Electromagnetic Thirring problems

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We consider systems of two concentric spherical shells—the interior carrying arbitrary charge q but no mass, the exterior carrying arbitrary mass M but no charge—and calculate the dragging (and partly antidragging) effects and the induced magnetic fields which are produced by (independent) rotations of these shells in first order. We compare with results from the literature which usually are based only on first order approximations in q and/or M, and we clear up a discrepancy between these results concerning their Machian interpretation. We examine some new interaction effects between strong electric and gravitational fields, and we study especially the collapse limit of this rotating two-shell system.

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I. INTRODUCTION

The standard Thirring problem describes the (nonlocal) influence of rotating masses on the inertial properties of space-time, especially the so-called dragging of inertial frames inside a rotating mass shell relative to the asymptotic frames. It is then a natural question to ask whether and how properties other than inertial ones are also influenced by rotating masses, and the first (noninertial and nongravitational) properties which come here to one's mind are surely electromagnetic phenomena. Since the coupled Einstein-Maxwell equations are structurally not much more complicated than the pure Einstein equations, one can also expect that an extension of the Thirring problem to electromagnetic phenomena is technically manageable. Indeed, in the period 1962—1971 some authors considered the influence of a rotating spherical mass shell on charges in its interior, where these charges usually are distributed also on a shell concentric to the mass shell. The phrase ''electromagnetic Thirring problem'' was coined by Ehlers and Rindler [1,2]. We use this expression in our title in the plural because we consider a much more comprehensive class of such two-shell systems than Ehlers and Rindler and the other authors did.

To our knowledge, the first paper on this issue was by Hofmann [3] who considers a charged shell within a rotating mass shell in the first orders of mass M, charge q, and angular velocity ω, and gets a magnetic dipole field induced by the rotating mass shell. Cohen [4] considers a similar system exactly in M, but explicitly states that the interior charged shell should have negligible mass. He calculates and discusses especially the case where the mass shell approaches its collapse limit, and can then be considered as an idealized substitute for the overall masses in our universe. In this limit he gets the completely Machian result that ''one cannot distinguish (even with electromagnetic fields reaching beyond the mass shell) whether the charged shell is rotating or the mass shell is rotating in the opposite direction.'' Ehlers and Rindler [1,2] claim to consider a similar system exactly in the charge q, and in first order of the mass, equivalently in first order of the gravitational constant. In the detailed calculations and discussions, they restrict themselves, however, also to the first order in q. They interpret the resulting magnetic field to be ''in fact Mach-negative or, at best, Mach-neutral.''

We treat these electromagnetic Thirring problems in more detail, with the goal to resolve, e.g., the obvious discrepancy between the results and interpretations of Cohen and Ehlers and Rindler by examining a more comprehensive class of charged two-shell systems which comprises the models of Cohen and Ehlers and Rindler as special cases. We consider a class of models consisting of two spherical shells of radii a and $R \geqslant a$, the first one carrying a nearly arbitrary charge q but no rest mass, the second one being nearly arbitrary massive but electrically neutral. The only restriction on the parameters is that the systems should be free of singularities and horizons. To these shells we apply small but otherwise arbitrary ''stirring'' angular velocities ω^I and ω^{II}, respectively corresponding angular momenta J^I and J^{II}. More precisely, we have the picture of a two-parameter family of solutions of the full Einstein-Maxwell equations depending smoothly on ω^I and ω^{II} from whose power series expansion in ω^I and ω^{II} we keep only the first order terms. This picture seems to be justified by calculations for the standard Thirring problem (one rotating, uncharged mass shell), where it was shown [5–7] that this problem has a unique solution in any order ω^n if we allow for a (centrifugally) deformed and differentially rotating mass shell such that the space-time inside the mass shell stays flat. In analogy, for the electromagnetic Thirring problem flat space-time inside the charged shell should be realizable with appropriate mass- and charge-dependent ''deformations'' of both shells in each order of ω^I and ω^{II}.

In Sec. II we give the solutions for this model class in zeroth order of ω^I and ω^{II} (static two-shell models). Obviously, these consist of three pieces of the Reissner-Nordström solution for the exterior region, the region between both shells, and the region inside the charged shell. Since, however, a globally continuous metric is desirable for the interpretation of the global dragging effects, and was also used in the papers [1–4], we have to transform the Reissner-Nordström pieces (separately) to, e.g., isotropic coordinates. For a Machian interpretation of the dragging effects, the exterior mass shell is often seen as an idealized substitute for part of or all of the cosmic masses. For this interpretation to be valid, a minimal condition seems to be that the energy-momentum tensor of this mass shell satisfy the weak energy condition. This analysis, which for arbitrary mass M and charge q is highly nontrivial but physically interesting by itself, is also performed in Sec. II.

In Sec. III we solve the coupled Einstein-Maxwell equations in first order of an angular velocity ω representing ω^I and/or ω^{II}. Surprisingly, the relevant differential equations are exactly solvable, and this not only for our rotational perturbations of the Reissner-Nordström metric but for general stationary perturbations with arbitrary angular momentum ''quantum number'' l. We then calculate the solutions explicitly i.e., we determine all integration constants, separately for the case of a rotating exterior shell, and for a rotating charged interior shell. For the general case where both shells have nonzero angular momentum, the results for the dragging effect and for the induced magnetic fields are just the sums of the above results, due to the linearity of the perturbation analysis of first order in ω. The interesting relations between the angular momenta and the angular velocities of the two shells are also explicitly worked out.

In order to compare our results with the work of [1–4], we have to make the appropriate approximations concerning the parameters M and/or q. In Sec. IV A we consider the terms of first order in q, and find generally agreement with the results and physical interpretation of Cohen [4] but we give the results for general M values, and not only for the collapse limit, and we consider also the magnetic field in the exterior region which approaches the Kerr-Newman field (in first order of q and ω) in the collapse limit. Section IV B provides the results for the dragging function in second order of q. These are of interest under the aspect of the influence of the electrostatic energy density on the curvature of space-time, and they prepare for the explicit disproof (in Sec. IV C) of the claim by Ehlers and Rindler [2] that the Thirring dragging effect and the Reissner-Nordström description of the charge effects act additively in first order of M. Furthermore, we find that an angular momentum J^I of the charged shell can, due to its violation of the usual energy conditions, lead to an ''antidragging'' phenomenon. In Sec. IV C we perform the weak field limit (first order approximation in M) on top of the results of Secs. IV A and IV B. Mathematically we find agreement with the results of [2,3] but we hope to convince the reader that these results are in perfect agreement with Machian expectations and by no means ''Mach-negative or, at best, Mach-neutral'' [2]. Section IV D presents some results for values of M and q which are not necessarily small. Especially we examine the collapse limit of a massive, highly charged two-shell system and find, e.g., that the important result by Brill and Cohen [8] that in this limit we have complete dragging of the inertial frames inside the mass shell extends to the inertial frames inside the mass shell of an electromagnetic Thirring system. In the exterior region of the system we find, as expected, the Kerr-Newman field in first order of ω.

II. STATIC TWO-SHELL MODEL

According to a generalization [9] of the Birkhoff theorem, a spherically symmetric solution of the Einstein-Maxwell equations is automatically static and asymptotically flat, and can be represented by the Reissner-Nordström metric

$$ds^2=-F(\rho)d\tau^2+F(\rho)^{-1}d\rho^2+\rho^2 d\Omega^2, \tag{1}$$

with $F(\rho)=1-2\mathcal{M}/\rho+q^2/\rho^2$, and $d\Omega^2=d\vartheta^2+\sin^2\vartheta d\varphi^2$. Therefore, our model of two concentric, spherically symmetric, charged mass shells is simply given by three pieces of this Reissner-Nordström metric: one for the region outside the exterior shell, one for the region between both shells, and one for the interior region.

However, a matching of these Reissner-Nordström metrics with different mass and charge parameters obviously would not be continuous at the shell positions. A global continuous metric is, however, desirable for the physical interpretation of the dragging effects in the later sections, and was also used Refs. [2–4] and [8]. It can be reached by a transformation of the metric (1) to the isotropic form

$$ds^2=-e^{2U(r)}dt^2+e^{2V(r)}(dr^2+r^2d\Omega^2). \tag{2}$$

Identification of Eqs. (2) and (1) results in

$$r(\rho)=\frac{1}{2D}(\sqrt{\rho^2-2\mathcal{M}\rho+q^2}+\rho-\mathcal{M}), \tag{3}$$

with an arbitrary constant D. For $F(\rho)>0$, i.e., outside of horizons, $r(\rho)$ is real. We simplify the following calculations somewhat by using dimensionless variables:

$$\alpha=\frac{M}{2R},\quad \beta=\frac{a}{R}\leqslant 1,\quad \gamma=\frac{q}{2R},\quad x=\frac{r}{R}. \tag{4}$$

Here, M denotes the mass parameter $\mathcal{M}$ in the exterior region, and $r=a$ is the position of the inner shell. A scaling of all parameters and variables by the radius R of the exterior shell seems appropriate because R should definitely be nonzero in our models whereas we may consider the limits $M\rightarrow 0$, $a\rightarrow 0$, and $q\rightarrow 0$. In the exterior region $x\geqslant 1$ we set $D=D_1=1$ (then r and ρ coincide asymptotically), and identify t with τ. Then the potentials read

$$V_1(x)=\log\left[\frac{(x+\alpha)^2-\gamma^2}{x^2}\right],\quad U_1(x)=\log\left[\frac{x^2-\alpha^2+\gamma^2}{(x+\alpha)^2-\gamma^2}\right]. \tag{5}$$

In the region $a\leqslant r\leqslant R$ between both shells we set $\mathcal{M}=\hat{M}$, and $\hat{\alpha}=\hat{M}/2R$, parameters which will be fixed only later on. The charge parameter should have the same value q as in the exterior region because the shell at $r=R$ is uncharged. It turns out (in accordance with [10]) that a nontrivial time transformation $t=C_2\tau$ is necessary in this region in order to guarantee continuity of the potential $U(r)$ at $r=R$. Denoting the constant D in Eq. (3) now by D_2, the potentials V and U in the region $\beta\leqslant x\leqslant 1$ read:

$$V_2(x)=\log\left[\frac{(D_2x+\hat{\alpha})^2-\gamma^2}{D_2x^2}\right],$$

$$U_2(x)=\log\left[\frac{D_2^2x^2-\hat{\alpha}^2+\gamma^2}{C_2((D_2x+\hat{\alpha})^2-\gamma^2)}\right]. \tag{6}$$

Continuity of $V(x)$ at $x=1$ produces a quadratic equation for D_2, with the solution

$$2D_2=(1+\alpha)^2-2\hat{\alpha}-\gamma^2 + \sqrt{[(1+\alpha)^2-2\hat{\alpha}-\gamma^2]^2+4(\gamma^2-\hat{\alpha}^2)}. \tag{7}$$

(We omit the negative sign of the square root, since we wish to have the result $D_2=1$ in the limits $q\to 0$ and $M,\hat{M}\ll R$.) Continuity of $U(x)$ at $x=1$ produces

$$C_2=\frac{D_2^2-\hat{\alpha}^2+\gamma^2}{D_2(1-\alpha^2+\gamma^2)}. \tag{8}$$

Of course there are restrictions on the parameters α, $\hat{\alpha}$, and γ such that D_2 and C_2 are real and positive. We will analyze these conditions after fixing $\hat{\alpha}$.

In the interior region $0\leqslant r\leqslant a$, we have to set $\mathcal{M}=0$ and $q=0$ in order to guarantee regularity at the origin $\rho=0$. The interior metric is then automatically flat. The transformations $r=\rho/D_3, t=C_3\tau$ produce in the region $0\leqslant x\leqslant\beta$ the potentials

$$V_3(x)=\log D_3=\text{const},\quad U_3(x)=-\log C_3=\text{const}, \tag{9}$$

and the continuity of $V(x)$ and $U(x)$ at $x=\beta$ leads to

$$D_3=\frac{(D_2\beta+\hat{\alpha})^2-\gamma^2}{D_2\beta^2},\quad C_3=C_2\frac{(D_2\beta+\hat{\alpha})^2-\gamma^2}{D_2^2\beta^2-\hat{\alpha}^2+\gamma^2}. \tag{10}$$

We now specify the parameter $\hat{\alpha}$: In our models, the function of the inner shell is mainly to provide a charge, and not so much to provide additional mass. It seems therefore reasonable to simplify our models (in accordance with [1,3,4]) by setting the rest mass density of the inner shell to zero. If we write Einstein's field equations in the form

$$G^\mu_\nu=8\pi(T^\mu_\nu+S^\mu_\nu), \tag{11}$$

with S^μ_ν being the electromagnetic energy-momentum tensor, then T^μ_ν in our two-shell models consists of two parts $\tau^\mu_\nu(a)\delta(r-a)$ and $\tau^\mu_\nu(R)\delta(r-R)$. Since in our isotropic metric form (2) the potentials $V(r)$ and $U(r)$ are by construction continuous at both shell positions, the components τ^μ_ν are essentially determined by the discontinuities of the radial derivatives of V and U denoted by V' and U':

$$8\pi\tau^0_0(a)=2e^{-2V_3(a)}[V_2'(a)-V_3'(a)], \tag{12}$$

$$8\pi\tau^2_2(a)=8\pi\tau^3_3(a) =e^{-2V_3(a)}[V_2'(a)-V_3'(a)+U_2'(a)-U_3'(a)]. \tag{13}$$

(Here and in the following, sometimes the indices 0,1,2,3 are used to denote, respectively, the variables t,r,ϑ,φ.) Equivalent relations are valid at the position $r=R$. The condition $\tau^0_0(a)=0$ reads now, due to $V_3(x)=\text{const}$, $dV_2(x)/dx|_{x=\beta}=0$ or $D_2=(\gamma^2-\hat{\alpha}^2)/\hat{\alpha}\beta$. Comparison with Eq. (7) and introduction of the abbreviations

$$\delta=(1+\alpha)^2-\gamma^2,\quad \Delta_\pm=\frac{1}{2}(\sqrt{\beta^2\delta^2+4\gamma^2(1-\beta)^2}\pm\beta\delta) \tag{14}$$

results in

$$\hat{\alpha}=\Delta_-/(1-\beta)^2. \tag{15}$$

[The second solution $\hat{\alpha}=-\Delta_+/(1-\beta)^2$ of the quadratic equation for $\hat{\alpha}$ is negative, and therefore has to be excluded.] If **a** denotes the invariant radius of the charged shell (coinciding with the Reissner-Nordström coordinate, $\mathbf{a}=\rho(r=a)=R\Delta_+$), its mass energy is $\hat{M}=2R\hat{\alpha}=q^2/2\mathbf{a}$, which, for small α and γ, reaches the limit $\hat{M}\to q^2/2a$, i.e., the energy of a charged shell with radius a in classical electrostatics. The condition $\tau^0_0(a)=0$ has the additional simplifying consequence that $1-2\hat{M}/\rho+q^2/\rho^2=1$ for $\rho=\rho(a)$. With the expression (15) for $\hat{\alpha}$, the constants D_i and C_i read

$$D_2=\frac{(1-\beta)^2\Delta_+-\Delta_-}{\beta(1-\beta)^2},\quad D_3=\frac{\Delta_+}{\beta}, \tag{16}$$

$$C_2=C_3=\frac{(1-\beta)\Delta_+-(1+\beta)\Delta_-}{\beta(1-\beta)(1-\alpha^2+\gamma^2)}. \tag{17}$$

Where necessary, one can also express γ^2 and δ through Δ_+, Δ_-, and β: $\gamma^2=\Delta_+\Delta_-/(1-\beta)^2, \delta=(\Delta_+-\Delta_-)/\beta$. The singularities of the expressions (16), (17) for $\beta\to 1$ are only of a formal nature. The equality $C_2=C_3$ means that there is, due to $\tau^0_0(a)=0$, no time change between the intermediate and interior regions. An extension of our work to the case $\tau^0_0(a)\neq 0$ would be possible but it would be algebraically considerably more involved.

In order that the metric (2) with the potentials (5), (6), and (9) really describes the static two-shell models, which we have in mind, and does, e.g., not have any horizons, the model parameters α,β,γ have to satisfy some inequalities: In order that $V_1(x)$ and $U_1(x)$ be real for all $x\geqslant 1$, we have to have

$$|\gamma|-1\leqslant\alpha\leqslant\sqrt{1+\gamma^2}. \tag{18}$$

The constants D_i and C_i from formulas (16), (17) obviously are all real. However, in order that the charged shell be interior to the mass shell, that the origin $r=0$ be interior to the charged shell, and that time run in the same direction in all regions, all these constants have to be (at least) non-negative. Whereas D_3 is automatically non-negative, the conditions $D_2 \geqslant 0$ and $C_2 = C_3 \geqslant 0$ sharpen the inequalities (18) to

$$\sqrt{\gamma(\gamma+2-\beta)} - 1 \leqslant \alpha \leqslant \sqrt{1+\gamma^2}. \tag{19}$$

According to the formulas (12), (13) we get, for the nonzero components of the matter tensor densities τ^{μ}_{ν},

$$\tau^2_2(a) = \tau^3_3(a) = -\frac{\beta\hat{\alpha}}{4\pi R \Delta^3_+}, \tag{20}$$

$$\tau^0_0(R) = \frac{(1+\alpha)(1-\beta) + \Delta_+ - \delta}{2\pi R \delta^3 (1-\beta)}, \tag{21}$$

$$\tau^2_2(R) = \tau^3_3(R) = \frac{1}{4\pi R \delta^2 (1-\alpha^2+\gamma^2)} \left(1 - \frac{D_2}{C_2}\right), \tag{22}$$

with D_2 and C_2 from Eqs. (16), (17). In the uncharged case $\gamma = 0$ we recover, observing the different definitions of α and τ^{μ}_{ν}, the results of [5]. In Secs. III and IV we will analyze a first order rotation of our two-shell systems, and we will try to discuss the results for dragging and induction of a magnetic field under Machian aspects. In order that this can be successful, we have to make sure that the mass shell at $r = R$—notwithstanding its unrealistic shell structure—can mimic in some way the overall masses of the universe. A minimal condition for this obviously is the weak energy condition. [Since the charged shell at $r = a$ has a totally different function in our models, it should not matter too much that, as a result of $\tau^0_0(a) = 0$ and $\tau^2_2(a) = \tau^3_3(a) \leqslant 0$, this shell nearly violates all energy conditions.] The weak energy condition (see, e.g., [11]) consists of two parts which, for the mass shell at $r = R$ and in our metric (2), read

$$\tau^0_0(R) \leqslant 0 \quad \text{and} \quad \tau^3_3(R) - \tau^0_0(R) \geqslant 0. \tag{23}$$

In Sec. III we will see that at least in one model class the induced magnetic field inside the charged shell is directly proportional to $[\tau^3_3(R) - \tau^0_0(R)]$, and therefore the sign of this expression is central to the physical interpretation of this field. The detailed analysis of the inequalities (23), especially of the second one, turns out to be algebraically quite involved. The condition $\tau^0_0(R) \leqslant 0$ leads to a further sharpening of the lower limit for the mass parameter α in inequality (19):

$$\alpha \geqslant \frac{1}{2}(\sqrt{\beta^2 + 4\gamma^2} - \beta) \quad \text{or}$$

$$\beta \geqslant \frac{\gamma^2 - \alpha^2}{\alpha} = \frac{2\alpha + 1 - \delta}{\alpha} =: \beta_1(\delta, \alpha). \tag{24}$$

In the overextreme Reissner-Nordström case $\gamma^2 > \alpha^2$ the stresses $\tau^2_2(R) = \tau^3_3(R)$ in the mass shell can become negative, and then the condition $\tau^3_3(R) - \tau^0_0(R) \geqslant 0$ can lead to a further sharpening of inequality (24): Eliminating the square root coming from $\Delta_\pm$ in Eq. (14), $\tau^3_3(R) - \tau^0_0(R) \geqslant 0$ leads to a sixth-order polynomial inequality for δ which, however, has the form $\delta(\delta - \delta_1) P_4(\delta; \alpha, \beta) \geqslant 0$, with $\delta_1 = 1 + \alpha(2 - \beta)$, and P_4 a fourth-order polynomial in δ. Here $P_4 \geqslant 0$ is equivalent to $\beta \geqslant \beta_2(\delta, \tilde{\alpha})$, with $\tilde{\alpha} = \alpha + 1$, $\beta_2(\delta, \tilde{\alpha}) = 1 - Q_1(\delta, \tilde{\alpha})/Q_2(\delta, \tilde{\alpha})$, and $Q_1(\delta, \tilde{\alpha}) = 3\delta(2\tilde{\alpha} - \delta)[(2\tilde{\alpha} - \delta)(\delta - \tilde{\alpha}) + 2\delta]$, $Q_2(\delta, \tilde{\alpha}) = (2\tilde{\alpha} - \delta)^2[2(2\tilde{\alpha} - \delta)^2 + 7\delta(\tilde{\alpha} - 1)] + 4\delta^2(2\tilde{\alpha} - 1 - \delta)$. For a typical value $\beta = 0.7$, Fig. 1 shows the (δ, α) region where $\beta \geqslant \beta_2(\delta, \tilde{\alpha})$ becomes effective, i.e., where $\beta_2(\delta, \alpha) > \beta_1(\delta, \alpha)$. The fact that for $\gamma^2 > \alpha^2$ the energy conditions (23) set lower limits to the radius $a = \beta R$ of the inner, charged shell is physically intuitive: For small β typically the mass energy $q^2/2\mathbf{a}$ of this shell gets large and can ''overcompensate'' the total mass $M = 2R\alpha$ of the two-shell system (with $\alpha^2 < \gamma^2$) in the way that the energy conditions for the mass shell at $r = R$ are violated.

The electromagnetic field tensor belonging to the Reissner-Nordström metric (1) has only a radial component $-F_{\tau\rho} = E_\rho = q/\rho^2$. Transformation of this component to our isotropic coordinates, under the condition that the charge q be concentrated solely on the inner shell at $r = a$, leads to $-F_{tr} = E_r = (q/r^2) e^{U-V} H(r-a)$, where $H(r-a)$ is the Heaviside function, and the potentials (5) or (6) have to be inserted in the respective regions. Herewith and with the inhomogeneous Maxwell equation $(1/\sqrt{-g})(\sqrt{-g} F^{t\mu})_{,\mu} = 4\pi j^t$, the charge density $\sigma = j^t$ at the inner shell $(r = a)$ can be calculated: For the metric (2), we have $\sqrt{-g} = e^{3V} e^U r^2 \sin\vartheta$, and only $F^{tr} = e^{-2(U+V)} E_r$ is nonzero. Therefore, $\sqrt{-g} F^{tr}$ is equal to $q \sin\vartheta$ (independent of r) for $r \geqslant R$ and for $a \leqslant r \leqslant R$, and zero for $0 \leqslant r < a$. The r derivative of this expression gives then, as is expected in our shell models, a δ-function-type charge density:

$$\sigma(x) = \frac{q\beta C_3}{4\pi R^3 \Delta^3_+} \delta(x - \beta), \tag{25}$$

with C_3 from Eq. (17). [This corresponds to $j^\tau = (q/4\pi\mathbf{a}^2)\delta(\rho - \mathbf{a})$ in the Reissner-Nordström variables.]

III. FIRST ORDER ROTATION OF THE SHELLS

Whereas a general stationary and axially symmetric metric for a system with matter requires at least four metric functions, depending on two variables, e.g., on r and ϑ, a first order rotational perturbation (with angular velocity parameter ω) of a spherically symmetric system requires only three metric functions, solely depending on r, because centrifugal deformations of the spherical system appear only in orders ω^2 and higher. We write the corresponding extension of our metric (2) for physical intuition in the form

$$ds^2 = -e^{2U(r)} dt^2 + e^{2V(r)} \{dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta \times [d\varphi - \omega A(r) dt]^2\}, \tag{26}$$

neglecting, however, in the following all terms of second and higher order in ω. Since as a result of the very definition of an angular velocity ω, a stationary rotating system is invari-

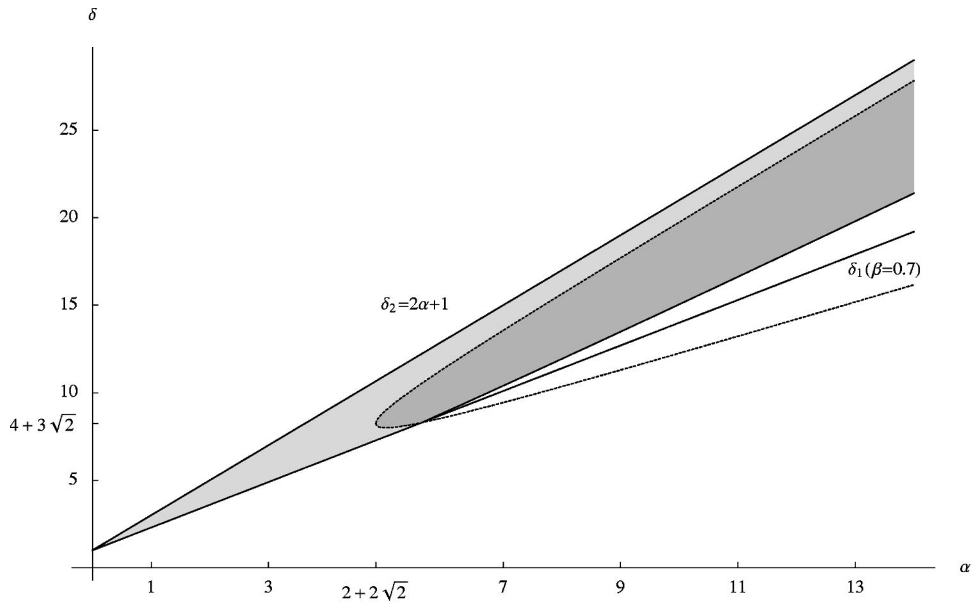

FIG. 1. The graphic shows the parameter domain of α and $\delta = (1+\alpha)^2 - \gamma^2$ in the overextreme Reissner-Nordström case $\gamma^2 > \alpha^2$, where the first and second parts of the weak-energy condition (23) for $\beta = 0.7$ are valid: the condition $\tau_0^0(R) \leqslant 0$ is satisfied for all values of (α, δ) between the lines δ_1 and δ_2. In order to ensure $\tau_3^3(R) - \tau_0^0(R) \geqslant 0$, α and δ have to be restricted to the grey shadowed regions. The darker one lies inside the (dashed) hyperbola branch $\beta_2(\delta, \tilde{\alpha}) = \beta_1(\delta, \tilde{\alpha})$, where the condition $\beta \geqslant \beta_2(\delta, \tilde{\alpha})$, following from $\tau_3^3(R) - \tau_0^0(R) \geqslant 0$, is more resrictive than $\beta \geqslant \beta_1(\delta, \tilde{\alpha})$ from Eq. (24). The picture looks essentially similiar for all other values of $\beta \in [0,1]$.

ant under the common substitutions $t \rightarrow -t$ and $\omega \rightarrow -\omega$, and since the metric functions U, V, and A are independent of t, they have to be even functions of ω. Since we assume that all these functions can be expanded in power series in ω, for our first order perturbation the functions $U(r)$ and $V(r)$ can be taken over unchanged from Sec. II, and $A(r)$ is independent of ω. Here and in the following, the symbol ω should denote a general angular velocity in our systems. Later on we will introduce more specific angular velocities ω^I for the interior shell and ω^{II} for the exterior shell. All these angular velocities are of the same order ω, or they are zero.

A. Integration of the field equations

In the following we often change between the variable ρ, in which the field equations and their solutions are simplest, and the variable r, necessary for the conditions at the shells and for the physical interpretation of the results. The radial dependence of the dragging function A is essentially given by the Einstein equation

$$G_3^0 = -\frac{\omega \sin^2\vartheta}{2\rho^2} \frac{d}{d\rho}\left(\rho^4 \frac{d}{d\rho} A(\rho)\right) = 8\pi (T_3^0 + S_3^0), \quad (27)$$

where $S_3^0 = (1/4\pi) F_\lambda^0 F_3^\lambda$ denotes the electromagnetic contribution to the energy-momentum tensor. Since the electric components of the field tensor $F_{\mu\nu}$ are time symmetric, they are even functions of ω, and therefore reduce in our first order perturbation in ω to the field E_r. The magnetic components of $F_{\mu\nu}$ are time antisymmetric, and therefore start with order-ω terms. However, since there are no electric currents in the r and ϑ directions in our models, the magnetic component $B_\varphi = F_{r\vartheta}$ is identically zero according to the inhomogeneous Maxwell equations. Therefore, in the first order of ω, there remains besides Eq. (27) only one nontrivial homogeneous Maxwell equation (with the overdot denoting the ϑ derivative)

$$B_r' + \dot{B}_\vartheta = 0 \quad (28)$$

and one nontrivial inhomogeneous Maxwell equation

$$4\pi j^\varphi = \frac{1}{\rho^2 \sin^2\vartheta} \frac{d}{d\rho}[F(\rho) B_\vartheta] - \frac{1}{\rho^4 \sin\vartheta}\left(\frac{B_\rho}{\sin\vartheta}\right)^{\cdot} + \frac{\omega}{\rho^2} \frac{d}{d\rho}(\rho^2 A E_\rho). \quad (29)$$

In the exterior region and in the intermediate region between both shells, Eq. (27) reads, with $E_\rho = q/\rho^2$,

$$-\frac{\omega \sin^2\vartheta}{4} \frac{d}{d\rho}\left(\rho^4 \frac{d}{d\rho} A(\rho)\right) = q B_\vartheta. \quad (30)$$

This equation, together with the fact that in the limit $q \rightarrow 0$ also the magnetic field should vanish in our models, suggests the ansatz $B_\vartheta = \omega q f(r) \sin^2\vartheta$, with a dimensionless function $f(r)$. Equation (28) then enforces the form $B_r = \omega q R g(r) \sin\vartheta \cos\vartheta$, with $f(r) = (-R/2) g'(r)$. Because of continuity across the charged shell, the forms for B_ϑ and B_r are also valid in the interior of this shell. Then Eqs. (27) and (29) constitute two coupled ordinary differential equations for the unknown functions A and g.

In the interior of the charged shell, these equations decouple and read

$$(r^4 A')' = 0, \quad g'' - \frac{2}{r^2} g = 0. \tag{31}$$

The solutions, which are regular at $r=0$, are given by

$$A_3(r) = \mu_3 / C_3, \quad g_3(r) = \eta_3 r^2 / R^2, \tag{32}$$

where μ_3 and η_3 are dimensionless constants, which have later to be fixed by continuity at the charged shell. (The factor C_3^{-1} is introduced for later convenience.) Because of $A_3(r) = \text{const}$, the interior region stays flat in first order perturbation in ω, as is physically to be expected. The magnetic field components B_r and B_ϑ represent in Cartesian coordinates a constant field $B_z = (\omega q/R)\eta_3$ along the z axis, as is well known for the interior of a charged, rotating shell from classical electrodynamics.

In the intermediate and exterior regions, due to $B_\vartheta \sim f(r) \sim g'(r)$, one integration of Eq. (30) is trivial:

$$\frac{d}{d\rho} A(\rho) = \frac{1}{\rho^4} [2q^2 R g(\rho) - 4\mathcal{M} R^2 \lambda], \tag{33}$$

with a dimensionless integration constant λ, which will be denoted by λ_1 and λ_2 for the exterior and intermediate regions, respectively. (The factor $-4\mathcal{M}R^2$ in front of λ is chosen with a view of the well-known exterior dragging term $A(r) = \frac{4}{3}\mathcal{M}R^2 r^{-3}$ in the standard Thirring problem.) Insertion of Eq. (33) into Eq. (29), together with $j^\varphi \equiv 0$ in the exterior and intermediate regions, results in the differential equation for $g(\rho)$:

$$\frac{d}{d\rho}\left[F(\rho)\frac{d}{d\rho} g(\rho)\right] - \frac{2}{\rho^2}\left(1 + \frac{2q^2}{\rho^2}\right) g(\rho) = -\frac{8\mathcal{M}R\lambda}{\rho^4}. \tag{34}$$

We write the general solution of this equation in the form

$$g(r) = \lambda \hat{g}(\rho(r)) + \eta \bar{g}(\rho(r)) + \zeta \bar{\bar{g}}(\rho(r)), \tag{35}$$

with dimensionless integration constants η, ζ, where $\hat{g}(\rho)$ is a special solution of the inhomogeneous equation (34), and $\bar{g}(\rho)$, $\bar{\bar{g}}(\rho)$ are fundamental solutions of the corresponding homogeneous equation. Luckily, there exist quite simple solutions $\hat{g}(\rho)$ and $\bar{g}(\rho)$ as polynomials in ρ and ρ^{-1}, and $\bar{\bar{g}}(\rho)$ can then be found by the d'Alembert's reduction procedure, $\bar{\bar{g}}(\rho) \sim \bar{g}(\rho) \int^\rho d\rho' [F(\rho')]^{-1} [\bar{g}(\rho')]^{-2}$:

$$\hat{g}(\rho) = \frac{4R}{3\rho}, \quad \bar{g}(\rho) = \frac{1}{R^2}\left(\rho^2 - 3q^2 + \frac{2q^4}{\mathcal{M}\rho}\right), \tag{36}$$

$$\bar{\bar{g}}(\rho) = \frac{3\mathcal{M}^2 R}{4(\mathcal{M}^2 - q^2)^2}\left[\frac{2q^2}{3\rho}\left(1 + \frac{2q^2}{\mathcal{M}^2}\right) - \rho - \mathcal{M} + R^2 \bar{g}(\rho) S(\rho; \mathcal{M}, q)\right], \tag{37}$$

with

$$S(\rho;\mathcal{M},q) = \begin{cases} \dfrac{1}{\sqrt{q^2 - \mathcal{M}^2}} \operatorname{arccot}\left(\dfrac{\rho - \mathcal{M}}{\sqrt{q^2 - \mathcal{M}^2}}\right) & \text{for} \quad q^2 > \mathcal{M}^2, \\ \dfrac{1}{2\sqrt{\mathcal{M}^2 - q^2}} \log\left(\dfrac{\rho - \mathcal{M} + \sqrt{\mathcal{M}^2 - q^2}}{\rho - \mathcal{M} - \sqrt{\mathcal{M}^2 - q^2}}\right) & \text{for} \quad q^2 < \mathcal{M}^2. \end{cases}$$

In the intermediate region we have $\mathcal{M} = \hat{M}$, and as a result of $D_2 = (\gamma^2 - \hat{\alpha}^2)/\hat{\alpha}\beta \geqslant 0$, $\hat{M}$ is never greater than $|q|$, and S reduces to the arccot case. Furthermore, the asymptotically diverging solution $\bar{g}(\rho)$ is missing in the exterior region: $\eta_1 = 0$. For small values of $\epsilon = (\sqrt{\mathcal{M}^2 - q^2})/(\rho - \mathcal{M})$ the function S has the series expansion

$$S = \frac{1}{\rho - \mathcal{M}}\left(1 + \frac{\epsilon^2}{3} + \frac{\epsilon^4}{5} + \cdots\right),$$

from which it is seen that we have chosen the normalization of $\bar{\bar{g}}(\rho)$ such that it behaves asymptotically as $R\rho^{-1}$ (independent of $\mathcal{M}$ and q). After transformation to Cartesian coordinates, all components of the magnetic field have the asymptotic behavior $B_i \sim \rho^{-3} \sim r^{-3}$, as is physically expected. On the other hand, in the limit $q^2 \to \mathcal{M}^2$, the expression (37) produces the relatively simple limit function

$$\bar{\bar{g}}(\rho)|_{q^2 = \mathcal{M}^2} = \frac{R(2\rho - \mathcal{M})(5\rho^2 - 5\mathcal{M}\rho + 2\mathcal{M}^2)}{10\rho(\rho - \mathcal{M})^3}. \tag{38}$$

The homogeneous solutions (36) and (38) have already been given by Bičák and Dvořák [12,13] within a more general analysis of stationary perturbations of the Reissner-Nordström metric. [There is, however, a misprint in the solution $\bar{g}(\rho)$.] The solutions (36), (37) have already been found by Briggs *et al.* [14]. Also for an arbitrary (natural) value of the angular momentum parameter l, there exists one homogeneous solution $\bar{g}(\rho)$ of the correspondingly modified equation (34) in the form of a polynomial $\Sigma_{k=-1}^{l+1} a_k \rho^k$, with $a_1 = 0$.

Having now available the general magnetic field function $g(r)$ in the intermediate and exterior regions, we can also calculate the general ''dragging function'' $A(r)$ in these regions by integrating Eq. (33). If we write $A(\rho)$ in the suggestive form

$$A(\rho)=C_iA(r) \\ =\frac{2q^2}{R^2}[\lambda\hat{A}(\rho(r))+\eta\overline{A}(\rho(r))+\zeta\overline{\overline{A}}(\rho(r))] \\ +\frac{4\mathcal{M}R^2\lambda}{3[\rho(r)]^3}+\mu, \tag{39}$$

with $C_1=1$ and $C_2=C_3$ from Eq. (17), we get

$$\hat{A}(\rho)=-\frac{R^4}{3\rho^4},\quad \overline{A}(\rho)=R\left(-\frac{1}{\rho}+\frac{q^2}{\rho^3}-\frac{q^4}{2\mathcal{M}\rho^4}\right), \tag{40}$$

$$\overline{\overline{A}}(\rho)=\frac{3\mathcal{M}R^4}{8(\mathcal{M}^2-q^2)^2}\Bigg[-\frac{1}{\rho}+\frac{\mathcal{M}}{\rho^2}+\frac{q^2+2\mathcal{M}^2}{3\rho^3} \\ -\frac{q^2(\mathcal{M}^2+2q^2)}{3\mathcal{M}\rho^4} \\ +\left(1+\frac{2\mathcal{M}}{R}\overline{A}(\rho)\right)S(\rho;\mathcal{M},q)\Bigg]. \tag{41}$$

In the extreme Reissner-Nordström case, $\overline{\overline{A}}(\rho)$ has the simpler form

$$\overline{\overline{A}}(\rho)|_{q^2=\mathcal{M}^2}=-\frac{R^4(5\rho^2-4\mathcal{M}\rho+\mathcal{M}^2)}{20\rho^4(\rho-\mathcal{M})^2}. \tag{42}$$

In the exterior region, we demand that $A(r)$ vanish asymptotically, expressing the fact that there is no dragging of inertial systems at infinity and that any rotation in our model is defined relative to (hypothetical) static observers at infinity, respectively. Therefore, together with the asymptotic decrease of the functions (40), (41), we have $\mu_1=0$. In the intermediate region we will have a nonzero (dimensionless) constant μ_2.

In the process of integration of the magnetic field function g and of the dragging factor A in the different regions, we had to introduce a total of eight nontrivial integration constants: μ_2, μ_3, λ_1, λ_2, η_2, η_3, ζ_1, and ζ_2. These constants have now to be fixed according to the physical and mathematical properties of our model systems. First, we have to define our two-shell systems in more detail. Especially we have to say what the ''real'' physical sources of the dragging effects and of the magnetic fields are. For this we have in mind the following picture: Imagine starting with the static two-shell models of Sec. II, and then to turn on slowly (adiabatically) a rotation of the shells by exerting independent torques (e.g., by appropriate handles) on the two shells at $r=a$ and $r=R$. As long as we consider only first order rotation effects, we see no mechanism how angular momentum could be transferred in this process from one shell to the other. (In higher orders of ω, nonspherical deformations of the shells have to be taken into account, and if their quadrupole moments change rapidly during a nonadiabatic switch-on process, gravitational waves are produced, and they will transfer angular momentum from one shell to the other.) It seems therefore reasonable to define the angular momenta J^I and J^{II}, imprinted on the systems in this way, as the two independent physical sources of the rotation effects. In contrast to the angular momenta, the angular velocities of the two shells are not in the same way independent, because the rotation of one shell leads by dragging to a nonzero angular velocity also of the other shell. (In this connection it should be said that the complicated interplay between the dragging of two rotating uncharged mass shell, as analyzed by Cohen and Brill [10], results mainly from their description in terms of the angular velocities instead of the angular momenta.) Since, however, a prescription of the values J^I and J^{II} does not seem to be a useful starting point for the determination of the integration constants, we find it advantageous to divide the general problem of rotation of both shells into the following two steps: We introduce nonsingular ''stirring'' angular velocities $\overline{\omega^I}$ and ω^{II} separately for the interior charged shell and for the exterior uncharged shell, respectively, where (in accordance with [4]) $\overline{\omega^I}=C_3\omega^I$ is measured in proper time $\tau=(1/C_3)t$. For the cases I ($\overline{\omega^I}\neq 0,\omega^{II}=0$) and II ($\omega^{II}\neq 0,\overline{\omega^I}=0$) we separately determine the integration constants, and herewith the dragging fields A^I,A^{II}, and the magnetic fields $\mathbf{B}^I$,$\mathbf{B}^{II}$. (The notation is here adapted to [2].) Thereafter we find unique and linear relations between $\overline{\omega^I}$ and $\overline{J^I}=C_3J^I$ and between ω^{II} and J^{II}, respectively, so that it is justified to start with the mathematically more useful parameters $\overline{\omega^I}$ and ω^{II} instead of the physical source parameters $\overline{J^I}$ and J^{II}. (In Sec. III B we will come back to the reasons why the relations between $\overline{\omega^I}$ and $\overline{J^I}$, and between ω^{II} and J^{II}, respectively, can only be given *a posteriori*.) Finally, for the general cases $\overline{J^I}\neq 0$ and $J^{II}\neq 0$, the discussion in the beginning of this paragraph, together with the general linearity of a first order perturbation of an exact solution, justifies writing the general dragging and magnetic fields as linear superpositions: $\omega A=\omega^I A^I+\omega^{II}A^{II}$,$\mathbf{B}=\mathbf{B}^I+\mathbf{B}^{II}$. (The same was done without much discussion in [2] and [4].) Since the detailed calculations are somewhat simpler for case II (rotation stirred by the exterior shell), we begin with this case.

B. Boundary conditions for a rotating exterior shell

According to the notation introduced in Sec. III A, all quantities in this section should in principle carry an upper index II. For simplicity we omit these indices in most formulas and add them only in the final expressions for the integration constants in the Appendix. The energy-momentum tensor $T^\mu_\nu=\tau^\mu_\nu(R)\delta(r-R)$ of the exterior shell has of course to satisfy the eigenvalue equations $T^\mu_\nu u^\nu=-\varrho_0 u^\mu$, where $u^\mu=u^0(R)(1,0,0,\omega(R))$ is the purely axial four-velocity vector of the shell matter, and ϱ_0 is the rest-energy density. Moreover, since our models shall consist of rigidly rotating shells, the components u^μ are constant, i.e., independent of ϑ (compare [5]). Comparison of the components $\mu=0$ and $\mu=3$ of the eigenvalue equations, together with the metric form (26), gives, in first order of the rotation,

$$\tau_3^0(R)=e^{2V(R)-2U(R)}R^2[\omega(R)-\omega A(R)] \times[\tau_3^3(R)-\tau_0^0(R)]\sin^2\vartheta, \tag{43}$$

with $\tau_0^0(R)$ and $\tau_3^3(R)$ from Eqs. (21), (22). From Eq. (43) it is seen, on the one hand, that $\tau_3^0(R)$ is zero if the angular velocity $\omega(R)$ of the exterior shell coincides with the dragging term $\omega A(R)$ coming from any other rotating sources, e.g., from a rotating interior shell. [In the language of Bardeen [15] this means that the exterior shell then consists of so-called zero-angular-momentum observers (ZAMOs).] This result can be understood as a mathematical confirmation of the argument in Sec. III A that in first order of rotation there can be no transfer of angular momentum from one shell to the other. On the other hand, if we really stick to our case II, where only the exterior shell rotates with angular velocity $\omega(R)=\omega^{II}\neq 0$ and no other rotating matter is in the game (especially $\overline{\omega^I}=0$), then Eq. (43) reads

$$\tau_3^0(R)=\omega^{II}e^{2V_1(R)-2U_1(R)}R^2[1-A_1(R)] \times[\tau_3^3(R)-\tau_0^0(R)]\sin^2\vartheta. \tag{44}$$

Now, the component $\tau_3^0(R)$ can, in analogy to the determination of the components τ_0^0 and $\tau_2^2=\tau_3^3$ in Sec. II [Eqs. (12), (13)], also be calculated from the Einstein equation (27),

$$\tau_3^0(R)=-\frac{1}{16\pi}\omega^{II}R^2e^{-2U_1(R)}[A_1'(R)-A_2'(R)]\sin^2\vartheta, \tag{45}$$

and with $A'(r)$ from Eq. (33) and with the continuity of $g(r)$ [to be substantiated in Eq. (49)]:

$$\tau_3^0(R)=\frac{\omega^{II}R(\alpha\lambda_1-\hat{\alpha}\lambda_2)}{2\pi\delta^2(1-\alpha^2+\gamma^2)}\sin^2\vartheta. \tag{46}$$

Equations (46) and (44) lead, with the abbreviation $\Delta\tau=-(2\pi R\delta^6)[\tau_3^3(R)-\tau_0^0(R)]/(1-\alpha^2+\gamma^2)$, to a first linear, inhomogeneous equation between the integration constants:

$$-[8(\gamma^2-\alpha\delta)\Delta\tau/3\delta^4+\alpha]\lambda_1+\hat{\alpha}\lambda_2+8\gamma^2\Delta\tau\overline{\overline{A}}_1(R\delta)\zeta_1 =\Delta\tau. \tag{47}$$

Since in the present case II the interior charged shell does not have an own angular momentum ($\overline{J^I}=0$ and $\overline{\omega^I}=0$) but is only dragged by the rotating exterior shell, we have, in analogy to Eq. (43), $\tau_3^0(a)=0$. Again this has to be compared with the result of an integration of the Einstein equation (27) from $r=a-\epsilon$ to $r=a+\epsilon$:

$$\tau_3^0(a)=-\frac{\omega^{II}R\beta C_3}{2\pi\Delta_+^3}(\gamma^2\beta^2\eta_3-\hat{\alpha}\lambda_2)\sin^2\vartheta=0. \tag{48}$$

Now we come to the continuity or discontinuity conditions for the magnetic field function $g(r)$ at the positions $r=R$ and $r=a$ of the two shells. Before going into details, we have to remark quite generally that the functions $\hat{g}$, $\overline{g}$, and $\overline{\overline{g}}$ in Eqs. (36), (37) are given in the variable ρ, whereas the continuity of the metric is only guaranteed in the variable r. Therefore we have to transform these functions and their derivatives to the variable r, according to Eq. (3). The position $r=R$ of the mass shell is in the variable ρ given by $\rho(R)=R\delta$, and this, as a result of the continuity of $V(r)$, coming both from the exterior and intermediate region. [This was already used in Eq. (47).] Similarly, the position $r=a$ of the charged shell is given (from both sides) by $\rho(a)=R\Delta_+$. In analogy to the boundary conditions of classical electrodynamics, saying that the magnetic field $\mathbf{B}$ at an interface (with normal vector $\mathbf{n}$) between two media has to satisfy $\mathbf{n}\cdot(\mathbf{B}_2-\mathbf{B}_1)=0$, our radial magnetic field function $g(r)$ has to be continuous at $r=R$ and $r=a$:

$$-\frac{4}{3\delta}\lambda_1+\frac{4}{3\delta}\lambda_2+\overline{g}_2(R\delta)\eta_2-\overline{\overline{g}}_1(R\delta)\zeta_1+\overline{\overline{g}}_2(R\delta)\zeta_2=0, \tag{49}$$

$$\frac{4}{3\Delta_+}\lambda_2+\overline{g}_2(R\Delta_+)\eta_2-\beta^2\eta_3+\overline{\overline{g}}_2(R\Delta_+)\zeta_2=0. \tag{50}$$

Similarly, the boundary condition $\mathbf{n}\times(\mathbf{B}_2-\mathbf{B}_1)=(4\pi/c)\mathbf{j}$ from classical electrodynamics, together with the fact that the shell at $r=R$ carries no charge and therefore no electric current, results in the continuity of the magnetic field component $B_\vartheta\sim f(r)$ and therefore in the continuity of $g'(r)$ at $r=R$:

$$\frac{4(1-\alpha^2+\gamma^2)}{3\delta^2}\lambda_1-\frac{4}{3\delta^2}\sqrt{F_2(R\delta)}\lambda_2+R\overline{g}_2'(R\delta)\eta_2 -R\overline{\overline{g}}_1'(R\delta)\zeta_1+R\overline{\overline{g}}_2'(R\delta)\zeta_2=0. \tag{51}$$

[Here and in the following, e.g., an expression $\overline{g}_2'(R\delta)$ is understood in the way that we take the derivative of $\overline{g}_2(\rho)$ from Eq. (36) with respect to ρ and multiply, due to the chain rule, by $d\rho/dr|_{R_-}$, similarly at the position $r=a$.] At the position $r=a$, a first sight gives the impression that $g'(r)$ should be discontinuous there because, due to dragging, the charged shell acquires a nonzero angular velocity $\omega^{II}A(a)$ and, connected with it, a nonzero electric current. It has, however, to be observed that the relation $\mathbf{n}\times(\mathbf{B}_2-\mathbf{B}_1)=(4\pi/c)\mathbf{j}$ is only valid in the local inertial frame, and this is just the frame corotating with angular velocity $\omega^{II}A(a)$. If then (in accordance with [2]) the charged shell consists of insulating material, its charge elements have the same angular velocity $\omega^{II}A(a)$ as the matter elements, so that in the presently considered case II there is no electric current relative to the local inertial frame, and the magnetic field $B_\vartheta\sim g'(r)$ is continuous across the charged shell:

$$R\overline{g_2'}(R\Delta_+)\eta_2-\frac{10}{3}\beta\eta_3+R\overline{\overline{g_2'}}(R\Delta_+)\zeta_2=0. \qquad (52)$$

[This can also be mathematically confirmed by comparing the current $j^{\varphi}(a)$ following from Eq. (29) with the expression $j^{\varphi}(a)=\omega^{II}A_3(a)\sigma(x)$, with $\sigma(x)$ from Eq. (25).]

After determining (in the Appendix) the integration constants from the linear equations (47)–(52), we can also explicitly calculate the total angular momentum J^{II} contained in this system. Because of the axial symmetry of the system, J^{II} is given by the integral

$$J^{II}=\int_0^{\infty}dr\int_0^{\pi}d\vartheta\int_0^{2\pi}d\varphi\sqrt{-g}[\tau_3^0(R)\delta(r-R)+S_3^{0II}]. \qquad (53)$$

With $\tau_3^0(R)$ from Eq. (46) and with

$$S_3^{0II}=-\frac{\omega^{II}}{8\pi}Rq^2\frac{g'(r)}{r^2}e^{-U-3V}H(r-a)\sin^2\vartheta,$$

we get the surprisingly simple result

$$J^{II}=\frac{2}{3}MR^2\omega^{II}\lambda_1^{II}. \qquad (54)$$

Therefore, the ‘‘driving’’ inhomogeneity λ_1^{II}, introduced in Eq. (33), describes the departure of the angular momentum J^{II} from the Newtonian value $\frac{2}{3}MR^2\omega^{II}$ of a shell with mass *M*, radius *R*, and angular velocity ω^{II}, due to strong gravitational and electromagnetic fields. It should not be overlooked that the constant λ_1^{II} from Eq. (A10) in the Appendix has quite a complicated dependence on the model parameters α, β, and γ which cannot be foreseen *a priori*, but can only be calculated by analyzing in detail all the junction conditions for the dragging field $A(r)$ and the magnetic field $g(r)$ at the two shells. This ‘‘difficulty’’ is mainly due to the nonlocalized electromagnetic contribution S_3^{0II} to the angular momentum J^{II}. And since this term contains (obviously) a factor q^2, the problem simplifies very much in an analysis of a similar system in first order of q which was performed by Cohen, Tiomno, and Wald [16].

C. Boundary conditions for a rotating charged interior shell

Many results from Sec. III B transfer essentially unchanged to the present case I, only that all quantities have now to be thought of carrying an upper index I instead of II, which, however, we again omit until the final expressions. The four-velocity of the interior shell elements reads $u^{\mu}=u^0(a)(1,0,0,\omega(a))$, with $\omega(a)=\omega^I$. Instead of Eq. (44) we have $\tau_3^0(R)=0$, because now the exterior mass shell has no angular momentum ($J^{II}=0$ and $\omega^{II}=0$) but is only dragged by the rotating interior shell. (A similar system was considered by Wald [17] in first order of q.) In contrast, now the expression $\tau_3^0(a)$ is nonzero, and is given in analogy to Eq. (44), and with $\tau_0^0(a)=0$ and $\tau_3^3(a)$ from Eq. (20), by

$$\tau_3^0(a)=-\frac{\omega^I R\hat{\alpha}\beta C_3(C_3-\mu_3)}{4\pi\Delta_+}\sin^2\vartheta. \qquad (55)$$

The continuity equations (49)–(51) for the magnetic field function $g(r)$ and its derivative at $r=R$ are of course unchanged. However, now the charged shell carries a ‘‘real’’ current, which is not only induced by dragging: $j^{\varphi}(a)=\omega^I\sigma(x)$ with $\sigma(x)$ from Eq. (25). This has to be compared with an evaluation of the inhomogeneous Maxwell equation (29) at $r=a$, to which obviously only the terms B'_{ϑ} and E'_r contribute localized currents proportional to $\delta(r-a)$:

$$j^{\varphi}(a)=\frac{\omega^I q\beta C_3}{4\pi R^3\Delta_+^3}\left[-\frac{\beta R^2}{2\Delta_+C_3}g''(r=a)+A_3(a)\delta(x-\beta)\right]. \qquad (56)$$

The total angular momentum $\overline{J^I}$ of the system I, measured in proper time τ, is given by the equivalent to Eq. (53), and has the result

$$\overline{J^I}=\frac{2}{3}\hat{M}R^2\overline{\omega^I}\lambda_2^I=\frac{2}{3}MR^2\overline{\omega^I}\lambda_1^I, \qquad (57)$$

which is formally the same as in Eq. (54) but of course with a constant λ_1^I totally different from λ_1^{II}. If both sources $\overline{J^I}$ and J^{II} are active at the same time, the combined angular velocities of the shells are

$$\omega(R)=\omega^{II}+C_3^{-1}\overline{\omega^I}A^I(R),\quad \overline{\omega}(a)=\overline{\omega^I}+C_3\omega^{II}A^{II}(a). \qquad (58)$$

These quantities are denoted by ω_s and $\overline{\omega_c}$ in the work of Cohen [4].

IV. RESULTS AND DISCUSSION

From Sec. III A the *r* dependence of the magnetic field function $g(r)$ and of the dragging function $A(r)$ is explicitly known in the whole space-time, and the Appendix gives explicitly all coefficients contained in $g(r)$ and $A(r)$. Therefore, all questions concerning the magnetic field and the dragging properties of our two-shell models can in principle be answered, and this in the whole ‘‘physical’’ region of the dimensionless parameters α,β,γ, given by inequalities (19), (24), by $\beta\geqslant\beta_2(\delta,\alpha)$, and by Fig. 1. It is, however, evident that most formulas are algebraically so involved that it is not easy to extract the physically interesting properties of our model systems in the general case, and even in the limiting cases $\beta\to 0$ and $\beta\to 1$ for the radius of the inner, charged shell the formulas do not simplify drastically.

It is therefore appropriate to consider first some approximations to our formulas. Because of the fact that in all astrophysical and cosmological circumstances charges seem to be small and also in order to make contact with the work of Hofmann [3], Cohen [4], and Ehlers and Rindler [2], a power series expansion of our formulas with respect to *q* is especially important and useful. Quite generally, the magnetic field is an odd function in *q* and therefore receives contributions from the orders $q^1,q^3,q^5,\ldots$, whereas the dragging term $A(r)$ is even in q, with contributions from orders $q^0, q^2, q^4, \ldots$. Since we have divided out a factor q in the definitions of the magnetic field functions, all our expressions for the functions $g_i(r)$ and $A_i(r)$, and for the integration constants appearing in them, have expansions in powers of q^2.

In Sec. IV A we consider in detail the terms of order q which provide the first nontrivial contributions to the magnetic field, and we compare our results with the work of Cohen [4]. The terms of order q^2 (in Sec. IV B) allow to study the interesting back reaction of the electrostatic energy $S_0^0(r)$ on the geometry, especially on the dragging function $A(r)$. In Sec. IV C we perform on top of the approximations of Secs. IV A and IV B a weak-field approximation; i.e., we keep only terms up to first order in M, and we compare these results with the work of Hofmann [3], and we perform a detailed analysis of the work of Ehlers and Rindler [2], especially under Machian aspects. In Sec. IV D we present some results without approximations in M and q, for instance for the interesting collapse limit.

A. Results in first order of the charge q

Inside the charged shell the function $g(r)$ is, according to Eq. (32), given by $g_3(r) = \eta_3 r^2/R^2$ which represents a constant field $B_z = (q/R)\omega\eta_3$ along the z axis. Since in our model class there appear the two "stirring" angular velocities $\overline{\omega^I}$ and ω^{II}, we write the magnetic field in the interior region in the form $B_z = (q/R)(\omega^I \eta_3^I + \omega^{II}\eta_3^{II}) = (q/R)(\overline{\omega^I}\tilde{\eta}_3^I + \omega^{II}\eta_3^{II})$, with the quantity $\tilde{\eta}_3^I = \eta_3^I/C_3$. In the region between both shells we have the representation of Eq. (35), and in zeroth order in q we get, from Eqs. (36), (37), $\hat{g}_2(\rho) = 4R/3\rho, \overline{g}_2(\rho) = \rho^2/R^2$, $\overline{\overline{g}}_2(\rho) = R/\rho = \frac{3}{4}\hat{g}_2(\rho)$. Consideration of the constants $\lambda_2, \eta_2, \zeta_2$ has now to be performed separately for the cases I (rotating charged interior shell) and II (rotating exterior shell). With the constants from the Appendix we see that the terms $\lambda_2^{II}\hat{g}_2(\rho)$ and $\zeta_2^{II}\overline{\overline{g}}_2(\rho)$ cancel in this order, with the result $g_2^{II}(r) = \eta_3^{II} r^2/R^2 = g_3^{II}(r)$: The magnetic field $\mathbf{B}^{II}$ is constant (in the z direction) in the whole region $r \leqslant R$, as is physically expected in this approximation because the field $\mathbf{B}^{II}$ is purely induced by dragging, and in first order of q the dragging function is constant in this region. In the case I of a rotating charged interior shell, the situation is, as expected, more complicated: The terms $\lambda_2^I\hat{g}_2(\rho)$ and $\zeta_2^I\overline{\overline{g}}_2(\rho)$ do not cancel, but produce together with $\eta_2^I\overline{g}_2(\rho)$ the magnetic field component

$$B_r^I = qR(1+\alpha)^2\overline{\omega^I} \times \left\{ \left[\frac{\tilde{\eta}_3^I}{(1+\alpha)^2} - \frac{2}{3\beta} \right] \frac{r^2}{R^2} + \frac{2\beta^2}{3}\frac{R}{r} \right\} \sin\vartheta\cos\vartheta. \tag{59}$$

Here, the second term represents (together with the accompanying component B_ϑ^I) the expected dipolar magnetic field outside a rotating charged sphere. The correction term to $\tilde{\eta}_3^I/(1+\alpha)^2$ in the first term (leading to a constant field B_z) has the function to cancel the second term at $r = a = \beta R$ and to guarantee in this way the continuity of B_r^I at $r = a$. In the exterior region we have $\eta_1 = 0$, $\hat{g}_1(\rho) = 4R/3\rho$, and

$$\overline{\overline{g}}_1(\rho) = -\frac{3R}{4M^2}\left[\rho + M + \frac{\rho^2}{2M}\log\left(1 - \frac{2M}{\rho}\right)\right] \approx \frac{R}{\rho}\left(1 + \frac{3M}{2\rho} + \frac{12M^2}{5\rho^2} + \cdots\right),$$

and

$$\rho = \frac{1}{r}\left(r + \frac{M}{2}\right)^2.$$

Therefore all Cartesian components of the magnetic fields $\mathbf{B}^I$ and $\mathbf{B}^{II}$ fall off asymptotically like r^{-3}, as expected.

Concerning the integration constants appearing in these first order magnetic field functions, we get the following:

$$\eta_3^{II} = \frac{3(1+\alpha)^2(2-\alpha)}{2\alpha(3-\alpha)} \frac{(1+\alpha)^2(1+\alpha^2) - \frac{8}{3}\alpha^2 - (1+\alpha)^4(1-\alpha)^2\frac{1}{2\alpha}\log\frac{1+\alpha}{1-\alpha}}{-1 + 2\alpha + \alpha^2 + (1+\alpha)^3(1-\alpha)\frac{1}{2\alpha}\log\frac{1+\alpha}{1-\alpha}}. \tag{60}$$

In the physical region $0 \leqslant \alpha \leqslant 1$ of the ratio $\alpha = M/2R$, η_3^{II} grows monotonically from the behavior $\frac{4}{3}\alpha$ at $\alpha = 0$ to the value 8 at $\alpha = 1$. Furthermore,

$$\lambda_1^{II} = \frac{3(1+\alpha)^5(2-\alpha)}{2(3-\alpha)}, \tag{61}$$

a quantity which increases monotonically from the value 1 at $\alpha = 0$ to the value 24 at $\alpha = 1$. The quantity ζ_1^{II} can be expressed through η_3^{II} in the form

$$\zeta_1^{II} = -\frac{4}{3}\alpha(1-\alpha)\eta_3^{II} - \frac{2(1+\alpha)^3(1-\alpha)^2(2-\alpha)}{3-\alpha}, \tag{62}$$

a quantity which decreases from the value $-\frac{4}{3}$ at $\alpha = 0$ to a minimum $\zeta_{1\,\text{min.}}^{II} \approx -1.56$ at $\alpha \approx 0.35$, and then reaches the value 0 at $\alpha=1$. Therefore, whereas the terms $\lambda_1^{II}\hat{g}_1(\rho)$ and $\zeta_1^{II}\overline{\overline{g}}_1(\rho)$ exactly cancel for $\alpha\rightarrow 0$, for $\alpha\gtrsim 0.5$, and especially for $\alpha\rightarrow 1$, the first term is absolutely dominant. Quite generally we observe that in the lowest order of q the magnetic field $\mathbf{B}^{II}$, since it is only induced by dragging and does not have ''real sources,'' is everywhere independent of $\beta=a/R$, i.e., independent of the distribution of the charge q within the mass shell, as long as it is spherically symmetric. [As a result of $S^{\mu}_{\nu}\sim q^2$ and $\tau^{\mu}_{\nu}(a)\sim q^2$, the exact position of the charged shell matters, however, in the higher orders of q.] The integration constant $\tilde{\eta}^I_3$ can in zeroth order in q be expressed through η^{II}_3 in the form

$$\tilde{\eta}^I_3=\frac{2(1+\alpha)^2}{3\beta}+\beta^2\left[\frac{4\alpha(1+\alpha)^2}{3(3+\alpha)}-\frac{(3-\alpha)}{(2-\alpha)(3+\alpha)}\,\eta^{II}_3\right]. \tag{63}$$

Furthermore, we get

$$\tilde{\zeta}^I_1=\frac{\zeta^I_1}{C_3}=\frac{8\alpha\beta^2(1+\alpha)^2(1-\alpha)}{3\left[-1+2\alpha+\alpha^2+(1+\alpha)^3(1-\alpha)\,\frac{1}{2\alpha}\log\frac{1+\alpha}{1-\alpha}\right]}, \tag{64}$$

whereas λ^I_1 vanishes in zeroth order of q. Therefore the magnetic field B^I_z inside the charged shell diverges like β^{-1} in the point charge limit $\beta\rightarrow 0$, as is expected from classical electrodynamics. In the region between both shells, the divergent term $2/3\beta$ in Eq. (59) is just canceled, and the remaining magnetic field, like the magnetic field in the exterior region, is proportional to $\beta^2=a^2/R^2$, as is again well known from classical electrodynamics. The α-dependent factors in front of these β^2 terms go for $\tilde{\eta}^I_3$ from 0 at $\alpha=0$ monotonically decreasing to $-8/3$ at $\alpha=1$, and for $\tilde{\zeta}^I_1$ from 2/3 at $\alpha=0$ through a maximum ≈ 0.74 at $\alpha\approx 0.32$ to zero at $\alpha=1$.

Of special interest is of course the limiting case $\alpha\rightarrow 1$ of the collapse of the exterior mass shell, since in this limit the mass shell represents a simple cosmological model of our universe, and as a result of the electromagnetic test-field approximation used here, the whole space inside the mass shell is flat. Therefore we can expect to see close analogies with results from classical electrodynamics. The collapse limit is also the only case for which Cohen [4] gives explicit results. In this limit the above formulas reduce to

$$B_r(r\leqslant a)=\frac{8}{3}q\frac{r^2}{a}\left\{\left[1-\left(\frac{a}{R}\right)^3\right]\overline{\omega}^I+3\frac{a}{R}\,\omega^{II}\right\}\sin\vartheta\cos\vartheta, \tag{65}$$

$$B_r(a\leqslant r\leqslant R)=\frac{8}{3}q\frac{a^2}{r}\left\{\left[1-\left(\frac{r}{R}\right)^3\right]\overline{\omega}^I+3\frac{r^3}{Ra^2}\,\omega^{II}\right\}\times\sin\vartheta\cos\vartheta, \tag{66}$$

$$B_r(r\geqslant R)=32q\frac{R^2 r}{(r+R)^2}\,\omega^{II}\sin\vartheta\cos\vartheta. \tag{67}$$

We compare the first terms proportional to $\overline{\omega}^I$ with Cohen's results in his Eqs. (44), (45) which are proportional to $\overline{\omega}_c-\overline{\omega}_s$. But according to our Eq. (58) we have $\overline{\omega}_c-\overline{\omega}_s=\overline{\omega}(a)-\overline{\omega}(R)=[1-A^I(R)]\overline{\omega}^I+(\mu^{II}_3-C_3)\omega^{II}$, and $A^I(R)$ is zero in lowest order of q and $\mu^{II}_3-C_3$ is zero in lowest order or q and for $\alpha\rightarrow 1$, so that $\overline{\omega}_c-\overline{\omega}_s$ coincides in this case with our $\overline{\omega}^I$. Transforming our isotropic coordinates to Cohen's frames, we see that our $\overline{\omega}^I$ terms in Eqs. (65), (66) exactly coincide with Cohen's results. The second terms proportional to ω^{II} in Eqs. (65)–(67) are missing in Cohen's work because he finally scales all quantities to the interior proper time

$$\tau=\frac{1}{C_3}t=\frac{1-\alpha}{1+\alpha}t,$$

and considers only the ''rotating universe'' represented by the interior of the mass shell. ($\omega^{II}=0$ if $\overline{\omega}^{II}$ stays finite.) It is nevertheless worthwhile to discuss also the magnetic field proportional to ω^{II}, especially in the exterior region. Taking into account the relation (54) between ω^{II} and the angular momentum J^{II} which reads in the collapse limit $J^{II}=32R^3\omega^{II}$, we see that $B_r(r\geqslant R)$ from Eq. (67) (together with the accompanying component B_ϑ) constitutes just the Kerr-Newman field in first order of q and ω (see, e.g., [18], Chaps. 33.2–33.3), as is physically expected according to the no hair theorem. The ω^{II} terms in Eqs. (65), (66) represent the continuation of this field to the interior of the mass shell.

Since the $\overline{\omega}^I$ terms in Eqs. (65), (66) exactly correspond to the results of Cohen [4], also the Machian interpretation of these results can be taken over unchanged; especially, an interior observer ''cannot distinguish (even with electromagnetic fields reaching beyond the mass shell) whether the charged shell is rotating or the mass shell is rotating in the opposite direction.'' For completeness, we should, however, add that Cohen's paper contains some misprints and minor errors: In his Eq. (4) the factor $(\overline{r}/\overline{r}_c)^3$ has to be substituted by $(\overline{r}/\overline{r}_c)^{-3}$. Equation (37) has to read $F(\overline{r})=2\overline{\alpha}(\overline{r}\psi^2)^{-1}+4\overline{\alpha}^2(\overline{r}\psi^2)^{-2}+\ln V$, in order that $n(\overline{r})$ fall off asymptotically like $\overline{r}^{-3}$ and coincide in the limit $\overline{\alpha}\rightarrow 0$ with Eq. (4). In Eq. (45) the quantity $p(\overline{r})$ has according to Eq. (28) to read $p(\overline{r})=(q\overline{r}_c^2/3\overline{r}^3)(\overline{\omega}_c-\overline{\omega}_s)\{3+2[(\overline{r}_0-\overline{r})/(\overline{r}_0+\overline{r})][1-(\overline{r}/\overline{r}_0)^3]\}$, an expression which coincides with Cohen's expression only for $\overline{r}=\overline{r}_0$. Figure 2 obviously contains a scale error. Since the ratio $\overline{r}_0/\overline{r}_c$ is chosen as 2.5, the ratio between the curved and flat space radial magnetic fields n for $\overline{r}<\overline{r}_c$ is $1-(\overline{r}_c/\overline{r}_0)^3\approx 0.936$ instead of Cohen's value 0.62. The last remark is not only a pedantic criticism of Cohen's figure but is of immediate ''observational'' relevance: If we ascribe to the cosmological model of a mass shell with $M=2R$ and $\overline{\alpha}=\overline{r}_0$, respectively, any, howsoever approximate reality, then the dimensions **a** and $\overline{r}_c$, respectively, of all laboratory and even solar system ''equipment'' satisfy $\overline{r}_c\ll\overline{r}_0$. Therefore the ''cosmological correction

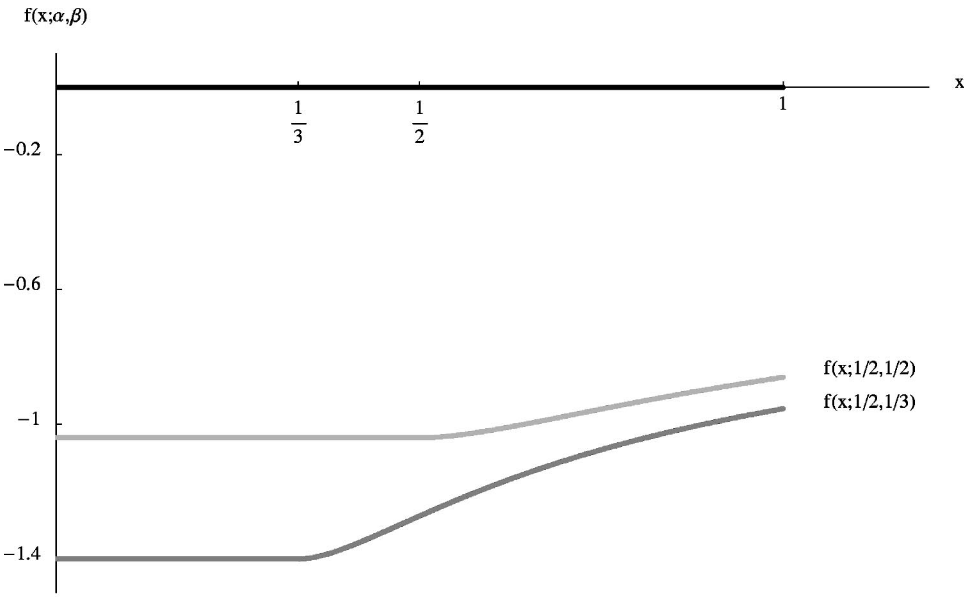

FIG. 2. A plot of the dimensionless and γ-independent function $f(x;\alpha,\beta) = \gamma^{-2}\,[(A_i^{\rm II}(x)/A_3^{\rm II}(\gamma=0))-1]$ for $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3},\frac{1}{2}$ in the region $0\leqslant x\leqslant 1$, $i=2,3$. The dark line shows the constant zeroth order dragging term of case II.

term'' $(\overline{r_c}/\overline{r_0})^3$, i.e., the relative difference between curved and flat space results, is, for good or bad, in all conceivable cases beyond measurability, and we get the perfectly Machian result that the magnetic field $\mathbf{B}^I$ well inside the mass shell is the field of a rotating charged shell as known from classical electrodynamics (with respect to an inertial observer):

$$B_\rho^I(\rho)=\begin{cases}\dfrac{2q\mathbf{a}^2}{3\rho^3}[\overline{\omega}(a)-\overline{\omega}(R)]\cos\vartheta & \text{for} \quad \mathbf{a}\leqslant\rho\ll R,\\[2ex] \dfrac{2q}{3\mathbf{a}}[\overline{\omega}(a)-\overline{\omega}(R)]\cos\vartheta & \text{for} \quad \rho\leqslant\mathbf{a}.\end{cases} \tag{68}$$

B. Results in second order of the charge q

As mentioned in the introduction to Sec. IV, in order q^2 there are only contributions to the dragging function $A(r)$. According to Eq. (39), it suffices to know the functions $\hat{A}_i(\rho)$, $\overline{A}_i(\rho)$, and $\overline{\overline{A}}_i(\rho)$ in zeroth order of q, equally the integration constants λ_2, η_2, ζ_1, and ζ_2. Only the constants λ_1, μ_2, and μ_3 have to be calculated up to second order in q. We start with case II (rotating exterior shell): Using the formulas from the Appendix, we get, for the constant dragging factor inside the charged shell,

$$A_3^{II}=\frac{4\alpha(2-\alpha)}{(1+\alpha)(3-\alpha)}\Bigg\{1-\frac{\gamma^2(1-\alpha)}{\alpha}\times\Bigg[\frac{3(2+9\alpha-4\alpha^2+\alpha^3)}{(1+\alpha)^4(2-\alpha)(3+\alpha)}+\frac{6(1-\alpha)^2}{\beta(1+\alpha)^4(2-\alpha)(3-\alpha)}-\frac{(3-\alpha)(3+5\alpha)}{6\alpha(1+\alpha)^5(2-\alpha)}\,\eta_3^{II}+\frac{4(3-\alpha+\alpha\beta^3)}{3\beta(1+\alpha)^6(2-\alpha)}\,\eta_3^{II}\Bigg]\Bigg\}. \tag{69}$$

The dominant term (for $\gamma=0$) coincides of course (in different notation) with the results in [5] and [8]. The correction terms of order γ^2 vanish in the collapse limit $\alpha\rightarrow 1$, and A_3^{II} attains the value 1. In the limit $\alpha\rightarrow 0$ some of the correction terms formally diverge and lead to a finite value $A_3^{II}(\alpha=0) = -8\gamma^2/3\beta$. This is to be understood in the following way: If for finite charge q we force the total mass M of our system to go to zero, the positive electrostatic energy $\hat{M}=q^2/2\mathbf{a}$ has to be compensated for by a negative mass density of the exterior mass shell [in violation of the energy condition (24)], which then leads to negative dragging $A_3^{II}(\alpha=0)= -4\hat{M}/3R=-8\gamma^2/3\beta$, and this dragging is greater, the smaller the radius of the charged shell is. The sum of the correction terms in Eq. (69) is always negative (the third correction term is equal or smaller than the absolute value of the first correction term), as is physically intuitive: If for fixed total mass M we increase the charge q from zero to a finite (small) value, this means that we ''substitute'' some mass density of the exterior mass shell by electrostatic energy, thereby reducing the overall dragging and again reducing the dragging, the more smaller the radius of the charged shell is.

In case I (rotating charged interior shell) the same argumentation concerning zeroth and second order terms in q leads to

$$\tilde{\mu}_3^I=\frac{\mu_3^I}{C_3} =\frac{8\gamma^2}{3(1+\alpha)^4(3+\alpha)}\Bigg\{\frac{3}{2}\beta^2(1+3\alpha)+\frac{1}{3}(3+\alpha+8\alpha\beta^3)\times\left(\frac{\alpha\beta}{1-\alpha}-\frac{1}{\beta^2}\right)-\frac{\beta^2(3-\alpha)\eta_3^{II}}{4\alpha(1+\alpha)^2(2-\alpha)}\times\left[3+16\alpha+\alpha^2+8\alpha\beta\left(\frac{\alpha\beta}{1-\alpha}-\frac{1}{\beta^2}\right)\right]\Bigg\}, \tag{70}$$

$$\tilde{\lambda}_2^I = \frac{\lambda_2^I}{C_3} = \frac{\beta^3(1+\alpha)^4}{(3+\alpha)}\left[\frac{3+\alpha}{6\beta}+\frac{4}{3}\alpha\beta^2-\frac{(3-\alpha)\beta^2}{(1+\alpha)^2(2-\alpha)}\eta_3^{II}\right]. \tag{71}$$

According to the metric form (26) the decisive dragging term inside the charged shell in case I is $\omega^I A_3^I dt = \overline{\omega^I}\tilde{\mu}_3^I d\tau$. With $\tilde{\mu}_3^I$ from Eq. (70) this term would diverge in the collapse limit $\alpha\to 1$. For a rotating charged shell inside a very massive or even collapsing mass shell it is, however, physically much more appropriate to relate the dragging term to the (invariant) angular momentum $\overline{J^I}$ instead of the angular velocity $\overline{\omega^I}$. Indeed Eq. (57) together with λ_2^I from Eq. (71) leads to a very unusual relation between $\overline{J^I}$ and $\overline{\omega^I}$ for our model system: (i) As a result of the masslessness of the charged shell, $\hat{\alpha}$ is proportional to γ^2; i.e., for small charge a finite angular momentum $\overline{J^I}$ produces an arbitrarily large angular velocity $\overline{\omega^I}$. (ii) λ_2^I diverges in the collapse limit $\alpha\to 1$; i.e., an infinite angular momentum $\overline{J^I}$ is necessary to produce a finite angular velocity, because the extremely massive exterior shell has also to be dragged along. (iii) The α and β dependence of λ_2^I has the effect that for $\alpha\gtrsim 0.7$ and β approaching the value 1, $\overline{J^I}$ and $\overline{\omega^I}$ have different signs, and in the collapse limit $\alpha=1$ the ratio $\overline{J^I}/\overline{\omega^I}$ is proportional to $(1-4\beta^3)$. If we now express the dragging term through the angular momentum $\overline{J^I}$, it reads $\frac{3}{4}\overline{J^I}R^{-3}\gamma^{-2}\beta(1-\alpha^2)(\mu_3^I/\lambda_2^I)d\tau$, so that at least ''difficulties'' (i) and (ii) are eliminated: The γ^2 factor in μ_3^I is canceled, and in the collapse limit $\alpha=1$ the factor $(1-4\beta^3)$ is exactly canceled, and the dragging term attains the finite and β independent form $(\overline{J^I}/32R^3)d\tau$. (Compare the relation $J^{II}=32R^3\omega^{II}$ for the exterior shell in the collapse limit.) For all $\alpha<1$ there are, however, small β values, below which the dragging term is negative; for $\alpha\lesssim 0.5$ the dragging term is even negative for all β, reaching the value $(-4\overline{J^I}/R^3\beta^3)d\tau$ for $\alpha\to 0$. Our interpretation of this unusual ''antidragging'' phenomenon is the following: The charged shell in our model has zero mass density $\tau_0^0(a)=0$, and, according to Eq. (20), negative ''pressure'' $\tau_2^2(a)=\tau_3^3(a)$ (balancing the Coulomb repulsion), and therefore nearly violates all energy conditions. In a forthcoming paper [19] it is proved that such a shell (without the exterior mass shell) produces a negative dragging term (antidragging) if it rotates, and this dragging is greater, the smaller the radius of the charged shell is. [Compare Eq. (20) which in Reissner-Nordström variables reads $T_3^3(\rho)=(-q^2/16\pi\mathbf{a}^3)\delta(\rho-\mathbf{a})$.] An example of antidragging was also given in [14].

In the region between both shells the deviation of $C_2A_2^{II}(r)$ from the constant μ_2^{II} begins according to Eq. (39), with terms of order q^2, as is intuitively clear because a r dependence of the dragging term can only be caused by the electrostatic energy density in this region:

$$A_2^{II}(r)=\frac{\mu_3^{II}}{C_3}+\frac{8\gamma^2(1-\alpha)\eta_3^{II}}{(1+\alpha)^7}\left[\frac{2R}{3a}-\left(1-\frac{a^2}{3r^2}\right)\frac{R}{r}\right]. \tag{72}$$

As a result of the continuity of $A^{II}(r)$ and $A'^{II}(r)$ at $r=a$, $A_2^{II}(r)$ starts at $r=a$ horizontally from the value A_3^{II} [Eq. (69)], then increases according to the function $h(r)=-(1-a^2/3r^2)R/r$, but always (until $r=R$) stays below the zero order term $A_3^{II}(q=0)=4\alpha(2-\alpha)/(1+\alpha)(3-\alpha)$. Now, an increasing dragging function $A_2^{II}(r)$ (in the region $a\leqslant r\leqslant R$) is at first sight hardly comprehensible. The explanation seems to come from the (positive and nonrotating) electrostatic energy density $S_0^0(r)$: Quite generally, the degree of dragging is determined, at least qualitatively, by the ratio between the rotating mass energy and the nonrotating mass energy. (E.g., in the standard Thirring problem: which part of the cosmic masses is sitting on a rotating mass shell.) If then (for fixed M) a small part of the (rotating) exterior shell is ''substituted'' by electrostatic energy, the dragging constant inside the charged shell is reduced: $A_3^{II}(q\neq 0)<A_3^{II}(q=0)$. For $r=r_0>a$, the part $S_0^0(r<r_0)$ has only a reduced effect because quite generally the dragging due to masses falls off like r^{-3} in their exterior. Therefore $A_2^{II}(r)$ increases in the region $a\leqslant r\leqslant R$. The value $A_2^{II}(r=R,q\neq 0)$ is, however, still smaller than $A_2^{II}(r=R,q=0)$ because there is still electrostatic energy S_0^0 outside the mass shell. Figure 2 shows the typical behavior of the function $A^{II}(r\leqslant R)$ in second order of the charge q in comparison with the zero order function for the parameters $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3},\frac{1}{2}$. In case I we have, in the region between both shells,

$$A_2^I(r)=\tilde{\mu}_3^I+\frac{8\gamma^2}{(1+\alpha)^4}\left\{\frac{R^2}{9a^2}-\frac{4\alpha}{3(3+\alpha)}\frac{a^2(r^2+a^2)}{Rr^3}+\frac{R^2a(r-a)}{6r^4}+\frac{(3-\alpha)\eta_3^{II}}{(1+\alpha)(2-\alpha)(3+\alpha)}\times\left[\frac{a^2(r^2+a^2)}{Rr^3}-\frac{2R}{3a}\right]\right\}. \tag{73}$$

$A_2^I(r)$ generally does not start horizontally at $r=a$; i.e., $A'^I_2(r)$ is discontinuous at $r=a$, due to $\tau_3^0(a)\neq 0$ [Eq. (55)]. At least for small α, $A_2^I(r)$ starts increasing at $r=a$. Outside the exterior mass shell all functions $A(r)$ decrease, according to Eq. (39), dominantly like $\{4MR^2/3[\rho(r)]^3\}\lambda_1$ with q^2-correction terms decreasing at least like $[\rho(r)]^{-4}$.

C. Results in the weak-field limit

In this section we restrict the results of Secs. IV A and IV B to the weak-field limit; i.e., we keep only linear terms in the mass M of the exterior mass shell. In this approximation a considerable simplification of all expressions and a simpler interpretation in terms of classical electrodynamics are to be expected. Furthermore, this approximation enables a direct comparison of our results with the work of Hofmann [3] and of Ehlers and Rindler [2]. It has, however, to be kept in mind that a consideration of finite q^2 contributions in the limit $M\rightarrow 0$ may violate the energy conditions for the exterior mass shell.

We begin with a consideration of the magnetic field $\mathbf{B}^{II}$. The approximations of the relevant integration constants read $\eta_3^{II}=\frac{4}{3}\alpha$, $\lambda_1^{II}=1+\frac{29}{6}\alpha$, $\zeta_1^{II}=-\frac{4}{3}(1+\frac{5}{6}\alpha)$. Herewith we get in the whole region $r\leqslant R$ the constant field

$$B_z^{II}(r\leqslant R)=\frac{2Mq}{3R^2}\,\omega^{II}. \tag{74}$$

In the exterior region $r>R$, the α-independent parts of λ_1^{II} and ζ_1^{II} cancel in the combination $\lambda_1^{II}\hat{g}_1(\rho)+\zeta_1^{II}\overline{\overline{g}}_1(\rho)$, and we are left with

$$B_r^{II}(r\geqslant R)=2Mq\left(\frac{4R}{3r}-\frac{R^2}{r^2}\right)\omega^{II}\sin\vartheta\cos\vartheta. \tag{75}$$

After transformation from our isotropic coordinates to Cartesian vector notation, these results coincide with the results of Hofmann [[3], Eqs. (23), (24)] and, observing the relation $\vec{\Omega}=(-4M/3R)\vec{\omega}$, with the results of Ehlers and Rindler [[2], Eqs. (4.11), (4.12)]. Notwithstanding this agreement in the mathematical results, we should like to add some critical remarks about the method of calculation of these authors and about parts of their physical discussion: In both papers [3] and [2] the electromagnetic field equations are written in a form such that on the left-hand side only the flat differential operators appear, and the curvature induced ''corrections'' appear on the right-hand side as so-called ''fictitious charges and currents.'' Besides the fact that this separation complicates all calculations, it has the misleading consequence that the (fictitious) charge and current distributions extend over the whole space, and have even jumps at the (uncharged) exterior mass shell. Quite generally, such a separation is against the spirit of general relativity. If these authors would perform their calculations in higher orders or even exact in the strength of the gravitational field (as we do in Secs. II and III), they had to introduce new fictitious sources in each order. But as is well known (see, e.g., [18], box 17.2), these terms finally sum up to a ''renormalization'' of the (unobservable) flat metric to the ''real'' curved metric, in our case the rotationally disturbed Reissner-Nordström metric. Hofmann [3] introduces his model as containing a charged sphere with radius a but later speaks about a point charge. It should, however, be clear that the model of a point charge is inconsistent already in classical electrodynamics. And indeed the resulting magnetic field [our Eqs. (74), (75)] is completely independent of the radius a because the charged shell does not carry a ''real current'' (in the local inertial system). Ehlers and Rindler state in their first short paper [1] explicitly that the charged shell has no material mass. In contrast, in the second paper [2] they claim that ''we need not restrict the relative magnitudes of the two shell masses and of the charge.'' However, their detailed calculations and results [e.g., their Eqs. (2.9) and (4.11), (4.12)] are only valid for a massless charged shell, and in first order of M and q. Neither Hofmann nor Ehlers and Rindler discuss the magnetic field $\mathbf{B}^{II}$ under Machian aspects. It should, however, be immediately clear that this field satisfies all Machian expectations: It is constant in the interior and has dipole character in the exterior, like the field of a rotating charged shell. And also the sign and the strength of this field are in full accord with Mach's ideas: For an inertial observer in the region $r\leqslant R$ the charged shell is nonrotating, and the asymptotic observers and the distant cosmic masses, respectively, are rotating with angular velocity $-(4M/3R)\boldsymbol{\omega}$. The magnetic field B_z^{II} of Eq. (74), produced by this system, is then the same as it would be produced in a ''Mach-equivalent situation'' by a shell with charge q and radius $4R/3$, rotating with angular velocity $(4M/3R)\boldsymbol{\omega}$ in a static cosmos. It should also be stressed that the existence of a magnetic field $\mathbf{B}^{II}$ in a system where there are nowhere localized currents (in the local inertial system) makes especially evident that this ''Machian'' field, induced by dragging due to the exterior mass shell and by rotating cosmic masses, respectively, has a nonlocal character. (As is well known—see, e.g., [20]—also in time-dependent systems Machian effects are connected with the nonlocal constraint equations.)

In case I (rotating charged shell) we have contributions in zeroth order of M: $\eta_3^I=2/3\beta$, $4\lambda_2^I=\zeta_1^I=\frac{2}{3}\beta^2$, $\eta_2^I=\lambda_1^I=0$, $\zeta_2^I=\frac{4}{3}\beta^2$, and therefore a M-independent field $\mathbf{B}^I$:

$$B_z^I(r\leqslant a)=\frac{2q}{3a}\,\omega^I,\quad B_r^I(r\geqslant a)=\frac{2qa^2}{3r}\,\omega^I\sin\vartheta\cos\vartheta. \tag{76}$$

Together with $B_\vartheta^I(r\geqslant a)$, this represents, as expected, just the magnetic field of a rotating shell with charge q and radius a from classical electrodynamics. In order to connect this with the work of Ehlers and Rindler [2], we have to observe that these authors do not consider two independent ''stirring'' angular velocities $\overline{\omega^I}$ and ω^{II} but only the case where these two sources work together in such a way that the charged shell stays at rest relative to the asymptotic observers. As explained, e.g., in Fig. 1.3 of Rindler's book [21], this case allows one to describe a Mach-equivalent view of the usual rotating charged shell in flat space-time. According to Eq. (58), $\overline{\omega}(a)=0$ leads in the weak-field limit to $\omega^I=(-4M/3R)\omega^{II}$ [compare Eq. (2.9) in [2]], and herewith Eq. (76) exactly coincide with the field $\mathbf{B}^I$ in Eq. (4.11) of [2]. We should, however, like to argue that also this field, like the field $\mathbf{B}^{II}$, satisfies all Machian expectations, and is by no means ''Mach-negative or, at best, Mach-neutral,'' as Ehlers and Rindler [2] state: On the one hand, a locally nonrotating observer (ZAMO) inside the mass shell (system S' in the notation of [2]) is dragged with angular velocity $(4M/3R)\omega^{II}$; i.e., he sees the charged shell rotating with velocity $(-4M/3R)\omega^{II}$, and expects therefore to measure exactly the magnetic field $\mathbf{B}^I$ of Eq. (76). On the other hand, an asymptotic observer (system S in the notation of [2]) sees a nonrotating charged shell, and therefore no current. But as he notices the rotating mass shell, he knows that according to the standard Thirring effect the interior shell should in fact rotate with angular velocity $(4M/3R)\omega^{II}$. So he concludes that in order to keep the inner shell at rest, this velocity has to be compensated by an angular velocity $\omega^I = (-4M/3R)\omega^{II}$, which in turn is connected with a nonzero angular momentum of the charged shell. So he is by no means surprised about the magnetic field $\mathbf{B}^I$ without observing any current $\mathbf{j}^I$. (The situation here is in some respect just the reverse of case II: there an asymptotic observer notices a current $\mathbf{j}^{II}$, whereas the local inertial observers see no currents producing $\mathbf{B}^{II}$.) Ehlers and Rindler did not analyze how the four-velocity $u^\mu(a) \sim (1,0,0,0)$ of the charged shell is actually realized, and therefore came to the wrong conclusion that the magnetic field $\mathbf{B}^I$ ''as expected on Machian grounds for the dragged frame S', should in fact arise in the 'wrong' frame, S.''

Concerning the terms in second order of the charge q from Sec. IV B in the weak-field limit, we find it especially worthwhile to consider the dragging function $A_2(r)$ in the intermediate region $a \leqslant r \leqslant R$. The total dragging term $\omega A_2(r)$ in the metric form (26), and in the case $\overline{\omega}(a)=0$, considered by Ehlers and Rindler [2], is given by

$$\omega A_2(r) = \omega^{II}\left\{ \mu_3^{II} - \frac{8\alpha}{3}\tilde{\mu}_3^I + \frac{64}{9}\gamma^2 \frac{R}{a} - \frac{64}{27}\gamma^2\alpha\frac{R^2}{a^2} - \frac{32}{9}\gamma^2\alpha\left[\frac{3R}{r} + \frac{Ra(R-a)}{r^3} - \frac{R^2a^2}{r^4}\right]\right\}. \quad (77)$$

We see that an r dependence of $A_2(r)$, as, e.g., shown in Fig. 2, survives also in the weak-field limit, and in the special model considered in [2], and it begins with γ^2 terms, representing the influence of the electromagnetic energy-momentum tensor S^μ_ν. This implies, contrary to the claim of Ehlers and Rindler, that the deviation of the full metric from the flat metric is not the sum of the corresponding contributions from the pure Thirring problem and the Reissner-Nordström problem.

Finally, we should like to make critical remarks about the terms of second order in the angular velocity ω which are calculated and discussed by Ehlers and Rindler [2]. We do not doubt the mathematical correctness of the relevant results for the quadrupolar corrections to the electric field. The question is, however, whether these calculations are physically relevant, especially under a Machian viewpoint, since they ''suffer'' from the notoriously wrong ''centrifugal forces'' in the work of Thirring and followers. (See footnote 4 in [2].) In contrast, it was shown in [5] that it is possible to realize flat space-time and, therefore, a correct centrifugal force inside a rotating mass shell if one allows for a nonspherical deformation of the shell in order ω^2. In [6] and [7] it was demonstrated in addition that this flatness can be preserved in any order ω^n if appropriate (uniquely determined) deformations and differential rotation corrections are introduced. In analogy, it should in principle be possible for our systems of a charged shell within a rotating mass shell to realize flatness inside the charged shell in any order ω^n by introducing appropriate M- and q-dependent deformations and differential rotation corrections for both shells. And we should like to argue that this realization of an electromagnetic Thirring problem is much more natural under Machian aspects than to stick artificially to exactly spherical, rigidly rotating shells also in higher orders of ω, with the consequence that, e.g., the quadrupolar electric field extends even to the interior of the charged shell (see Fig. 2 in [2]).

D. Results exact in mass M and charge q

As remarked in the introduction to Sec. IV, the formulas for the magnetic fields $\mathbf{B}^I$ and $\mathbf{B}^{II}$ and for the dragging functions $A^I(r)$ and $A^{II}(r)$ in the case of general values M and q are too complicated for extracting much obvious physical interpretation. Therefore we restrict ourselves here to a comment on the general case II (rotating exterior shell) and to a broader discussion of the important case of the collapse limit of the two-shell system. (It may, however, be that there exist other physically interesting examples or effects within this class of strong-field rotating two-shell models.)

We observe that in case II all eight integration constants are proportional to the expression $\Delta\tau \sim \tau_3^3(R) - \tau_0^0(R)$, appearing in the energy conditions (23). Therefore, the magnetic field $\mathbf{B}^{II}$ and the dragging function A^{II} are zero for $\Delta\tau = 0$, and change sign if $\Delta\tau$ changes sign; e.g., A^{II} changes from dragging to antidragging. This emphasizes the importance of the discussion of the energy conditions for the mass shell in Sec. II. A similar behavior shows up in the analysis of the Thirring problem with cosmological boundary conditions [22].

Now we come to the collapse limit of our two-shell system, i.e., to the case where a horizon appears at the position $r=R$. In our metric form (2) and with the expression (5) for $U_1(x)$, this obviously happens for $\gamma^2 = \alpha^2 - 1 + \epsilon$ and, $\epsilon \to 0$. In this limit some quantities (partly from the Appendix) diverge: $\Delta\tau$ and $(\mathrm{Det})^I$ like ϵ^{-2}, C_2, C_3, $\tau_3^3(R)$, $R\delta\overline{\overline{g_1'}}(R\delta)$, N, $\tilde{N}$ and $3\delta(\mathrm{Det})^{II}/\Delta\tau$ like ϵ^{-1}, and $\overline{\overline{g_1'}}(R\delta)$ and $\overline{\overline{A_1}}(R\delta)$ like $\log\epsilon^{-1}$. Inserting these results into the formulas for the integration constants in the Appendix, we see that in both cases I (rotating charged shell) and II (rotating mass shell) the constants ζ_1 are zero whereas the constants ζ_2, λ_2 and $\eta_i (i=2,3)$ attain finite values. (In contrast, we have seen in Secs. IV A and IV B that in the limit of weak charges q, e.g., the constant $\tilde{\eta}_3^I = \eta_3^I/C_3$ and, equally, the constants $\tilde{\eta}_2^I$, $\tilde{\lambda}_2^I$, and $\tilde{\zeta}_2^I$ stay finite in the collapse limit. This ''conflict'' obviously shows that the collapse limit and the limit of small charges are not interchangeable.) Concerning the constants μ_i, we see that the expressions $\mu_i - C_3$ are finite; i.e., the constants μ_i diverge like ϵ^{-1} in the collapse limit. From Eqs. (32) and (39) it results then that in both cases I and II we have

$$A(r \leqslant R) \equiv 1. \quad (78)$$

If both ''stirring'' angular velocities $\overline{\omega^I}$ and ω^{II} are active (have finite, nonzero values), we have $\omega A = \omega^I A^I + \omega^{II} A^{II} = (\overline{\omega^I}/C_3)A^I + \omega^{II}A^{II} = \omega^{II}A^{II}$ in the collapse limit. We therefore see that the important result by Brill and Cohen [8], that inside a rotating collapsing mass shell one has total dragging of the inertial systems, transfers also to our highly charged two-shell system. By a similar reasoning ($\omega^I = 0$, if $\overline{\omega^I}$ is finite, and scaling of all magnetic fields by proper time

$\tau = t/C_3 \sim \epsilon t$ instead of coordinate time t inside the mass shell) we see that all ''physical'' magnetic fields vanish inside the mass shell in the collapse limit, and therefore the detailed (finite) values of the constants λ_2, η_2, η_3, and ζ_2 are not of much interest in this limit. (Confined to the first order of q, a comparable discussion was performed in [16] and [17]: the ''intrinsic'' magnetic dipole moment vanishes in the collapse limit, and only the ''induced'' dipole moment survives.)

In the exterior region the essential nonzero integration constant in the collapse limit is $\lambda_1^{II} = 6(1+\alpha)^2$, which, with Eq. (54), gives, for the angular momentum per mass parameter (usually denoted by a), $J^{II}/M = 4(1+\alpha)^2 R^2 \omega^{II}$. With $g_1^{II}(\rho) = 4\lambda_1^{II} R/3\rho$ from Eq. (35) in the collapse limit, the magnetic field components read then

$$B_\rho^{II}(\rho) = \frac{J^{II}}{M}\frac{2q}{\rho}\sin\vartheta\cos\vartheta, \quad B_\vartheta^{II}(\rho) = \frac{J^{II}}{M}\frac{q}{\rho^2}\sin^2\vartheta. \tag{79}$$

Similarly, Eq. (39) results in this limit in

$$A_1^{II}(\rho) = \frac{8R^2(1+\alpha)^2}{\rho^3}\left(M - \frac{q^2}{2\rho}\right). \tag{80}$$

Observing that up to the first order in the angular velocity the Reissner-Nordström coordinate ρ coincides with the Boyer-Lindquist radial coordinate, we see that Eqs. (79), (80) exactly represent the Kerr-Newman field in lowest order of J^{II} (see, e.g., [18], Chaps. 33.2–33.3), as is expected according to the no-hair theorem.

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APPENDIX: DETERMINATION OF THE INTEGRATION CONSTANTS

Before we give explicit solutions of the system of the eight linear continuity and discontinuity conditions for the eight integration constants μ_2, μ_3, λ_1, λ_2, η_2, η_3, ζ_1, and ζ_2 in both cases I and II, we introduce some useful abbreviations:

$$P = \frac{\beta^2\Delta_+^2}{4(\Delta_+^2-\gamma^2)^2}\left[3 + \frac{\Delta_+^2+2\gamma^2}{2\gamma\sqrt{\Delta_+^2-\gamma^2}}\mathrm{arccot}\frac{\Delta_+^2-2\gamma^2}{2\gamma\sqrt{\Delta_+^2-\gamma^2}}\right],$$

$$Q = -\frac{4\beta^2}{3\Delta_+}(\Delta_+^2+2\gamma^2), \tag{A1}$$

$$\tilde{P} = -\frac{\Delta_+^3}{12(\Delta_+^2-\gamma^2)^2}\left[5 - \frac{2\gamma^2}{\Delta_+^2} + \frac{3\Delta_+^2}{2\gamma\sqrt{\Delta_+^2-\gamma^2}}\mathrm{arccot}\frac{\Delta_+^2-2\gamma^2}{2\gamma\sqrt{\Delta_+^2-\gamma^2}}\right],$$

$$\tilde{Q} = \frac{4}{3}\Delta_+^2, \tag{A2}$$

$$K = \frac{4\beta^2\Delta_+}{3\delta} + P\bar{g}_2(R\delta) + Q\bar{\bar{g}}_2(R\delta), \tag{A3}$$

$$L = -\frac{4\beta^2\Delta_+}{3}\sqrt{F_2(R\delta)} + PR\delta\bar{g}_2'(R\delta) + QR\delta\bar{\bar{g}}_2'(R\delta), \tag{A4}$$

$$\tilde{K} = \tilde{P}\bar{g}_2(R\delta) + \tilde{Q}\bar{\bar{g}}_2(R\delta) + \frac{2\Delta_+^2}{3\alpha\delta}(\hat{\alpha}-\alpha), \tag{A5}$$

$$\tilde{L} = \tilde{P}R\delta\bar{g}_2'(R\delta) + \tilde{Q}R\delta\bar{\bar{g}}_2'(R\delta) + \frac{2\Delta_+^2}{3}\left[\sqrt{F_2(R\delta)} - \frac{\hat{\alpha}(1-\alpha^2+\gamma^2)}{\alpha\delta}\right], \tag{A6}$$

$$N = \gamma^2 P[\bar{A}_2(R\Delta_+) - \bar{A}_2(R\delta)] + \gamma^2 Q[\bar{\bar{A}}_2(R\Delta_+) - \bar{\bar{A}}_2(R\delta)] + \frac{\beta^2\gamma^2}{3\delta^4}\left[C_3\left(\delta - \frac{\gamma^2}{\alpha}\right) - (\delta - \Delta_+)\right], \tag{A7}$$

$$\tilde{N} = \frac{1}{8} + \gamma^2\tilde{P}[\bar{A}_2(R\Delta_+) - \bar{A}_2(R\delta)] + \gamma^2\tilde{Q}[\bar{\bar{A}}_2(R\Delta_+) - \bar{\bar{A}}_2(R\delta)] - \frac{\gamma^2\Delta_+}{6\delta^4}\left[C_3\left(\delta - \frac{\gamma^2}{\alpha}\right) - (\delta - \Delta_+)\right]. \tag{A8}$$

We start as in Sec. III B with case II: The linear homogeneous equations (48), (50), and (52) can be easily solved for λ_2, η_2, and ζ_2 as functions of η_3 (noticing that the determinant of the system of equations (50) and (52) is just the Wronskian of Eq. (34): $(\bar{g}_2 R\bar{\bar{g}}_2' - \bar{\bar{g}}_2 R\bar{g}_2')|_{\rho=R\Delta_+} = -3\Delta_+/\beta)$:

$$\lambda_2^{II} = \beta^2\Delta_+\eta_3^{II}, \quad \eta_2^{II} = P\eta_3^{II}, \quad \zeta_2^{II} = Q\eta_3^{II}. \tag{A9}$$

Equations (47), (49), and (51) constitute then a complete system of linear equations for the constants λ_1, ζ_1, and η_3, with the solutions

$$\lambda_1^{II} = \frac{-\Delta\tau}{4(\mathrm{Det})^{II}}[L\bar{\bar{g}}_1(R\delta) - KR\delta\bar{\bar{g}}_1'(R\delta)], \tag{A10}$$

$$\zeta_1^{II} = \frac{\Delta\tau}{3\delta(\mathrm{Det})^{II}}[(1-\alpha^2+\gamma^2)K+L], \tag{A11}$$

$$\eta_3^{II} = \frac{\Delta\tau}{3\delta(\mathrm{Det})^{II}}[(1-\alpha^2+\gamma^2)\overline{\overline{g}}_1(R\delta) + R\delta\overline{\overline{g}}_1'(R\delta)], \tag{A12}$$

where $(\mathrm{Det})^{II}$ is the 3×3 determinant of the system

$$\frac{3\delta(\mathrm{Det})^{II}}{\Delta\tau} = \left[\frac{2(\gamma^2-\alpha\delta)}{\delta^3} + \frac{3\alpha\delta}{4\Delta\tau}\right][L\overline{\overline{g}}_1(R\delta) - KR\delta\overline{\overline{g}}_1'(R\delta)] + \frac{\beta^2\gamma^2}{\Delta\tau}[(1-\alpha^2+\gamma^2)\overline{\overline{g}}_1(R\delta) + R\delta\overline{\overline{g}}_1'(R\delta)] + 8\gamma^2\overline{\overline{A}}_1(R\delta)[(1-\alpha^2+\gamma^2)K+L]. \tag{A13}$$

The remaining integration constants μ_2 and μ_3 can easily be calculated from the continuity conditions for the dragging function $A(r)$ at the positions $r=R$ and $r=a$:

$$\frac{1}{8}\mu_2^{II} = C_3\frac{\alpha\delta-\gamma^2}{3\delta^4}\lambda_1^{II} + C_3\gamma^2\overline{\overline{A}}_1(R\delta)\zeta_1^{II} - \gamma^2\left[\frac{\beta^2(\delta-\Delta_+)}{3\delta^4} + P\overline{A}_2(R\delta) + Q\overline{\overline{A}}_2(R\delta)\right]\eta_3^{II}, \tag{A14}$$

$$\frac{1}{8}\mu_3^{II} = \frac{1}{8}\mu_2^{II} + \gamma^2[P\overline{A}_2(R\Delta_+) + Q\overline{\overline{A}}_2(R\Delta_+)]\eta_3^{II}. \tag{A15}$$

In case I we get, from Eq. (46), $\lambda_1^I = (\hat{\alpha}/\alpha)\lambda_2^I$. Similar to the situation in case II, the linear homogeneous equations (50), (55) [together with the equivalent to Eq. (45)], and (56) can again be solved for λ_2, η_2, and ζ_2 as functions of η_3 and $(C_3-\mu_3)$:

$$\lambda_2^I = \beta^2\Delta_+\eta_3^I - \frac{\Delta_+^2}{2}(C_3-\mu_3^I),$$

$$\eta_2^I = P\eta_3^I + \tilde{P}(C_3-\mu_3^I), \quad \zeta_2^I = Q\eta_3^I + \tilde{Q}(C_3-\mu_3^I). \tag{A16}$$

Since in Eqs. (49) and (51) then also the integration constant μ_3 appears, these equations have to be combined with the equivalent of Eq. (A15):

$$N\eta_3 + \gamma^2C_3\overline{\overline{A}}_1(R\delta)\zeta_1 + \tilde{N}(C_3-\mu_3) = \frac{1}{8}C_3. \tag{A17}$$

These three equations constitute a complete system of linear equations for the constants ζ_1, η_3, and μ_3,

$$\zeta_1^I = \frac{C_3}{8(\mathrm{Det})^I}\left\{(\tilde{L}K-\tilde{K}L) - \frac{4\beta^2\gamma^2}{3\alpha\delta}[(1-\alpha^2+\gamma^2)\tilde{K}+\tilde{L}]\right\}, \tag{A18}$$

$$\eta_3^I = \frac{C_3}{8(\mathrm{Det})^I}[\tilde{L}\overline{\overline{g}}_1(R\delta) - \tilde{K}R\delta\overline{\overline{g}}_1'(R\delta)], \tag{A19}$$

$$\mu_3^I = C_3 + \frac{C_3}{8(\mathrm{Det})^I}\left\{[L\overline{\overline{g}}_1(R\delta) - KR\delta\overline{\overline{g}}_1'(R\delta)] + \frac{4\beta^2\gamma^2}{3\alpha\delta}[(1-\alpha^2+\gamma^2)\overline{\overline{g}}_1(R\delta) + R\delta\overline{\overline{g}}_1'(R\delta)]\right\}, \tag{A20}$$

where $(\mathrm{Det})^I$ is again the 3×3 determinant of this system of equations:

$$(\mathrm{Det})^I = C_3\gamma^2\overline{\overline{A}}_1(R\delta) \times\left\{(\tilde{L}K-\tilde{K}L) - \frac{4\beta^2\gamma^2}{3\alpha\delta}[(1-\alpha^2+\gamma^2)\tilde{K}+\tilde{L}]\right\} - \frac{4\beta^2\gamma^2}{3\alpha\delta}\tilde{N}[(1-\alpha^2+\gamma^2)\overline{\overline{g}}_1(R\delta) + R\delta\overline{\overline{g}}_1'(R\delta)] + N[\tilde{L}\overline{\overline{g}}_1(R\delta) - \tilde{K}R\delta\overline{\overline{g}}_1'(R\delta)] - \tilde{N}[L\overline{\overline{g}}_1(R\delta) - KR\delta\overline{\overline{g}}_1'(R\delta)]. \tag{A21}$$

It is noteworthy that the constants K, L, $\tilde{K}$, and $\tilde{L}$ are free of arccot terms, but all integration constants $\lambda_i, \eta_i, \zeta_i, \mu_i$ contain such terms. The remaining constant μ_2^I can in principle be calculated from the equivalent of Eq. (A14).

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