to a local observer, say, in Damour's local tetrad frame from that of standard Maxwell theory. Finally, we have a word of caution to avoid a possible confusion the potential readers might have. Namely, again we point out that in all the calculations involved in this work to read off physical components of tensors such as Maxwell field tensor and current 4-vector, we strictly used the quasiorthonormal tetrad given in Eqs. (8) and (9) and nothing else. In this sense, our choice of local tetrad frame was slightly different from that in the original work of Damour [6] in which he introduced, particularly on the 2-dimensional, $v=$const section of the event horizon, some other orthonormal basis [slightly different from $\{e_2^{\mu}, e_3^{\mu}\}$ given in Eq. (8)] specially adapted to the ''intrinsic geometry'' of the $v=$const section of the horizon and used them to project out physical components of tensors. As such, any deviation of the results one may find in the expressions for the electric field, magnetic field and surface charge and current densities appeared in the text of the present work from their counterparts in the original work of Damour can be attributed to this slightly different choices of the local tetrad vectors. This discrepancy, however, is insensitive to the physical nature of this study of the horizon boundary conditions that we try to deliver in the present work.

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