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Embedding variables in finite dimensional models

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Global problems associated with the transformation from the Arnowitt-Deser-Misner (ADM) to the Kuchař variables are studied. Two models are considered: The Friedmann cosmology with scalar matter and the torus sector of the 2+1 gravity. For the Friedmann model, transformations to the Kuchař description corresponding to three different popular time coordinates are shown to exist on the whole ADM phase space, which becomes a proper subset of the Kuchař phase spaces. The 2+1 gravity model is shown to admit a description by embedding variables everywhere, even at the points with additional symmetry. The transformation from the Kuchař to the ADM description is, however, a many-to-one transformation there, and so the two descriptions are inequivalent for this model, too. The most interesting result is that the new constraint surface is free from the conical singularity and the new dynamical equations are linearization stable. However, some residual pathology persists in the Kuchař description.

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I. INTRODUCTION

Generally covariant systems are quite popular in the theoretical physics of today. Each such model contains one or more spacetime-like objects. For example, in string theory, we find target spacetime as well as string (and membrane) sheets. The variables that specify points in phase space are then tensor (density) fields on Cauchy surfaces in some of the spacetimes. For example, in general relativity, the first and second fundamental forms of the Cauchy surface are used, or rather some modifications thereof, the so-called Arnowitt-Deser-Misner (ADM) variables $q_{kl}(x)$ and $\pi^{kl}(x)$ [1]. We call the canonical formalism based on these variables the *ADM description*.

As early as 1962 it was recognized [2] that the ADM variables contain a mixture of two types of information. The first has to do with the physical, gauge independent state of the system. The second just tells us where in the spacetime the Cauchy surface lies.

The mathematical language of this idea has been worked out by Kuchař [3]. The variables that describe the position of the Cauchy surface are so-called *embeddings*: maps of the form $X:\Sigma \mapsto \mathcal{M}$ of the Cauchy manifold Σ into the spacetime manifold $\mathcal{M}$. The gauge invariant, true physical degrees of freedom can be described by variables of the so-called Heisenberg picture [4]. They are observables in the sense of Dirac [5]. The momenta P conjugate to the embeddings X are simultaneously the new constraint functions. We call the canonical formalism based on these variables the *Kuchař description*.

One advantage of Kuchař variables is that they enable a four-dimensional, spacetime formulation of canonical theory: all Cauchy surfaces are admitted in the canonical description of the dynamics (''bubble time'' or ''many-finger time'' [3]). This is to be compared with the one-parameter time evolution based on a particular choice of a one-dimensional family of Cauchy surfaces in each solution spacetime, the so-called foliation. A foliation is a particular 3+1 split of four-dimensional spacetime. The original ADM reduction program (for a current version, cf. [6]) was based on such a split. Kuchař's approach allows one to write down explicitly the action of the four-dimensional diffeomorphism group [7]—the gauge group of the model.

A canonical transformation from the ADM to the embedding variables, their conjugate momenta and the observables will be called a *Kuchař decomposition* or *Kuchař transformation*. The Kuchař transformation turned out to be a difficult task. It was managed only for a few special models [8–11]. Moreover, some general, negative results were published. In [12] and [13], simple models were constructed that did not allow a global Kuchař decomposition. Torre [14] showed that the decomposition, which, in fact, brought the system to the form of the so-called ''already parametrized system,'' was impossible at some points of the constraint surface of general relativity. These were the points that, as Cauchy data, evolved to spacetimes with additional Killing vectors. Thus, even the existence of Kuchař decomposition was questioned.

Some progress in this situation has been achieved in [15] (see also [16]). The conditions for the existence of the Kuchař transformation have been clarified. First, each Kuchař decomposition is associated, and in fact determined, by a choice of gauge. The Kuchař coordinate chart can cover only such part of the constraint surface for which a common gauge fixing exists. Second, all points of the constraint surface must be excluded that evolve to spacetimes with *any* isometries, not just with Killing vectors. And, finally, even if these conditions are satisfied, the existence could only be shown for a neighborhood of the constraint surface, not for the whole ADM phase space.

The aim of the present paper is to start a study of the conditions mentioned in the previous paragraph. This would be rather difficult in a general context. We shall, therefore, start by studying finite-dimensional, ''minisuperspace'' models. For such models, the spacetime manifold $\mathcal{M}$ is effectively one dimensional and the Cauchy manifold is just a point, so the space of embeddings can be identified with $\mathcal{M}$—a finite-dimensional space. The models chosen are completely solvable. This enables us to construct Kuchař transformations explicitly (the proof in [15] is not constructive).

The plan of the paper is as follows. In Sec. II, we consider the Friedmann cosmological model driven by a zero-rest-mass, conformally coupled scalar field. This model has been studied in some detail in [17]. First, we specify the gauge needed for a Kuchař transformation. In the one-dimensional spacetime model, it can be called the ''choice of time''; the time coordinate is, in fact, an embedding variable. It is advantageous to decompose the choice into two steps. The first one specifies the lapse N as a function on the ADM phase space $\mathcal{P}$. Then the canonical equations of the Hamiltonian, $P:=N\mathcal{H}$, define the so-called *trajectories* everywhere in $\mathcal{P}$; $\mathcal{H}$ is the constraint function. The second step is a choice of the surface transversal to the trajectories as the origin of time. We study three choices of time: *conformal*, *proper*, and constant mean curvature (CMC) time and try to find the corresponding Kuchař coordinates on the whole of $\mathcal{P}$.

The model of Sec. III is the torus sector of the 2+1 gravity theory, partially reduced so that a minisuperspace model results. This has been carried out in [18], from where we adopt our starting formulas. The model is interesting for several reasons. Its constraint set $\mathcal{C}$ does contain points associated with higher symmetry—the static tori. $\mathcal{C}$ has a bifurcation and conical singularity at these points. The conical singularity is a feature associated with additional Killing vectors; see [19]. It is also the cause of the so-called *linearization instability* [20]. $\mathcal{C}$ has no well-defined differential structure at these points. This is a difficulty not only for the transformation to Kuchař variables, but also for the definition of the ADM physical phase space (see [21]). Finally, this model does not admit a globally transversal surface. We can, therefore, study this topological obstruction, too.

All these problems disappear if we truncate the model by excising the points associated with the static tori as has been done in [18] and [22]. The truncated model consists of two separated parts. Each part admits a globally transversal surface, a global chart of Kuchař variables, and a nice physical phase space. In the present paper, we are trying to extend both parts of the truncated model.

Section III B investigates important properties of the physical phase space of the extended model. We construct an atlas for the physical phase space from a chosen family of transversal surfaces in the constraint set. In this way, a smooth manifold (in fact, analytic) can be obtained.

In Sec. III C, we turn to the embedding variables. Strictly speaking, the negative results of [14] and [15] only imply that the ADM variables cannot be transformed into Kuchař ones at the points with higher symmetry. This does not mean that there is no Kuchař description including solutions with additional symmetry. However, if it exists, it cannot be equivalent to the ADM description.

Our atlas for the physical phase space serves as a starting point. The transversal surfaces defining it can be extended from the constraint surface to a part of the ADM phase space. There is a patch of Kuchař coordinates for each transversal surface. In this way, we obtain a Kuchař description of the whole model. The transformation from the Kuchař description to the ADM one becomes singular, many to one, at the points of higher symmetry (the trajectories containing these points are zero dimensional in the ADM description and one dimensional in Kuchař description). However, all classical solutions of the new system coincide completely with the corresponding solutions of the old one. Yet the new constraint surface is free from the bifurcation and the conical singularity. The new dynamical equations are *linear*. This has an obvious but amusing consequence: they are linearization stable.

The results are discussed in Sec. IV. There is also an attempt at a synthesis of the results from both minisuperspace models.

II. FRIEDMANN MODEL WITH CONFORMALLY COUPLED SCALAR FIELD

In this section, we shall study the spatially closed Friedmann cosmological model with a particular matter content: a zero-rest mass, conformally coupled scalar field.

The action has the form

$$S=\int dt(p_a\dot{a}+p_\phi\dot{\phi}-N\mathcal{H}),$$

where $a(t)$ is the scale factor of the Robertson-Walker line element,

$$ds^2=-N^2dt^2+a(t)^2\left(\frac{dr^2}{1-r^2}+r^2(d\vartheta^2+\sin\vartheta^2d\varphi^2)\right), \quad (1)$$

N is the lapse function, ϕ is defined in terms of the original scalar field Φ by

$$\phi:=\frac{2\sqrt{2}\mathrm{G}}{3}a\Phi,$$

G is the Newton constant, and $\mathcal{H}$ is the Hamiltonian constraint:

$$\mathcal{H}=\frac{1}{2a}(-p_a^2+p_\phi^2-a^2+\phi^2).$$

For more details see [17].

The ADM phase space $\mathcal{P}$ is four dimensional, covered by the canonical chart a, ϕ, p_a and p_ϕ with ranges

$$a\in(0,\infty), \quad \phi\in(-\infty,\infty),$$

$$p_a\in(-\infty,\infty), \quad p_\phi\in(-\infty,\infty).$$

The constraint surface $\mathcal{C}$ is the three-dimensional ''cone''

$$-p_a^2+p_\phi^2-a^2+\phi^2=0.$$

The background manifold $\mathcal{M}$ is one dimensional, $\mathcal{M}=\mathbf{R}$. For its complete definition, a choice of time coordinate T is needed [15]. The Cauchy manifold is represented by a zero-dimensional manifold (a point) Σ, and the space Emb$(\Sigma,\mathcal{M})$ of embeddings $T:\Sigma\mapsto\mathcal{M}$ can be identified with $\mathcal{M}$:

$$\mathrm{Emb}(\Sigma,\mathcal{M})=\mathcal{M}.$$

A set of Kuchař variables consists then of the time variable T, its conjugate momentum P, which is proportional to the Hamiltonian constraint, and two Dirac observables, which are constants of motion. For the present case, it is not difficult to find the transformation to such variables if T is chosen so that the equations of motion simplify.

A. Conformal time

A suitable choice of time is connected with the following value of the lapse:

$$N = a. \tag{2}$$

Equation (1) shows that T is a *conformal time* then. The conjugate variable is

$$P := N\mathcal{H} = \frac{1}{2}(-p_a^2 + p_\phi^2 - a^2 + \phi^2). \tag{3}$$

The time coordinate is not yet completely specified. Some surface is to be chosen as the origin $T=0$.

The equations of motion corresponding to the Hamiltonian P are

$$\dot{a} = -p_a, \quad \dot{p}_a = a, \tag{4}$$

$$\dot{\phi} = p_\phi, \quad \dot{p}_\phi = -\phi. \tag{5}$$

It follows that $\dot{p}_a$ is positive everywhere in $\mathcal{P}$, and we can choose the surface defined by $p_a = 0$ as $T=0$. The resulting general solution to the equations of motion is

$$a = A\cos T, \quad p_a = A\sin T, \tag{6}$$

$$\phi = B\cos(T+C), \quad p_\phi = -B\sin(T+C), \tag{7}$$

where A, B, and C are constants. We can express P by these constants:

$$P = \frac{1}{2}(B^2 - A^2).$$

The functions T, P, B, and C form a complete set of independent variables. Equations (6) and (7) can be written by means of these variables if $A = \sqrt{B^2 - 2P}$ is substituted for A. They can then be considered as transformation equations from the variables a, p_a, ϕ, and p_ϕ to T, P, B, and C. Let us express the Liouville form in terms of the new variables. A simple calculation reveals

$$p_a da + p_\phi d\phi = P dT + \frac{1}{2}B^2 dC + d\left(-P\sin T\cos T + \frac{1}{2}B^2 \sin T\cos T - \frac{1}{2}B^2 \sin(T+C)\cos(T+C)\right).$$

To improve the right-hand side, we introduce the functions q and p by

$$q = B\cos C, \quad p = -B\sin C;$$

this implies that

$$p\,dq = \frac{1}{2}B^2 dC + d\left(-\frac{1}{2}B^2\sin C\cos C\right).$$

Hence,

$$a = \sqrt{q^2+p^2-2P}\cos T, \quad p_a = \sqrt{q^2+p^2-2P}\sin T, \tag{8}$$

$$\phi = q\cos T + p\sin T, \quad p_\phi = -q\sin T + p\cos T \tag{9}$$

is a canonical transformation. The meaning of the variables q and p can be inferred from Eq. (9): $q = \phi|_{T=0}$ and $p = \pi_\phi|_{T=0}$. These are values of the field ϕ and its momentum π_ϕ at the surface of maximal expansion.

The transformation defined by Eqs. (8) and (9) maps the following subset of $\mathbf{R}^4$ with natural coordinates T, P, q, and p onto $\mathcal{P}$:

$$(q,p) \in \mathbf{R}^2 \backslash \{0\}, \quad T \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

$$P \in \left(-\infty, \frac{1}{2}(q^2+p^2)\right). \tag{10}$$

T is the embedding variable corresponding to our choice of gauge. Its conjugate variable P is proportional to the constraint function. The remaining variables q and p are Dirac observables. They span the physical phase space Γ. Hence, the new action reads

$$S = \int dt(P\dot{T} + p\dot{q} - N'P), \tag{11}$$

where $N' = aN$ is the new lapse function. The action has the Kuchař form.

The boundaries defined by Eq. (10) have the following meaning. $T = -\pi/2$ is the big bang and $T = \pi/2$ the big crunch singularity of the solution to Einstein equations for our model, if $P=0$. The points are still singular for $P \neq 0$, but this is a property of the present gauge (N can be chosen such that the solutions of the resulting equations of motion outside the constraint surface are regular). The boundary $P = (q^2+p^2)/2$ corresponds to $a(T)=0$ for all T. This ''solution'' does not define any spacetime. Finally, the point $q = p = 0$ corresponds to the scalar field being identically zero. Then, again, there is no spacetime solution for $P=0$.

The existence of bounds on the embeddings and their conjugate momenta seems to be an important general feature of Kuchař transformation. Reference [15] already mentioned one kind of such bound: the embeddings must be everywhere space-like for each given geometry. In the present case, only very special embeddings are allowed, which are automatically space like. On the other hand, our findings on the bound that must be satisfied by P are rather unexpected and new. To understand it, let us recall that a Kuchař transformation is described in [15] as a map $\chi:\Gamma\times T^*\,\mathrm{Emb}(\Sigma,\mathcal{M})\mapsto\mathcal{P}$. Here χ is a symplectic diffeomorphism and its existence has been shown (under certain conditions) only in an open subset $\mathcal{U}$ of $T^*\,\mathrm{Emb}(\Sigma,\mathcal{M})$ such that $\chi(\mathcal{U})$ is a neighborhood of the constraint surface $\mathcal{C}$ in $\mathcal{P}$. One would expect that $\chi(\mathcal{U})$ is a proper subset of $\mathcal{P}$ so that the transformation exists only for limited values of the ADM variables, because nothing more has been proved in [15] but there is still some uncertainty. On the other hand, $\mathcal{U}$ must be a proper subset of $T^*\,\mathrm{Emb}(\Sigma,\mathcal{M})$, so there are always some bounds on $X\in\mathrm{Emb}(\Sigma,\mathcal{M})$ and $P\in T^*_X\,\mathrm{Emb}(\Sigma,\mathcal{M})$.

In our case, $\mathrm{Emb}(\Sigma,\mathcal{M})=\mathcal{M}=\mathbf{R}$, and we also use the letter T rather than X to denote an embedding. Then $T^*\,\mathrm{Emb}(\Sigma,\mathcal{M})=\mathbf{R}$. Our result is that $\chi(\mathcal{U})=\mathcal{P}$ so that there are only bounds on P and T, not on the ADM variables. The interpretation is that the whole ADM phase space $\mathcal{P}$ is a proper subspace of the Kuchař phase space $\Gamma\times T^*\,\mathrm{Emb}(\Sigma,\mathcal{M})$.

Let us observe that the points of $\Gamma\times T^*\,\mathrm{Emb}(\Sigma,\mathcal{M})$ that do not satisfy the bound (10) for P do not define any reasonable initial data for the spacetime and the scalar field. However, one can use the action (11) in the whole space $\Gamma\times T^*\,\mathrm{Emb}(\Sigma,\mathcal{M})$ without any harm. All points of the constraint surface satisfy the bounds, so the solution of the equations of motion within the ADM framework coincide with those within the so-extended Kuchař framework.

B. Transversal surface

Let us study the geometrical structures that underlie the calculation of the previous section.

The first step has been a choice of fixed phase-space function for the lapse N. This has determined the true Hamiltonian P by Eq. (3). The Hamiltonian vector field ξ_P is given by the right-hand sides of Eqs. (4) and (5). It is important to observe that the direction of ξ_P is independent of N at the constraint surface $\mathcal{C}$; it is only the parametrization of the integral curves of ξ_P that changes with N. Outside $\mathcal{C}$, however, even the direction of ξ_P depends on N, and the resulting integral curves form different foliations of $\mathcal{P}$ for different N.

The variable T is to be conjugate to P. This implies the condition

$$\xi_P T=1. \tag{12}$$

Hence, any parameter of the integral curves of ξ_P can be chosen as T.

Let us denote the remaining two variables that we are looking for by X and Y. They are to form a canonical chart together with T and P. It follows that they have vanishing Poisson brackets with P:

$$\xi_P X=\xi_P Y=0. \tag{13}$$

Thus, X and Y are "integrals of motion." Observe that condition (13) depends on the choice of N outside $\mathcal{C}$. At $\mathcal{C}$, it is, however, independent of it, and it implies that X and Y are Dirac observables. As also $\xi_P P=0$, we conclude that the functions P, X, and Y form a complete set of independent integrals of motion.

The conditions (12) and (13) do not determine the functions T, X, and Y. We can fix T using the following idea. Let T be some function satisfying Eq. (12). Then the equation $T=\mathrm{const}$ defines a surface in $\mathcal{P}$ at least for some value of the constant. This surface must intersect each integral curve of ξ_P at most once (there can be curves along which T does not attain the value of the constant). Moreover, the tangent spaces to the surface $T=\mathrm{const}$ and that to the integral curves of ξ_P must have only the zero vector in common at every point of the surface. We call a surface that satisfies both conditions *transversal* and *globally transversal* if it intersects all integral curves of ξ_P. Suppose that the vector field ξ_P admits a globally transversal surface $\mathcal{T}$. Then the function T can be chosen so that it vanishes at $\mathcal{T}$; by that, the function is completely determined.

Let us turn to the functions X and Y. They must have vanishing Poisson brackets with T. Hence they have to satisfy the conditions

$$\xi_T X=\xi_T Y=0, \tag{14}$$

where ξ_T is the Hamiltonian vector field of T. Observe that the Lie brackets between ξ_P and ξ_T vanish,

$$[\xi_T,\xi_P]=0,$$

because $\{T,P\}=1$. Our construction of the functions X and Y is based on this observation.

Let $\mathcal{T}$ be a globally transversal surface. Consider the two-dimensional surface $\mathcal{T}\cap\mathcal{C}$. The pullback ω of the symplectic form Ω from $\mathcal{P}$ to $\mathcal{T}\cap\mathcal{C}$ is again symplectic (nondegenerate). The symplectic manifold $(\mathcal{T}\cap\mathcal{C},\omega)$ can be identified with the physical phase space Γ.

Let us choose two coordinates x and y on $\mathcal{T}\cap\mathcal{C}$ satisfying

$$\{x,y\}_\omega=1.$$

We extend these functions in two steps to the whole of $\mathcal{P}$. First, we use the condition (14) to extend them to $\mathcal{T}$. Equation (14) can be considered as a differential equation on $\mathcal{T}$: as $\xi_T T=0$, the vector field ξ_T is tangential to $\mathcal{T}$. Let the functions X and Y at $\mathcal{T}$ satisfy the differential equations (14) together with the initial conditions

$$X|_{\mathcal{T}\cap\mathcal{C}}=x,\quad Y|_{\mathcal{T}\cap\mathcal{C}}=y.$$

This is sensible because the surface $\mathcal{T}\cap\mathcal{C}$ is transversal to ξ_T in $\mathcal{T}$. The reason is that $\xi_T P=-1$ and $\mathcal{T}\cap\mathcal{C}$ is defined by $P=0$.

The second step is to use the differential equations (13) with the initial conditions at $\mathcal{T}$ given by the values of $X|_{\mathcal{T}}$ and $Y|_{\mathcal{T}}$ as obtained in the previous step. The two steps result in a pair of functions X and Y that automatically satisfy the remaining Poisson bracket conditions. This follows from the Jacobi identity as follows.

At $\mathcal{T}$, we have

$$\begin{aligned}\xi_T\{X,Y\}&=\{\{X,Y\},T\}\\&=-\{\{T,X\},Y\}-\{\{Y,T\},X\}\\&=\{\xi_T X,Y\}-\{\xi_T Y,X\}=0\end{aligned} \tag{15}$$

because of Eq. (14). Equation (14) implies that

$$\{X,T\}|_{\mathcal{T}}=0, \tag{16}$$

$$\{Y,T\}|_{\mathcal{T}}=0, \tag{17}$$

and $\{X,Y\}|_{\mathcal{T}}$ is constant along the integral curves of ξ_T.

Similarly, $\xi_P\{X,Y\}=0$, $\xi_P\{X,T\}=0$, and $\xi_P\{T,Y\}=0$ for the propagation along ξ_P in the second step. From the values given by Eqs. (16) and (17), and from Eq. (13), we obtain that

$$\{X,T\}=\{Y,T\}=\{X,P\}=\{Y,P\}=0 \tag{18}$$

everywhere in $\mathcal{P}$. We also have that $\{T,P\}=1$, so it remains only to show that $\{X,Y\}=1$ everywhere. Now, T, P, X, and Y are independent functions in a neighborhood of the surface $\mathcal{T}\cap\mathcal{C}$ and can be chosen as coordinates there. The only nonzero components of the symplectic form Ω in these coordinates are $\Omega_{PT}=-\Omega_{TP}=1$ and $\Omega_{YX}=-\Omega_{XY}$ because of Eq. (18). The surface $\mathcal{T}\cap\mathcal{C}$ is defined by the embedding relations

$$X=x,\quad Y=y,\quad P=0,\quad T=0.$$

Hence, the pullback ω is

$$\omega_{yx}=\Omega_{YX}|_{P=T=0}$$

and so $\{X,Y\}|_{\mathcal{T}\cap\mathcal{C}}=1$ because $\omega_{yx}=1$. The desired result follows, for the brackets $\{X,Y\}$ must be constant along ξ_T and ξ_P.

Let us summarize. The construction of the previous section is based on three choices: the function P, the transversal surface $\mathcal{T}$, and the coordinates x and y on $\mathcal{T}\cap\mathcal{C}$. The Poisson bracket conditions on the functions T, P, X, and Y imply differential equations that propagate the functions from $\mathcal{T}\cap\mathcal{C}$ to $\mathcal{T}$ and from $\mathcal{T}$ to $\mathcal{P}$. The result is unique, given the three choices.

The propagation from $\mathcal{T}\cap\mathcal{C}$ to $\mathcal{T}$ has been only implicit in the calculation of Sec. II B because the functions B and C that satisfy Eq. (14) have been guessed.

From the commutativity of the vector fields ξ_P and ξ_T, it follows that the propagation is independent of the way chosen. Hence, we could propagate first from $\mathcal{T}\cap\mathcal{C}$ to $\mathcal{C}$ along ξ_P and then from $\mathcal{C}$ to $\mathcal{P}$ along ξ_T with the same result. To do that, there has first to be given the function T everywhere on $\mathcal{P}$ instead of P. We can therefore call the method of the previous section P-way and the alternative method T-way. We shall use the T-way in the next section.

C. Bergmann-Komar transformation

The conformal time T of Sec. II A has a well-defined value at each point of any spacetime solution. In general, the transformation from T to some different ''time coordinate'' T' will be solution dependent. For our model, it can depend on the variables P, q, and p. Such transformations have been introduced and studied by Bergmann and Komar [23].

As we have shown in Sec. II A, a choice of time can be done in two steps: that of lapse function and that of transversal surface. If N is not changed, the Hamiltonian P will be preserved. The solution arcs (the point sets defined by the trajectories) will then be the same *everywhere* in $\mathcal{P}$. A change of transversal surface leads only to a reparametrization of the trajectories. Hence, such a transformation can be considered as a genuine, solution-dependent, reparametrization of the trajectories everywhere. In our case, it has the general form

$$T'=T+\tilde{T}(P,q,p),\quad p'=\tilde{P}(P,q,p), \tag{19}$$

$$q'=\tilde{q}(P,q,p),\quad p'=\tilde{p}(P,q,p). \tag{20}$$

Such transformations form a subgroup. They do not change the character of the time. In our case, it remains conformal time. Still, it is a Bergmann-Komar transformation that cannot be implemented, in general, by a unitary transformation in the quantum theory (see [24]).

If we change N, the character of time changes. In this section, we are going to study two such examples: the *proper* and the *constant-mean-external-curvature* times. These are two relatively popular choices in cosmology.

In fact, a change of N leads to a more radical change of the trajectories outside the constraint surface than just a reparametrization. The notion of ''solution spacetime'' is gauge independent only at $\mathcal{C}$. Yet the trajectories are important for us everywhere in $\mathcal{P}$: they define the Kuchař coordinates there.

The trajectories are solutions to the canonical equations of the Hamiltonian $P=N\mathcal{H}$. They are so uniquely defined through all points of $\mathcal{P}$. The fact that the solution arcs depend on N outside $\mathcal{C}$ has to do with the way that the dynamics of a generally covariant system is usually formulated. The classical dynamics of such a system is completely determined by the constraint surface (but cf. [25]). The form of the constraint functions is irrelevant as long as they define the same constraint surface.

How are the trajectories outside the constraint surface to be interpreted? To be sure, each gauge and any of the corresponding trajectories define a unique Robertson-Walker spacetime and scalar field on it. The gauge supplies the lapse function $N(q,p,P,T)$ and then each of the corresponding trajectories determines a unique scale factor $a(q,p,P,T)$ as well as the scalar field $\phi(q,p,P,T)$. In this manner, there is a spacetime with the metric

$$ds^2=-N^2(q,p,P,T)dT^2+a^2(q,p,P,T)\left(\frac{dr^2}{1-r^2}+r^2d\Omega^2\right) \tag{21}$$

for each trajectory determined by the constant values of q, p, and P. The scalar field $\phi(q,p,P,T)$ can be considered as a field on this spacetime. Of course, the relation between the momenta p_a and p_ϕ on the one hand and the velocities

da/dT and $d\phi/dT$ on the other will *not*, in general, coincide with those obtained from the second-order formalism, e.g., by

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{a}}.$$

This does not seem, however, to disturb the interpretation based on the existence of the solution spacetime (21).

1. Proper time

Here, we calculate the transformation from T, P, q, and p to Kuchař variables corresponding to the proper time T'.

The lapse function N that is associated with the proper time has the value $N=1$. We obtain that $a dT = dT'$. At the constraint surface, $P=0$, and Eq. (8) implies

$$dT' = \sqrt{q^2+p^2}\cos T dT. \tag{22}$$

The T' curves at the constraint surface consist of the same points as the T curves. Hence, the values of q' and p' are again constant along them, $q'=q$ and $p'=p$, and Eq. (22) has the integral

$$T' = \sqrt{q^2+p^2}\sin T, \tag{23}$$

where we have chosen the same transversal surface $\mathcal{T}\cap\mathcal{C}$ as in Sec. II A.

In this way, the function T' is determined at $\mathcal{C}$. To proceed with the calculation in the T-way, we have to extend the function to the outside of $\mathcal{C}$. As was explained above, the particular value of the extension does not have any physical meaning and can be chosen just by convenience. A suitable choice is Eq. (23) everywhere (i.e., T' independent of P). Then, the transversal surface $\mathcal{T}$ of Sec. II A is preserved.

The next step of the T-way is the propagation of the functions P', q', and p' by the differential equations

$$\xi_{T'}P' = -1, \quad \xi_{T'}q' = 0, \quad \xi_{T'}p' = 0 \tag{24}$$

out of $\mathcal{C}$, where we have the initial conditions

$$P'|_{\mathcal{C}} = 0, \quad q'|_{\mathcal{C}} = q, \quad p'|_{\mathcal{C}} = p. \tag{25}$$

Then, the required values of the Poisson brackets,

$$\{T',P'\} = 1, \quad \{q',T'\} = 0, \quad \{p',T'\} = 0,$$

are granted. And for a similar reason as in Sec. II B, all other Poisson brackets will also have the desired values.

From Eq. (23), we easily obtain

$$\xi_{T'} = -\sqrt{q^2+p^2}\cos T\frac{\partial}{\partial P} + \frac{p\sin T}{\sqrt{q^2+p^2}}\frac{\partial}{\partial q} - \frac{q\sin T}{\sqrt{q^2+p^2}}\frac{\partial}{\partial p}.$$

Equations (24) can be solved by the method of characteristics. The characteristic equations read

$$\frac{\partial T}{\partial P'} = 0, \quad \frac{\partial P}{\partial P'} = \sqrt{q^2+p^2}\cos T, \tag{26}$$

$$\frac{\partial q}{\partial P'} = -\frac{p\sin T}{\sqrt{q^2+p^2}}, \quad \frac{\partial p}{\partial P'} = \frac{q\sin T}{\sqrt{q^2+p^2}}. \tag{27}$$

We have already used the fact that the parameter of the characteristic curves can be chosen to be $-P'$. This follows from the first equation of Eqs. (24).

We can see immediately that T is an integral of the system and one verifies easily that $\sqrt{q^2+p^2}$ is another one. Then, the integration of the system is straightforward and we obtain

$$T = T_0, \quad P = P'\sqrt{q_0^2+p_0^2}\cos T_0, \tag{28}$$

$$q = q_0\cos(\nu_0 P') - p_0\sin(\nu_0 P'),$$

$$p = q_0\sin(\nu_0 P') + p_0\cos(\nu_0 P'), \tag{29}$$

where

$$\nu_0 = \frac{\sin T_0}{\sqrt{q_0^2+p_0^2}}$$

and we have used the fact that $P=P'=0$ at $\mathcal{C}$. The integration constants T_0, q_0, and p_0 are the values of the coordinates T, q, and p at the point where the characteristic intersects the constraint surface $\mathcal{C}$.

Everywhere along the characteristic passing through the point (T_0,q_0,p_0), the functions q' and p' must have the values

$$q' = q_0, \quad p' = p_0. \tag{30}$$

This is a consequence of Eq. (24). The function T' is constant along each characteristic because of the trivial equation $\xi_{T'}T' = 0$. Hence, the value of T' along the characteristic (28) and (29) is

$$T' = \sqrt{q'^2+p'^2}\sin T_0. \tag{31}$$

If we substitute Eqs. (30) and (31) into Eqs. (28) and (29), we obtain

$$P = P'\sqrt{q'^2+p'^2-T'^2}, \tag{32}$$

$$q = q'\cos(\nu' P') - p'\sin(\nu' P'), \tag{33}$$

$$p = q'\sin(\nu' P') + p'\cos(\nu' P'), \tag{34}$$

where

$$\nu' := \frac{T'}{q'^2+p'^2}.$$

Equations (33), (34), and (23) yield

$$T = \arcsin\frac{T'}{\sqrt{q'^2+p'^2}}. \tag{35}$$

Equations (32)–(35) are the desired transformation formulas between the two coordinate systems. From the construction, it follows that the transformation is canonical; this can be verified by direct calculation.

The inverse transformation is given by Eq. (23) together with

$$P' = \frac{P}{\sqrt{q^2+p^2}\cos T}, \tag{36}$$

$$q' = q\cos(\nu P) + p\sin(\nu P), \tag{37}$$

$$p' = -q\sin(\nu P) + p\cos(\nu P), \tag{38}$$

where

$$\nu := \frac{\tan T}{q^2+p^2}.$$

The transformation functions (23) and (36)–(38) are differentiable everywhere inside $\mathcal{P}$, that is, for values of the coordinates T, P, q, and p within the bounds (10). The corresponding ranges of the coordinates T', P', q', and p' are

$$(q',p') \in \mathbf{R}\backslash\{0\},$$

$$T' \in \left(-\sqrt{q'^2+p'^2}, \sqrt{q'^2+p'^2}\right),$$

$$P' \in \left(-\infty, \frac{1}{2}\sqrt{\frac{q'^2+p'^2}{q'^2+p'^2-T'^2}}\right).$$

Finally, we observe that the transformation is *not* of the form (19) and (20).

2. CMC time

The external mean curvature L of the surfaces $t=$ const has the following value for the metric (1):

$$L = -\frac{1}{3a}\frac{da}{Ndt}.$$

For the conformal time T, $N=a$, and we obtain, from Eq. (8),

$$L = \frac{1}{3\sqrt{q^2+p^2-2P}}\frac{\sin T}{\cos^2 T}.$$

In this section, we shall choose the time function T'' to be equal to L at the constraint surface $P=0$. We call this coordinate constant mean curvature (CMC) time. Again, we shall extend this function to the whole of $\mathcal{P}$ so that it is independent of P:

$$T'' = \frac{1}{3\sqrt{q^2+p^2}}\frac{\sin T}{\cos^2 T}.$$

The same method as in Sec. II C leads to the transformation formulas

$$P'' = 3\sqrt{q^2+p^2}\frac{\cos^3 T}{1+\sin^2 T}\, P,$$

$$q'' = q\cos(\tilde{\nu}P) + p\sin(\tilde{\nu}P),$$

$$p'' = -q\sin(\tilde{\nu}P) + p\cos(\tilde{\nu}P),$$

where

$$\tilde{\nu} := \frac{1}{q^2+p^2}\frac{\sin T\cos T}{1+\sin^2 T}.$$

The transformation is again differentiable everywhere in $\mathcal{P}$. The range of the coordinate T'' in $\mathcal{P}$ is the whole real axis, those of q'' and p'' remain the same as those of q and p, and the range of P'' can be described by its boundary, defined parametrically as follows:

$$P''_{\text{boundary}} = \frac{3}{2}[(q'')^2+(p'')^2]^{3/2}\frac{\cos^3 T}{1+\sin^2 T},$$

$$T''_{\text{boundary}} = \frac{1}{3\sqrt{(q'')^2+(p'')^2}}\frac{\sin T}{\cos^2 T},$$

where $T \in (-\pi/2, +\pi/2)$.

III. TORUS SECTOR OF 2+1 GRAVITY

Our second model is the partially reduced torus sector of the 2+1 gravity without sources and with zero cosmological constant. We shall use the form of the metric

$$ds^2 = -N^2dt^2 + e^{q^3-q^1}(du^1)^2 + e^{q^3+q^1}(du^2+q^2du^1)^2 \tag{39}$$

and the action

$$S = \int dt(p_i\dot{q}^i - N\mathcal{H}), \tag{40}$$

where

$$\mathcal{H} = \frac{1}{2}e^{-q^3}(p_3^2 - p_1^2 - e^{-2q^1}p_2^2), \tag{41}$$

as written down by Moncrief [18]. Here, t, u^1, and u^2 are coordinates on the three-dimensional spacetime of topology $\mathbf{R}\times S^1\times S^1$ chosen such that $t=$const are the CMC surfaces and $u^A \in (0,2\pi)$ for $A=1,2$ are coordinates on the torus such that $x^A=$ const are closed geodesics of the space metric. Such coordinates can always be chosen [18]; using this ''spatially homogeneous gauge,'' Moncrief reduced the field model to a mechanical model with a finite number of degrees of freedom. These are represented by the Teichmüller parameters q^1 and q^2. The coordinate q^3 is related to the surface area $\mathcal{F}$ of the $T=$const surface by the formula

$$\mathcal{F} = 4\pi^2 e^{q^3}.$$

This model is very interesting because it admits solutions with higher symmetry. All solution spacetimes are spatially homogeneous, invariant with respect to the Abelian group $(u^1,u^2)\mapsto(u^1+\Delta u^1,u^2+\Delta u^2)$; the time evolution of the tori leads to expansion or contraction. However, if $p_1=p_2=0$, then also $p_3=0$ and we obtain static tori. This is an additional symmetry. Observe that the constraint surface defined by $\mathcal{H}=0$ has a conical singularity at these points.

Our aim is to find out if the Kuchař description can incorporate the points of higher symmetry. The strategy will be to transform the model to the Kuchař variables everywhere except at the points of higher symmetry. There, the transformation becomes singular. We shall try to extend the resulting Kuchař description. If we manage, then the extended Kuchař description will not be equivalent to the original Moncrief one because the transformation between them is singular at the points with symmetry.

A. Constraint surface

The phase space $\mathcal{P}$ of the model with the action (40) is $\mathbf{R}^6$ and the canonical chart $(q^1,q^2,q^3,p_1,p_2,p_3)$ covers the whole manifold. The symplectic form is $\Omega=d\Theta$, where the Liouville form Θ reads

$$\Theta=p_i dq^i, \quad i=1,2,3.$$

The constraint surface $\mathcal{C}$ is defined by the constraint function $\mathcal{H}$ of Eq. (41). Its manifold structure is $\mathbf{R}^3\times C$, where $\mathbf{R}^3$ is covered by the coordinates q^1, q^2, and q^3, and C is a two-cone. The tip $\mathcal{S}$ of the cone $p_1=p_2=p_3=0$ is a three-dimensional surface. At the points of $\mathcal{S}$, $\mathcal{H}$ and the gradient of $\mathcal{H}$ both vanish. The canonical transformation generated by the function $\mathcal{H}$ is trivial at $\mathcal{S}$. This corresponds to the triviality of evolution of initial data in static toroidal spacetimes. Thus, each point of $\mathcal{S}$ is a whole trajectory of the Hamiltonian action. At all other points of $\mathcal{C}$, grad $\mathcal{H}$ is nonvanishing and so the trajectories are one dimensional.

The constraint manifold is an embedded hypersurface locally, at each point of $\mathcal{C}\backslash\mathcal{S}$. There, we have a differential structure and the pullback $\Omega_{\mathcal{C}}$ of Ω to $\mathcal{C}$. Here $\Omega_{\mathcal{C}}$ is only a presymplectic form because it is degenerate. At the points of $\mathcal{S}$, no such structure is well defined.

The submanifold $\mathcal{C}\backslash\mathcal{S}$ is the constraint surface of the truncated model; it consists of two components. As extensions of these two parts of the truncated model, we introduce two subsets $\mathcal{C}^+$ and $\mathcal{C}^-$ of $\mathcal{C}$; all points of $\mathcal{C}^+$ ($\mathcal{C}^-$) satisfy the inequality $p_3\geqslant 0$ ($p_3\leqslant 0$). Here $\mathcal{C}^+$ and $\mathcal{C}^-$ are topological (C^0) surfaces. The maps $\varphi_\pm:\mathbf{R}^5\mapsto\mathcal{C}^\pm$ defined by

$$\varphi_\pm(x^1,x^2,x^3,y_1,y_2)=(q^1,q^2,q^3,p_1,p_2)$$

such that $q^i=x^i$ for all $i=1,2,3$, $p_1=y_1$, $p_2=y_2$, and $p_3=\pm\sqrt{y_1^2+e^{-2x^1}y_2^2}$ are both homeomorphisms. They are not differentiable at $y_1=y_2=0$. Hence, $\mathcal{C}^\pm$ are topological surfaces with conical singularities at $y_1=y_2=0$. The constraint set $\mathcal{C}$ is, however, more singular than that. It has a bifurcation at $\mathcal{S}$, where both $\mathcal{C}^+$ and $\mathcal{C}^-$ coincide. The presymplectic form $\Omega_{\mathcal{C}}$ on $\mathcal{C}^\pm\backslash\mathcal{S}$ is simply $dy_1\wedge dx^1+dy_2\wedge dx^2$.

The bifurcation is connected with the way in which the time reversal acts on the ADM variables. Let us first do a few general remarks concerning the time reversal. The trajectories at the constraint surface can be considered as classes of an equivalence relation [26]; two initial data are equivalent if they evolve into maximal solutions that are isometric to each other. We have, however, to restrict this isometry to the component of unity of the diffeomorphism group. In particular, it has to preserve all orientations. Thus, two data from different trajectories can still evolve to isometric spacetimes, but they must then have different orientations.

For our models, only time orientation exists. We observe that the map $\mathbf{T}(q^i,p_i)=(q^i,-p_i)$ is antisymplectic, takes $\mathcal{C}^+$ into $\mathcal{C}^-$, and vice versa. $\mathbf{T}$ coincides with the change of initial data that is brought about by the time reversal in the solution spacetimes. The time reversal maps, e.g., an expanding spacetime onto a contracting one. $\mathbf{T}$ has a well-defined projection $\mathbf{T}_p$ to the physical phase space. $\mathbf{T}_p$ is trivial at $\mathcal{S}$. The reason is that the two possible time orientations of a static spacetime cannot be distinguished by their ADM data. We have, therefore, some motivation to consider all points of $\mathcal{C}^+$ (noncontracting spacetimes) as physically different from all points of $\mathcal{C}^-$ (nonexpanding spacetimes). This will remove the bifurcation an will leave us just with the conical singularities.

B. Physical phase space

We first construct the two components $\Gamma_\pm$ of the truncated physical phase space corresponding to $\mathcal{C}^\pm\backslash\mathcal{S}$. The truncated space is defined as the quotient manifold $\Gamma_\pm:=(\mathcal{C}^\pm\backslash\mathcal{S})$/trajectories. Let us calculate the trajectories.

To integrate the canonical equations that are implied by the action (40), we choose a particular gauge. This gauge will be useful for other aims, too. The value of the associated lapse function is

$$N=e^{q^3}. \tag{42}$$

The corresponding time coordinate T has the following relation to the proper time τ along the solution spacetimes:

$$e^{q^3}dT=d\tau. \tag{43}$$

The canonical equations for the Hamiltonian,

$$P=N\mathcal{H}=\frac{1}{2}(p_3^2-p_1^2-e^{-2q^1}p_2^2), \tag{44}$$

can be written in the following form:

$$\dot{q}^1=-p_1, \quad \dot{p}_1=-e^{-2q^1}p_2^2, \tag{45}$$

$$\dot{q}^2=-e^{-2q^1}p_2, \quad \dot{p}_2=0, \tag{46}$$

$$\dot{q}^3=p_3, \quad \dot{p}_3=0. \tag{47}$$

At the constraint surface, $P=0$, but P is an integral of these equations everywhere in $\mathcal{P}$. Second Eq. (47) implies then that

$$K := \sqrt{p_1^2 + e^{-2q^1} p_2^2} \tag{48}$$

is also an integral. A straightforward but lengthy calculation gives a general solution to Eqs. (45)–(47) everywhere in $\mathcal{P}$:

$$q^1 = q_0^1 + \ln\left(\frac{K - p_1^0}{2K} e^{KT} + \frac{K + p_1^0}{2K} e^{-KT} \right), \tag{49}$$

$$p_1 = -K \frac{(K - p_1^0) e^{KT} - (K + p_1^0) e^{-KT}}{(K - p_1^0) e^{KT} + (K + p_1^0) e^{-KT}}, \tag{50}$$

$$q^2 = q_0^2 - e^{-2q^1} p_2^0 \frac{e^{KT} - e^{-KT}}{(K - p_1^0) e^{KT} + (K + p_1^0) e^{-KT}}, \tag{51}$$

$$p_2 = p_2^0, \tag{52}$$

$$q^3 = p_3^0 T + q_0^3, \tag{53}$$

$$p_3 = p_3^0; \tag{54}$$

this solution runs through the point (q_0^i, p_i^0) for $T = 0$. At $\mathcal{C}^{\pm}$, we have

$$p_3^0 = \pm K_0,$$

where

$$K_0 := \sqrt{(p_1^0)^2 + e^{-2q_0^1} (p_2^0)^2}.$$

The subset $\mathcal{S}^{\pm}$ of $\mathcal{C}^{\pm}$ is defined by $K = 0$. Then Eqs. (49)–(54) become

$$q^i = q_0^i, \quad p_i = p_i^0.$$

The range of the time coordinate T is $(-\infty, \infty)$. Equations (43) and (53) show, however, that $q^3 \to -\infty$ is a singularity: it is reached in a finite proper time. We obtain easily, from Eq. (43),

$$\tau_{\text{sing}} - \tau_0 = -\frac{e^{q^3}}{p_3},$$

where τ_0 is the value of proper time at the point $T = 0$.

An important property of Eqs. (44)–(47) is the so-called *linearization instability* [19,20] at the points of $\mathcal{S}$, where $p_1 = p_2 = p_3 = 0$. If we expand these equations around the static solutions, the constraint

$$p_3^2 - p_1^2 - e^{-2q^1} p_2^2 = 0$$

becomes trivial, $0 = 0$, in the first order. The first nontrivial contribution to it is the second-order one:

$$(\delta_1 p_3)^2 - (\delta_1 p_1)^2 - e^{-2q_0^1} (\delta_1 p_2)^2 = 0.$$

This equation does not, however, contain any second-order correction $\delta_2 q^i$ and $\delta_2 p_i$. It is a second-order condition for the first-order corrections $\delta_1 q^i$ and $\delta_1 p_i$. Thus, some solutions of the first-order equations (''linearized equations'') are spurious.

In the set $\mathcal{C}^{\pm} \backslash \mathcal{S}$, the integral K is positive. Equation (53) then implies that q^3 is a strictly increasing function of T on $\mathcal{C}^{+} \backslash \mathcal{S}$ and well defined for $T \in (-\infty, \infty)$. The range of the function is again $(-\infty, \infty)$. Similarly, q^3 is strictly decreasing on $\mathcal{C}^{-} \backslash \mathcal{S}$. Hence, for both cases, the surface $\mathcal{T}^{\pm}$ defined by $q^3 = q_0^3$ intersect *each* trajectory exactly *once*, in a *transversal* direction. It is, therefore, a transversal surface for any value of $q_0^3 \in (-\infty, \infty)$. The transversal surface can be described as the following embedding of the manifold $\mathbf{R}^2 \times (\mathbf{R}^2 \backslash \{0\})$ with coordinates x^1, x^2, y_1, and y_2, where $(x^1, x^2) \in \mathbf{R}^2$ and $(y^1, y^2) \in \mathbf{R}^2 \backslash \{0\}$, into $\mathcal{C}_{\pm}$:

$$q^1 = x^1, \quad q^2 = x^2, \quad p_1 = y_1, \quad p_2 = y_2, \tag{55}$$

$$q^3 = q_0^3, \quad p_3 = \pm \sqrt{y_1^2 + e^{-2x^1} y_2^2}. \tag{56}$$

The pullback $\omega_{\pm}$ of the presymplectic form $\Omega_{\mathcal{C}}$ to $\Gamma_{\pm}$,

$$\omega_{\pm} = dy_1 \wedge dx^1 + dy_2 \wedge dx^2, \tag{57}$$

is nondegenerate.

Let us consider two such sections $\mathcal{T}_1^{\pm}$ and $\mathcal{T}_2^{\pm}$, defined by $q^3 = q_1^3$ and $q^3 = q_2^3$, respectively. Let the coordinates on these sections analogous to those defined by Eqs. (55) and (56) be x_1^1, x_1^2, y_1^1, y_2^1, and x_2^1, x_2^2, y_1^2, y_2^2, respectively. Then at each point of $\mathcal{T}_1^{\pm}$ a unique trajectory starts and it intersects $\mathcal{T}_2^{\pm}$. This defines a point of $\mathcal{T}_2^{\pm}$ for each point of $\mathcal{T}_1^{\pm}$, and we obtain a map $\phi^{\pm}_{q_1^3 q_2^3}$ between $\mathcal{T}_1^{\pm}$ and $\mathcal{T}_2^{\pm}$. We easily find the map from the solution (49)–(54) in terms of coordinates:

$$x_2^1 = x_1^1 + \ln\left(\frac{K_1 - y_1^1}{2K_1} e^{\pm \Delta q^3} + \frac{K_1 + y_1^1}{2K_1} e^{\mp \Delta q^3} \right), \tag{58}$$

$$y_1^2 = -K_1 \frac{(K_1 - y_1^1) e^{\pm \Delta q^3} - (K_1 + y_1^1) e^{\mp \Delta q^3}}{(K_1 - y_1^1) e^{\pm \Delta q^3} + (K_1 + y_1^1) e^{\mp \Delta q^3}}, \tag{59}$$

$$x_2^2 = x_1^2 - e^{-2q_1^1} y_2^1 \frac{e^{\pm \Delta q^3} - e^{\mp \Delta q^3}}{(K_1 - y_1^1) e^{\pm \Delta q^3} + (K_1 + y_1^1) e^{\mp \Delta q^3}}, \tag{60}$$

$$y_2^2 = y_2^1, \tag{61}$$

where $\Delta q^3 := q_2^3 - q_1^3$ and $K_1 := \sqrt{(y_1^1)^2 + e^{-2x_1^1} (y_2^1)^2}$. The map $\phi^{\pm}_{q_1^3 q_2^3}$ is, of course, a symplectic diffeomorphism (as one can also verify by a direct calculation). The truncated physical phase space $\Gamma_+ \cup \Gamma_-$ can be considered as the set of all transversal surfaces, all points of which are identified by the maps analogous to $\phi^{\pm}_{q_1^3 q_2^3}$.

Our next aim is to extend the Kuchař description to the solution spacetimes with additional symmetry. The corresponding separation between the physical and the gauge degrees of freedom requires a well-defined physical phase space as a necessary ingredient. Our first step must, therefore, be an extension of the truncated spaces Γ_+ and Γ_- to the points of $\mathcal{S}^+$ and $\mathcal{S}^-$.

Let us consider the quotient *sets* $\mathcal{C}^{\pm}$/trajectories and denote the corresponding projection maps by $\pi_{\pm}$. The quotient topology is defined as the finest one on $\mathcal{C}^{\pm}$/trajectories that makes $\pi_{\pm}$ continuous. Hence, $\pi_{\pm}$ is open. Let us denote the resulting topological spaces by $\bar{\Gamma}_{\pm}$. They are paracompact, locally compact, but not Hausdorff. The real problem is, however, that the topological spaces $\bar{\Gamma}_{\pm}$ do not carry any natural differential structure.

It is, however, possible to introduce a differentiable structure on $\bar{\Gamma}_{\pm}$ that is inherited directly from $\mathcal{P}$. An atlas for $\bar{\Gamma}_{\pm}$ can be defined by transversal surfaces as follows. Let us extend each transversal surface $\mathcal{T}_{\kappa}^{\pm}$, defined by $q^3=\kappa$, to $\bar{\mathcal{T}}_{\kappa}^{\pm}$ by adding all points of $\mathcal{S}^{\pm}$ that satisfy the same equation. This is the two-dimensional subset $(q^1,q^2)\in\mathbf{R}^2$, $q^3=\kappa$. The coordinates on $\bar{\mathcal{T}}_{\kappa}^{\pm}$ can be chosen as $(x^1,x^2,y_1,y_2)\in\mathbf{R}^4$ and the embedding formulas coincide with Eqs. (55) and (56). The sets $\bar{\mathcal{T}}_{\kappa}^{\pm}$ are topological submanifolds of $\mathcal{C}_{\pm}$ and *differentiable submanifolds* of $\mathcal{P}$. The symplectic form $\omega_{\kappa}^{\pm}$ given by Eq. (57) is uniquely extensible to $\bar{\mathcal{T}}_{\kappa}^{\pm}$ by continuity.

For a given κ, each point of $\bar{\mathcal{T}}_{\kappa}^{\pm}$ represents a unique trajectory, but all points of $\bar{\mathcal{T}}_{\kappa}^{\pm}$ do not represent all trajectories. Those points of $\mathcal{S}^{\pm}$ that do not satisfy the equation $q^3=\kappa$ are trajectories that do not intersect $\bar{\mathcal{T}}_{\kappa}^{\pm}$. Hence, to represent all trajectories, we need *all* transversal surfaces, $\kappa\in(-\infty,\infty)$. The points of $\bar{\mathcal{T}}_{\kappa}^{\pm}$ that do not lie in $\mathcal{S}^{\pm}$ represent trajectories that intersect all other transversal surfaces. Hence, to represent each trajectory by just one point, we have to identify the transversal surfaces by the maps $\phi_{\kappa_1\kappa_2}^{\pm}$. The surfaces $\bar{\mathcal{T}}_{\kappa}^{\pm}$ for all $\kappa\in(-\infty,\infty)$, together with the maps $\phi_{\kappa_1\kappa_2}^{\pm}$ form the desired atlas, which we denote by $\mathcal{A}$.

As it is usual for manifolds, its topology can be defined by a basis that is a union of the bases for each chart. Thus, the atlas $\mathcal{A}$ also defines a topology on $\bar{\Gamma}_{\pm}$—let us call it the $\mathcal{A}$ topology. However, the $\mathcal{A}$ topology and the quotient one do not coincide. For example, the $\mathcal{A}$ topology is not *paracompact*. To cover $\bar{\Gamma}_{\pm}$, we need an uncountable set of charts. Then the basis of the topology of $\bar{\Gamma}_{\pm}$ is not countable and $\bar{\Gamma}_{\pm}$ is not paracompact. The $\mathcal{A}$ topology also fails to describe the ''nearness'' between different κ levels of $\mathcal{S}^{\pm}$ properly. Indeed, each κ level of $\mathcal{S}^{\pm}$ is contained in a different chart, $\bar{\mathcal{T}}_{\kappa}^{\pm}$. Therefore, $\mathcal{S}^{\pm}\cap\bar{\mathcal{T}}_{\kappa}^{\pm}$ is contained in an open set that does not intersect any other κ level. Then, a sequence of points of $\mathcal{S}^{\pm}$ that do not lie in $\mathcal{S}^{\pm}\cap\bar{\mathcal{T}}_{\kappa}^{\pm}$ can never converge to any point of $\mathcal{S}^{\pm}\cap\bar{\mathcal{T}}_{\kappa}^{\pm}$ in the $\mathcal{A}$ topology.

To see that $\bar{\Gamma}_{\pm}$ is not Hausdorff (in the $\mathcal{A}$ topology as well as in the quotient one), let us consider two transversal surfaces $\bar{\mathcal{T}}_1^{\pm}$ and $\bar{\mathcal{T}}_2^{\pm}$ defined by $q^3=q_1^3$ and $q^3=q_2^3$, respectively. Let $\{Q_{1n}^{\pm}\}\subset\bar{\mathcal{T}}_1^{\pm}$ be a point sequence in $\bar{\mathcal{T}}_1^{\pm}$ with coordinates

$$Q_{1n}^{\pm}:=(x_1^1,x_1^2,y_{1n}^1,y_{2n}^1).$$

Let $y_{1n}^1\neq 0$ and $y_{2n}^1\neq 0$ for all positive integers n and

$$\lim_{n\to\infty} y_{1n}^1=0, \quad \lim_{n\to\infty} y_{2n}^1=0.$$

This sequence converges in $\bar{\mathcal{T}}_1^{\pm}$ to the point

$$Q_1^{\pm}=(x_1^1,x_1^2,0,0)\in\mathcal{S}_{\pm}.$$

All points of the sequence lie outside $\mathcal{S}^{\pm}$ and so can be identified with points $Q_{2n}^{\pm}$ of $\bar{\mathcal{T}}_2^{\pm}$ that are determined by Eqs. (58)–(61). Their coordinates are

$$x_{2n}^1=x_1^1+\ln(\cosh\Delta q^3-\sin\alpha_n\sinh\Delta q^3),$$

$$x_{2n}^2=x_1^2-\frac{\cos\alpha_n\sinh\Delta q^3}{\cosh\Delta q^3-\sin\alpha_n\sinh\Delta q^3},$$

$$y_{1n}^2=\pm K_n^1\frac{-\sinh\Delta q^3+\sin\alpha_n\cosh\Delta q^3}{\cosh\Delta q^3-\sin\alpha_n\sinh\Delta q^3},$$

$$y_{2n}^2=y_{2n}^1,$$

where α_n is defined by

$$\sin\alpha_n=\frac{y_{1n}^1}{K_n^1},\ \cos\alpha_n=\frac{y_{2n}^1 e^{-x_1^1}}{K_n^1},$$

and

$$K_n^1:=\sqrt{(y_{1n}^1)^2+e^{-2x_{1n}^1}(y_{2n}^1)^2}.$$

The sequence $\{Q_{2n}^{\pm}\}$ converges to the point $Q_{2\alpha}^{\pm}\in\mathcal{S}^{\pm}$ that is given by $(x_2^1,x_2^2,0,0)$ if and only if $\lim_{n\to\infty}\alpha_n=\alpha$ exists. Then

$$x_2^1=x_1^1+\ln(\cosh\Delta q^3-\sin\alpha\sinh\Delta q^3), \tag{62}$$

$$x_{2n}^2=x_1^2-\frac{\cos\alpha\sinh\Delta q^3}{\cosh\Delta q^3-\sin\alpha\sinh\Delta q^3}. \tag{63}$$

Each sequence $\{Q_{2n}^{\pm}\}$ with a converging α_n has then a unique limit in $\bar{\mathcal{T}}_2^{\pm}$; the possible limit points of all converging sequences fill out a closed curve in $\mathcal{S}^{\pm}\cap\bar{\mathcal{T}}_2^{\pm}$ defined by Eqs. (62) and (63) for $\alpha\in S^1$. Each point $Q_{2\alpha}^{\pm}$ is different from $Q_1^{\pm}$ because its coordinates q^3 in $\mathcal{C}^{\pm}$ differ. Thus, one sequence can have two different limits and the space cannot be Hausdorff.

An important property of the atlas can easily be shown: it is not unique. Indeed, we could slightly deform the transversal surfaces $\bar{\mathcal{T}}_{\kappa}^{\pm}$ so that they remain transversal in $\mathcal{C}^{\pm}\backslash\mathcal{S}^{\pm}$ and so that their intersections with $\mathcal{S}^{\pm}$ remain two dimensional. Let the new transversal surfaces be defined by some equation of the form $f(q^1,q^2,q^3,p_1,p_2)=\kappa$, $\kappa\in(-\infty,\infty)$. The intersection with $\mathcal{S}^{\pm}$ is given by

$$f(q^1,q^2,q^3,0,0)=\kappa$$

and it will generically intersect the surface $q^3=\kappa$ in a curve

$$f(q^1,q^2,\kappa,0,0)=\kappa.$$

Thus, the intersection between some transversal surface of the first family with some transversal surface of the second family will *not* be open. It follows that we cannot simply add the new family of transversal surfaces to the old atlas, and so the manifold defined by each of the two atlases will be different.

Any of these atlases can serve as a basis for a construction of Kuchař decomposition. This will be shown in the next section.

C. Transformation to embedding variables

To construct the transformation we shall use the P-way as described in Sec. II. The Hamiltonian that we choose corresponds to the value of the lapse function (42). The Hamiltonian itself has the form (44), the canonical equations are (45)–(47), and the general solution to these equations is given by Eqs. (49)–(54).

The next step is a choice of transversal surface in $\mathcal{P}$. Our choice is a straightforward extension of the transversal surfaces $\overline{\mathcal{T}}_\kappa^\pm$ of Sec. III B to $\mathcal{P}$ by the equation $q^3=\kappa$. Let us denote the result by $\tilde{\mathcal{T}}_\kappa^\pm$. It is transversal everywhere in $\mathcal{P}\backslash\mathcal{C}$ as long as $p_3\neq 0$ because of Eq. (53).

Equations (44) and (48) imply that

$$p_3^0=\pm\sqrt{2P+K^2}.$$

The function p_3^0 remains nonzero in the part of $\mathcal{P}$ that is determined by the following inequality:

$$P>-\frac{1}{2}K^2. \tag{64}$$

The trajectories lying at its boundary have T-independent surface area e^{q^3}, but they are not static if $P\neq 0$: q^1 and q^2 are evolving in a nontrivial way. At $\mathcal{C}^\pm$, where $P=0$ (and so $K=0$ at the boundary), there are no problems because the corresponding trajectories are just points. However, the trajectories at $P<0$, $K=\sqrt{-2P}$ are not points and they are lying in the surfaces $q^3=$const. Thus, $\tilde{\mathcal{T}}_\kappa^\pm$ ceases to be transversal at this boundary.

It is helpful to realize that $\mathcal{C}$ divides $\mathcal{P}$ into three disjoint parts similarly as the light cone divides Minkowski spacetime into the future interior, the past interior, and the exterior of the light cone. Thus, we have $\mathcal{P}^+$ defined by $P>0$ and $p_3>0$, $\mathcal{P}^-$ by $P>0$ and $p_3<0$ and $\mathcal{P}^0$ by $P<0$. The surface $p_3=0$ separates $\mathcal{P}$ into two halves, one with $p_3>0$ and one with $p_3<0$. The transversal surfaces $\tilde{\mathcal{T}}_\kappa^+$ cover $\mathcal{P}^+$, $\mathcal{C}^+$, and the $p_3>0$ part of $\mathcal{P}^0$. Let us denote this set by $\tilde{\mathcal{P}}^+$. Similarly, the surfaces $\tilde{\mathcal{T}}_\kappa^-$ cover $\mathcal{P}^-$, $\mathcal{C}^-$, and the $p_3<0$ part of $\mathcal{P}^0$; this set will be denoted by $\tilde{\mathcal{P}}^-$. Observe that $\tilde{\mathcal{P}}^\pm$ is not an open subset of $\mathcal{P}$; it has the boundary $\mathcal{S}^\pm$. At all points of $\mathcal{C}^\pm\backslash\mathcal{S}^\pm$, the surfaces $\tilde{\mathcal{T}}_\kappa^\pm$ are well defined at both sides of $\mathcal{C}^\pm$. At $\mathcal{S}^\pm$, they are defined only in the $\mathcal{P}^\pm$ side of the surface $\mathcal{C}^\pm$.

For each κ, the solution (49)–(54) with $q_0^3=\kappa$ and $p_3^0=\pm\sqrt{2P+K^2}$ covers a certain part $\tilde{\mathcal{P}}_\kappa^\pm$ of $\tilde{\mathcal{P}}^\pm$. We are going to use Eqs. (49)–(54) to define maps from $\tilde{\mathcal{T}}_\kappa^\pm\times\mathbf{R}$ into $\tilde{\mathcal{P}}_\kappa^\pm$ that we call $\psi_\kappa^\pm$. Let the coordinates on $\tilde{\mathcal{T}}_\kappa^\pm$ be x^1, x^2, y_1, y_2, P and that on $\mathbf{R}$ be T. Equations (49)–(54) have to be rewritten, to define $\psi_\kappa^\pm$, in such a way that $q_0^3=\kappa$, $q_0^1=x^1$, $q_0^2=x^2$, $p_1^0=y_1$ and $p_2^0=y_2$:

$$q^1=x^1+\ln\left(\frac{Y-y_1}{2Y}e^{YT}+\frac{Y+y_1}{2Y}e^{-YT}\right), \tag{65}$$

$$p_1=-Y\frac{(Y-y_1)e^{YT}-(Y+y_1)e^{-YT}}{(Y-y_1)e^{YT}+(Y+y_1)e^{-YT}}, \tag{66}$$

$$q^2=x^2-e^{-2x^1}y_2\frac{e^{YT}-e^{-YT}}{(Y-y_1)e^{YT}+(Y+y_1)e^{-YT}}, \tag{67}$$

$$p_2=y_2, \tag{68}$$

$$q^3=\pm T\sqrt{2P+Y^2}+\kappa, \tag{69}$$

$$p_3=\pm\sqrt{2P+Y^2}, \tag{70}$$

where

$$Y=\sqrt{y_1^2+e^{-2x^1}y_2^2}. \tag{71}$$

The map $\psi_\kappa^\pm$ is invertible on $\tilde{\mathcal{P}}_\kappa^\pm\backslash\mathcal{S}_\kappa^\pm$ where $p_3\neq 0$. The inverse transformation is described by the following equations:

$$T=\frac{q^3-\kappa}{p_3}, \tag{72}$$

$$P=\frac{1}{2}(p_3^2-p_1^2-e^{-2q^1}p_2^2), \tag{73}$$

$$x^1=q^1+\ln\frac{(K+p_1)e^{KT}+(K-p_1)e^{-KT}}{2K}, \tag{74}$$

$$y_1=K\frac{(K+p_1)e^{KT}-(K-p_1)e^{-KT}}{(K+p_1)e^{KT}+(K-p_1)e^{-KT}}, \tag{75}$$

$$x^2=q^2-e^{-2q^1}p_2\frac{e^{KT}-e^{-KT}}{(K+p_1)e^{KT}+(K-p_1)e^{-KT}}, \tag{76}$$

$$y_2=p_2, \tag{77}$$

where K is defined by Eq. (48) and the substitution (72) is to be made for T in Eqs. (74)–(76).

The functions T and P given by Eqs. (72) and (73) are singular at $\mathcal{S}^{\pm}$ where $p_3=p_1=p_2=0$:

$$dT=\frac{1}{p_3}dq^3-\frac{q^3-\kappa}{p_3}dp_3,$$

$$dP=e^{-2q^1}p_2^2dq^1-p_1dp_1-e^{-2q^1}p_2dp_2+p_3dp_3.$$

dT diverges and dP goes to zero. Still, the pullback Ω_κ of the symplectic form Ω by $\psi_\kappa^{\pm}$ from $\mathcal{P}_\kappa^{\pm}\backslash\mathcal{S}_\kappa^{\pm}$ remains regular at these points. An easy calculation reveals that

$$\Omega_\kappa=dP\wedge dT+dy_1\wedge dx^1+dy_2\wedge dx^2.$$

Hence, there is a trivial extension of Ω_κ to the whole of $\tilde{\mathcal{T}}_\kappa^{\pm}\times\mathbf{R}$. This seems to be a general pattern that may hold for all conical singularities.

The range of $\psi_\kappa^{\pm}$ is not the whole of $\tilde{\mathcal{P}}^{\pm}$: it contains all trajectories of $\tilde{\mathcal{P}}^{\pm}\backslash\mathcal{S}^{\pm}$, but it does not contain the point trajectories of $\mathcal{S}^{\pm}$ that do not satisfy the equation $q^3=\kappa$. To cover the whole of $\tilde{\mathcal{P}}^{\pm}$, we have to use $\psi_\kappa^{\pm}$ for all $\kappa\in(-\infty,\infty)$.

Let, for two different κ's, κ_1 and κ_2, the corresponding maps be $\psi_1^{\pm}$ and $\psi_2^{\pm}$, and let their domains have coordinates $(x_1^1,x_1^2,y_1^1,y_2^1,P_1,T^1)$ and $(x_2^1,x_2^2,y_1^2,y_2^2,P_2,T^2)$, respectively. The maps $\psi_1^{\pm}$ and $\psi_2^{\pm}$ are invertible and C^∞ where their ranges overlap, so they define a map $(\psi_2^{\pm})^{-1}\circ\psi_1^{\pm}$ on $(\psi_1^{\pm})^{-1}(\mathcal{P}_1^{\pm}\cap\mathcal{P}_2^{\pm})$. This map can be explicitly calculated from Eqs. (65)–(70) and (72)–(77); it turns out to be a C^∞ symplectomorphism. Hence, the manifold $\tilde{\mathcal{P}}_K^{\pm}$ that results by pasting together all charts $\mathcal{T}_\kappa^{\pm}\times\mathbf{R}$ by these maps is a C^∞ symplectic manifold.

$\tilde{\mathcal{P}}_K^{\pm}$ is Hausdorff, as any sequence that converges to some point of $\tilde{\mathcal{T}}_1^{\pm}$ in the chart corresponding to κ_1 will diverge in the chart corresponding to $\kappa\neq\kappa_1$, say, $\kappa=\kappa_2$. This can be seen from the relation between T_2 and T_1 obtained from Eqs. (72) and (69):

$$T_2=T_1\mp\frac{\kappa_2-\kappa_1}{\sqrt{2P_1+Y_1^2}}.$$

Observe that $P_1=P_2$ and so the constraint set $\mathcal{C}_K^{\pm}$ in $\tilde{\mathcal{P}}_K^{\pm}$ can be described by the single equation

$$P=0.$$

It is a smooth (Hausdorff) submanifold of $\tilde{\mathcal{P}}_K^{\pm}$.

$\mathcal{C}_K^{\pm}$ can be considered as a fiber bundle with the basis $\tilde{\Gamma}^{\pm}$ and fiber $\mathbf{R}$. It is defined by the trivializations $\tilde{\mathcal{T}}_\kappa^{\pm}\times\mathbf{R}$ and by pasting maps of the type $(\psi_2^{\pm})^{-1}\circ\psi_1^{\pm}$ restricted to $P=0$.

In this way, we have constructed a kind of Kuchař description for the torus sector that includes the static tori. The construction is not unique, and the result is also somewhat strange. The origin of the problem is in the pathological structure of the physical phase space, which is shared with the ADM description.

The Kuchař charts cover only the part $\tilde{\mathcal{P}}^{+}\cup\tilde{\mathcal{P}}^{-}$ of the original phase space $\mathcal{P}$; the points of $\mathcal{S}$ are covered two times. They have to satisfy the inequalities

$$T\in(-\infty,\infty),\quad (x^1,x^2,y_1,y_2)\in\mathbf{R}^4$$

and

$$P\in\left(-\frac{1}{2}(y_1^2+e^{-2x^1}y_2^2),\infty\right)\cup\{0\}$$

in each chart.

The new dynamical equations for the variables T, P, x^1, x^2, y_1, and y_2 in each chart are very simple:

$$\dot{x}^1=\dot{x}^2=\dot{y}_1=\dot{y}_1=\dot{P}=0$$

and

$$\dot{T}=N',\quad P=0,$$

where N' is the new Lagrange multiplier that enforces the new constraint. These equations are manifestly linearization stable (because they are linear). The deeper reason for the stability is the absence of conical singularities in the new description.

IV. CONCLUDING REMARKS

We have studied some global properties of the transformation from the ADM to the Kuchař description in two minisuperspace models. We have found in all cases that the Kuchař description is globally inequivalent to the ADM description. The solutions to the two corresponding sets of dynamical equations, however, always completely coincide.

The first interesting feature that we have met are the nontrivial boundaries for Kuchař variables. As yet, three different kinds of boundaries have been detected. First, there are boundaries due to singularities in solutions of Einstein's equations. It does not seem sensible to propose any general method of dealing with these singularities in the classical version of the theory. The hope is that the quantum theory will cure them in some way (for an example, see [27]). We just ignore these boundaries.

Second, we have found bounds for the variables conjugate to embeddings, in our case P, in all models. The meaning of the bounds is simply that the function P does not attain all values on the ADM phase space (Sec. II), or on a suitable part of it (Sec. III). Our standpoint here simply is that nothing seems to prevent an extension of Kuchař phase space to all values of P. The dynamics is not changed by this extension. This is the reason why we did not make any comment when the bounds became too narrow so that a part of the constraint surface appeared ''bare'' from one side (the set $\mathcal{S}^{\pm}$ of the torus sector of 2+1 gravity model). In fact, we have seen that most claims concerning the structure of the set $\mathcal{P}\backslash\mathcal{C}$ are either trivial or gauge dependent. It seems, therefore, that this structure is not relevant to physical properties of the system (although it can be used for some methodical purposes). This is one more reason to consider its extensions as harmless.

Finally, there is a boundary for the embedding variables due to non-space-like character of some embeddings. This type of boundary has not been encountered here, but it is quite analogous to the P boundary. It seems that an extension of the Kuchař description to all values of embeddings may again be harmless. Let us consider the extension $\overline{\mathcal{P}_K}$ of the Kuchař phase space that contains all embeddings of the space $\mathrm{Emb}(\Sigma,\mathcal{M})$ for each point of the physical phase space Γ and all values of the momenta from the space $T^*_X\,\mathrm{Emb}(\Sigma,\mathcal{M})$ for each pair of points of $\Gamma\times\mathrm{Emb}(\Sigma,\mathcal{M})$. The inequalities that both the embeddings X and their conjugate momenta P have to satisfy are then telling us that the ADM phase space $\mathcal{P}$ is a proper subset of $\overline{\mathcal{P}_K}$. Does it mean that the ADM phase space is too small, or that the Kuchař phase space is unnecessarily large?

There is one argument in favor of the first claim. Isham and Kuchař [7] have studied the action of the diffeomorphism group in a phase space of ADM type. They observed that, given any fixed diffeomorphism $\varphi\in\mathrm{Diff}\,\mathcal{M}$, there is a Cauchy surface Σ in any solution spacetime such that $\varphi(\Sigma)$ is not space like. If the ADM variables associated with the surface Σ are q_{kl} and π^{kl}, then the representative of φ acting on the phase space must map the point (q_{kl},π^{kl}) out of the phase space. They have concluded that only its Lie algebra but not the group $\mathrm{Diff}\,\mathcal{M}$ itself has a well-defined action on the phase space. This can be compared with the situation in the Yang-Mills field theory, where the full gauge group has a well-defined action on the phase space. We also easily recognize that $\mathrm{Diff}\,\mathcal{M}$ acts without hindrance on $T^*\,\mathrm{Emb}(\Sigma,\mathcal{M})$ and so it has a well-defined action on the extended Kuchař phase space $\overline{\mathcal{P}_K}$.

Other problems that we have studied are connected with the points that correspond to solutions of higher symmetry. For a model—the torus sector of 2+1 gravity—we have found a description by Kuchař variables including such points. The new description is smooth: there is no bifurcation, no conical singularity, and no linearization instability. It is not equivalent to the ADM description given in [18]. This point ought to be stressed: ''passing'' to the Kuchař description is *not* just a coordinate transformation on the phase space. A mere coordinate transformation could not remove bifurcation, conical singularity, or linearization instability.

The essence of the problem with the bifurcation, the conical singularity and the linearization instability in the case of the ADM description is that the fields q_{kl} and π^{kl} cannot distinguish between two Cauchy surfaces that are linked by an isometry. Two such Cauchy surfaces then define one and the same point in the ADM phase space. However, the two Cauchy surfaces can surely be distinguished from each other by an observer living in the corresponding solution spacetime. Hence, the ADM description is not true if there are *any* symmetries. This is to be compared with the description by the embedding variables, which is true, so to speak, by definition. The conical singularity and the linearization instability are consequences of this untrue description only if some additional conditions are satisfied. First, the solutions with the symmetry must form a subset of all solutions that also include solutions without the symmetry (the static tori are solutions as well as the expanding and contracting tori are). Second, the symmetry must be continuous (Killing vectors).

The bifurcation of the ADM constraint surface is caused by an additional discrete symmetry: the time reversal. The static spacetimes are invariant with respect to it; the expanding and contracting spacetimes are not. The ADM description identifies the two possible time orientations of the static spacetimes, but it always distinguishes the points corresponding to the contracting from those corresponding to the expanding spacetimes. In the phase space, the two surfaces, one for the noncontracting and the other for the nonexpanding spacetimes, are then identified at the points corresponding to the static spacetimes. In this way, the bifurcation of the constraint surface $\mathcal{C}$ within the ADM description comes about.

In the case of the torus model, we have also seen that there is no global gauge at the constraint surface $\mathcal{C}$. It may be possible to choose one smooth lapse function N with a domain that includes the whole of $\mathcal{C}$, but there is no global *transversal surface*. This leads to a nontrivial fiber-bundle structure for the constraint surface within the Kuchař description. Each transversal surface defines a trivialization of this bundle, but there is no global trivialization. Such a construction has been considered in [15]. In fact, even in the cases that admit a global gauge, the gauge is not unique. Thus, it is the bundle that represents the gauge invariant structure of the constraint surface in all cases. Although the bundle is trivial if a global gauge exists, it does not possess any *canonical* trivialization. This has been explained in [15].

The present paper focuses on the transformation between the ADM and the Kuchař descriptions. This necessarily leads to a comparison of just these two. One should not, however, forget that there are many other descriptions. In this respect, it may be interesting to observe that the problem with additional symmetry is not characteristic for the ADM approach only but it also afflicts the configuration space of the (usual) second order (Lagrangian) approach.

Our results are of course only valid for the two particular models. Of these, the torus may be the most pathological case that exists. The Kuchař description of less pathological models with additional symmetry may, therefore, be regular. If not and if the residual pathology is very disturbing, one can still truncate the model. More cases ought to be looked at, and some attempts at proofs of some general theorems ought to be done. This is left for future papers.

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[1] R. Arnowitt, S. Deser and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
[2] R. F. Baierlein, D. H. Sharp, and J. A. Wheeler, Phys. Rev. **126**, 1864 (1962).
[3] K. V. Kuchař, J. Math. Phys. **13**, 768 (1972).
[4] K. V. Kuchař, in *Highlights in Gravitation and Cosmology*, edited by B. R. Iyer *et al.* (Cambridge University Press, Cambridge, England, 1988).
[5] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).
[6] A. E. Fischer and V. Moncrief, Nucl. Phys. B (Proc. Suppl.) **88**, 83 (2000).
[7] C. J. Isham and K. V. Kuchař, Ann. Phys. (N.Y.) **164**, 288 (1985); **164**, 316 (1985).
[8] K. V. Kuchař, Phys. Rev. D **4**, 955 (1971).
[9] K. V. Kuchař, Phys. Rev. D **50**, 3961 (1994).
[10] J. D. Brown and K. V. Kuchař, Phys. Rev. D **51**, 5600 (1995).
[11] K. V. Kuchař, J. D. Romano, and M. Varadarajan, Phys. Rev. D **55**, 795 (1997).
[12] P. Hájíček, J. Math. Phys. **30**, 2488 (1989).
[13] M. Schön and P. Hájíček, Class. Quantum Grav. **7**, 861 (1990).
[14] C. G. Torre, Phys. Rev. D **46**, R3231 (1992).
[15] P. Hájíček and J. Kijowski, Phys. Rev. D **61**, 024037 (2000).
[16] P. Hájíček and C. Kiefer, ''Embedding Variables in the Canonical Theory of Gravitating Shells,'' hep-th/007004.
[17] C. Kiefer, Nucl. Phys. **B341**, 273 (1990).
[18] V. Moncrief, J. Math. Phys. **30**, 2907 (1989); **31**, 2978 (1990).
[19] J. Marsden, *Lectures on Geometric Methods in Mathematical Physics*, CMBS Vol. 37 (SIAM, Philadelphia, 1981).
[20] V. Moncrief, J. Math. Phys. **16**, 493 (1975).
[21] V. Moncrief, J. Math. Phys. **16**, 1556 (1975).
[22] P. Hájíček, J. Math. Phys. **39**, 4824 (1998).
[23] P. G. Bergmann and A. B. Komar, Int. J. Theor. Phys. **5**, 15 (1972).
[24] P. Hájíček, Nucl. Phys. B (Proc. Suppl.) **80**, (2000) (CD-ROM Suppl.).
[25] P. Hájíček and K. V. Kuchař, Phys. Rev. D **41**, 1091 (1990).
[26] A. E. Fischer and J. E. Marsden, *General Relativity. An Einstein Centenary Survey*, edited by by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[27] P. Hájíček, ''Unitary Dynamics of Spherical Null Gravitating Shells,'' hep-th/0007005.