CPT anomaly in two-dimensional chiral $U(1)$ gauge theories

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The *CPT* anomaly, which was first seen in perturbation theory for certain four-dimensional chiral gauge theories, is also present in the exact result for a class of two-dimensional chiral $U(1)$ gauge theories on the torus. Specifically, the chiral determinant for periodic fermion fields changes sign under a *CPT* transformation of the background gauge field. There is, in fact, an anomaly of Lorentz invariance, which allows for the *CPT* theorem to be circumvented.

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I. INTRODUCTION

Recently, a *CPT* anomaly has been found in certain four-dimensional chiral gauge theories, with the topology and spin structure of the spacetime manifold playing a crucial role [1]. The well-known *CPT* theorem [2] is circumvented by the breakdown of Lorentz invariance at the quantum level [1,3]. The calculation of Ref. [1] was done perturbatively and more or less the same type of anomaly was expected to occur in appropriate higher- and lower-dimensional chiral gauge theories. Here, we consider the two-dimensional chiral $U(1)$ gauge theory over the torus, for which the chiral determinant is known *exactly* [4–6]. The aim of this paper is to determine whether or not the exact result contains the *CPT* anomaly and perhaps to learn more about the anomaly itself [7].

II. CHIRAL DETERMINANT

We consider in this Brief Report two-dimensional Euclidean chiral $U(1)$ gauge theory, defined over the torus T^2. For simplicity, we take a particular torus (modulus $\tau=i$), with Cartesian coordinates $x^\mu \in [0,L]$, $\mu=1,2$, and Euclidean metric $g_{\mu\nu}=\delta_{\mu\nu}$. The theory has the fermionic action

$$S[A,\overline{\psi},\psi]=-\int_0^L \mathrm{d}x^1 \int_0^L \mathrm{d}x^2\, \overline{\psi}\, \sigma^\mu(\partial_\mu + iA_\mu)\psi, \tag{1}$$

with $\sigma^1=1$ and $\sigma^2=i$. The boundary conditions for the real gauge potential $A(x)\equiv A_\mu(x)\mathrm{d}x^\mu$ and the 1-component Weyl field $\psi(x)$ are *both* taken to be periodic:

$$A(x^1+mL,x^2+nL)=A(x^1,x^2),$$
$$\psi(x^1+mL,x^2+nL)=\psi(x^1,x^2), \tag{2}$$

for arbitrary integers m and n.

The two-dimensional gauge potential in the trivial topological sector can be decomposed as follows [4]:

$$A_\mu(x)=\epsilon_{\mu\nu} g^{\nu\rho}\partial_\rho \phi(x) + 2\pi h_\mu/L + \partial_\mu \chi(x), \tag{3}$$

with $\phi(x)$ and $\chi(x)$ real periodic functions and h_μ real constants (the harmonic pieces of the gauge potential). Here, $\chi(x)$ corresponds to the gauge degree of freedom. Furthermore, the gauge potential $A_\mu(x)$ is taken to be smooth, i.e. without delta-function singularities.

The chiral determinant (the exponential of minus the Euclidean effective action) is then given by the following functional integral:

$$D^{\mathrm{PP}}[A]\equiv \exp(-\Gamma^{\mathrm{PP}}[A]) = \int_{\mathrm{PP}} \mathcal{D}\psi \mathcal{D}\overline{\psi}\, \exp(-S[A,\overline{\psi},\psi]), \tag{4}$$

where PP indicates the doubly-periodic boundary condition (2) on the fermion field. This chiral determinant has been calculated using various regularization methods. See Refs. [4–6] and references therein. Reference [6], in particular, introduces a local counterterm to restore translation invariance and obtains the following result [9]:

$$D^{\mathrm{PP}}[A]=\hat{\vartheta}\left(h_1+\frac{1}{2},h_2+\frac{1}{2}\right)\exp\left(\frac{i\pi}{2}(h_1-h_2)\right)$$
$$\times \exp\left(\frac{1}{4\pi}\int \mathrm{d}^2x\,(\phi\partial^2\phi + i\phi\partial^2\chi)\right), \tag{5}$$

with, for real variables k_1 and k_2, the definition [5,6]

$$\hat{\vartheta}(k_1,k_2)\equiv \exp[-\pi(k_2)^2 + i\pi k_1 k_2]\,\vartheta(k_1+ik_2;i)/\eta(i), \tag{6}$$

in terms of the Riemann theta function and Dedekind eta function

$$\vartheta(z;\tau)\equiv \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z),$$

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$$\eta(\tau) \equiv \exp(\pi i \tau/12) \prod_{m=1}^{\infty} [1 - \exp(2\pi i m \tau)]. \tag{7}$$

The result (5) holds for the chiral determinant of a single positive chirality (right-moving) Weyl fermion of unit charge; cf. Eq. (1). If the charge is q_{R1} instead, then the variables h_μ, $\phi(x)$, and $\chi(x)$ in Eq. (5) each need to be multiplied by a factor q_{R1}. For a negative chirality (left-moving) Weyl fermion of charge q_{L1}, one also has to take the complex conjugate of the whole expression (5). For the 345-model (three chiral fermions with charges $q_{R1}=3$, $q_{R2}=4$, and $q_{L3}=5$), one obtains the following chiral determinant [6]:

$$D_{345}^{\mathrm{PP}}[A] = D^{\mathrm{PP}}[3A] D^{\mathrm{PP}}[4A] (D^{\mathrm{PP}}[5A])^*. \tag{8}$$

The chiral determinant (8) of the 345-model is gauge invariant. Indeed, it is straightforward to verify both the χ independence and the invariance under large gauge transformations $h_\mu \to h_\mu + n_\mu$ for arbitrary integers n_μ [10]. We will first focus on this particular chiral model. Other chiral models will be discussed later.

III. *CPT* NONINVARIANCE

The question, now, is how the gauge-invariant chiral determinant (8) of the 345-model behaves under a *CPT* transformation of the background gauge field:

$$A_\mu(x) \to A_\mu^{CPT}(x) \equiv -A_\mu(-x). \tag{9}$$

Using the elementary properties of the theta function [11], one finds

$$D_{345}^{\mathrm{PP}}[A^{CPT}] = -D_{345}^{\mathrm{PP}}[A], \tag{10}$$

with each of the three chiral fermions contributing a multiplicative factor -1 on the right-hand side. Hence, the effective action of the chiral $U(1)$ gauge theory with PP spin structure over the torus *changes* under a *CPT* transformation (9) of the background gauge field, provided the total number (N_F) of charged chiral fermions of the theory is *odd* (e.g. $N_F=3$ for the 345-model). The result (10) thus provides conclusive evidence for a *CPT* anomaly of the chiral model considered.

The asymmetry (10) implies the vanishing of the chiral determinant (8) for $A_\mu(x)=0$. For gauge fields (3) with $\phi(x)=\chi(x)=0$ and infinitesimal harmonic pieces h_μ, one has, in fact,

$$D_{345}^{\mathrm{PP}}[h_1,h_2] = c(h_1 + i h_2)(h_1^2 + h_2^2) + \mathrm{O}(h^5), \tag{11}$$

with a nonvanishing complex constant c. This result follows from the observation that the analytic function $\vartheta(z;i)$ appearing in Eq. (5) has a simple zero at $z=(1+i)/2$. More directly, the holomorphic factor (h_1+ih_2) in Eq. (11) corresponds to one of the eigenvalues of the Weyl operator $\sigma^\mu(\partial_\mu + iA_\mu)$ with doubly-periodic boundary condition and constant gauge potential, as do the holomorphic and antiholomorphic factors contained in $(h_1^2+h_2^2)$. Equation (11) agrees, of course, with the general result (10) on *CPT* violation. But the real importance of Eq. (11) is that, for this special case, the *origin* of the two-dimensional *CPT* anomaly can be identified explicitly, namely one particular eigenvalue of the Weyl operator. (See [12] for further details.)

The chiral determinant [6] of the 345-model over the torus is *CPT* invariant for the other spin structures AA, PA, and AP, where (A)P stands for (anti-)periodic boundary conditions on the fermion fields (the three fermion species being treated equally). This appears to be related to the observation that the *CPT* anomaly is not expected for the AA spin structure [1,3] and the fact that the chiral determinants [6] for the AA, PA, and AP spin structures transform into each other under modular transformations (global diffeomorphisms; cf. Ref. [4]), whereas the chiral determinant of the PP spin structure is invariant up to a phase. It is important to realize that this extra requirement of modular invariance for the AA, PA, and AP spin structures restricts the type of theories considered and also possible regularization methods [13]. For the general question of how to sum over the different spin structures, see, for example, the discussion in Refs. [15,16]. In our case, the two-dimensional *CPT* anomaly would be present as long as the PP spin structure appears in the sum.

IV. LORENTZ NONINVARIANCE

Given that *CPT* invariance no longer holds for the 345-model with doubly-periodic spin structure over the torus, $SO(1,1)$ Lorentz invariance, or rather $SO(2)$ invariance for the Euclidean theory, is expected to be broken as well [1,3]. Concretely, this can be tested by comparing the (translation-invariant) chiral determinant (8) for two different, *localized* gauge fields which are related by a Lorentz transformation [17].

Consider, for example, a gauge potential $\tilde{A}_\mu(x)$ which, up to periodicity, is allowed to be nonzero only for $|x^\mu - L/2| < l$, with a fixed length $l \ll L/2$, and which has infinitesimal, but nonvanishing, harmonic pieces $\tilde{h}_\mu \equiv (2\pi L)^{-1} \int d^2x \tilde{A}_\mu(x)$. In other words, the gauge potential $\tilde{A}_\mu(x)$ has local support (set by l) and produces small, but nonzero, averages $\tilde{h}_\mu$ (typically of order l/L). According to Eq. (11), the chiral determinant for this gauge field is then proportional to $(\tilde{h}_1 + i\tilde{h}_2) = \sigma^\mu \tilde{h}_\mu$. Similarly, the chiral determinant for the $SO(2)$ Lorentz transformed (''boosted'') gauge potential,

$$\begin{pmatrix} \tilde{A}'_1(x) \\ \tilde{A}'_2(x) \end{pmatrix} = \Lambda \begin{pmatrix} \tilde{A}_1(\Lambda x) \\ \tilde{A}_2(\Lambda x) \end{pmatrix}, \quad \Lambda \equiv \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{12}$$

is proportional to $\sigma^\mu \tilde{h}'_\mu$. But these two particular factors differ by a phase factor $\exp(i\alpha)$, as can be readily verified. All other factors of the two chiral determinants being equal, this then implies

$$D^{\rm PP}_{345}[\tilde{A}'] = \exp(i\alpha) D^{\rm PP}_{345}[\tilde{A}]. \tag{13}$$

Note that Eq. (13), for $\alpha=\pi$, agrees with the previous result (10). Also note that the noninvariance of the factor $\sigma^{\mu}\tilde{h}_{\mu}$ in the chiral determinant directly carries over to the theory with Minkowskian metric $g_{\mu\nu}=\text{diag}\ (+1,-1)$. In short, the Lorentz invariance of the chiral determinant (8) for the localized gauge field $\tilde{A}_{\mu}(x)$ is broken through its $\tilde{h}_{\mu}$ dependence. [The term $\int d^2x \tilde{\phi}\partial^2\tilde{\phi}$ from Eq. (5) is, of course, Lorentz invariant.]

As far as the gauge potential is concerned, this localized configuration $\tilde{A}_{\mu}(x)$ could also have been embedded in the Euclidean plane $\mathbb{R}^2$. The Lorentz noninvariance of the effective gauge field action comes from the chiral fermions which are sensitive to the topology of the torus T^2. More physically, the periodic boundary conditions *predispose* the chiral fermions of the 345-model to select specific $\tilde{h}_{\mu}$-dependent terms from the local interaction with the gauge field. These special terms in the effective action then make the local dynamics of the (classical) gauge field $\tilde{A}_{\mu}(x)$ Lorentz noninvariant.

V. OTHER CHIRAL MODELS

Up until now, we have focused on the 345-model, which has an odd number of charged chiral fermions ($N_F=3$). A chiral model with even N_F does not have the *CPT* anomaly discussed above, but can still be Lorentz noninvariant. An example for $N_F=10$ would be the $1^9 3$-model, which has ten chiral fermions with charges $q_{Ri}=1$, for $i=1,\ldots,9$, and $q_{L10}=3$. For this model, the chiral determinant (11) becomes

$$D^{\rm PP}_{1^9 3}[h_1,h_2] = c'(h_1+ih_2)^8(h_1^2+h_2^2) + \text{O}(h^{12}), \tag{14}$$

which is invariant under the *CPT* transformation (9), but changes under the $SO(2)$ Lorentz transformation (12) by a phase factor $\exp(i8\alpha)$. On the other hand, a chiral model with even N_F can also be Lorentz invariant, in the sense discussed above. An example would be the chiral model with $N_F=6$ chiral fermions of charges $\{q_R\}=\{3,4,13\}$ and $\{q_L\}=\{5,5,12\}$, for which the chiral determinant is $c''(h_1^2+h_2^2)^3$ to lowest order. (Vectorlike models, which have $\{q_R\}=\{q_L\}$, are always Lorentz invariant.) Clearly, a deeper understanding of what distinguishes these gauge-invariant chiral models remains to be desired.

VI. DISCUSSION

For the two-dimensional chiral $U(1)$ gauge theory with an odd number N_F of charged chiral fermions defined over the torus, we have thus seen that the *CPT* noninvariance of the effective gauge field action $\Gamma^{\rm PP}[A]$ is carried by the harmonic pieces h_{μ} of the gauge fields $A_{\mu}(x)$. These h_{μ} are of the same type as the local Chern-Simons-like terms encountered previously in four dimensions [1,3]. Indeed, the Chern-Simons one-form for an one-dimensional Abelian $U(1)$ gauge field $a(x)$ is given by

$$\omega_{\rm CS}[a] \equiv (2\pi)^{-1} a(x)\,dx. \tag{15}$$

One possible two-dimensional Chern-Simons-like term is then the average over the x^2 coordinate of $2\pi i$ times the genuine Chern-Simons term for the x^1 space S^1, namely

$$\begin{aligned}\Gamma^{S^1\times S^1}_{\text{CS-like},1}[A] &\equiv \int_0^L \frac{dx^2}{L}\left(2\pi i\int_{S^1}\omega_{\rm CS}[A_1]\right)\\ &= i\int_0^L dx^1\int_0^L dx^2 A_1(x^1,x^2)/L\\ &= 2\pi i h_1,\end{aligned} \tag{16}$$

where h_1 is defined by Eq. (3). The other two-dimensional Chern-Simons-like term (based on the genuine Chern-Simons term for the x^2 space) equals $2\pi i h_2$. Hence, Chern-Simons-like terms play a role for the *CPT* anomaly in both two and four dimensions. There is, however, a difference, in that the four-dimensional Chern-Simons-like term immediately affects the gauge field propagation, with the vacuum becoming optically active [1,18].

In closing, we remark that the *CPT* noninvariance found here appears to be not directly related to the purely gravitational anomaly which afflicts Weyl fermions in two dimensions ($4k+2$ dimensions in general) [8]. The gravitational anomaly (breakdown of general coordinate invariance) of the two-dimensional 345-model, say, shows up for deviations from the Euclidean metric $g_{\mu\nu}=\delta_{\mu\nu}$, but in our case the metric is perfectly Euclidean and, still, the effective gauge field action $\Gamma^{\rm PP}[A]$ is *CPT* noninvariant. Instead of local spacetime fluctuations, it is the spacetime topology (and spin structure) that is relevant to the *CPT* anomaly. The *CPT* anomaly resembles in this respect the so-called topological Casimir effect [19].

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[9] To arrive at Eq. (5) we have chosen factors $P[b_1] = +\frac{1}{2}$ and $P[b_2] = +\frac{1}{2}$ in Eq. (2.18) of Ref. [6] but other signs are also possible, as long as they are fixed once and for all.

[10] For the 345-model with PP spin structure, the chiral determinant from Ref. [5] would not be invariant under arbitrary large gauge transformations. This shows that the extra phase factor $\exp[i\pi(h_1 - h_2)/2]$ in Eq. (5), which traces back to the local counterterm of Ref. [6], also plays a role in maintaining the Abelian gauge invariance of the full chiral determinant (8).

[11] The relevant properties of $\vartheta(z;\tau)$ are its periodicity under $z \to z+1$ and quasi-periodicity under $z \to z+\tau$, together with the symmetry $\vartheta(-z;\tau) = \vartheta(z;\tau)$. We take the opportunity to correct Eq. (4.13) of Ref. [6], which must have an extra factor $\exp(i\pi n_1 n_2)$ on the right-hand side.

[12] For constant gauge potentials $A_\mu(x) = 2\pi h_\mu / L$, the single chiral determinant $D^{\mathrm{PP}}[h_1, h_2]$ from Eq. (4) is, formally, proportional to $\Pi(n_1 + h_1 + i n_2 + i h_2)$, with the product running over the integers n_1 and n_2. The *CPT*-odd factor $(h_1 + i h_2)$ is manifest for $n_1 = n_2 = 0$ [which corresponds to the eigenvector $\psi(x) = \mathrm{const}$ of the Weyl operator], whereas the other contributions combine into *CPT*-even factors whose product still needs to be regularized appropriately. The full chiral determinant (8) of the 345-model then has factors $3(h_1 + i h_2)$, $4(h_1 + i h_2)$, and $5(h_1 - i h_2)$, which combine to give the result (11).

[13] For the two-dimensional chiral $U(1)$ gauge theory on a torus, we have also calculated the chiral determinant for constant gauge potentials $A_1(x) = 2\pi h_1 / L$ and $A_2(x) = 0$, using the partial regularization method adopted in Ref. [1]. For this setup, the Fourier modes of the x^2 direction decouple and each of them can be regarded as a one-dimensional Dirac fermion in x^1 space. The regularization method then amounts to introducing one-dimensional Pauli-Villars fields over x^1 space for each Fourier mode from the x^2 direction. The result for the chiral determinant differs in general from the one obtained in Ref. [6] and behaves differently under modular transformations. In particular, *CPT* violation is observed if the boundary conditions on the fermion fields are taken to be periodic in the x^2 direction and either periodic or antiperiodic in the x^1 direction, whereas the result from Ref. [6] is *CPT* violating only for the doubly-periodic (PP) spin structure, as shown by Eq. (10). The apparent regularization dependence of the theory deserves further study; cf. Ref. [14]. Still, the existence of a *CPT* anomaly for the PP spin structure is essentially independent of the ultraviolet regularization, as discussed below Eq. (11) and in [12].

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