$N=3$ chiral supergravity compatible with the reality condition and higher N chiral Lagrangian density

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We obtain $N=3$ chiral supergravity (SUGRA) compatible with the reality condition by applying the prescription of constructing the chiral Lagrangian density from usual SUGRA. The $N=3$ chiral Lagrangian density in first-order form, which leads to Ashtekar's canonical formulation, is determined so that it reproduces the second-order Lagrangian density of usual SUGRA especially by adding appropriate four-fermion contact terms. We show that the four-fermion contact terms added in the first-order chiral Lagrangian density are the nonminimal terms required from the invariance under first-order supersymmetry transformations. We also discuss the case of higher N theories, especially for $N=4$ and $N=8$.

I. INTRODUCTION

The supersymmetric extension of Ashtekar's canonical gravity [1,2] has been developed since the first construction of $N=1$ chiral[1] supergravity (SUGRA) [3]. In particular, the extended chiral SUGRA was constructed in the context of two-form gravity [4,5] for $N=2$ [6,7] and $N=3,4$ theories [8], and was also constructed by closely following the method of usual SUGRA for $N=2$ theory [9]. Furthermore the canonical formulation of SUGRA in terms of the Ashtekar variable was explicitly derived up to $N=2$ theory from the method of two-form SUGRA [6,7] and also from that of usual SUGRA [10]. However, for $N\geq 3$ chiral SUGRA, a straightforward derivation of the canonical formulation in terms of the Ashtekar variables has not yet been done.

In this paper we construct $N=3$ chiral SUGRA compatible with the reality condition, which is the lowest N theory involving a spin-1/2 field in addition to spin-2 (gravitational), spin-3/2 and spin-1 fields, by closely following the method of the usual SUGRA as a preliminary to derive the canonical formulation of $N=3$ SUGRA in terms of the Ashtekar variable. Furthermore, we discuss the construction of higher N chiral SUGRA, in particular, the construction of the chiral Lagrangian density for $N=4$ and $N=8$ theories.

When we construct the chiral SUGRA, we assume at first that the tetrad is complex and construct such a chiral Lagrangian as analytic in complex field variables as briefly mentioned in Ref. [3]. This means that right- and left-handed supersymmetric (SUSY) transformations introduced in the chiral SUGRA are independent of each other even in the second-order formalism. This fact makes it more transparent to confirm the SUSY invariance, particularly the right-handed one. Once we construct the chiral Lagrangian

density, we impose the reality condition.[2]

In order to construct the chiral Lagrangian density in first-order form, we apply the prescription proposed in the case of $N=2$ chiral SUGRA [9]. First the chiral Lagrangian density in the second-order formalism, $\mathcal{L}^{(+)}[$second order$]$, with a (complex) tetrad is obtained from the Lagrangian density of the usual SUGRA, $\mathcal{L}_{\text{usual SUGRA}}$ (second order), by complexifying spinor fields (spin-3/2 and -1/2 fields) as follows.

(a) Replace Rarita-Schwinger fields[3] ψ^I_μ and their conjugates $\bar\psi^I_\mu$ with

$$\psi^I_\mu \rightarrow \psi^I_{R\mu} + \tilde\psi^I_{L\mu},$$

$$\bar\psi^I_\mu \rightarrow \bar\psi^I_{L\mu} + \bar{\tilde\psi}^I_{R\mu}, \tag{1.1}$$

by using two independent sets of Rarita-Scwinger fields ψ^I_μ and $\tilde\psi^I_\mu$.

(b) Rewrite the kinetic term of $e\,\epsilon^{\mu\nu\rho\sigma}\bar\psi^I_{L\mu}\gamma_\rho\nabla_\sigma\tilde\psi^I_{L\nu}$ into $e\,\epsilon^{\mu\nu\rho\sigma}\bar{\tilde\psi}^I_{R\mu}\gamma_\rho\nabla_\sigma\psi^I_{R\nu}$ plus a total derivative by partial integration, where ∇_σ denotes the ordinary covariant derivative in general relativity.

(c) Apply the prescription (a) and (b) to spin-1/2 fields.

Then the chiral Lagrangian density in first-order form, $\mathcal{L}^{(+)}$, is determined by the following prescription.

(d) Replace the ∇_σ to the $D^{(+)}_\sigma$ which is defined in Eq. (2.2) later.

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[1]''Chiral'' means in this paper that only right-handed (or left-handed) spinor fields are coupled to the spin connection in the kinetic term of spinor fields.

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[2]After imposing the reality condition, the phase space in the canonical formulation consists of the real triad and the complex connection as discussed, for example, in Refs. [1,2,11,12] for general relativity, and in Refs. [3,13] for ($N=1$) chiral SUGRA. This phase space leads to non-Hermitian operator with respect to the connection variable in quantizing the theory (see, for example, Refs. [11,13]).

[3]Rarita-Schwinger fields denoted by ψ^I_μ and $\tilde\psi^I_\mu$, and spin-1/2 fields denoted by χ (or χ^I, χ^{IJK}) and $\tilde\chi$ (or $\tilde\chi^I, \tilde\chi^{IJK}$) represent Majorana spinors. Throughout this paper capital letters $I, J, \ldots$, denote the number of Rarita-Schwinger fields, and we shall follow the notation and convention of Ref. [9].

(e) Add appropriate four-fermion contact terms so that the $\mathcal{L}^{(+)}$ reproduce the $\mathcal{L}^{(+)}$[second order].

The four-fermion contact terms added in $\mathcal{L}^{(+)}$ by means of the prescription (e) are also required from the invariance under first-order SUSY transformations up to $N=3$ chiral SUGRA as will be explained later: We expect that the same result will be the case for higher N. The $\mathcal{L}^{(+)}$ in terms of the real spin contents is obtained by imposing the reality condition (e.g., $\overline{e_\mu^i}=e_\mu^i$ for the tetrad, and $\widetilde{\psi}_\mu^I=\psi_\mu^I$ for Rarita-Schwinger fields).

In the case of $N=2$ chiral SUGRA [9] under the reality condition, the chiral Lagrangian density in first-order form constructed by the above prescription (a)–(e) differs from the first-order Lagrangian density of the usual $N=2$ SUGRA by

$$(\mathcal{L}_{N=2}^{(+)} - \mathcal{L}_{N=2 \text{ usual SUGRA}}) \quad \text{(first order)}$$

$$= -\frac{i}{8\kappa^2} e\, \epsilon^{\mu\nu\rho\sigma}(T_{\lambda\mu\nu} + i\kappa^2 \bar{\psi}_\mu^I \gamma_\lambda \psi_\nu^I) T^\lambda_{\rho\sigma}$$

$$+ \frac{i}{8}\kappa^2 e\, \epsilon^{\mu\nu\rho\sigma}(\bar{\psi}_{L\mu}^I \psi_{R\nu}^J) \bar{\psi}_{R\rho}^K \psi_{L\sigma}^L \epsilon^{IJ}\epsilon^{KL}$$

$$- \frac{i}{4\kappa^2}\partial_\mu\{e\, \epsilon^{\mu\nu\rho\sigma}(T_{\nu\rho\sigma} + i\kappa^2 \bar{\psi}_\rho^I \gamma_\nu \psi_\sigma^I)\}, \quad (1.2)$$

where $T_{\lambda\mu\nu}$ stands for the torsion tensor. The last imaginary boundary term corresponds to a certain Chern-Simons boundary term given by Macías [14] and Mielke *et al.* [15] as a generating function of the canonical transformation. However, in the first-order formalism, the second four-fermion contact term added by the prescription (e) does not appear in $N=1$ chiral SUGRA [3,14,15]. Indeed, we showed that this new second term is the nonminimal one required from the invariance under first-order SUSY transformations [9]. In the second-order formalism, the first term does not vanish by itself in contrast with the $N=1$ theory, but cancels with the second four-fermion contact term by using a Fierz transformation. In $N=3$ chiral SUGRA, an additional four-fermion contact term quadratic with respect to both spin-3/2 and -1/2 fields is also required in the chiral Lagrangian density as explained in the next section.

This paper is organized as follows. In Sec. II we construct $N=3$ chiral SUGRA compatible with the reality condition by applying the above prescription (a)–(e). The invariance of the field equation for vector fields under duality transformations in $N=3$ chiral SUGRA is shown in Sec. III. In Sec. IV we discuss the case of higher N theories, especially for $N=4$ and $N=8$. The conclusion is given in Sec. V.

II. $N=3$ CHIRAL SUGRA COMPATIBLE WITH THE REALITY CONDITION

Let us construct $N=3$ chiral SUGRA by means of the prescription (a)–(e) explained in the Introduction. The usual $N=3$ SUGRA [16,17] has spin contents $(2,\frac{3}{2},\frac{3}{2},\frac{3}{2},1,1,1,\frac{1}{2})$. Corresponding to these spin contents, the independent variables in $N=3$ chiral SUGRA are a (complex) tetrad e_μ^i, two

independent sets of Rarita-Schwinger fields $(\psi_\mu^I, \widetilde{\psi}_\mu^I)$ ($I=1,2,3$), two independent spin-1/2 fields $(\chi,\widetilde{\chi})$, (complex) vector fields A_μ^I, in addition to the self-dual connection $A_{ij\mu}^{(+)}$ which is also treated as one of the independent variables in the first-order formalism. Then the $N=3$ chiral Lagrangian density in first-order form is given by

$$\mathcal{L}_{N=3}^{(+)} = -\frac{i}{2\kappa^2} e\, \epsilon^{\mu\nu\rho\sigma} e_\mu^i e_\nu^j R_{ij\rho\sigma}^{(+)} - e\, \epsilon^{\mu\nu\rho\sigma} \bar{\bar{\psi}}_{R\mu}^I \gamma_\rho D_\sigma^{(+)} \psi_{R\nu}^I$$

$$+ ie\,\bar{\bar{\chi}}_R \gamma^\mu D_\mu^{(+)} \chi_R - \frac{e}{4}(F_{\mu\nu}^I)^2 + \frac{\kappa}{2\sqrt{2}} e\{(F^{(-)I\mu\nu}$$

$$+ \hat{F}^{(-)I\mu\nu})\bar{\psi}_{L\mu}^J \psi_{R\nu}^K + (F^{(+)I\mu\nu}$$

$$+ \hat{F}^{(+)I\mu\nu})\bar{\bar{\psi}}_{R\mu}^J \widetilde{\psi}_{L\nu}^K\} \epsilon^{IJK} - \frac{i}{2}\kappa e$$

$$\times \left\{ \hat{F}_{\mu\nu}^I - \frac{\kappa}{2}(\bar{\psi}_{L\mu}^I \gamma_\nu \widetilde{\chi}_L + \bar{\bar{\psi}}_{R\mu}^I \gamma_\nu \chi_R) \right\} (\bar{\psi}_{L\lambda}^I S^{\mu\nu} \gamma^\lambda \widetilde{\chi}_L$$

$$+ \bar{\bar{\psi}}_{R\lambda}^I S^{\mu\nu} \gamma^\lambda \chi_R)$$

$$+ \frac{i}{8}\kappa^2 e\, \epsilon^{\mu\nu\rho\sigma}(\bar{\psi}_{L\mu}^J \psi_{R\nu}^K) \bar{\bar{\psi}}_{R\rho}^L \widetilde{\psi}_{L\sigma}^M \epsilon^{IJK} \epsilon^{ILM}$$

$$+ \frac{\kappa^2}{2} e (\bar{\bar{\psi}}_{R\mu}^I \gamma^{[\mu} \psi_{R\nu}^I) \bar{\bar{\chi}}_R \gamma^{\nu]} \chi_R, \quad (2.1)$$

which is globally O(3) invariant. Here $e:=\det(e_\mu^i)$, ϵ^{IJK} denotes a totally antisymmetric tensor, and $F_{\mu\nu}^{(\pm)I}:=(1/2)(F_{\mu\nu}^I \mp i\widetilde{F}_{\mu\nu}^I)$ with $F_{\mu\nu}^I = 2\partial_{[\mu}A_{\nu]}^I$ and $\widetilde{F}_{\mu\nu}^I = (1/2)\epsilon_{\mu\nu\rho\sigma}F^{I\rho\sigma}$. The covariant derivative $D_\mu^{(+)}$ and the curvature $R^{(+)ij}{}_{\mu\nu}$ are

$$D_\mu^{(+)} := \partial_\mu + \frac{i}{2}A_{ij\mu}^{(+)} S^{ij},$$

$$R^{(+)ij}{}_{\mu\nu} := 2(\partial_{[\mu}A^{(+)ij}{}_{\nu]} + A^{(+)i}{}_{k[\mu}A^{(+)kj}{}_{\nu]}), \quad (2.2)$$

while $\hat{F}_{\mu\nu}^I$ in the chiral Lagrangian density (2.1) is defined as

$$\hat{F}_{\mu\nu}^I := F_{\mu\nu}^I - \frac{\kappa}{\sqrt{2}}(\bar{\psi}_{L\mu}^J \psi_{R\nu}^K + \bar{\bar{\psi}}_{R\mu}^J \widetilde{\psi}_{L\nu}^K)\epsilon^{IJK}. \quad (2.3)$$

In Eq. (2.1), the A_μ^I-dependent terms and the four-fermion contact terms except for the last two contact terms correspond to those obtained in the usual SUGRA. Note that Eq. (2.1) is reduced to the $N=2$ chiral Lagrangian density [9], if we put the condition

$$(\chi,\widetilde{\chi})=0, \quad (\psi_\mu^3,\widetilde{\psi}_\mu^3)=0, \quad F_{\mu\nu}^1=0=F_{\mu\nu}^2, \quad (2.4)$$

in Eq. (2.1), and if the e_μ^i, $(\psi_\mu^I,\widetilde{\psi}_\mu^I)$ ($I=1,2$) and $F_{\mu\nu}(:=F_{\mu\nu}^3)$ are taken as the field variables.

We show that the chiral Lagrangian density (2.1) is invariant under the local SUSY transformations. The invari-

ance up to the order κ is confirmed by means of the following right- and left-handed SUSY transformations in the first-order formalism, and entirely determines the form of the chiral Lagrangian density. In particular, the last two contact terms in Eq. (2.1), which are added by the prescription (e) of constructing the chiral Lagrangian density, are the nonminimal terms required from the invariance of order κ under the right-handed SUSY transformations generated by α^I given by

$$\delta_R e^i_\mu = i\kappa\,\bar{\alpha}^I_L\gamma^i\tilde{\psi}^I_{L\mu}, \quad \delta_R A^I_\mu = \sqrt{2}\,\epsilon^{IJK}\bar{\alpha}^J_L\psi^K_{R\mu} + \bar{\alpha}^I_L\gamma_\mu\tilde{\chi}_L,$$

$$\delta_R \psi^I_{R\mu} = \frac{2}{\kappa}D^{(+)}_\mu\alpha^I_R - \frac{i}{2}\kappa(\bar{\tilde{\chi}}_R\gamma^\lambda\chi_R)\gamma_\lambda\gamma_\mu\alpha^I_R,$$

$$\delta_R \tilde{\psi}^I_{L\mu} = -\frac{1}{\sqrt{2}}\epsilon^{IJK}\bar{F}^{(-)J}_{\rho\sigma}S^{\rho\sigma}\gamma_\mu\alpha^K_R$$

$$\qquad\qquad -\frac{i}{\sqrt{2}}\kappa\epsilon^{IJK}(\bar{\psi}^J_{L\mu}\gamma_\nu\tilde{\chi}_L)\gamma^\nu\alpha^K_R,$$

$$\delta_R \chi_R = \bar{F}^{(+)I}_{\mu\nu}S^{\mu\nu}\alpha^I_R, \quad \delta_R\tilde{\chi}_L = 0, \qquad (2.5)$$

and under the left-handed SUSY transformations generated by $\tilde{\alpha}^I$ given by

$$\delta_L e^i_\mu = i\kappa\,\bar{\tilde{\alpha}}^I_R\gamma^i\psi^I_{R\mu}, \quad \delta_L A^I_\mu = \sqrt{2}\,\epsilon^{IJK}\bar{\tilde{\alpha}}^J_R\tilde{\psi}^K_{L\mu} + \bar{\tilde{\alpha}}^I_R\gamma_\mu\chi_R,$$

$$\delta_L \tilde{\psi}^I_{L\mu} = \frac{2}{\kappa}D^{(-)}_\mu\tilde{\alpha}^I_L + \frac{i}{2}\kappa(\bar{\tilde{\chi}}_R\gamma^\lambda\chi_R)\gamma_\lambda\gamma_\mu\tilde{\alpha}^I_L,$$

$$\delta_L \psi^I_{R\mu} = -\frac{1}{\sqrt{2}}\epsilon^{IJK}\bar{F}^{(+)J}_{\rho\sigma}S^{\rho\sigma}\gamma_\mu\tilde{\alpha}^K_L$$

$$\qquad\qquad -\frac{i}{\sqrt{2}}\kappa\epsilon^{IJK}(\bar{\tilde{\psi}}^J_{R\mu}\gamma_\nu\chi_R)\gamma^\nu\tilde{\alpha}^K_L,$$

$$\delta_L \chi_R = 0, \quad \delta_L\tilde{\chi}_L = \bar{F}^{(-)I}_{\mu\nu}S^{\mu\nu}\tilde{\alpha}^I_L, \qquad (2.6)$$

where $\bar{F}^I_{\mu\nu}$ in Eqs. (2.5) and (2.6) is defined as

$$\bar{F}^I_{\mu\nu} := \hat{F}^I_{\mu\nu} - \kappa(\bar{\psi}^I_{L\mu}\gamma_\nu\tilde{\chi}_L + \bar{\tilde{\psi}}^I_{R\mu}\gamma_\nu\chi_R). \qquad (2.7)$$

In addition, we choose the right- and left-handed SUSY transformations of $A^{(+)}_{ij\mu}$ as [18]

$$\delta_R A^{(+)}_{ij\mu} = 0,$$

$$\delta_L A^{(+)}_{ij\mu} = \text{self-dual part of } \{-\kappa(B_{\mu ij} - e_{\mu[i}B^m{}_{|m|j]})\},$$

$$\qquad\qquad (2.8)$$

respectively, with

$$B^{\lambda\mu\nu} := \epsilon^{\mu\nu\rho\sigma}\bar{\tilde{\alpha}}^I_R\gamma^\lambda D^{(+)}_\rho\psi^I_{R\sigma}. \qquad (2.9)$$

Although the $A^{(-)}_{ij\mu}$ appears at $\delta_L\tilde{\psi}^I_{L\mu}$ in the left-handed SUSY transformations of Eq. (2.6), it is not an independent variable but a quantity given by e^i_μ, $(\psi^I_\mu, \tilde{\psi}^I_\mu)$, and $(\chi, \tilde{\chi})$; namely, the $A^{(-)}_{ij\mu}$ is fixed as the sum of the antiself-dual part of the Ricci rotation coefficients $A_{ij\mu}(e)$ and that of $K_{ij\mu}$ given by

$$K_{ij\mu} = \frac{i}{2}\kappa^2(\bar{\tilde{\psi}}^I_{R[i}\gamma_{|\mu|}\psi^I_{Rj]} + \bar{\tilde{\psi}}^I_{R[i}\gamma_{|j|}\psi^I_{R\mu]} - \bar{\tilde{\psi}}^I_{R[j}\gamma_{|i|}\psi^I_{R\mu]})$$

$$\qquad + \frac{\kappa^2}{4}\epsilon_{ij\mu\nu}\bar{\tilde{\chi}}_R\gamma^\nu\chi_R. \qquad (2.10)$$

At order κ^2 and κ^3, on the other hand, the transformations (2.8) should be corrected to recover the SUSY invariance in the first-order formalism. However, this task will be complicated as is expected from the usual SUGRA [16], and therefore we turn to the second-order formalism in order to minimize complication. Then it can be shown by a straightforward calculation that the chiral Lagrangian density (2.1) is invariant under the SUSY transformations of Eqs. (2.5) and (2.6), provided that the $A^{(+)}_{ij\mu}$ is fixed as

$$A^{(+)}_{ij\mu} = A^{(+)}_{ij\mu}(e) + K^{(+)}_{ij\mu} \qquad (2.11)$$

by solving the equation $\delta\mathcal{L}^{(+)}_{N=3}/\delta A^{(+)}_{ij\mu} = 0$ with respect to $A^{(+)}_{ij\mu}$.

Here we also note that the last two four-fermion contact terms in Eq. (2.1), which do not appear in the first-order Lagrangian density [16,17,19] of the usual SUGRA, are necessary to reproduce the second-order Lagrangian density of the usual $N=3$ SUGRA, when the reality condition

$$\overline{e^i_\mu} = e^i_\mu, \quad \tilde{\psi}^I_\mu = \psi^I_\mu, \quad \tilde{\chi} = \chi \text{ and } \overline{A^I_\mu} = A^I_\mu, \quad (2.12)$$

is imposed.[4] Indeed, if we use the solution (2.11) in the first three terms in Eq. (2.1), then these terms give rise to a number of four-fermion contact terms, which involve the following terms:

$$\frac{i}{8\kappa^2}e\,\epsilon^{\mu\nu\rho\sigma}T_{\lambda\mu\nu}T^\lambda{}_{\rho\sigma} = -\frac{i}{16}\kappa^2 e\,\epsilon^{\mu\nu\rho\sigma}$$

$$\times(\bar{\tilde{\psi}}^J_{R\mu}\gamma_\lambda\psi^L_{R\nu})\bar{\tilde{\psi}}^K_{R\rho}\gamma^\lambda\psi^M_{R\sigma}\epsilon^{IJK}\epsilon^{ILM}$$

$$-\frac{\kappa^2}{2}e(\bar{\tilde{\psi}}^I_{R\mu}\gamma^{[\mu}\psi^I_{R\nu})\bar{\tilde{\chi}}_R\gamma^{\nu]}\chi_R,$$

$$\qquad\qquad (2.13)$$

where the torsion tensor is defined by $T^i{}_{\mu\nu} = -2D_{[\mu}e^i_{\nu]}$ with $D_\mu e^i_\nu = \partial_\mu e^i_\nu + A^i{}_{j\mu}e^j_\nu$. The second last term in Eq. (2.1), on the other hand, can be rewritten as

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[4]The bars of e^i_μ and A^I_μ in Eq. (2.12) mean the complex conjugate.

$$\frac{i}{8}\kappa^2 e\,\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}^J_{L\mu}\psi^K_{R\nu})\bar{\bar{\psi}}^L_{R\rho}\tilde{\psi}^M_{L\sigma}\epsilon^{IJK}\epsilon^{ILM}$$

$$=\frac{i}{16}\kappa^2 e\,\epsilon^{\mu\nu\rho\sigma}(\bar{\bar{\psi}}^J_{R\mu}\gamma_\lambda\psi^K_{R\nu})\bar{\bar{\psi}}^L_{R\rho}\gamma^\lambda\psi^M_{R\sigma}\epsilon^{IJK}\epsilon^{ILM}$$

$$(2.14)$$

by using a Fierz transformation, and the sum of the last two terms in Eq. (2.1) exactly cancels out the terms of Eq. (2.13). When the reality condition (2.12) is imposed in the second-order formalism, the last two terms in Eq. (2.1) and the terms of Eq. (2.13) are purely imaginary up to boundary terms, but they cancel each other. Therefore the $\mathcal{L}^{(+)}_{N=3}$ (second order) of $N=3$ chiral SUGRA with the reality condition (2.12) is reduced to that of the usual one up to imaginary boundary terms, namely, we have[5]

$$\mathcal{L}^{(+)}_{N=3}\ \ (\text{second order})=\mathcal{L}_{N=3\ \text{usual SUGRA}}\ (\text{second order})$$

$$+\frac{1}{8}\partial_\mu(e\,\epsilon^{\mu\nu\rho\sigma}\bar{\psi}^I_\rho\gamma_\nu\psi^I_\sigma$$

$$+ie\bar{\chi}\gamma_5\gamma^\mu\chi).\qquad(2.15)$$

This second-order Lagrangian density (2.15) is invariant under the right- and left-handed SUSY transformations of Eqs. (2.5) and (2.6), which are now complex conjugate of each other in the second-order formalism under the reality condition (2.12).

Let us explain how to gauge [20] the global $O(3)$ invariance of the chiral Lagrangian density (2.1). First we introduce a minimal coupling of $\psi^I_{R\mu}$ with A^I_μ, which automatically requires a spin-3/2 masslike term and a cosmological term in order to ensure the SUSY invariance of the Lagrangian, and we also replace the Abelian field strength $F^I_{\mu\nu}$ with the non-Abelian one

$$F'^I_{\mu\nu}:=F^I_{\mu\nu}+\lambda\,\epsilon^{IJK}A^J_\mu A^K_\nu\qquad(2.16)$$

with the gauge coupling constant λ: The three terms added to Eq. (2.1) in order to gauge the $O(3)$ invariance are then written as

$$\mathcal{L}_{\text{cosm}}=-\lambda e\,\epsilon^{\mu\nu\rho\sigma}\bar{\bar{\psi}}^I_{R\mu}\gamma_\rho\psi^K_{R\nu}A^J_\sigma\epsilon^{IJK}$$

$$-\sqrt{2}i\kappa^{-1}\lambda e(\bar{\bar{\psi}}^I_{R\mu}S^{\mu\nu}\tilde{\psi}^I_{L\nu}+\bar{\psi}^I_{L\mu}S^{\mu\nu}\psi^I_{R\nu})-\Lambda\kappa^{-2}e,$$

$$(2.17)$$

where the cosmological constant Λ is related to λ as $\Lambda=-6\kappa^{-2}\lambda^2$.

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[5]The imaginary boundary terms in Eq. (2.15) correspond to the Chern-Simons type boundary terms [14,15] as appeared in Eq. (1.2) evaluated in the second-order formalism.

III. DUALITY INVARIANCE IN $N=3$ CHIRAL SUGRA

In the extended usual SUGRA without gauging the global $O(N)$ invariance, the field equation for vector fields is invariant under duality transformations [21–23] which generalize those of the free Maxwell equations, while the Lagrangian changes its form in a specific way under these transformations.

As we have already noted, the chiral Lagrangian density (2.1) possesses global $O(3)$ invariance. We show in this section that the field equation for vector fields derived from the nongauged, chiral Lagrangian density (2.1) is invariant under duality transformations. The field equation for A^I_μ can be written as

$$\partial_\mu(e\tilde{G}^{I\mu\nu})=0,\qquad(3.1)$$

where the $\tilde{G}^{I\mu\nu}$ are defined by

$$\tilde{G}^{I\mu\nu}:=\frac{2}{e}\frac{\partial\mathcal{L}^{(+)}_{N=3}}{\partial F^I_{\mu\nu}}=-F^{I\mu\nu}+\kappa(H^{(+)I\mu\nu}+I^{(-)I\mu\nu})$$

$$(3.2)$$

with $H^{(+)I\mu\nu}$ and $I^{(-)I\mu\nu}$ being given by

$$H^{(+)I\mu\nu}=\sqrt{2}\,\epsilon^{IJK}(\bar{\bar{\psi}}^{J\mu}_R\tilde{\psi}^{K\nu}_L)^{(+)}-i\bar{\psi}^I_{L\lambda}S^{\mu\nu}\gamma^\lambda\tilde{\chi}_L,$$

$$I^{(-)I\mu\nu}=\sqrt{2}\,\epsilon^{IJK}(\bar{\psi}^{J\mu}_L\psi^{K\nu}_R)^{(-)}-i\bar{\bar{\psi}}^I_{R\lambda}S^{\mu\nu}\gamma^\lambda\chi_R.$$

$$(3.3)$$

In addition, the $\tilde{F}^I_{\mu\nu}$ satisfies the Bianchi identity

$$\partial_\mu(e\tilde{F}^{I\mu\nu})=0.\qquad(3.4)$$

Equations (3.1) and (3.4) are invariant under the following (global) duality transformations:

$$\delta e^i_\mu=0,\quad \delta\psi^I_{R\mu}=-i\Lambda^{IJ}\psi^J_{R\mu},\quad \delta\tilde{\psi}^I_{L\mu}=i\Lambda^{IJ}\tilde{\psi}^J_{L\mu},$$

$$\delta\chi_R=0,\quad \delta\tilde{\chi}_L=0,$$

$$\delta\begin{pmatrix}F^{I\mu\nu}\\G^{I\mu\nu}\end{pmatrix}=\begin{pmatrix}0 & \Lambda^{IJ}\\-\Lambda^{IJ} & 0\end{pmatrix}\begin{pmatrix}F^{J\mu\nu}\\G^{J\mu\nu}\end{pmatrix}\qquad(3.5)$$

with the constant parameters Λ^{IJ} which are assumed to be complex, symmetric $(\Lambda^{IJ}=\Lambda^{JI})$ and traceless $(\Lambda^{II}=0)$: When the reality condition (2.12) is imposed, however, Λ^{IJ} are supposed to be real. The transformations of Eq. (3.5) can be rewritten in terms of the bases $(H^{(+)I\mu\nu},I^{(-)I\mu\nu})$ and $(F^{I\mu\nu}+iG^{I\mu\nu},F^{I\mu\nu}-iG^{I\mu\nu})$ as

$$\delta\begin{pmatrix}H^{(+)I\mu\nu}\\I^{(-)I\mu\nu}\end{pmatrix}=-i\begin{pmatrix}\Lambda^{IJ} & 0\\0 & -\Lambda^{IJ}\end{pmatrix}\begin{pmatrix}H^{(+)J\mu\nu}\\I^{(-)J\mu\nu}\end{pmatrix},$$

$$\delta\begin{pmatrix}F^{I\mu\nu}+iG^{I\mu\nu}\\F^{I\mu\nu}-iG^{I\mu\nu}\end{pmatrix}=-i\begin{pmatrix}\Lambda^{IJ} & 0\\0 & -\Lambda^{IJ}\end{pmatrix}\begin{pmatrix}F^{J\mu\nu}+iG^{J\mu\nu}\\F^{J\mu\nu}-iG^{J\mu\nu}\end{pmatrix}.$$

$$(3.6)$$

Since the transformations of Eq. (3.6) combined with the O(3) transformations becomes the SU(3) group [21], the duality symmetry based on Eq. (3.6) is an example for ''compact'' duality symmetries [22]. Also, the transformations (3.6) reduce to (global) U(1) transformations in $N=2$ chiral SUGRA if Eq. (2.4) and $\Lambda^{11}=\Lambda^{22}=-(1/2)\Lambda^{33}$ are satisfied.

The $N=3$ chiral Lagrangian density (2.1) is expressed by using $(F^{I\mu\nu},G^{I\mu\nu})$ and $(H^{(+)I\mu\nu},I^{(-)I\mu\nu})$ as

$$\mathcal{L}_{N=3}^{(+)}=\frac{e}{4}(F_{\mu\nu}^I\widetilde{G}^{I\mu\nu})+\frac{e}{8}\{(F_{\mu\nu}^{(+)I}-iG_{\mu\nu}^{(+)I})H^{(+)I\mu\nu}+(F_{\mu\nu}^{(-)I}$$
$$+iG_{\mu\nu}^{(-)I})I^{(-)I\mu\nu}\}+(A_\mu^I\text{-independent terms}).\quad(3.7)$$

The second term is obviously invariant under the transformation of Eq. (3.6), and the invariance of the A_μ^I-independent terms can also be confirmed by using Eq. (3.5). Therefore, the $\mathcal{L}_{N=3}^{(+)}$ transforms under Eq. (3.6) in a definite way as

$$\delta\mathcal{L}_{N=3}^{(+)}=\delta\left(\frac{e}{4}F_{\mu\nu}^I\widetilde{G}^{I\mu\nu}\right)$$
$$=-\frac{e}{4}(F_{\mu\nu}^I\Lambda^{IJ}\widetilde{F}^{J\mu\nu}-\widetilde{G}_{\mu\nu}^I\Lambda^{IJ}G^{J\mu\nu}),\quad(3.8)$$

which is same as that of the usual SUGRA except that the parameters Λ^{IJ} are now complex.

IV. CONSTRUCTION OF THE HIGHER N CHIRAL LAGRANGIAN DENSITY

In this section let us first construct the $N=4$ chiral Lagrangian density by applying the prescription (a)–(e) explained in the Introduction. In the usual $N=4$ SUGRA, the field contents are a tetrad field, four Rarita-Schwinger fields, six vector fields, four spin-1/2 fields, and two (real) scalar fields. Since there are scalar fields in the theory, the duality symmetry group becomes ''noncompact'' [22] in contrast with the $N=2,3$ theories. Indeed, the duality symmetry group of the usual $N=4$ SUGRA is SU(4)×SU(1,1) [24], and the two scalar fields are described by the SU(1,1)/U(1) nonlinear sigma model [22,23].

In $N=4$ chiral SUGRA, we introduce at first the (complex) tetrad e_μ^i, two independent sets of Rarita-Schwinger fields $(\psi_\mu^I,\widetilde{\psi}_\mu^I)$ $(I=1,2,3,4)$, two independent sets of spin-1/2 fields $(\chi^I,\widetilde{\chi}^I)$, (complex) vector fields $A_\mu^{IJ}(=-A_\mu^{JI})$, and complex scalar fields[6] $(^\pm\phi_1,^\pm\phi_2)$ as the field variables. The self-dual connection $A_{ij\mu}^{(+)}$ is also treated as one of the independent variables in the first-order formalism. If we apply the prescription of constructing the chiral Lagrangian density from the usual $N=4$ SUGRA as in the case of $N=3$, then the obtained $N=4$ chiral Lagrangian density in first-order

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[6]In $N=4$ chiral SUGRA, we define complex scalar fields as $^\pm\phi_1=p\pm iq$ and $^\pm\phi_2=r\pm is$ with p, q, r, and s being assumed to be complex, respectively. The reality condition for these scalar fields will be taken as $(\overline{\phi_1},\overline{\phi_2})=(^+\phi_1,^+\phi_2)$.

form can be written schematically as

$$\mathcal{L}_{N=4}^{(+)}=-\frac{i}{2\kappa^2}e\,\epsilon^{\mu\nu\rho\sigma}e_\mu^ie_\nu^jR_{ij\rho\sigma}^{(+)}$$

$$-e\,\epsilon^{\mu\nu\rho\sigma}\overline{\widetilde{\psi}}_{R\mu}^I\gamma_\rho D_\sigma^{(+)}\psi_{R\nu}^I+ie\,\overline{\widetilde{\chi}}_R^I\gamma^\mu D_\mu^{(+)}\chi_R^I$$

$$+\mathcal{L}_{N=4}\text{ (scalar kinetic term}+A_\mu^{IJ}\text{-dependent terms)}$$

$$+(A_\mu^{IJ}\text{-independent terms})$$

$$+\frac{i}{16}\kappa^2 e\,\epsilon^{\mu\nu\rho\sigma}(\overline{\psi}_{L\mu}^K\psi_{R\nu}^L)\overline{\widetilde{\psi}}_{R\rho}^M\widetilde{\psi}_{L\sigma}^N\epsilon^{IJKL}\epsilon^{IJMN}$$

$$+\frac{\kappa^2}{2}e(\overline{\widetilde{\psi}}_{R\mu}^I\gamma^{[\mu}\psi_{R\nu}^I)\overline{\widetilde{\chi}}_R^J\gamma^{\nu]}\chi_R^J\quad(4.1)$$

with

$$\mathcal{L}_{N=4}\text{ (scalar kinetic term}+A_\mu^{IJ}\text{-dependent terms)}$$

$$=\frac{1}{2}\frac{\partial_\mu{}^+z\partial^\mu{}^-z}{(1-{}^+z{}^-z)}-\frac{e}{4}F_{\mu\nu}^{IJ}K_{IJ,KL}F^{KL\mu\nu}$$

$$+(F_{\mu\nu}^{IJ}\text{-proportional terms}).\quad(4.2)$$

The first term in Eq. (4.2) is the kinetic term of the scalar fields corresponding to the SU(1,1)/U(1) nonlinear sigma model, and $^\pm z$ in this term is defined as U(1) invariant variable constructed from $(^\pm\phi_1,^\pm\phi_2)$, i.e.,

$$^\pm z:={}^\pm\phi_2(^\pm\phi_1)^{-1}.\quad(4.3)$$

The function $K_{IJ,KL}(=K_{KL,IJ})$ in the second term of Eq. (4.2), on the other hand, is given by

$$K_{IJ,KL}=\frac{1+{}^-z^2}{1-{}^-z^2}\delta_{I[K}\delta_{|J|L]}-\frac{2\,{}^-z}{1-{}^-z^2}\frac{1}{2}\epsilon_{IJKL},\quad(4.4)$$

which is determined from a specific transformation property of $K_{IJ,KL}$ [22,23] under the duality transformations. Equation (4.2) and the A_μ^{IJ}-independent terms in Eq. (4.1) correspond to those obtained in the usual $N=4$ SUGRA [24,25]. In order to prove the SUSY invariance of the chiral Lagrangian density (4.1) under right- and left-handed SUSY transformations, we will need a straightforward calculation.

The last two four-fermion contact terms in Eq. (4.1) has the same role as in $N=2,3$ chiral SUGRA, namely, those terms ensure the first-order SUSY invariance and are also necessary to reproduce the Lagrangian density of the usual $N=4$ SUGRA, when the reality condition

$$\overline{e_\mu^i}=e_\mu^i,\quad\overline{\widetilde{\psi}_\mu^I}=\psi_\mu^I,\quad\overline{\widetilde{\chi}^I}=\chi^I,$$

$$\overline{A_\mu^{IJ}}=A_\mu^{IJ},\quad\text{and}\quad(\overline{\phi_1},\overline{\phi_2})=(^+\phi_1,^+\phi_2),\quad(4.5)$$

is imposed. Indeed, the last two terms in Eq. (4.1) exactly cancel with two of terms obtained in the second-order formalism by solving the equation $\delta\mathcal{L}_{N=4}^{(+)}/\delta A_{ij\mu}^{(+)}=0$ with respect to $A_{ij\mu}^{(+)}$. Then the $\mathcal{L}_{N=4}^{(+)}$ (second order) of $N=4$ chiral

SUGRA with the reality condition (4.5) is reduced to that of the usual one up to imaginary boundary terms as

$$\mathcal{L}_{N=4}^{(+)} \text{ (second order)} = \mathcal{L}_{N=4 \text{ usual SUGRA}} \text{ (second order)}$$

$$+ \frac{1}{8} \partial_\mu (e \, \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\rho^I \gamma_\nu \psi_\sigma^I$$

$$+ i e \bar{\chi}^I \gamma_5 \gamma^\mu \chi^I). \tag{4.6}$$

This second-order Lagrangian density (4.6) is invariant under the SUSY transformations of the usual $N=4$ SUGRA.

Next we discuss the construction of the $N=8$ chiral Lagrangian density also by applying the prescription (a)–(e) explained in the Introduction. Here we note the characteristic features of the chiral Lagrangian density constructed so far. The $N=3$ and 4 chiral Lagrangian densities of Eqs. (2.1) and (4.1) have different forms from those of the usual SUGRA, in particular, with respect to the following points: First, only the gravitational and spinor (spin-3/2 and -1/2) kinetic terms in the chiral Lagrangian density are written in terms of the self-dual connection $A_{ij\mu}^{(+)}$. Then the appropriate four-fermion contact terms, which are required from the invariance under first-order SUSY transformations at order κ at least up to $N=3$ chiral SUGRA, are added in the chiral Lagrangian density by the prescription (e). In view of these points, the prescription of constructing the chiral Lagrangian density is easily extended to $N=8$ SUGRA [26,27].

The field contents of the usual $N=8$ SUGRA are a tetrad field, eight Rarita-Schwinger fields, 28 vector fields, 56 spin-1/2 fields, and 35 complex scalar fields. If we introduce at first the (complex) tetrad e^i_μ, two independent sets of Rarita-Schwinger fields $(\psi_\mu^I, \tilde{\psi}_\mu^I)$ $(I=1,\dots,8)$, two independent sets of spin-1/2 fields $(\chi^{IJK}, \tilde{\chi}^{IJK})$,[7] then the gravitational and spinor kinetic terms written by the $A_{ij\mu}^{(+)}$ in the $N=8$ chiral Lagrangian density $\mathcal{L}_{N=8}^{(+)}$ are written as

$$\mathcal{L}_{N=8}^{(+)} \text{ (gravitational and spinor kinetic terms)}$$

$$= -\frac{i}{2\kappa^2} e \, \epsilon^{\mu\nu\rho\sigma} e^i_\mu e^j_\nu R_{ij\rho\sigma}^{(+)} - e \, \epsilon^{\mu\nu\rho\sigma} \bar{\bar{\psi}}_{R\mu}^I \gamma_\rho D_\sigma^{(+)} \psi_{R\nu}^I$$

$$+ \frac{i}{6} e \bar{\bar{\chi}}_R^{IJK} \gamma^\mu D_\mu^{(+)} \chi_R^{IJK}. \tag{4.7}$$

The four-fermion contact terms added in $\mathcal{L}_{N=8}^{(+)}$ by means of the prescription (e), on the other hand, are chosen as

$$\mathcal{L}_{N=8}^{(+)} \text{ [contact terms by the prescription (e)]}$$

$$= \frac{i}{8 \times 6!} \kappa^2 e \, \epsilon^{\mu\nu\rho\sigma} (\bar{\psi}_{L\mu}^P \psi_{R\nu}^Q) \bar{\bar{\psi}}_{R\rho}^R \tilde{\psi}_{L\sigma}^S$$

$$\times \epsilon^{IJKLMNPQ} \epsilon^{IJKLMNRS}$$

$$+ \frac{\kappa^2}{12} e (\bar{\bar{\psi}}_{R\mu}^I \gamma^{[\mu} \psi_{R\nu}^I) \bar{\bar{\chi}}_R^{JMN} \gamma^{\nu]} \chi_R^{JMN}, \tag{4.8}$$

which will also be required from the invariance under first-order SUSY transformations at order κ. The terms other than Eqs. (4.7) and (4.8) correspond to those obtained in the usual $N=8$ SUGRA [26,27]. In order to prove the SUSY invariance of $\mathcal{L}_{N=8}^{(+)}$ under right- and left-handed SUSY transformations, we will also need a straightforward calculation.

By means of the four-fermion contact terms of Eq. (4.8), the $\mathcal{L}_{N=8}^{(+)}$ (second order) of $N=8$ chiral SUGRA with the reality condition is also reduced to that of the usual one up to imaginary boundary terms as

$$\mathcal{L}_{N=8}^{(+)} \text{ (second order)}$$

$$= \mathcal{L}_{N=8 \text{ usual SUGRA}} \text{ (second order)}$$

$$+ \frac{1}{8} \partial_\mu \left(e \, \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\rho^I \gamma_\nu \psi_\sigma^I + \frac{i}{6} e \bar{\chi}^{IJK} \gamma_5 \gamma^\mu \chi^{IJK} \right), \tag{4.9}$$

which is invariant under the SUSY transformations of the usual $N=8$ SUGRA. The imaginary boundary terms in Eq. (4.9) are same as those of the $N=3,4$ chiral SUGRA.

V. CONCLUSION

In this paper we obtained $N=3$ chiral SUGRA compatible with the reality condition by applying the prescription of constructing the chiral Lagrangian density from the usual $N=3$ SUGRA. The $N=3$ chiral Lagrangian density in first-order form of Eq. (2.1) was determined so that it reproduces the $\mathcal{L}_{N=3}^{(+)}$ (second order) of Eq. (2.15) by adding the appropriate four-fermion contact terms, and showed that those four-fermion contact terms added in Eq. (2.1) are the nonminimal terms required from the invariance under the first-order SUSY transformations at order κ. We also showed that the field equation for the vector fields derived from Eq. (2.1) is invariant under the (compact) duality transformations.

Furthermore, we constructed the $N=4$ chiral Lagrangian density, in which the duality symmetry group is (noncompact) SU(4)×SU(1,1), and we also discussed the construction of the $N=8$ chiral Lagrangian density. In the higher N chiral Lagrangian density we added appropriate four-fermion contact terms as in the case of $N=3$, which will be required from the invariance under the first-order SUSY transformations at order κ. We will need a straightforward calculation in order to prove the SUSY invariance of the higher N chiral Lagrangian density under right- and left-handed SUSY transformations.

Finally we briefly discuss the polynomiality of constraints in the canonical formulation of the chiral SUGRA. There appear, in the chiral SUGRA, right- and left-handed SUSY constraints in addition to Gauss-law, U(1) gauge (for $N \geq 2$), vector and Hamiltonian constraints, which reflect the invariance of the chiral Lagrangian density. In the $N=1$ theory [3], all the constraints are indeed written in polynomial form in terms of the canonical variables of the Ashtekar formulation. In the $N=2$ theory [6,10], although only the left-handed SUSY constraint (and the Hamiltonian constraint as stated in Ref. [6]) has the nonpolynomial factor as in the

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[7]The χ^{IJK} denotes totally antisymmetric spinor, i.e., $\chi^{IJK} = \chi^{[IJK]}$.

case of the Einstein-Maxwell theory in the Ashtekar variable [28], the rescaled left-handed SUSY constraint by multiplying this factor becomes polynomial. In the $N=3$ theory derived from the $N=3$ chiral Lagrangian density (2.1) with the reality condition (2.12), it can be verified that both right- and left-handed SUSY constraints have the same nonpolynomial factor as appears only in the left-handed SUSY constraint of the $N=2$ theory. However, the polynomiality of these SUSY constraints is also recovered by multiplying this factor to the constraints. The constraint algebra of the $N=3$ theory is now under investigation and will be reported elsewhere.

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