Neutrino masses and mixing angles in a realistic string-inspired model

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We analyze a supersymmetric string-inspired model of all fermion masses and mixing angles based on the Pati-Salam $SU(4)\times SU(2)_L\times SU(2)_R$ gauge group supplemented by a $U(1)_X$ flavor symmetry. The model involves third family Yukawa unification and predicts the top quark mass and the ratio of the vacuum expectation values $\tan\beta$. The model also provides a successful description of the CKM matrix and predicts the masses of the down and strange quarks. However, our main focus is on the neutrino masses and MNS mixing angles, and we show how the recent atmospheric neutrino mixing observed by Super-Kamiokande and the MSW solution to the solar neutrino problem lead to important information about the flavor structure of the model near the string scale. We show how single right-handed neutrino dominance may be implemented by the use of ''Clebsch zeros,'' leading to the LMA MSW solution, corresponding to bimaximal mixing. The LOW MSW and SMA MSW solutions are also discussed.

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I. INTRODUCTION

The problem of understanding the quark and lepton masses and mixing angles represents one of the major unsolved questions of the standard model. Recently additional information on the fermion mass spectrum has come from the measurement of the atmospheric neutrino masses and mixing angles by Super-Kamiokande [1]. The most recent data disfavors mixing involving a sterile neutrino, and finds a good fit for $\nu_\mu\to\nu_\tau$ mixing with $\sin^2(2\theta_{23})>0.88$ and a mass square splitting Δm^2_{23} in the 1.5–5×10^{-3} eV^2 range at 90% C.L. [2]. Super-Kamiokande has also provided additional support for solar neutrino mixing. The most recent Super-Kamiokande data do not show a significant day-night asymmetry and show an energy independent neutrino spectrum; thus it also disfavors the sterile neutrino mixing hypothesis, the just-so vacuum oscillation hypothesis, and the small mixing angle (SMA) Mikheyev-Smirnov-Wolfenstein (MSW) [3] solution [4]. The preferred solution at the present time seems to be the large mixing angle (LMA) MSW solution, although a similar solution with a low mass splitting (LOW) solution is also possible. A typical point in the LMA MSW region is $\sin^2(2\theta_{12})\approx0.75$ and $\Delta m^2_{12}\approx2.5\times10^{-5}$ eV^2 [5].

If one accepts the recent data as evidence for neutrino masses and mixing angles, then the obvious question is how these can be accommodated in the standard model or one of its supersymmetric extensions. The simplest possibility to account for the smallness of the neutrino masses is the seesaw mechanism [6] in which one introduces right-handed neutrinos which acquire very large Majorana masses at a super-heavy mass scale. When one integrates out the right-handed neutrinos the ''normal sized'' Dirac Yukawa couplings, which connect the left-handed to the right-handed neutrinos, are transformed into very small couplings which generate very light effective left-handed physical Majorana neutrino masses. Given the seesaw mechanism, it is natural to expect that the spectrum of the neutrino masses will be hierarchical, since the Dirac Yukawa couplings in the charged fermion sector are observed to be hierarchical, and if they are related to the Dirac neutrino Yukawa couplings, then they should also be hierarchical, leading to hierarchical light Majorana masses.[1]

Having assumed the seesaw mechanism and a hierarchical neutrino mass spectrum, the next question is how such large (almost maximal) lepton mixing angles such as θ_{23} could emerge. There are several possibilities that have been suggested in the literature. One possibility is that it happens as a result of the off-diagonal 23 entries in the left-handed Majorana matrix being large and the determinant of the 23 sub-matrix being accidentally small, leading to a neutrino mass hierarchy with large neutrino mixing angles [8]. Another possibility is that the neutrino mixing angles start out small at some high energy scale and then get magnified by renormalization group (RG) running down to low energies [9]. A third possibility is that the off-diagonal elements of the left-handed neutrino Majorana matrix are large, but the 23 sub-determinant of the matrix is small for a physical reason, as would be the case if a single right-handed neutrino were providing the dominant contribution to the 23 sub-matrix [10–12]. We shall refer to these three approaches as the accidental, the magnification and the single right-handed neutrino dominance (SRHND) mechanisms, respectively. As we shall see, in the model under consideration, only the SRHND mechanism provides a successful description of the atmospheric neutrino data, and the results in this paper will rely on this mechanism.

A promising approach to understanding the fermion mass spectrum is within the framework of supersymmetric (SUSY) unified theories. Within the framework of such theories the quark and lepton masses and mixing angles become related to each other, and it begins to be possible to understand the spectrum. The simplest grand unified theory (GUT) is $SU(5)$ but this theory in its minimal version does not contain any right-handed neutrinos. Nevertheless, three right-handed neutrinos may be added, and in this theory it is

[1]However, this is not guaranteed due to the unknown structure of the heavy Majorana matrix, and for example an inverted neutrino mass hierarchy could result although this relies on some non-hierarchical couplings in the Dirac Yukawa matrix [7].

possible to have a large 23 element[2] on the Dirac neutrino Yukawa matrix without introducing a large 23 element into any of the charged fermion Yukawa matrices. The problem of maintaining a 23 neutrino mass hierarchy in these models may be solved for example by assuming SRHND [13]. Another possibility within the framework of $SU(5)$ is to maintain all the off-diagonal elements to be small, but require the 22 and 32 elements of the Dirac neutrino Yukawa matrix to be equal and the second right-handed neutrino to be dominant, in which case SRHND again leads to a large 23 neutrino mixing angle with hierarchical neutrino masses [14]. However, the drawback of $SU(5)$ is that it does not predict any right-handed neutrinos, which must be added as an afterthought.

From the point of view of neutrino masses, the most natural GUTs are those like $SO(10)$ that naturally predict right-handed neutrinos. However, within the framework of $SO(10)$ the quark masses and mixing angles are related to the lepton masses and mixing angles, and the existence of large neutrino mixing angles is not expected in the minimal versions of the theory in which the Higgs doublets are in one (or two) $\mathbf{10}'$ [ten dimensional representations of $SO(10)$] and each matter family is in a **16**. Nevertheless, various possibilities have been proposed in $SO(10)$ in order to account for the large neutrino mixing angles. Within the framework of minimal $SO(10)$ with third family Yukawa unification, it has been suggested that if two operators with different Clebsch coefficients contribute with similar strength, then, with a suitable choice of phases, in the case of the lepton Yukawa matrices one may have large numerical 23 elements, which add up to give a large lepton mixing angle, while for the quarks the 23 elements can be small due to approximate cancellation of the two contributing operators [15]. This is an example of the accidental mechanism mentioned above, where in addition one requires the quark mixing angles to be small by accident, although it remains to be seen if the LMA MSW solution can be understood in this framework. Moving away from minimal $SO(10)$, one may invoke a non-minimal Higgs sector in which one Higgs doublet arises from a **10** and one from a **16**, and in this framework it is possible to understand atmospheric neutrino mixing [16]. Alternatively, one may invoke a non-minimal matter sector in which parts of a quark and lepton family arise from a **16** and other parts from a **10**, and in these models one may account for atmospheric and solar neutrinos via an inverted mass hierarchy mechanism [17].

In the present paper we shall discuss neutrino masses and mixing angles in a particular string-inspired *minimal* model based on the Pati-Salam $SU(4)\times SU(2)_L\times SU(2)_R$ (422) group [18]. As in $SO(10)$ the presence of the gauged $SU(2)_R$ predicts the existence of three right-handed neutrinos. However, unlike $SO(10)$, there is no Higgs doublet-triplet splitting problem since in the minimal model both Higgs doublets are contained in a (1,2,2) representation. Moreover, since the left-handed quarks and leptons are in the (4,2,1) and the right-handed quarks and leptons in the (4,1,2) representations, the model also leads to third family Yukawa unification as in minimal $SO(10)$. Although the Pati-Salam gauge group is not unified at the field theory level, it readily emerges from string constructions either in the perturbative fermionic constructions [19] or in the more recent type I string constructions [20], unlike $SO(10)$ which typically requires large Higgs representations which do not arise from the simplest string constructions. The question of fermion masses and mixing angles in the string-inspired Pati-Salam model has already been discussed for the case of charged fermions [21,22] and later for the case of neutrinos [23]. For the neutrino study [23] it was assumed that the heavy Majorana neutrino mass matrix was proportional to the unit matrix, and only small neutrino mixing angles were considered. Later on, a $U(1)_X$ family symmetry was added to the model, in order to understand the horizontal hierarchies, although in this case the neutrino spectrum was not analyzed at all [24].

The purpose of the present paper is to discuss neutrino masses and mixing angles in the string-inspired Pati-Salam model supplemented by a $U(1)_X$ flavor symmetry. The model involves third family Yukawa unification and predicts the top quark mass and the ratio of the vacuum expectation values $\tan\beta$, as we recently discussed in Ref. [25]. It is already known that the model can provide a successful description of the Cabibbo-Kobayashi-Maskawa (CKM) matrix and predicts the down and strange quark masses, although our present analysis differs from that presented previously [24] partly due to recent refinements in third family Yukawa unification [25], but mainly as a result of recent Super-Kamiokande data which has important implications for the flavor structure of the model. In fact our main focus here is on the neutrino masses and mixing angles which were not previously discussed at all in this framework. We assume a minimal version of the model, and avoid the use of the accidental cancellation mechanism, which in any case has difficulties in accounting for bi-maximal neutrino mixing. We also show that the mixing angle magnification mechanism can only provide limited increases in the mixing angles, due to the fact that the unified third family Yukawa coupling is only approximately equal to 0.7 [25] and is therefore too small to have a dramatic effect. Instead, we rely on the SRHND mechanism, and we show how this mechanism may be implemented in the 422 model by appropriate use of operators with ''Clebsch zeros'' resulting in a natural explanation for atmospheric neutrinos via a hierarchical mass spectrum. We specifically focus on the LMA MSW solution in the text, with the LOW and SMA MSW solutions relegated to Appendixes.

The layout of the remainder of the paper is as follows. In Sec. II we briefly review the seesaw mechanism in the minimal supersymmetric standard model (MSSM) [26] with right-handed neutrinos. In Sec. III we introduce the string-inspired Pati-Salam model, and in Sec. IV we introduce an Abelian anomalous gauge $U(1)_X$ family symmetry into the model, and show how horizontal Yukawa hierarchies may be generated. In Sec. V we describe our operator approach to

[2]We use the left-right (LR) convention for Yukawa matrices in this paper.

fermion masses, including the heavy Majorana neutrino masses. Section VI contains the main results of the paper. In this section we show how a particular choice of $U(1)_X$ family charges and operators with certain Clebsch coefficients can lead to a successful description of quark and lepton masses and mixing angles, and in particular describe atmospheric and solar neutrinos via SRHND. SRHND is reviewed in Appendix B. Although the neutrino masses and mixing angles correspond to the usual LMA MSW solution, in Appendix D we show how a modification of the heavy Majorana mass matrix can lead to a large mixing angle MSW solution with a LOW mass splitting. In Appendix E we present a different choice of $U(1)_X$ charges and operators which can lead to the SMA MSW solution.

II. MSSM WITH RIGHT-HANDED NEUTRINOS

The superpotential of the MSSM with right-handed neutrinos is given by

$$\mathcal{W}=\mathcal{W}_{MSSM}+\mathcal{W}_{\nu^c} \tag{1}$$

$$\mathcal{W}_{MSSM}=q_A(\lambda_u)_{AB}u^c_Bh_u-q_A(\lambda_d)_{AB}d^c_Bh_d -l_A(\lambda_e)_{AB}e^c_Bh_d+\mu h_uh_d \tag{2}$$

$$\mathcal{W}_{\nu^c}=l_A(\lambda_\nu)_{AB}\nu^c_Bh_u+\tfrac{1}{2}\nu^c_A(M_{RR})_{AB}\nu^c_B \tag{3}$$

where $A,B=1,\ldots,3$ are family indices, u^c, d^c, e^c and ν^c are the right-handed $SU(2)_L$ singlet superfields, $q=(u,d)$ and $l=(\nu,e)$ are the $SU(2)_L$ quark and lepton doublets, and h_u (h_d) is the up (down) Higgs boson doublet. The Dirac neutrino coupling and the heavy Majorana mass for the right-handed neutrinos are denoted by λ_ν and M_{RR} respectively. When the neutral components of the two MSSM Higgs bosons $h^0_{u,d}$ acquire their vacuum expectation values (VEVs) $v_{2,1}$ ($\tan\beta=v_2/v_1\sim 40$–50) the superpotential in Eq. (1) generates the following sum of mass terms:

$$\mathcal{L}_{U,D,E}=-U(\lambda_uv_2)U^c-D(\lambda_dv_1)D^c-E(\lambda_ev_1)E^c+\text{H.c.} \tag{4}$$

$$\mathcal{L}_N=-N(\lambda_\nu v_2)N^c-\tfrac{1}{2}N^cM_{RR}N^c+\text{H.c.} \tag{5}$$

where the uppercase letters now denote the fermionic components of the superfields in $\mathcal{W}$; for example, u contains $(U,\tilde{u})\equiv(U_L,\tilde{u}_L)$ and u^c contains $(U^c,\tilde{u}^c)\equiv(U^*_R,\tilde{u}^*_R)$. The Yukawa matrices in Eq. (4) can be diagonalized by biunitary transformations S and T defined by

$$T^{u*}\lambda_uS^{uT}=\lambda'_u,\quad T^{d*}\lambda_dS^{dT}=\lambda'_d,\quad T^{e*}\lambda_eS^{eT}=\lambda'_e. \tag{6}$$

Thus the physical (primed) states $U'_{R,L}$ are related to the gauge eigenstates $U_{R,L}$ by $U'_R=S^uU_R$ and $U'_L=T^uU_L$, etc. In this model, the left-handed neutrino masses are generated via the seesaw mechanism [6] by the terms in Eq. (5) which can be re-arranged into a 2×2 block matrix in the following way:

$$\mathcal{L}_N=-\frac{1}{2}(NN^c)\begin{pmatrix}0 & m_{LR}\\ m^T_{LR} & M_{RR}\end{pmatrix}\begin{pmatrix}N\\ N^c\end{pmatrix}+\text{H.c.} \tag{7}$$

where $m_{LR}=\lambda_\nu v_2$. Thus, after the heavy N^c fields are integrated out, the light left-handed neutrinos N effectively acquire a small mass given by

$$m_{LL}=m_{LR}M^{-1}_{RR}m^T_{LR}. \tag{8}$$

Finally, the diagonalization of m_{LL},

$$T^{N*}m_{LL}T^{N\dagger}_L=\text{diag}(m_{\nu_1},m_{\nu_2},m_{\nu_3}), \tag{9}$$

allows the determination of the masses of the physical neutrinos m_{ν_A} and enables the physical neutrino states $N'=(\nu_1,\nu_2,\nu_3)$ to be related to the neutrino gauge fields $N=(\nu_e,\nu_\mu,\nu_\tau)$ by $N'=T^NN$.

Taking into account the above conventions, we now proceed to give expressions for the CKM matrix [27] (V^{CKM}) and the corresponding lepton analogue, the Maki-Nakawaga-Sakata (MNS) matrix [28] (V^{MNS}). Their definitions derive from the charged current interactions[3]

$$-\frac{g}{\sqrt{2}}W^+_\mu\overline{\Psi}_U\gamma^\mu P_L\Psi_D\to-\frac{g}{\sqrt{2}}W^+_\mu\Psi_{U'}\gamma^\mu P_LV^{CKM}\Psi_{D'} \tag{10}$$

$$-\frac{g}{\sqrt{2}}W^-_\mu\overline{\Psi}_E\gamma^\mu P_L\Psi_N\to-\frac{g}{\sqrt{2}}W^-_\mu\overline{\Psi}_{E'}\gamma^\mu P_LV^{MNS}\Psi_{N'} \tag{11}$$

which imply

$$V^{CKM}=T^uT^{d\dagger},\quad V^{MNS}=T^eT^{N\dagger}. \tag{12}$$

In what follows we will assume that the matrices in Eq. (12) are real.[4] Thus, we will write V^{MNS} in terms of three rotation matrices

$$V^{MNS}=R_{23}R_{13}R_{12} \tag{13}$$

given by

$$R_{23}=\begin{pmatrix}1&0&0\\0&c_{23}&s_{23}\\0&-s_{23}&c_{23}\end{pmatrix},\quad R_{13}=\begin{pmatrix}c_{13}&0&s_{13}\\0&1&0\\-s_{13}&0&c_{13}\end{pmatrix},$$

$$R_{12}=\begin{pmatrix}c_{12}&s_{12}&0\\-s_{12}&c_{12}&0\\0&0&1\end{pmatrix} \tag{14}$$

where $s_{AB}=\sin\theta_{AB}$ and $c_{AB}=\cos\theta_{AB}$ refer to the lepton mixing angles between the A and B generations. Using Eq. (14) in Eq. (13) gives

[3]The four component fermion fields Ψ are given by $\Psi_F=(F,-i\sigma^2F^{c*})$ for $F=U,D,E$ and $\Psi_N=(N,-i\sigma^2N^*)$ for the neutrinos.

[4]We shall not address the question of CP violation in this paper.

$$V^{MNS}=\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13} & c_{12}c_{23}-s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13} & -c_{12}s_{23}-s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix}. \quad (15)$$

It is also practical to have expressions for the θ_{AB} angles in terms of the V^{MNS} entries. Inverting Eq. (15) we find that[5]

$$\sin\theta_{13}=V_{e3}, \quad \sin\theta_{23}=\frac{V_{\mu 3}}{\sqrt{1-V_{e3}^2}}, \quad \sin\theta_{12}=\frac{V_{e2}}{\sqrt{1-V_{e3}^2}}. \quad (16)$$

Finally we note that while the above expressions were derived in the context of three neutrino species, the analysis of the experimental results assumed only two; thus a direct comparison of mixing angles is not exactly valid.

III. THE PATI-SALAM MODEL

Here we briefly summarize the parts of the Pati-Salam model [18] that are relevant for our analysis. For a more complete discussion see Ref. [19]. The SM fermions, together with the right-handed neutrinos, are conveniently accommodated in the following $F=(4,2,1)$ and $F^c=(\overline{4},1,\overline{2})$ representations:

$$F_A=\begin{pmatrix} u & u & u & \nu \\ d & d & d & e \end{pmatrix}_A, \quad F_B^c=\begin{pmatrix} d^c & d^c & d^c & e^c \\ u^c & u^c & u^c & \nu^c \end{pmatrix}_B. \quad (17)$$

The MSSM Higgs bosons fields are contained in $h=(1,\overline{2},2)$:

$$h=\begin{pmatrix} h_d^- & h_u^0 \\ h_d^0 & h_u^+ \end{pmatrix} \quad (18)$$

whereas the heavy Higgs bosons $\overline{H}=(\overline{4},1,\overline{2})$ and $H=(4,1,2)$ are denoted by:

$$\overline{H}=\begin{pmatrix} \overline{H}_d & \overline{H}_d & \overline{H}_d & \overline{H}_e \\ \overline{H}_u & \overline{H}_u & \overline{H}_u & \overline{H}_\nu \end{pmatrix}, \quad H=\begin{pmatrix} H_d & H_d & H_d & H_e \\ H_u & H_u & H_u & H_\nu \end{pmatrix}. \quad (19)$$

In addition to the Higgs fields in Eqs. (18),(19) the model also involves an $SU(4)$ sextet field $D=(6,1,1)=(D_3,D_3^c)$.

The superpotential of the minimal 422 model is

$$\mathcal{W}=F\lambda F^c h+\lambda_h Shh+\lambda_S S(\overline{H}H-M_H^2)+\lambda_H HHD + \lambda_{\overline{H}}\overline{H}\overline{H}D+F^c\lambda' F^c\frac{HH}{M_\phi} \quad (20)$$

where S denotes a gauge singlet superfield, the λ's are real dimensionless parameters and $M_H\sim M_X\sim 10^{16}$ GeV. Additionally, $M_\phi\sim 10^{18}$ GeV denotes the VEV of extra matter that has been integrated out from the model at high energy.[6] As a result of the superpotential terms involving the singlet S, the Higgs fields develop VEVs $\langle H\rangle=\langle H_\nu\rangle\sim M_X$ and $\langle\overline{H}\rangle=\langle\overline{H}_\nu\rangle\sim M_X$ which lead to the symmetry breaking

$$SU(4)\otimes SU(2)_L\otimes SU(2)_R\rightarrow SU(3)_c\otimes SU(2)_L\otimes U(1)_Y. \quad (21)$$

The singlet S itself also naturally develops a small VEV of the order of the SUSY breaking scale [29] so that the $\lambda_h S$ term in Eq. (20) gives an effective μ parameter of the correct order of magnitude. Under Eq. (21) the Higgs field h in Eq. (18) splits into the familiar MSSM doublets h_u and h_d whose neutral components subsequently develop weak scale VEVs $\langle h_u^0\rangle=v_2$ and $\langle h_d^0\rangle=v_1$ with $\tan\beta=v_2/v_1$. The neutrino fields ν^c acquire a large mass $M_{RR}\sim\lambda'\langle HH\rangle/M_\phi\sim 10^{14}$ GeV through the non-renormalizable term in $\mathcal{W}$ which, together with the Dirac ν^c-ν interaction (proportional to $\lambda\langle h_u^0\rangle$), gives rise to a 2×2 matrix that generates, via a seesaw mechanism [6], a suppressed mass for the left-handed neutrino states. The D field does not develop a VEV but the terms HHD and $\overline{H}\overline{H}D$ combine the color triplet parts of H, $\overline{H}$ and D into acceptable GUT scale mass terms [19].

IV. ABELIAN FLAVOR SYMMETRY

The pattern of fermion masses and mixing angles is one of the fundamental problems in particle physics that has not yet been understood. The importance of this unsolved puzzle is demonstrated by the numerous works published in the literature over the past years (see Refs. [30–36] for a ''short'' list). In the standard model (SM) the quark and lepton masses and the CKM matrix are input parameters fixed by laboratory experiments. Surprisingly, however, their values, though unconstrained and *a priori* arbitrary, do display a

[5] $V_{e2}=V_{12}^{MNS}$, $V_{e3}=V_{13}^{MNS}$ and $V_{\mu 3}=V_{23}^{MNS}$.

[6] The full model involves four gauge singlet fields ϕ that have renormalizable interactions given by $\lambda_\phi F^c H\phi$ and $\lambda_3\phi\phi\phi$. When one of the ϕ fields acquires a large VEV $\langle\phi\rangle=M_\phi$ the mentioned interactions generate, after the ϕ fields are integrated out via a seesaw type mechanism, the last non-renormalizable term in Eq. (20).

certain degree of organization. The fermion masses are highly hierarchical and the CKM matrix can be described in terms of the small Wolfenstein expansion parameter $\lambda \sim |V_{12}| \sim 0.22$ [37]. These results suggest that a broken flavor symmetry might be playing an important role in the setting of the structure of the Yukawa matrices.

In this work we will assume that the ''vertical'' gauge group is supplemented by an additional $U(1)_X$ ''horizontal'' flavor symmetry that constrains the nature of the couplings of quarks and leptons to SM singlet fields θ and $\overline{\theta}$. The family symmetry, however, is broken at some high energy scale $V_\theta > M_X$ by the VEVs of the θ, $\overline{\theta}$ fields which under the $U(1)_X$ group have charges $X_\theta = -1$ and $X_{\overline{\theta}} = +1$. As a consequence of the $U(1)_X$ symmetry breaking, the low energy effective theory includes Dirac interactions between the F and F^c fields of the following form:

$$F_A F_B^c h \left(\frac{\theta}{M_\theta}\right)^{p_{AB}} \rightarrow F_A F_B^c h \left(\frac{\langle\theta\rangle}{M_\theta}\right)^{p_{AB}} \sim F_A F_B^c h \epsilon^{p_{AB}} \quad (22)$$

$$F_A F_B^c h \left(\frac{\overline{\theta}}{M_\theta}\right)^{p_{AB}} \rightarrow F_A F_B^c h \left(\frac{\langle\overline{\theta}\rangle}{M_\theta}\right)^{p_{AB}} \sim F_A F_B^c h \epsilon^{p_{AB}} \quad (23)$$

where p_{AB} is the modulus of the sum of the $U(1)_X$ charges of the F_A, F_B^c and h fields, i.e., $p_{AB} = |X_{AB}| = |X_{F_A} + X_{F_B^c} + X_h|$. Thus Eq. (22) holds if $X_{AB} > 0$ whereas Eq. (23) holds if $X_{AB} < 0$. The non-renormalizable terms in Eqs. (22),(23) might originate from interactions between the F and θ fields with additional exotic vector matter with mass $M_\theta > M_X$ that lead to ''spaghetti'' diagrams as discussed in Ref. [38].[7] In summary, the equations above show that, in the context of a $U(1)_X$ symmetry, the observed hierarchy in the fermion masses and mixing angles might be the result of the flavor charges carried by the fields of the 422 model which act to suppress the Yukawa couplings by some ϵ power.

The introduction of the $U(1)_X$ symmetry provides a way to relate the various flavor parameters of the model, thus making it more predictive. However, one should be careful. Generally the $U(1)_X$ group is potentially dangerous since it can introduce, through triangle diagrams, mixed anomalies with the SM gauge group.[8] In the last part of this section we review the constraints imposed on X charges of the fields of our model enforced by the requirement of anomaly cancellation [32].

The mixed anomalies that we shall consider are[9]

$$SU(3)^2 U(1)_X: \; A_3 = \sum_{A=1}^{3} (2X_{q_A} + X_{u_A^c} + X_{d_A^c}) \quad (24)$$

$$SU(2)^2 U(1)_X: \; A_2 = \sum_{A=1}^{3} (3X_{q_A} + X_{l_A}) + X_{h_u} + X_{h_d} \quad (25)$$

$$U(1)_Y^2 U(1)_X: \; A_1 = \sum_{A=1}^{3} \left(\frac{1}{3}X_{q_A} + \frac{8}{3}X_{u_A^c} + \frac{2}{3}X_{d_A^c} + X_{l_A} + 2X_{e_A^c}\right) + X_{h_u} + X_{h_d} \quad (26)$$

$$U(1)_Y U(1)_X^2: \; A_1' = 2\left[\sum_{A=1}^{3} (X_{q_A}^2 - 2X_{u_A^c}^2 + X_{d_A^c}^2 - X_{l_A}^2 + X_{e_A^c}^2) + X_{h_u}^2 - X_{h_d}^2\right]. \quad (27)$$

For example, A_3 corresponds to the anomalous term generated by the Feynman diagram that has two $SU(3)$ gluons and one $U(1)_X$ gauge boson attached to the triangle vertices. We note that the first three anomalies A_3, A_2, and A_1 are linear in the trace of the charges, i.e., $X_f = \Sigma_{A=1}^3 X_{f_A}$, where f is any of the q, u^c, d^c, l, e^c fields; thus they constrain only the family independent (FI) part of the $U(1)_X$ charges. On the other hand, A_1' is quadratic in the X charges; thus it generally constrains the FI and family dependent (FD) part of the $U(1)_X$ charges.

In this paper we will assume that the cancellation of anomalies results from the Green-Schwartz (GS) mechanism [39]. This is possible if the A_3, A_2, and A_1 anomalies are in the ratio $A_3 : A_2 : A_1 = k_3 : k_2 : k_1$ where the k_i are the Kac-Moody levels of the $SU(3)$, $SU(2)$, and $U(1)_Y$ gauge groups that determine the boundary conditions for the gauge couplings at the string scale $g_3^2 k_3 : g_2^2 k_2 : g_1^2 k_1$. Hence, using the canonical GUT normalization for the gauge couplings [that successfully predicts $\sin^2(\theta_W) = 3/8$ [40]], anomalies can be canceled if we require that

$$A_3 = A_2 = \tfrac{5}{3} A_1. \quad (28)$$

As a consequence of the two constraints implicit in Eq. (28), the set of solutions for the X charges appearing in Eqs. (24)–(26) is given by [32]

$$X_{e^c} = \sum_{A=1}^{3} X_{e_A^c} = x, \quad X_l = \sum_{A=1}^{3} X_{l_A} = y, \quad X_{h_u} = -z,$$

$$X_q = \sum_{A=1}^{3} X_{q_A} = x + u, \quad X_{d^c} = \sum_{A=1}^{3} X_{d_A^c} = y + v,$$

$$X_{h_d} = +z + (u + v),$$

$$X_{u^c} = \sum_{A=1}^{3} X_{u_A^c} = x + 2u \quad (29)$$

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[7] In this work we will assume that the VEVs of the θ fields $\langle\theta\rangle = \langle\overline{\theta}\rangle = V_\theta$ and the mass of the extra vector fields M_θ are related in such a way that $V_\theta / M_\theta = \epsilon = 0.22$.

[8] The cancellation of anomalies requires the vanishing of the trace $\mathrm{Tr}(T^a\{T^b, T^c\}) = 0$ where $T^{a,b,c}$ are any of the group generators which stand at the three gauge boson vertices of the triangle diagrams.

[9] We will not include the analysis of the $U(1)_X^3$ or of the gravitational anomaly because they depend exclusively on SM singlet fields.

where x,y,z,u,v are free parameters. However, not all the solutions in Eqs. (29) are valid after $A_1'=0$ is enforced. In fact, as we said before, generally A_1' constrains both the FI and FD charges of $U(1)_X$. By this we mean that, if we conveniently write the charge of the f_A field X_{f_A} as a sum of a FI part X_f plus a FD part X'_{f_A}, i.e., $X_{f_A}=\frac{1}{3}X_f+X'_{f_A}$, then $A_1'=0$ is a complicated equation on all X_f, X'_{f_A}, and X_{h_u}, X_{h_d} charges. However, it is easy to see that, if all the left-handed fields and if all the right-handed fields have the same FD charges, i.e., $X'_{q_A}=X'_{l_A}$ and $X'_{u^c_A}=X'_{d^c_A}=X'_{e^c_A}$, as is the case of the 422 model, then $A_1'=0$ is an equation on the FI charges only:

$$A_1'=\tfrac{2}{3}(X_q^2-2X_{u^c}^2+X_{d^c}^2-X_l^2+X_{e^c}^2+3X_{h_u}^2-3X_{h_d}^2)=0. \tag{30}$$

Thus, a simple solution to all the anomaly constraints is given by Eq. (29) with $u=v=0$. Finally, we must add that since the Pati-Salam model unifies all the left- and right-handed quark and lepton fields in the F/F^c multiplets, and the MSSM Higgs fields h_u, h_d in the h Higgs bi-doublet, we must also have $x=y$ and $z=0$. Thus, anomaly cancellation in the 422 model via the GS mechanism is possible if the traces of the $U(1)_X$ charges of the F and F^c fields are equal, i.e., $X_F=\Sigma_{A=1}^3 X_{F_A}\equiv\Sigma_{A=1}^3 X_{F^c_A}=X_{F^c}$.

V. OPERATOR APPROACH TO FERMION MASSES

In the simplest formulation of the 422 model extended by a $U(1)_X$ horizontal symmetry all the Yukawa couplings originate from a single matrix. The Abelian $U(1)_X$ symmetry introduced in the previous section mainly serves one purpose: it establishes a hierarchy between the flavor dependent couplings. Thus, it provides no precise or predictive information about the relationships between the different Yukawa coupling matrices. As a result, all the SM fermions of a given family have identical Yukawa couplings at the unification scale. Naturally, when the fermion masses run from the M_X to the M_Z scale they lead to quark and lepton masses that are incompatible with the experimental data.

The idea of Yukawa unification, though unsuccessful in its most simpler form, is not, however, a complete failure. As a matter of fact, it turns out that third family Yukawa unification works rather well. It is well known that the GUT boundary condition for the Yukawa couplings,

$$\lambda_t(M_X)=\lambda_b(M_X)=\lambda_\tau(M_X)=\lambda_{\nu_\tau}(M_X), \tag{31}$$

leads to a large pole top quark mass prediction $M_t\sim175$ GeV and $\tan\beta\sim m_t/m_b$. On the other hand, the first and second family fermion masses can be predicted if special relations between the ''vertical'' intra-generation Yukawa couplings at M_X hold. For example, the Georgi-Jarlskog (GJ) [41] relation between the muon and strange Yukawa couplings $\lambda_\mu\sim3\lambda_s$ successfully reproduces the low energy experimental $m_s/m_\mu\sim1$ mass ratio. In the context of GUT theories the appearance of numerical factors relating the couplings of the up-down-lepton Yukawa matrices might originate from non-renormalizable operators involving the interaction between the fermions and the heavy Higgs bosons that break the GUT symmetry [42,43].

In the Pati-Salam model, we will have in mind operators of the following form [24]:

$$F_AF_B^ch\left(\frac{H\overline{H}}{M_V^2}\right)^n\left(\frac{\theta}{M_\theta}\right)^{p_{AB}},\quad F_AF_B^ch\left(\frac{H\overline{H}}{M_V^2}\right)^n\left(\frac{\overline{\theta}}{M_\theta}\right)^{p_{AB}}. \tag{32}$$

The idea is that when the H and θ fields develop their VEVs such operators reduce to effective Yukawa couplings with small coefficients. For example, if F_2, F_2^c, and h carry a charge $X_{F_2}=0$, $X_{F_2^c}=2$, and $X_h=0$ under $U(1)_X$ symmetry, then Eq. (32) (with $n=1$) generates the following terms:

$$(x_uu_2u_2^ch_u^0+x_dd_2d_2^ch_d^0+x_ee_2e_2^ch_d^0+x_\nu\nu_2\nu_2^ch_u^0)\,\delta\epsilon^2 \tag{33}$$

where $\delta=\langle H\rangle\langle\overline{H}\rangle/M_V^2$ and $\epsilon=\langle\theta\rangle/M_\theta$ are small dimensionless parameters,[10] u_2,d_2,e_2,ν_2 are the charm, strange, muon, muon neutrino superfields, and x_f $(f=u,d,e,\nu)$ are Clebsch factors that depend on the group theoretical contractions between the fields in Eq. (32) [21,22]. In Table XIV (Appendix A) we present a complete list of all x_f values that result from $n=1$ operators in the 422 model [24] normalized by

$$x_u^2+x_d^2+x_e^2+x_\nu^2=4. \tag{34}$$

It is interesting to point out that different operators imply zero Clebsch coefficients for different x_f's. For example, Class-I operators are rather special since of all x_f's only one is non-zero (and significantly large). The Class-II operators have $x_u=x_\nu=0$ while Class-III have $x_d=x_e=0$. Additionally Class-IV operators have $x_u=x_d=0$ and Class-V have $x_e=x_\nu=0$. Finally Class-VI operators have all x_f's different from zero. The variety of the operator Clebsch coefficients is to be welcome since, as we will see, they open the possibility of avoiding the disastrous fermion mass predictions characteristic of the minimal 422 model with a unified renormalizable interaction.

Finally we shall mention the origin of the heavy Majorana neutrino mass matrix. Generally M_{RR} results from non-renormalizable operators of the form[11]

[10]In analogy with the M_θ parameter, the M_V scale represents the mass of extra matter that has been integrated out from our model (in this case at an energy slightly above the unification scale) such that $\delta=\langle H\rangle\langle\overline{H}\rangle/M_V^2=0.22$.

[11]Generally the M_θ scales involved in the Dirac F^cF and Majorana F^cF^c interactions in Eq. (32) and Eq. (35) respectively need not necessarily be identical [44]. In this work, however, for the sake of simplicity we will assume they are equal.

$$F_A^c F_B^c \left(\frac{HH}{M_\phi}\right)\left(\frac{H\overline{H}}{M_V^2}\right)^n \left(\frac{\theta}{M_\theta}\right)^{q_{AB}} \rightarrow M_X \left(\frac{M_X}{M_\phi}\right) \nu_A^c \nu_B^c \delta^n \epsilon^{q_{AB}} \tag{35}$$

where $q_{AB}=|X_{F_A^c}+X_{F_B^c}+\sigma|$ and $\sigma=2X_H$. Three important differences distinguish Eq. (35) from Eq. (32). First, we note that while Eq. (32) allows for renormalizable operators, M_{RR} as given by Eq. (35) is always the result of nonrenormalizable operators. Second, we note that the combination of the HH fields in Eq. (35) introduces an additional free parameter σ that may be fixed at our convenience. Third, we observe that while Eq. (32) is able to generate precise relationships between the up-down-lepton Yukawa couplings (via Clebsch factors), Eq. (35) is an expression that constrains only the hierarchy of M_{RR} [via the $U(1)_X$ symmetry]; as a result, it is less predictive.

VI. NEUTRINO MASSES AND MIXING ANGLES

In this section we show how the $U(1)_X$ horizontal family symmetry of Sec. IV can be combined with the operator approach of Sec. V to give predictions for the fermion masses and mixing angles in the 422 model. In particular we are interested in the predictions for the neutrino masses and mixing angles for the LMA MSW solution to the solar neutrino problem. The LOW and the SMA MSW solutions are discussed in Appendixes C and D. We start by listing in Table I the quark and charged lepton experimental data used in our analysis.[12]

We used $m_u(1\text{ GeV})$, $m_d(1\text{ GeV})$ to denote the running masses of the up and down quarks at $Q=1$ GeV;[13] $m_c(M_c)$, $m_s(M_c)$,$m_b(M_b)$ the running masses of the charm, strange and bottom quarks at their pole masses ($M_c=1.6$ GeV, $M_b=4.8$ GeV); M_t the top pole mass; and $M_{e,\mu,\tau}$ ($m_{e,\mu,\tau}$) the well-known pole (running) charged lepton masses. We converted the above pole masses to running masses using the expressions in Ref. [46] with

$$\alpha_s(M_Z)=0.120, \quad \alpha_e^{-1}(M_Z)=127.8. \tag{36}$$

Finally the CKM matrix at $Q=M_Z$ was fixed by[14]

$$|V_{12}|=0.2215, \quad |V_{23}|=0.040, \quad |V_{13}|=0.0035. \tag{37}$$

It is important to note that, in fact, not all the parameters above were taken as input. Indeed, $M_t\sim175$ GeV is a prediction that results from third family Yukawa unification $\lambda_t=\lambda_b=\lambda_\tau$ at the GUT scale. Moreover, as we will see, our model is also able to predict the masses of the down and charm quarks; thus their values listed above should be taken merely as a guide and/or convenient initial estimates.

[12]The numbers inside the curly brackets indicate the experimental ranges according to Ref. [45].

[13]All running masses are given in the modified minimal subtraction ($\overline{\text{MS}}$) scheme.

[14]The experimental ranges are $|V_{12}|=0.219$ to 0.226, $|V_{23}|=0.037$ to 0.043 and $|V_{13}|=0.002$ to 0.005 [45].

TABLE I. Experimental (input) values of quark and lepton masses used in this paper (see main text for full explanation on adopted notation).

$m_u(1\text{ GeV})=4.7$ MeV	$(1.35-6.75)$ MeV
$m_c(M_c)=1.21$ GeV	(1.15–1.35) GeV
$M_t\sim$ 175 GeV	(170–180) GeV
$m_d(1\text{ GeV})\sim6.0$ MeV	(4–12) MeV
$m_s(M_s)\sim160$ MeV	(100–230) MeV
$m_b(M_b)=4.15$ GeV	(4.0–4.4) GeV
$M_e=0.511$ MeV	$m_e(M_e)=0.496$ MeV
$M_\mu=105.7$ MeV	$m_\mu(M_\mu)=104.6$ MeV
$M_\tau=1777.0$ MeV	$m_\tau(M_\tau)=1772.8$ MeV

We now turn to the neutrino experimental data. The results from the Super-Kamiokande Collaboration [1,2] indicate that the atmospheric neutrino anomaly can be understood in terms of $\nu_\mu\leftrightarrow\nu_\tau$ oscillations with

$$\sin^2(2\theta_{23})>0.88, \quad 1.5\times10^{-3}\text{ eV}^2<\Delta m_{23}^2<5\times10^{-3}\text{ eV}^2 \tag{38}$$

at 90% confidence level. On the other hand, the LMA MSW solution to the solar neutrino deficit suggests that [5]

$$\sin^2(2\theta_{12})\sim0.75, \quad \Delta m_{12}^2\sim2.5\times10^{-5}\text{ eV}^2. \tag{39}$$

Assuming that the neutrino spectrum is hierarchical, i.e., $\Delta m_{23}^2=|m_{\nu_3}^2-m_{\nu_2}^2|\sim m_{\nu_3}^2$ and $\Delta m_{12}^2=|m_{\nu_2}^2-m_{\nu_1}^2|\sim m_{\nu_2}^2$, the values in Eqs. (38),(39) give

$$\sin(\theta_{23})>0.57, \quad m_{\nu_3}\sim0.05\text{ eV},$$
$$\sin(\theta_{12})\sim0.50, \quad m_{\nu_2}\sim0.005\text{ eV}. \tag{40}$$

The latest results from the CHOOZ experiment also show that, over the interesting Δm_{23}^2 range suggested by the Super-Kamiokande data, $\sin^2(2\theta_{13})<0.10$ at 90% C.L. [47].

The experimental data in Table I constrain the parameters of our model at low energy. However, the GUT symmetry is broken at an energy $M_X\sim10^{16}$ GeV. Thus, before we start our analysis we should correct the fermion masses and mixing angles for the radiative corrections that result from the running of the renormalization group equations (RGEs) between the $Q=M_Z$ and $Q=M_X$ scales. The implementation of the RGEs, the decoupling of SUSY particles and the boundary conditions is a complicated subject whose detailed description is beyond the scope of this work. Here we will only mention that we used 2-loop RGEs in the gauge and Yukawa couplings and refer the interested reader to Ref. [25] where the issue of Yukawa unification in the 422 model is discussed. As a result of the running of the RGEs, subject to third family Yukawa unification at M_X, the low energy input values for the fermion masses in Table I effectively constrain the eigenvalues of the Yukawa couplings at $Q=M_X$. These are presented in Table II.

At this point it is convenient to re-write the data in Table II in terms of the Wolfenstein [37] expansion parameter λ

TABLE II. Values of the Yukawa couplings and CKM entries at the unification scale (M_X) that result from the running of the low energy (input) data of Table I between the relevant low energy scales and M_X.

$\lambda_u(M_X)=4.738\times10^{-6}$	$\lambda_c(M_X)=1.529\times10^{-3}$	$\lambda_t(M_X)=0.677$
$\lambda_d(M_X)\sim3.208\times10^{-4}$	$\lambda_s(M_X)\sim9.612\times10^{-3}$	$\lambda_b(M_X)=0.677$
$\lambda_e(M_X)=1.490\times10^{-4}$	$\lambda_\mu(M_X)=3.154\times10^{-2}$	$\lambda_\tau(M_X)=0.677$
$\|V_{12}(M_X)\|=0.2215$	$\|V_{23}(M_X)\|=0.032$	$\|V_{13}(M_X)\|=0.0028$

$=0.22\sim|V_{12}|$ (see Table III). The expressions in Table III neatly summarize the hierarchy of the quark and charged lepton sectors at M_X that we aim to reproduce and predict.

It is now time to specify the structure of the LMA model in more detail. We start by indicating the nature of the (non-)renormalizable operators responsible for the structure of the Dirac and neutrino Majorana matrices:

$$\lambda_{AB}:\ F_3F_3^ch+F_AF_B^ch\frac{H\bar{H}}{M_V^2}\times\left[1+\left(\frac{H\bar{H}}{M_V^2}\right)+\left(\frac{H\bar{H}}{M_V^2}\right)^2+\cdots\right]\left(\frac{\theta}{M_\theta}\right)^{p_{AB}} \tag{41}$$

$$(M_{RR})_{AB}:\ \left\{F_3^cF_3^c+F_A^cF_B^c\left[\frac{H\bar{H}}{M_V^2}+\cdots\right]\right\}\frac{HH}{M_\phi}\left(\frac{\theta}{M_\theta}\right)^{q_{AB}}. \tag{42}$$

The first term in Eq. (41) is renormalizable; thus it implies third family Yukawa unification at M_X. The second term, which we shall assume to be present for $AB\neq33$, on the other hand, is a sum of non-renormalizable operators. For the sake of simplicity we will consider that the $H\bar{H}/M_V^2$ part of λ_{AB} that lies outside the square brackets in Eq. (41) has non-trivial gauge contractions with the $F_AF_B^ch$ fields next to it, thereby generating the Clebsch factors in Table XIV (Appendix A). On the other hand, the $(H\bar{H}/M_V^2)^{1,2}$ factors inside the square brackets will form gauge singlet terms that will be responsible for the appearance of higher δ powers in the entries of λ_{AB}. The M_{RR} matrix, as given by Eq. (42), depends only on non-renormalizable operators because gauge invariance demands that every combination of F^cF^c fields must be paired with at least a couple of HH fields. However, we will assume that the only $n=1$ operator in M_{RR} is placed on the 33 entry. All other entries of M_{RR} result from $n=2$ operators.[15]

We can see from Eqs. (41),(42) that the structure of the Yukawa and Majorana matrices can be decomposed into a ''vertical'' δ component and a ''horizontal'' ϵ component. Thus we write :

$$(\lambda_f)_{AB}\sim(\lambda^\delta)_{AB}(\lambda^\epsilon)_{AB},\quad (M_{RR})_{AB}\sim(M_{RR}^\delta)_{AB}(M_{RR}^\epsilon)_{AB}. \tag{43}$$

The hierarchies of λ^ϵ and M_{RR}^ϵ are fixed by the choice of the $U(1)_X$ charges. Using the results of Ref. [11], we can write the most general form of the unified $(\lambda^\epsilon)_{AB}$ matrix in the 422 model, constrained by the absence of anomalies, in terms of only four independent parameters $\bar{X}_{F_1}$, $\bar{X}_{F_2}$, $\bar{X}_{F_1^c}$, and $\bar{X}_{F_2^c}$:[16]

$$\lambda^\epsilon=\begin{pmatrix}\epsilon^{|\bar{X}_{F_1}+\bar{X}_{F_1^c}|} & \epsilon^{|\bar{X}_{F_1}+\bar{X}_{F_2^c}|} & \epsilon^{|\bar{X}_{F_1}|}\\ \epsilon^{|\bar{X}_{F_2}+\bar{X}_{F_1^c}|} & \epsilon^{|\bar{X}_{F_2}+\bar{X}_{F_2^c}|} & \epsilon^{|\bar{X}_{F_2}|}\\ \epsilon^{|\bar{X}_{F_1^c}|} & \epsilon^{|\bar{X}_{F_2^c}|} & 1\end{pmatrix}. \tag{44}$$

From the equation above it is easy to see that the values of $\bar{X}_{F_2}$, $\bar{X}_{F_2^c}$, $\bar{X}_{F_1}$ and $\bar{X}_{F_1^c}$ are closely related to the large neutrino θ_{23} angle, the second family Yukawa couplings, the V_{12} CKM angle, and the masses of the lightest fermions respectively. In the first row of Table IV we list our choices for the $\bar{X}$ parameters which we will, from now on, refer to as $U(1)_{\bar{X}}$ charges. In the second rows we indicate the values of the physical (anomaly free) $U(1)_X=U(1)_{FD}+U(1)_{FI}$ charges of the fields of our model. In the third and fourth rows we list the values of the family dependent (traceless) and family independent (unphysical) charges that sum up to give $U(1)_X$.

We note that the $U(1)_{\bar{X}}$ and $U(1)_X$ charges are ''equivalent'' in the sense that they determine equal family structures for the Yukawa and neutrino Majorana matrices.[17]

TABLE III. Values of the Yukawa couplings and CKM entries at the unification scale, as in Table II, expanded in powers of the Wolfenstein parameter $\lambda=0.22$.

$\lambda_u(M_X)=\lambda^{8.097}$	$\lambda_c(M_X)=\lambda^{4.282}$	$\lambda_t(M_X)=\lambda^{0.257}$
$\lambda_d(M_X)\sim\lambda^{5.313}$	$\lambda_s(M_X)\sim\lambda^{3.068}$	$\lambda_b(M_X)=\lambda^{0.257}$
$\lambda_e(M_X)=\lambda^{5.820}$	$\lambda_\mu(M_X)=\lambda^{2.283}$	$\lambda_\tau(M_X)=\lambda^{0.257}$
$\|V_{12}(M_X)\|=\lambda^{0.996}$	$\|V_{23}(M_X)\|=\lambda^{2.273}$	$\|V_{13}(M_X)\|=\lambda^{3.882}$

[15]We note that these assumptions about the nature of the Majorana matrix are unique to the LMA MSW solution. The SMA MSW and LOW solutions discussed in Appendixes C and D are characterized by a Majorana matrix filled with $n=1$ operators only.

[16]In [11] these charges were called $\alpha,\beta,\gamma,\delta$. Roughly, this corresponds to choosing a basis of charges that has $\bar{X}_{F_3}=\bar{X}_{F_3^c}=\bar{X}_h=0$.

[17]However, only the $U(1)_X$ symmetry is anomaly free.

TABLE IV. List of $U(1)$ flavor charges that determine the family structure of the Yukawa and neutrino Majorana matrices of the LMA model. The first set, indicated by $U(1)_{\bar{X}}$, refers to the values of the $\bar{X}$ parameters that determine the hierarchy of λ^{ϵ} in Eq. (44). The second set, $U(1)_X = U(1)_{FD} + U(1)_{FI}$, corresponds to the anomaly free physical flavor charges of the model. The third set $U(1)_{FD}$ indicates the charges of the family dependent (traceless) component of $U(1)_X$ and $U(1)_{FI}$ refers to the family independent component of $U(1)_X$.

	X_{F_1}	X_{F_2}	X_{F_3}	$X_{F_1^c}$	$X_{F_2^c}$	$X_{F_3^c}$	X_h	X_H	$X_{\bar{H}}$
$U(1)_{\bar{X}}$	1	0	0	4	2	0	0	0	0
$U(1)_X$	$\frac{11}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{19}{6}$	$\frac{7}{6}$	$-\frac{5}{6}$	0	$\frac{5}{6}$	$-\frac{5}{6}$
$U(1)_{FD}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	2	0	-2	$\frac{7}{3}$	2	-2
$U(1)_{FI}$	$\frac{7}{6}$	$\frac{7}{6}$	$\frac{7}{6}$	$\frac{7}{6}$	$\frac{7}{6}$	$\frac{7}{6}$	$-\frac{7}{3}$	$-\frac{7}{6}$	$\frac{7}{6}$

The charges in Table IV fix the ϵ structure of λ^{ϵ} and M_{RR}^{ϵ} to be

$$\lambda^{\epsilon} \sim \begin{pmatrix} \epsilon^5 & \epsilon^3 & \epsilon \\ \epsilon^4 & \epsilon^2 & 1 \\ \epsilon^4 & \epsilon^2 & 1 \end{pmatrix}, \quad M_{RR}^{\epsilon} \sim \begin{pmatrix} \epsilon^8 & \epsilon^6 & \epsilon^4 \\ \epsilon^6 & \epsilon^4 & \epsilon^2 \\ \epsilon^4 & \epsilon^2 & 1 \end{pmatrix}. \quad (45)$$

Comparing λ^{ϵ} above with the hierarchy of the Yukawa couplings listed in Table III we see that the $U(1)_X$ symmetry (by itself) cannot explain the pattern of all fermion masses and mixing angles. For example, although the symmetry allows a large 23 entry suitable for generating large 23 mixing from the neutrino Yukawa matrix, it also allows similarly large 23 entries in the charged lepton and quark Yukawa matrices which are not welcome. In order to overcome this we shall assume that although a renormalizable operator in the 23 position is allowed by the $U(1)_X$ symmetry, it is forbidden by some unspecified string symmetry which however allows a 23 operator containing one factor of $(H\bar{H})$. We shall further select a 23 operator which will involve a Clebsch factor of zero for the charged lepton and quark entries, with only its neutrino component having a non-zero contribution, thereby generating a large 23 mixing from the neutrino sector, with only small 23 mixing in the charged lepton and quark sectors arising from operators containing higher powers of $(H\bar{H})^n$, with $n>1$.[18]

The existence of such operators with ‘‘Clebsch zeros’’ is clearly crucial for the success of our approach.

In general, we shall show that by a suitable choice of non-renormalizable operators, which determine the λ^{δ} ‘‘vertical’’ structure of λ_f, we can obtain a successful description of all quark and lepton masses and mixing angles. For example, let us consider λ^{δ} and M_{RR}^{δ} given by the following operator matrices:

$$\lambda^{\delta} \sim \begin{pmatrix} \mathcal{O}^R + \mathcal{O}''^V & \mathcal{O}^J + \mathcal{O}'^Q & \mathcal{O}^g + \mathcal{O}'^f \\ \mathcal{O}^G + \mathcal{O}''^K & \mathcal{O}^W + \mathcal{O}'^H & \mathcal{O}^I + \mathcal{O}'^W \\ \mathcal{O}^R + \mathcal{O}''^V & \mathcal{O}^M + \mathcal{O}'^K & 1 \end{pmatrix},$$

TABLE V. Approximate structure of the Yukawa and neutrino Majorana matrices resulting from Eqs. (47),(48) when the numerical effect of the Clebsch and of the order-1 a,A parameters is neglected.

$$\lambda_u(M_X) \sim \begin{pmatrix} \delta^3\epsilon^5 & \delta^2\epsilon^3 & \delta^2\epsilon \\ \delta^3\epsilon^4 & \delta^2\epsilon^2 & 0 \\ \delta^3\epsilon^4 & \delta^2\epsilon^2 & 1 \end{pmatrix} \sim \begin{pmatrix} \lambda^8 & \lambda^5 & \lambda^3 \\ \lambda^7 & \lambda^4 & 0 \\ \lambda^7 & \lambda^4 & 1 \end{pmatrix}$$

$$\lambda_d(M_X) \sim \begin{pmatrix} \delta\epsilon^5 & \delta^2\epsilon^3 & \delta^2\epsilon \\ \delta\epsilon^4 & \delta\epsilon^2 & \delta^2 \\ \delta\epsilon^4 & \delta\epsilon^2 & 1 \end{pmatrix} \sim \begin{pmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^3 & \lambda^2 \\ \lambda^5 & \lambda^3 & 1 \end{pmatrix}$$

$$\lambda_e(M_X) \sim \begin{pmatrix} \delta\epsilon^5 & \delta\epsilon^3 & \delta\epsilon \\ \delta\epsilon^4 & \delta\epsilon^2 & \delta^2 \\ \delta\epsilon^4 & \delta\epsilon^2 & 1 \end{pmatrix} \sim \begin{pmatrix} \lambda^6 & \lambda^4 & \lambda^2 \\ \lambda^5 & \lambda^3 & \lambda^2 \\ \lambda^5 & \lambda^3 & 1 \end{pmatrix}$$

$$\lambda_\nu(M_X) \sim \begin{pmatrix} \delta^3\epsilon^5 & \delta\epsilon^3 & \delta\epsilon \\ \delta^3\epsilon^4 & \delta^2\epsilon^2 & \delta \\ \delta^3\epsilon^4 & \delta^2\epsilon^2 & 1 \end{pmatrix} \sim \begin{pmatrix} \lambda^8 & \lambda^4 & \lambda^2 \\ \lambda^7 & \lambda^4 & \lambda \\ \lambda^7 & \lambda^4 & 1 \end{pmatrix}$$

$$M_{RR}(M_X) \sim \begin{pmatrix} \delta\epsilon^8 & \delta\epsilon^6 & \delta\epsilon^4 \\ \delta\epsilon^6 & \delta\epsilon^4 & \delta\epsilon^2 \\ \delta\epsilon^4 & \delta\epsilon^2 & 1 \end{pmatrix} \sim \begin{pmatrix} \lambda^9 & \lambda^7 & \lambda^5 \\ \lambda^7 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^3 & 1 \end{pmatrix}$$

TABLE VI. Numerical values of the order-1 a,A parameters that parametrize the Yukawa and neutrino Majorana matrices in Eqs. (47),(48).

$$a = \begin{pmatrix} -1.285 & 1.000 & 1.000 \\ 1.000 & -1.420 & 1.000 \\ -1.000 & 1.000 & 0.677 \end{pmatrix}$$

$$a' = \begin{pmatrix} 0.000 & 0.700 & 1.015 \\ 0.000 & 0.782 & 0.705 \\ 0.000 & -1.000 & 0.000 \end{pmatrix}$$

$$a'' = \begin{pmatrix} 0.907 & 0.000 & 0.000 \\ 1.000 & 0.000 & 0.000 \\ 1.000 & 0.000 & 0.000 \end{pmatrix}$$

$$A = \begin{pmatrix} 1.071 & 0.733 & 0.653 \\ 0.733 & 1.072 & 0.567 \\ 0.653 & 0.576 & 1.000 \end{pmatrix}$$

[18]Note that in the Pati-Salam model the 23 entry in the neutrino Dirac Yukawa matrix is associated with the 23 entries in the quark Yukawa matrices, unlike in $SU(5)$, for example, where it is associated with the 32 entry in the down quark Yukawa matrix, which may of course be large since it does not contribute to the CKM matrix. Also note that in the presence of phases it is possible to obtain small quark mixing angles by canceling two large entries.

$$M_{RR}^{\delta}\sim\begin{pmatrix}\mathcal{O} & \mathcal{O} & \mathcal{O}\\ \mathcal{O} & \mathcal{O} & \mathcal{O}\\ \mathcal{O} & \mathcal{O} & 1\end{pmatrix} \tag{46}$$

where $\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{O}''$ are $n=1$, $n=2$ and (very small) $n=3$ operators respectively[19] where n is defined in Eq. (32) and refers to the powers of $(H\bar{H})^n$. Using Eqs. (45),(46) in Eq. (43) gives

$$\lambda_f(M_X)=\begin{pmatrix} x_f^R a_{11}\delta\epsilon^5 & x_f^J a_{12}\delta\epsilon^3 & x_f^g a_{13}\delta\epsilon^2\\ x_f^G a_{21}\delta\epsilon^4 & x_f^W a_{22}\delta\epsilon^2 & x_f^I a_{23}\delta\\ x_f^R a_{31}\delta\epsilon^4 & x_f^M a_{32}\delta\epsilon^2 & a_{33}\end{pmatrix} + \begin{pmatrix} 0 & x_f^Q a'_{12}\delta^2\epsilon^3 & x_f^f a'_{13}\delta^2\epsilon^2\\ 0 & x_f^H a'_{22}\delta^2\epsilon^2 & x_f^W a'_{23}\delta^2\\ 0 & x_f^K a'_{32}\delta^2\epsilon^2 & 0\end{pmatrix} + \begin{pmatrix} x_f^V a''_{11}\delta^3\epsilon^5 & 0 & 0\\ x_f^K a''_{21}\delta^3\epsilon^4 & 0 & 0\\ x_f^V a''_{23}\delta^3\epsilon^4 & 0 & 0\end{pmatrix} \tag{47}$$

$$\frac{M_{RR}(M_X)}{M_{RR}(M_X)_{33}}=\begin{pmatrix} A_{11}\delta\epsilon^8 & A_{12}\delta\epsilon^6 & A_{13}\delta\epsilon^4\\ A_{21}\delta\epsilon^6 & A_{22}\delta\epsilon^4 & A_{23}\delta\epsilon^2\\ A_{31}\delta\epsilon^4 & A_{32}\delta\epsilon^2 & A_{33}\end{pmatrix} \tag{48}$$

where the subscript f stands for any of the u,d,e,ν indices, $x_f^{\mathcal{O}}$ is the Clebsch of the $\mathcal{O}$ operator of the f-type fermion, and the a's (A's) are order-1 f-independent Yukawa (Majorana) parameters that parametrize λ_f (M_{RR}). The first matrix on the right-hand side of Eq. (47) contains the leading $n=1$ operators giving contributions of order δ, while the second and third matrices contain the $n=2$ and $n=3$ operators which give contributions of order δ^2 and δ^3 and provide the leading contributions in the cases where the $n=1$ operators involve Clebsch zeros.

The effective matrices resulting from Eqs. (47),(48) are approximately given in Table V. This table shows an interesting structure for the Yukawa matrices. We find that $\lambda_u\sim\lambda^8$, $\lambda_c\sim\lambda^4$ and $\lambda_d\sim\lambda_e\sim\lambda^6$, $\lambda_s\sim\lambda_\mu\sim\lambda^3$. Furthermore, the CKM matrix has $|V_{12}|\sim\lambda$, $|V_{23}|\sim\lambda^2$, $|V_{13}|\sim\lambda^3$. Comparing these approximate results with the data in Table III we see that only the λ_d, λ_μ couplings need substantial (one λ-power) corrections. On the other hand, the neutrino sector described by λ_ν and M_{RR} in Table V is clearly dominated by the right-handed tau neutrino and predicts $(\lambda_\nu)_{12}\sim(\lambda_\nu)_{22}$ which according to Eq. (B18) successfully generates a large θ_{12} solar neutrino angle. However, the subdominant perturbation to m_{LL} in Eq. (8) resulting from λ_ν and M_{RR} in Table V is too small to correctly predict the neutrino mass ratio $m_{\nu_2}/m_{\nu_3}\sim\lambda^{1.5}$ required by Eq. (40). These approximate predictions can be further improved because Table V does not include the numerical effects of the operator Clebsches and of the order-1 a,A factors.

TABLE VII. Numerical values for the entries of the Yukawa ($\lambda_u,\lambda_d,\lambda_e,\lambda_\nu$) and neutrino Majorana (M_{RR}) matrices at the unification scale M_X (M_{RR} is given in GeV mass units).

$$\lambda_u(M_X)=\begin{pmatrix}7.034\times10^{-6} & 4.079\times10^{-4} & 4.324\times10^{-3}\\ 3.991\times10^{-5} & 1.466\times10^{-3} & 0.000\\ 3.528\times10^{-5} & -3.748\times10^{-3} & 0.677\end{pmatrix}$$

$$\lambda_d(M_X)=\begin{pmatrix}-2.331\times10^{-4} & -4.079\times10^{-4} & 8.648\times10^{-3}\\ 4.609\times10^{-4} & -8.827\times10^{-3} & 2.157\times10^{-2}\\ -8.246\times10^{-4} & 1.506\times10^{-2} & 0.677\end{pmatrix}$$

$$\lambda_e(M_X)=\begin{pmatrix}-1.748\times10^{-4} & 3.884\times10^{-3} & 8.574\times10^{-2}\\ 9.219\times10^{-4} & 3.015\times10^{-2} & -6.472\times10^{-2}\\ -6.184\times10^{-4} & 1.501\times10^{-2} & 0.677\end{pmatrix}$$

$$\lambda_\nu(M_X)=\begin{pmatrix}7.034\times10^{-6} & 2.401\times10^{-3} & 7.710\times10^{-2}\\ 2.993\times10^{-5} & 2.932\times10^{-3} & 0.440\\ 3.528\times10^{-5} & -2.811\times10^{-3} & 0.677\end{pmatrix}$$

$$M_{RR}(M_X)=\begin{pmatrix}3.991\times10^{8} & 5.652\times10^{9} & 1.040\times10^{11}\\ 5.652\times10^{9} & 1.706\times10^{11} & 1.866\times10^{12}\\ 1.040\times10^{11} & 1.866\times10^{12} & 3.090\times10^{14}\end{pmatrix}$$

The success of our model (in the SM sector) depends on the ability to find suitable solutions for the a's in Eq. (47) which simultaneously can account for all the hierarchies in Table III. Generally we will require that $0.5<|a_{AB}|,|a'_{AB}|,|a''_{AB}|<2.0$ for all $A,B=1,2,3$. At first, it looks that such a solution is trivial since Eq. (47) depends on 16 parameters,[20] while the expressions in Table III are a set of 9 constraints (on the first and second family Yukawa couplings and CKM entries). However, we should not forget that $\mathcal{O}\gg\mathcal{O}'\gg\mathcal{O}''$ and that the CKM matrix constrains only the 12, 13 and 23 entries of λ_f. As a consequence, we find that the parameters in Table III are mainly sensitive to $a_{22}\leftrightarrow\lambda_{s,\mu}$, $a_{11}\leftrightarrow\lambda_{d,e}$, an independent combination of $(a''_{11},a''_{21},a''_{31})\leftrightarrow\lambda_u$, $a'_{22}\leftrightarrow\lambda_c$ and $a'_{12}\leftrightarrow V_{12}$, $a'_{23}\leftrightarrow V_{23}$, $a'_{13}\leftrightarrow V_{13}$, which allows two predictions to be made: λ_d and λ_s. Thus we fitted the $a_{22},a_{11},a''_{11},a'_{22},a'_{12},a'_{23},a'_{13}$ dependence[21] of λ_f to the $\lambda_\mu,\lambda_e,\lambda_u,\lambda_c$ and V_{12},V_{23},V_{13} experimental constraints in Table III. The results are shown in Table VI.[22]

Thus, using Eq. (47) with the a's of Table VI and the

[19]The $n=3$ operators can, to a very good approximation, be neglected. Their inclusion here serves only to fill the 11,21,31 entries of the $\lambda_{u,\nu}$ Yukawa matrices, thereby ensuring (for example) that the up quark is given a very small mass.

[20]We note that a_{33} is fixed by quadruple Yukawa unification at M_X, i.e., $\lambda_t=\lambda_b=\lambda_\tau=\lambda_{\nu_\tau}$.

[21]Fixing all other a's to be ±1.

[22]The values of the A parameters in Eq. (48) are not constrained by the experimental data; thus we chose them to be ''arbitrary'' numbers in the $0.5<A_{AB}<2.0$ range.

Clebsch factors of Table XIV (Appendix A) we get the numerical results for $\lambda_f(M_X)$ shown in Table VII. In Table VIII we present the results of Table VII expanded in powers of $\delta=\epsilon=\lambda=0.22$.

We can analyze the effect of the operator Clebsch coefficients by comparing Table V against Table VIII. We see that the $\mathcal{O}^W$ operator[23] in the 22 entry of λ_f split $(\lambda_d)_{22}=(\lambda_e)_{22}\sim\lambda^3$ in Table V into $|(\lambda_d)_{22}|=\lambda^{3.124}$ and $(\lambda_e)_{22}=\lambda^{2.313}$ in Table VIII, thus allowing for a proper GJ ratio $\lambda_\mu/\lambda_s\sim3$ at M_X. Similarly, the operators in the 12 block have allowed for a more appropriate λ_d/λ_e Yukawa ratio. Numerically we have the following predictions for the lightest eigenvalues of the down-Yukawa matrix at M_X: $\lambda_d=\lambda^{5.469}$ and $\lambda_s=\lambda^{3.087}$.

The effect of the Clebsch factors also modified the neutrino Yukawa matrix $\lambda_\nu(M_X)$ in Table V. As a result of the $\mathcal{O}^I$ operator in the 23 position of λ_f, which has a large $x_\nu^I=2$ Clebsch coefficient, we are now able to predict a large θ_{23} atmospheric neutrino mixing angle. Indeed using Eq. (B3) we can roughly estimate that $\tan\theta_{23}\sim0.44/0.68=0.65$, implying $\sin^2(2\theta_{23})\sim0.83$.

It is interesting to check that λ_ν and M_{RR} in Table VIII do lead to a m_{LL} matrix dominated by the right-handed tau neutrino. (For convenience the mechanism of SRHND is reviewed in Appendix B.) As a result of the small mixing angles of M_{RR} it is convenient to work in a basis where M_{RR} is diagonal. Furthermore, it is practical to scale[24] λ_ν such that the 23 and 33 entries are $(\lambda_\nu)_{23}\sim(\lambda_\nu)_{33}\sim1$, and approximate the normalized entries to (semi-)integer λ powers. Thus using

$$M_{RR}\sim\begin{pmatrix}\lambda^{9.0}&0&0\\0&\lambda^{5.0}&0\\0&0&1\end{pmatrix},\quad m_{RL}\sim\begin{pmatrix}\lambda^{7.5}&\lambda^{3.5}&\lambda^{1.5}\\\lambda^{6.5}&\lambda^{3.5}&1\\\lambda^{6.5}&\lambda^{3.5}&1\end{pmatrix}\tag{49}$$

in Eq. (8) gives

$$m_{LL}\sim\begin{pmatrix}\lambda^{3.0}+\lambda^{2.0}+\lambda^{6.0}&\lambda^{1.5}+\lambda^{2.0}+\lambda^{5.0}&\lambda^{1.5}+\lambda^{2.0}+\lambda^{5.0}\\\lambda^{1.5}+\lambda^{2.0}+\lambda^{5.0}&1+\lambda^{2.0}+\lambda^{4.0}&1+\lambda^{2.0}+\lambda^{4.0}\\\lambda^{1.5}+\lambda^{2.0}+\lambda^{5.0}&1+\lambda^{2.0}+\lambda^{4.0}&1+\lambda^{2.0}+\lambda^{4.0}\end{pmatrix}\tag{50}$$

where the first (second and third) term in each entry corresponds to the third (second and first) family neutrino contribution ν_τ^c (ν_μ^c and ν_e^c) coming from M_{RR}. Clearly Eq. (50) shows that even though, in this case, ν_τ^c is the heaviest right-handed neutrino, it nevertheless dominates the 23 block, and that the sub-dominant contribution from ν_μ^c induces λ^2 perturbations in m_{LL} that are compatible with the m_{ν_2}/m_{ν_3} mass ratio.

Using the MSSM RGEs adapted and properly extended to take into account the presence and successive decoupling of the right-handed neutrinos between the $Q=M_X$ and $Q=M_Z$ scales (see Appendix C) we find that the Yukawa and neutrino Majorana matrices at low energy are given by Table IX.

Thus, inserting the results of Table IX into Eq. (8) and Eq. (12) we get the mass matrix for the left-handed neutrinos m_{LL} and the V^{MNS} mixing matrix shown in Table X. The predictions for the neutrino masses and squared mass splittings are shown in Table XI. In Table XII we examine how

TABLE VIII. Values for the entries of the Yukawa $(\lambda_u,\lambda_d,\lambda_e,\lambda_\nu)$ and neutrino Majorana (M_{RR}) matrices at the unification scale M_X as given in Table VII but expanded in powers of $\delta=\epsilon=\lambda=0.22$ [note $M_{RR}(M_X)_{33}=3.090\times10^{14}$ GeV].

$$\lambda_u(M_X)\sim\begin{pmatrix}\lambda^{7.836}&\lambda^{5.154}&\lambda^{3.595}\\\lambda^{6.690}&\lambda^{4.309}&0\\\lambda^{6.771}&-\lambda^{3.690}&\lambda^{0.257}\end{pmatrix}$$

$$\lambda_d(M_X)\sim\begin{pmatrix}-\lambda^{5.524}&-\lambda^{5.154}&\lambda^{3.137}\\\lambda^{5.074}&-\lambda^{3.124}&\lambda^{2.534}\\-\lambda^{4.690}&\lambda^{2.771}&\lambda^{0.257}\end{pmatrix}$$

$$\lambda_e(M_X)\sim\begin{pmatrix}-\lambda^{5.714}&\lambda^{3.666}&\lambda^{1.622}\\\lambda^{4.616}&\lambda^{2.313}&-\lambda^{1.808}\\-\lambda^{4.880}&\lambda^{2.771}&\lambda^{0.257}\end{pmatrix}$$

$$\lambda_\nu(M_X)\sim\begin{pmatrix}\lambda^{7.836}&\lambda^{3.984}&\lambda^{1.693}\\\lambda^{6.880}&\lambda^{3.852}&\lambda^{0.542}\\\lambda^{6.771}&-\lambda^{3.880}&\lambda^{0.257}\end{pmatrix}$$

$$\frac{M_{RR}(M_X)}{M_{RR}(M_X)_{33}}\sim\begin{pmatrix}\lambda^{8.955}&\lambda^{7.205}&\lambda^{5.281}\\\lambda^{7.205}&\lambda^{4.954}&\lambda^{3.375}\\\lambda^{5.281}&\lambda^{3.375}&1\end{pmatrix}$$

TABLE IX. Numerical values for the entries of the Yukawa $(\lambda_u,\lambda_d,\lambda_e,\lambda_\nu)$ and neutrino Majorana (M_{RR}) matrices at the $Q=M_Z$ scale (M_{RR} is given in GeV mass units).

$$\lambda_u(M_Z)=\begin{pmatrix}1.478\times10^{-5}&8.920\times10^{-4}&5.058\times10^{-3}\\8.484\times10^{-5}&3.143\times10^{-3}&-3.553\times10^{-3}\\4.687\times10^{-5}&-5.015\times10^{-3}&0.905\end{pmatrix}$$

$$\lambda_d(M_Z)=\begin{pmatrix}-4.605\times10^{-4}&-9.389\times10^{-4}&1.175\times10^{-2}\\9.379\times10^{-4}&-1.795\times10^{-2}&3.124\times10^{-2}\\-1.057\times10^{-3}&1.934\times10^{-2}&0.862\end{pmatrix}$$

$$\lambda_e(M_Z)=\begin{pmatrix}-1.526\times10^{-4}&3.445\times10^{-3}&6.611\times10^{-2}\\8.832\times10^{-4}&2.931\times10^{-2}&-5.446\times10^{-2}\\-4.626\times10^{-4}&1.196\times10^{-2}&0.522\end{pmatrix}$$

$$\lambda_\nu(M_Z)=\begin{pmatrix}6.110\times10^{-6}&2.271\times10^{-3}&6.339\times10^{-2}\\2.699\times10^{-5}&2.737\times10^{-3}&0.394\\2.911\times10^{-5}&-2.428\times10^{-3}&0.562\end{pmatrix}$$

$$M_{RR}(M_Z)=\begin{pmatrix}3.987\times10^{8}&5.651\times10^{9}&9.981\times10^{10}\\5.651\times10^{9}&1.707\times10^{11}&1.807\times10^{12}\\9.981\times10^{10}&1.807\times10^{12}&2.879\times10^{14}\end{pmatrix}$$

[23]Which has Clebsch coefficients $x_e^W=-3x_d^W$.

[24]$(\lambda_\nu)_{AB}\to(\lambda_\nu)_{AB}/k$ with $k=\frac{1}{2}[(\lambda_\nu)_{23}+(\lambda_\nu)_{33}]=\lambda^{0.384}$.

TABLE X. Predicted values for the left-handed neutrino mass matrix $m_{LL}(M_Z)$ in units of $m_{LL}(M_Z)_{33}=3.954\times10^{-3}$ eV and for the MNS neutrino mixing matrix $V^{MNS}(M_Z)$.

$$\frac{m_{LL}(M_Z)}{m_{LL}(M_Z)_{33}}=\begin{pmatrix} 4.792\times10^{-2} & 1.007\times10^{-1} & 3.458\times10^{-2} \\ 1.007\times10^{-1} & 4.610\times10^{-1} & 5.659\times10^{-1} \\ 3.458\times10^{-2} & 5.659\times10^{-1} & 1 \end{pmatrix}$$

$$V^{MNS}(M_Z)=\begin{pmatrix} 0.8290 & 0.5532 & -0.0819 \\ -0.3948 & 0.6827 & 0.6149 \\ 0.3961 & -0.4774 & 0.7843 \end{pmatrix}$$

the neutrino mixing angles evolve between the unification $Q=M_X\sim3\times10^{16}$ GeV, the right-handed tau neutrino mass $Q=M_{\nu_3}\sim3\times10^{14}$ GeV and the M_Z scale. We see that the effect of the radiative corrections has increased the magnitude of $\sin\theta_{12}$, $\sin\theta_{23}$ and $\sin\theta_{13}$ by 2.5%, 6.4% and 2.4% respectively. These corrections agree with the results found in Ref. [48]. Finally, we present in Table XIII the predictions for the down and strange quark masses.

We would like to conclude this section by noting that the predictions for the neutrino parameters, in particular for the neutrino Δm^2_{12} squared mass splitting, should be taken carefully. Generally we expect at least 20% (theoretical) errors in the quoted values which, for example, arise from our inability to fix order-1 factors in the entries of $M_{RR}(M_X)$.

VII. CONCLUSION

We have discussed a theory of all fermion masses and mixing angles based on a particular string-inspired *minimal* model based on the Pati-Salam group $SU(4)\times SU(2)_L\times SU(2)_R$ [18] supplemented by a gauged $U(1)_X$ family symmetry. We argued that this gauge group preserves the attractive features of $SO(10)$ such as predicting three right-handed neutrinos and Yukawa unification, while avoiding the doublet-triplet splitting problem. Although it is not a unified gauge group at the field theory level, it naturally arises from string constructions and so in principle may be fully unified with gravity.

Earlier work in collaboration with one of us [24] had already shown that the model can provide a successful description of the charged fermion masses and the CKM matrix. The use of the $U(1)_X$ family symmetry to provide the horizontal mass splittings combined with the Clebsch factors arising from the $(H\bar{H})^n$ insertion in the operators has already been shown to provide a powerful approach to the fermion mass spectrum in this model [24]. The present analysis differs from that presented previously partly due to the recent refinements in third family Yukawa unification [25], but mainly due to the recent data from Super-Kamiokande which imply that the 23 operator should be allowed by the $U(1)_X$ family symmetry. We have therefore extended our previous analysis to the atmospheric and solar neutrino masses and mixing angles, and showed that all three MSW solutions to the solar neutrino data may be accommodated, namely the LMA MSW region discussed in the main text as well as the LOW MSW and the SMA MSW regions discussed in the Appendixes.

TABLE XI. Predicted values for the left-handed neutrino masses $m_{\nu_{1,2,3}}$ and squared mass splittings $\Delta m^2_{12}=|m^2_{\nu_2}-m^2_{\nu_1}|$, $\Delta m^2_{23}=|m^2_{\nu_3}-m^2_{\nu_2}|$.

$m_{\nu_1}=4.84\times10^{-8}$ eV	$m_{\nu_2}=5.79\times10^{-3}$ eV	$m_{\nu_3}=5.39\times10^{-2}$ eV
$\Delta m^2_{12}=3.35\times10^{-5}$ eV^2	$\Delta m^2_{23}=2.87\times10^{-3}$ eV^2	

TABLE XII. Running of the neutrino mixing angles at the unification $Q=M_X$, the right-handed tau neutrino mass $Q=M_{\nu_3}$ and Z boson mass $Q=M_Z$ energy scales.

	$Q=M_X\sim3\times10^{16}$ GeV	
$\sin\theta_{12}=0.541$	$\sin\theta_{23}=0.578$	$\sin\theta_{13}=-0.080$
$\sin^2(2\theta_{12})=0.828$	$\sin^2(2\theta_{23})=0.890$	$\sin^2(2\theta_{13})=0.025$
	$Q=M_{\nu_3}\sim3\times10^{14}$ GeV	
$\sin\theta_{12}=0.543$	$\sin\theta_{23}=0.590$	$\sin\theta_{13}=-0.082$
$\sin^2(2\theta_{12})=0.832$	$\sin^2(2\theta_{23})=0.908$	$\sin^2(2\theta_{13})=0.027$
	$Q=M_Z$	
$\sin\theta_{12}=0.555$	$\sin\theta_{23}=0.617$	$\sin\theta_{13}=-0.082$
$\sin^2(2\theta_{12})=0.853$	$\sin^2(2\theta_{23})=0.943$	$\sin^2(2\theta_{13})=0.027$

The approach to neutrino masses and mixing angles followed here makes use of the SRHND mechanism [10–12] in which one of the right-handed neutrinos (the ν^c_τ) gives the dominant contribution to the 23 block of the light effective Majorana matrix. This mechanism avoids reliance on accidental cancellations, and does not rely on excessive magnification of mixing angles, although a mild enhancement was observed in the numerical results in agreement with that observed in [48]. Crucial to the implementation of SRHND in this model is the assumption that the renormalizable 23 operator is forbidden by unspecified string selection rules, and the leading 23 operator contains $(H\bar{H})$ and involves ''Clebsch zeros,'' which give a zero contribution to the charged lepton and quark Yukawa matrices, but a non-zero contribution to the neutrino Yukawa matrix, thereby allowing small V_{cb} but large 23 mixing in the lepton sector.

The analysis in this paper is essentially ''bottom up.'' A particular choice of $U(1)_X$ family symmetry charges was used to give the horizontal mass splittings, and the vertical mass splittings were achieved by particular choices of operators corresponding to different Clebsch factors in the leading contributions to each entry of the Yukawa matrix. It would be very nice to understand these choices from the point of view of a ''top-down'' string construction, such as the type I string construction which has recently led to the Pati-Salam

TABLE XIII. Predictions for the running $\overline{\text{MS}}$ masses of the down (m_d) and strange (m_s) quarks at $Q=1$ GeV and $Q=M_s$, respectively (M_s indicates the strange pole quark mass).

$m_d(1\text{ GeV})=4.9$ MeV	$m_s(M_s)=156$ MeV

TABLE XIV. List of Clebsch factors resulting from all possible $n=1$ operators, as given by Eq. (32), in the Pati-Salam model.

Class	$\mathcal{O}$	x_u	x_d	x_e	x_ν
I	$\mathcal{O}^N$	2.0000	0.0000	0.0000	0.0000
I	$\mathcal{O}^E$	0.0000	2.0000	0.0000	0.0000
I	$\mathcal{O}^i$	0.0000	0.0000	2.0000	0.0000
I	$\mathcal{O}^I$	0.0000	0.0000	0.0000	2.0000
II	$\mathcal{O}^M$	0.0000	1.4142	1.4142	0.0000
II	$\mathcal{O}^G$	0.0000	0.8944	1.7889	0.0000
II	$\mathcal{O}^R$	0.0000	1.6000	1.2000	0.0000
II	$\mathcal{O}^W$	0.0000	0.6325	−1.8974	0.0000
III	$\mathcal{O}^V$	1.4142	0.0000	0.0000	1.4142
III	$\mathcal{O}^O$	0.8944	0.0000	0.0000	1.7889
III	$\mathcal{O}^K$	1.6000	0.0000	0.0000	1.2000
III	$\mathcal{O}^Z$	0.6325	0.0000	0.0000	−1.8974
IV	$\mathcal{O}^J$	0.0000	0.0000	1.7889	0.8944
IV	$\mathcal{O}^g$	0.0000	0.0000	1.4142	1.4142
IV	$\mathcal{O}^h$	0.0000	0.0000	−1.4142	1.4142
IV	$\mathcal{O}^j$	0.0000	0.0000	0.8944	1.7889
V	$\mathcal{O}^F$	1.4142	−1.4142	0.0000	0.0000
V	$\mathcal{O}^a$	1.4142	1.4142	0.0000	0.0000
V	$\mathcal{O}^b$	1.7889	0.8944	0.0000	0.0000
V	$\mathcal{O}^c$	0.8944	1.7889	0.0000	0.0000
VI	$\mathcal{O}^H$	0.8000	0.4000	0.8000	1.6000
VI	$\mathcal{O}^f$	0.4000	0.8000	1.6000	0.8000
VI	$\mathcal{O}^S$	0.7155	1.4311	1.0733	0.5367
VI	$\mathcal{O}^L$	1.4311	0.7155	0.5367	1.0733
VI	$\mathcal{O}^T$	0.5657	0.2828	0.2828	0.5657
VI	$\mathcal{O}^U$	0.2828	0.5657	0.5657	0.2828
VI	$\mathcal{O}^X$	0.5657	0.2828	−0.8485	−1.6971
VI	$\mathcal{O}^Y$	0.2828	0.5657	−1.6971	−0.8485
VI	$\mathcal{O}^D$	0.4472	−0.4472	1.3416	−1.3416
VI	$\mathcal{O}^e$	0.6325	−0.6325	−1.2649	1.2649
VI	$\mathcal{O}^B$	1.0000	−1.0000	−1.0000	1.0000
VI	$\mathcal{O}^Q$	1.1314	−1.1314	−0.8485	0.8485
VI	$\mathcal{O}^P$	1.1314	1.1314	0.8485	0.8485
VI	$\mathcal{O}^A$	1.0000	1.0000	1.0000	1.0000
VI	$\mathcal{O}^d$	0.6325	0.6325	1.2649	1.2649
VI	$\mathcal{O}^C$	0.4472	0.4472	−1.3416	−1.3416

gauge group with three chiral families [20]. We believe that only by a combination of top-down and bottom-up approaches (such as that presented here) will a completely successful string theory of fermion masses and mixing angles emerge. We have shown that the recent discovery of neutrino mass by Super-Kamiokande provides precious information about the flavor structure of such a future string theory.

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APPENDIX A

A list of Clebsch factors is given in Table XIV.

APPENDIX B

In this appendix we summarize the theory behind SRHND and review the analytic results presented in Ref. [12] for the case of the diagonal dominated right-handed neutrino mass matrix M_{RR}. Third family single right-handed neutrino dominance [10–12] is a mechanism that can explain the large atmospheric (θ_{23}) and the solar LMA MSW (θ_{12}) neutrino mixing angles and a small θ_{13}. SRHND relies on the possibility that the neutrino mass matrix (m_{LL}) is dominated by contributions coming solely from a single right-handed neutrino (for example ν_τ^c.) In this scheme a maximal θ_{23} angle arises when the tau right-handed neutrino ν_τ^c couples to the left-handed muon ν_μ and tau neutrino ν_τ with equal strength. Similarly, if ν_μ^c couples to ν_e and to ν_μ with comparable strength, then θ_{12} is large. The role of the (subdominant) muon neutrino is also important since it provides small perturbations to the m_{LL} matrix (which otherwise has one heavy and two massless eigenstates), thus leading to a neutrino mass splitting $\Delta m_{12}^2=|m_{\nu_2}^2-m_{\nu_1}^2|$ compatible with experiment.

The seesaw formula for the left-handed neutrino matrix in Eq. (32) depends explicitly on M_{RR}. Although M_{RR} might have a non-trivial structure, we find instructive to start our analysis by considering the very simple case of M_{RR} given by

$$M_{RR}^{-1}\sim \mathrm{diag}(M_{\nu_1}^{-1},M_{\nu_2}^{-1},M_{\nu_3}^{-1})\sim \mathrm{diag}(0,0,M_{\nu_3}^{-1}) \quad \text{(B1)}$$

which effectively corresponds to taking $M_{\nu_1},M_{\nu_2}\gg M_{\nu_3}$. Replacing Eq. (B1) in Eq. (8) we find that[25]

$$m_{LL}=v_2^2\lambda_\nu M_{RR}^{-1}\lambda_\nu^T\sim\lambda M_{RR}^{-1}\lambda^T \sim M_{\nu_3}^{-1}\begin{pmatrix}\lambda_{13}^2 & \lambda_{13}\lambda_{23} & \lambda_{13}\lambda_{33}\\ \lambda_{13}\lambda_{23} & \lambda_{22}^2 & \lambda_{23}\lambda_{33}\\ \lambda_{13}\lambda_{33} & \lambda_{23}\lambda_{33} & \lambda_{33}^2\end{pmatrix}. \quad \text{(B2)}$$

The m_{LL} matrix above is easily diagonalized by the matrices[26] R_{23},R_{13},R_{12} in Eq. (14) with rotation angles given by

$$s_{23}=\frac{\lambda_{23}}{A} \quad c_{23}=\frac{\lambda_{33}}{A}, \quad A^2=\lambda_{33}^2+\lambda_{23}^2,$$

$$s_{13}=\frac{\lambda_{13}}{B} \quad c_{13}=\frac{A}{B}, \quad B^2=\lambda_{33}^2+\lambda_{23}^2+\lambda_{13}^2, \quad \text{(B3)}$$

which successively act on m_{LL} as follows:

$$m_{LL}'''=R_{12}^\dagger R_{13}^\dagger R_{23}^\dagger m_{LL}R_{23}R_{13}R_{12}=\mathrm{diag}(m_{\nu_1},m_{\nu_2},m_{\nu_3}). \quad \text{(B4)}$$

It is also convenient to define the following primed matrices:

[25]In this appendix we will use the following simplified notation: $m_{LR}=\lambda_\nu v_2\equiv\lambda v_2\sim\lambda$.

[26]Note that R_{12} for m_{LL} in Eq. (B2) is undetermined.

$$m'_{LL}=R_{23}^{\dagger}m_{LL}R_{23}, \quad m''_{LL}=R_{13}^{\dagger}m'_{LL}R_{13}, \quad m'''_{LL}=R_{12}^{\dagger}m''_{LL}R_{12} \tag{B5}$$

which, for m_{LL} as in Eq. (B2), are explicitly given by

$$m'_{LL}\sim M_{\nu_3}^{-1}\begin{pmatrix}\lambda_{13}^2 & 0 & \lambda_{13}A\\ 0 & 1 & 0\\ \lambda_{13}A & 0 & A^2\end{pmatrix},$$

$$m''_{LL}\equiv m'''_{LL}\sim M_{\nu_3}^{-1}\begin{pmatrix}0&0&0\\0&0&0\\0&0&B^2\end{pmatrix}. \tag{B6}$$

We can see from Eq. (B3) that if $\lambda_{23}=\lambda_{33}$, then a maximal $\theta_{23}=45°$ angle results. Moreover, if $\lambda_{13}\ll\lambda_{23},\lambda_{33}$, then θ_{13} is small. Although SRHND, in the limiting case of Eq. (B1), is successful in predicting a maximal atmospheric neutrino angle, it fails to account for a viable neutrino spectrum. Indeed, from Eq. (B6), we see that the two lightest neutrinos are massless, $m_{\nu_1}=m_{\nu_2}=0$. Moreover, the solar neutrino angle θ_{12} is undetermined. These two problems can be solved by allowing the right-handed muon neutrino ν_μ^c to play a sub-dominant and perturbative role in the structure of m_{LL} in Eq. (B2).

We now turn to the more realist model in which M_{RR} can be approximated by[27]

$$M_{RR}^{-1}\sim \mathrm{diag}(M_{\nu_1}^{-1},M_{\nu_2}^{-1},M_{\nu_3}^{-1})\sim \mathrm{diag}(0,M_{\nu_2}^{-1},M_{\nu_3}^{-1}). \tag{B7}$$

Using Eq. (B7) in Eq. (8) we find that

$$m_{LL}\sim M_{\nu_3}^{-1}\begin{pmatrix}\lambda_{13}^2 & \lambda_{13}\lambda_{23} & \lambda_{13}\lambda_{33}\\ \lambda_{13}\lambda_{23} & \lambda_{23}^2 & \lambda_{23}\lambda_{33}\\ \lambda_{13}\lambda_{33} & \lambda_{23}\lambda_{33} & \lambda_{33}^2\end{pmatrix} + M_{\nu_2}^{-1}\begin{pmatrix}\lambda_{12}^2 & \lambda_{12}\lambda_{22} & \lambda_{12}\lambda_{32}\\ \lambda_{12}\lambda_{22} & \lambda_{22}^2 & \lambda_{22}\lambda_{32}\\ \lambda_{12}\lambda_{32} & \lambda_{22}\lambda_{32} & \lambda_{32}^2\end{pmatrix}. \tag{B8}$$

Given that we assumed SRHND by the ν_τ^c neutrino, it follows that the contributions to the 23 block of m_{LL} in Eq. (B8) arising from the terms proportional to $M_{\nu_3}^{-1}$ dominate over the ones proportional to $M_{\nu_2}^{-1}$.[28] Clearly, the rotations R_{12}, R_{13} parametrized by the angles in Eq. (B3) diagonalize m_{LL} in Eq. (B8) up to terms of order $\mathcal{O}(M_{\nu_2}^{-1})$. Thus the new primed matrices m'_{LL} and m''_{LL} are given by

$$m'_{LL}\sim M_{\nu_3}^{-1}\begin{pmatrix}\lambda_{13}^2 & 0 & \lambda_{13}A\\ 0&0&0\\ \lambda_{13}A & 0 & A^2\end{pmatrix} + M_{\nu_2}^{-1}\begin{pmatrix}\lambda_{12}^2 & \lambda_{12}\frac{C^2}{A} & \lambda_{12}\frac{D^2}{A}\\ \lambda_{12}\frac{C^2}{A} & \frac{C^4}{A^2} & \frac{C^2D^2}{A^2}\\ \lambda_{12}\frac{D^2}{A} & \frac{C^2D^2}{A^2} & \frac{D^4}{A^2}\end{pmatrix} \tag{B9}$$

$$m''_{LL}\sim M_{\nu_3}^{-1}\begin{pmatrix}0&0&0\\0&0&0\\0&0&B^2\end{pmatrix} + M_{\nu_2}^{-1}\begin{pmatrix}\frac{E^6}{A^2B^2} & \frac{C^2E^3}{A^2B} & \frac{F^2E^3}{AB^2}\\ \frac{C^2E^3}{A^2B} & \frac{C^4}{A^2} & \frac{C^2F^2}{AB}\\ \frac{F^2E^3}{AB^2} & \frac{C^2F^2}{AB} & \frac{F^4}{B^2}\end{pmatrix} \tag{B10}$$

where

$$C^2=\lambda_{22}\lambda_{33}-\lambda_{32}\lambda_{23} \tag{B11}$$

$$D^2=\lambda_{33}\lambda_{32}+\lambda_{22}\lambda_{23} \tag{B12}$$

$$E^3=\lambda_{12}(\lambda_{33}^2+\lambda_{23}^2)-\lambda_{13}(\lambda_{33}\lambda_{32}+\lambda_{22}\lambda_{23}) \tag{B13}$$

$$F^2=\lambda_{33}\lambda_{32}+\lambda_{22}\lambda_{23}+\lambda_{12}\lambda_{13}. \tag{B14}$$

The diagonalization of the 12 block of m''_{LL} in Eq. (B10) is achieved by a R_{12} matrix parametrized by the following θ_{12} rotation angle:

$$s_{12}=\frac{E^3}{\sqrt{E^6+B^2C^4}}, \quad c_{12}=\frac{BC^2}{\sqrt{E^6+B^2C^4}}. \tag{B15}$$

Thus we find

[27]Note that although Eq. (B7) still looks very simple, it can, in many cases, provide a good qualitative description of the physics involving the heaviest neutrinos. Indeed, if M_{RR} is diagonal dominated and if m_{RL} is highly hierarchical, then the limiting case of Eq. (B7) applies.

[28]Note that this does not necessarily imply that $M_{\nu_3}^{-1}$ is larger than $M_{\nu_2}^{-1}$ since the Yukawa couplings must also be taken into account.

$$m'''_{LL} \sim M_{\nu_3}^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B^2 \end{pmatrix} + M_{\nu_2}^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{E^6+B^2C^4}{A^2B^2} & \frac{F^2\sqrt{E^6+B^2C^4}}{AB^2} \\ 0 & \frac{F^2\sqrt{E^6+B^2C^4}}{AB^2} & \frac{F^4}{B^2} \end{pmatrix}. \tag{B16}$$

It is interesting to note that the R_{12} rotation has not only diagonalized the 12 block of m''_{LL} but also put zeros in the 13,31 entries of m'''_{LL}. The reason is because m''_{LL} displays a special structure. Indeed, their elements obey

$$t_{12} = \frac{s_{12}}{c_{12}} = \frac{(m''_{LL})_{12}}{(m''_{LL})_{22}} = \frac{(m''_{LL})_{11}}{(m''_{LL})_{12}} = \frac{(m''_{LL})_{13}}{(m''_{LL})_{23}} = \frac{E^3}{BC^2}. \tag{B17}$$

Explicitly $t_{12} = \tan\theta_{12}$ is given by

$$t_{12} = \frac{\lambda_{12}(\lambda_{33}^2+\lambda_{23}^2) - \lambda_{13}(\lambda_{33}\lambda_{32}+\lambda_{22}\lambda_{23})}{(\lambda_{22}\lambda_{33}-\lambda_{32}\lambda_{23})\sqrt{\lambda_{33}^2+\lambda_{23}^2+\lambda_{13}^2}} \sim \frac{\lambda_{12}}{\lambda_{22}}. \tag{B18}$$

From Eq. (B18) we see that, although t_{12} generally depends on the second and third family neutrino Yukawa couplings, if λ_{33} is much bigger than the other Yukawa couplings, then $t_{12} \sim \lambda_{12}/\lambda_{22}$. This means that the θ_{12} angle is set not by the dominant neutrino couplings, but by the sub-dominant ν_μ^c neutrino couplings to the ν_e and ν_μ neutrinos. Thus while a large atmospheric neutrino mixing angle θ_{23} can be achieved by requiring $\lambda_{23} \sim \lambda_{33}$, a large MSW solar neutrino angle θ_{12} results from $\lambda_{12} \sim \lambda_{22}$. Moreover, Eqs. (B3) and (B18) show that bi-maximal θ_{23}, θ_{12} mixing can be achieved with a small θ_{13} angle as long as $\lambda_{13} \ll \lambda_{23}, \lambda_{33}$. The neutrino mass spectrum can be read from m'''_{LL} in Eq. (B16). We find a massless neutrino state $m_{\nu_1} = 0$, plus a light state with mass $m_{\nu_2} \sim \lambda_{22}^2/M_{\nu_2}$ and a heavy neutrino with mass $m_{\nu_3} \sim \lambda_{33}^2/M_{\nu_3}$.

APPENDIX C

In this appendix we briefly review some technical issues related to the presence of the right-handed neutrinos. First, we show how the decoupling of the neutrinos affect the one-loop RGEs for the Yukawa couplings in the MSSM$+\nu^c$ model:

$$16\pi^2 \frac{d\lambda_u}{dt} = [3\mathrm{Tr}\ U + \mathrm{Tr}\ N + 3U + D - G^u]\lambda_u \tag{C1}$$

$$16\pi^2 \frac{d\lambda_d}{dt} = [3\mathrm{Tr}\ D + \mathrm{Tr}\ E + 3D + U - G^d]\lambda_d \tag{C2}$$

$$16\pi^2 \frac{d\lambda_e}{dt} = [3\mathrm{Tr}\ D + \mathrm{Tr}\ E + 3E + N - G^e]\lambda_e \tag{C3}$$

$$16\pi^2 \frac{d\lambda_\nu}{dt} = [3\mathrm{Tr}\ U + \mathrm{Tr}\ N + 3N + E - G^\nu]\lambda_\nu \tag{C4}$$

where $t = \ln(Q)$,

$$\begin{aligned} U &= \lambda_u\lambda_u^\dagger, \quad G^u = \frac{26}{30}g_1^2 + 3g_2^2 + \frac{16}{3}g_3^2, \\ D &= \lambda_d\lambda_d^\dagger, \quad G^d = \frac{14}{30}g_1^2 + 3g_2^2 + \frac{16}{3}g_3^2, \\ E &= \lambda_e\lambda_e^\dagger, \quad G^e = \frac{18}{10}g_1^2 + 3g_2^2, \\ N &= \lambda_\nu\lambda_\nu^\dagger, \quad G^\nu = \frac{6}{10}g_1^2 + 3g_2^2, \end{aligned} \tag{C5}$$

and $\mathrm{Tr}U = U_{11} + U_{22} + U_{33}$, etc. The general idea behind the process of decoupling the right-handed neutrinos (in the ''step'' approximation) is that a Feynman diagram that includes a specific flavor of a right-handed neutrino ν_A^c, with mass M_{ν_A}, in an internal line only makes a contribution to the RGEs in Eqs. (C1)–(C4) for energies Q bigger than M_{ν_A}. Thus, the procedure depends on properly adapting the N parameter in Eq. (C5). We shall now make this statement more precise. Let us assume that the neutrino Majorana matrix M_{RR} is diagonalized by the following transformation:

$$S^{\nu^c\dagger} M_{RR} S^{\nu^c} = M'_{RR} = \mathrm{diag}(M_{\nu_1}, M_{\nu_2}, M_{\nu_3}). \tag{C6}$$

Then, the decoupling of the right-handed neutrinos in Eqs. (C1)–(C4) can be accounted by replacing N in Eq. (C5) by N_θ given by

$$N = \lambda_\nu\lambda_\nu^\dagger = \lambda_\nu S^{\nu^c} S^{\nu^c\dagger}\lambda_\nu^\dagger \rightarrow \lambda_\nu S^{\nu^c}\Theta S^{\nu^c\dagger}\lambda_\nu^\dagger = N_\theta \tag{C7}$$

where $\Theta(Q)$ is a energy dependent diagonal matrix defined by

$$\Theta(Q) = \mathrm{diag}(\theta(Q-M_{\nu_1}), \theta(Q-M_{\nu_2}), \theta(Q-M_{\nu_3})) \tag{C8}$$

with $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$.

The second issue that we would like to address concerns the effect of a large $(\lambda_\nu)_{23}$ coupling on third family Yukawa unification and, as a consequence, for example, on the prediction for the top quark mass. We claim that the effect is small. To see why let us assume that the only large Yukawa couplings in Eqs. (C1)–(C4) are $\lambda_t=(\lambda_u)_{33}$, $\lambda_b=(\lambda_d)_{33}$, $\lambda_\tau=(\lambda_e)_{33}$ and $\lambda_{\nu_\tau}=(\lambda_\nu)_{33}$, $\lambda_{23}=(\lambda_\nu)_{23}$. In this limit, the RGEs simplify to

$$16\pi^2\frac{d\lambda_t}{dt}=\lambda_t(6\lambda_t^2+\lambda_b^2+\lambda_{\nu_\tau}^2+\lambda_{23}^2-G^u) \tag{C9}$$

$$16\pi^2\frac{d\lambda_b}{dt}=\lambda_b(6\lambda_b^2+\lambda_t^2+\lambda_\tau^2-G^d) \tag{C10}$$

$$16\pi^2\frac{d\lambda_\tau}{dt}=\lambda_\tau(4\lambda_\tau^2+3\lambda_b^2+\lambda_{\nu_\tau}^2-G^e) \tag{C11}$$

$$16\pi^2\frac{d\lambda_{\nu_\tau}}{dt}=\lambda_{\nu_\tau}(4\lambda_{\nu_\tau}^2+4\lambda_{23}^2+3\lambda_t^2+\lambda_\tau^2-G^\nu). \tag{C12}$$

From Eqs. (C10),(C11) we see that the presence of the λ_{23} coupling does not affect the RGEs of $\lambda_{b,\tau}$. Moreover, the effect of λ_{23} on the RGE of λ_t is small ($1/8\sim 12\%$). The only RGE that is significantly affected by λ_{23} is the RGE of λ_{ν_τ}. However, since the correct prediction for the heaviest left-handed neutrino $m_{\nu_3}\sim 0.05$ eV requires that $M_{\nu_3}>10^{13}$ GeV, the $\lambda_{\nu_\tau}^2$ and λ_{23}^2 terms in Eqs. (C9),(C12) are only present in a rather short energy range, i.e., between 10^{13} GeV$<Q<M_X\sim 10^{16}$ GeV. As a consequence the presence or absence of the neutrino Yukawa couplings, as far as third family Yukawa unification is concerned, is not important.[29]

Finally we find interesting to comment on the radiative corrections to the neutrino atmospheric mixing angle θ_{23} between the GUT and the M_Z scales. It is well known [9] that the running of $\sin^2(2\theta_{23})$ can be understood from the following evolution equation:

$$16\pi^2\frac{1}{\sin^2(2\theta_{23})}\frac{d\sin^2(2\theta_{23})}{dt}=-2(\lambda_\tau^2-\lambda_\mu^2)\frac{(m_{LL})_{33}^2-(m_{LL})_{22}^2}{[(m_{LL})_{33}-(m_{LL})_{22}]^2+4(m_{LL})_{23}^2} \tag{C13}$$

which displays a resonance peak at $(m_{LL})_{33}\sim(m_{LL})_{22}$ when $(m_{LL})_{23}$ is small. Generally, it is possible that $(m_{LL})_{33}$ starts at $Q=M_X$ bigger than $(m_{LL})_{22}$ but as a result of the third family Yukawa radiative effects, to be driven to smaller values faster than $(m_{LL})_{22}$. As a result, even if the initial values of $(m_{LL})_{33}$ and $(m_{LL})_{22}$ at M_X are different, they may at some point become comparable. If this is the case, then a large θ_{23} angle can be generated radiatively from a small tree level θ_{23} at M_X. This mechanism, of amplifying θ_{23} radiatively, as been studied for example in Refs. [8,9]. However, in these works, and as can be seen from Eq. (C13), the amplification is only efficient if at least the λ_τ Yukawa coupling is large (about 2 or 3). In our model, since we demanded top-bottom-tau Yukawa unification, the value of the third family Yukawa coupling is rather small (~ 0.7); thus the $\sin^2(2\theta_{23})$ is stable under radiative corrections.

[29]Numerically we found that when $(\lambda_\nu)_{23}$ is allowed to take values comparable with $(\lambda_\nu)_{33}$ the prediction for the top quark mass roughly decreased by 1 GeV, the value of $\tan\beta$ decreased by 0.5 and the value of the unified third family Yukawa coupling at the unification scale decreased by 0.015.

APPENDIX D

In this appendix we show that is easy to convert the results of the LMA MSW solution found in the main body of this paper into results for the LOW solution which is also characterized by maximal $\nu_e\to\nu_\mu$ oscillations but smaller Δm_{12}^2 [49,50]:

$$\text{LOW}:\quad \sin^2(2\theta_{12})\sim 1,\quad \Delta m_{12}^2\sim 10^{-7}\ \text{eV}^2. \tag{D1}$$

The reason why we can adapt the LMA results is because, as we showed in Appendix B, in the SRHND approach, the θ_{12} and θ_{23} neutrino mixing angles come ‘‘solely’’ from the neutrino Yukawa matrix. On the other hand, the neutrino mass spectrum depends on the hierarchies of λ_ν and M_{RR}. Thus, as long as we keep within the SRHND scenario, we can change M_{RR} to fit the LOW Δm_{12}^2 solution without that implying a significant change in θ_{12} and θ_{23}.

Let us consider a LOW model with the same $U(1)_X$ flavor charges and the same operator matrix for λ_f as in the LMA model, given by Table IV and Eq. (47), but with a M_{RR} matrix with the following structure:

$$\frac{M_{RR}(M_X)}{M_{RR}(M_X)_{33}}=\begin{pmatrix} A_{11}\epsilon^8 & A_{12}\epsilon^6 & A_{13}\epsilon^4\\ A_{21}\epsilon^6 & B_{22}\epsilon^4 & A_{23}\epsilon^2\\ A_{31}\epsilon^4 & A_{32}\epsilon^2 & A_{33}\end{pmatrix}. \tag{D2}$$

Comparing Eq. (D2) with Eq. (48) we see that these two equations differ only by their ‘‘vertical’’ δ component [we assumed that Eq. (D2) has $M_{RR}^\delta\sim\mathbf{1}$] and by the numerical factor $B_{22}=1.821\neq 1.072=A_{22}$. We note that the removal the δ factor in the 22 entry of M_{RR} and the increase of the $B_{22}>A_{22}$ coefficient act to decrease Δm_{12}^2.

The Majorana matrix $M_{RR}(M_Z)$ and the neutrino Yukawa matrix $\lambda_\nu(M_Z)$ in the LOW model resulting from $M_{RR}(M_X)$ in Eq. (D2) and the Yukawa matrices $\lambda_f(M_X)$ given by Table VII [recall that we take $\lambda_f^{LOW}(M_X)=\lambda_f^{LMA}(M_X)$] are shown in Table XV. In Table XVI we present the predicted values for the left-handed neutrino matrix and the MNS matrix in the LOW model. The results for the neutrino masses in the LOW model are given in Table XVII. Finally in Table XVIII we show the values of the neutrino mixing angles.

APPENDIX E

In this appendix we briefly present a model that explores the possibility of a SMA MSW solution to the solar neutrino

TABLE XV. Numerical values for the entries of the neutrino Yukawa (λ_ν) and Majorana (M_{RR}) matrices at the $Q=M_Z$ scale in the LOW model (M_{RR} is given in GeV mass units).

$$\lambda_\nu(M_Z)=\begin{pmatrix} 6.120\times10^{-6} & 2.271\times10^{-3} & 6.368\times10^{-2} \\ 2.710\times10^{-5} & 2.736\times10^{-3} & 0.396 \\ 2.925\times10^{-5} & -2.429\times10^{-3} & 0.565 \end{pmatrix}$$

$$M_{RR}(M_Z)=\begin{pmatrix} 2.848\times10^{9} & 4.036\times10^{10} & 7.218\times10^{11} \\ 4.036\times10^{10} & 2.071\times10^{12} & 1.287\times10^{13} \\ 7.218\times10^{11} & 1.287\times10^{13} & 4.551\times10^{14} \end{pmatrix}$$

TABLE XVI. Predicted values for the left-handed neutrino mass matrix $m_{LL}(M_Z)$ in units of $m_{LL}(M_Z)_{33}=3.567\times10^{-3}$ eV and for the MNS neutrino mixing matrix $V^{MNS}(M_Z)$ in the LOW model.

$$\frac{m_{LL}(M_Z)}{m_{LL}(M_Z)_{33}}=\begin{pmatrix} 1.334\times10^{-2} & 7.342\times10^{-2} & 9.957\times10^{-2} \\ 7.342\times10^{-2} & 4.694\times10^{-1} & 6.776\times10^{-1} \\ 9.957\times10^{-2} & 6.776\times10^{-1} & 1 \end{pmatrix}$$

$$V^{MNS}(M_Z)=\begin{pmatrix} 0.8291 & 0.5559 & -0.0597 \\ -0.3955 & 0.6586 & 0.6402 \\ 0.3952 & -0.5072 & 0.7659 \end{pmatrix}$$

TABLE XVII. Predicted values for the left-handed neutrino masses $m_{\nu_{1,2,3}}$ and squared mass splittings $\Delta m^2_{12}=|m^2_{\nu_2}-m^2_{\nu_1}|$, $\Delta m^2_{23}=|m^2_{\nu_3}-m^2_{\nu_2}|$ in the LOW model.

$m_{\nu_1}=7.29\times10^{-9}$ eV $m_{\nu_2}=3.54\times10^{-4}$ eV $m_{\nu_3}=5.25\times10^{-2}$ eV

$\Delta m^2_{12}=1.25\times10^{-7}$ eV^2 $\Delta m^2_{23}=2.76\times10^{-3}$ eV^2

TABLE XVIII. Running of the neutrino mixing angles at the unification $Q=M_X$, the right-handed tau neutrino mass $Q=M_{\nu_3}$ and Z boson mass $Q=M_Z$ energy scales in the LOW model.

	$Q=M_X\sim3\times10^{16}$ GeV	
$\sin\theta_{12}=0.543$	$\sin\theta_{23}=0.607$	$\sin\theta_{13}=-0.056$
$\sin^2(2\theta_{12})=0.832$	$\sin^2(2\theta_{23})=0.931$	$\sin^2(2\theta_{13})=0.013$
	$Q=M_{\nu_3}\sim5\times10^{14}$ GeV	
$\sin\theta_{12}=0.545$	$\sin\theta_{23}=0.617$	$\sin\theta_{13}=-0.058$
$\sin^2(2\theta_{12})=0.836$	$\sin^2(2\theta_{23})=0.943$	$\sin^2(2\theta_{13})=0.013$
	$Q=M_Z$	
$\sin\theta_{12}=0.557$	$\sin\theta_{23}=0.641$	$\sin\theta_{13}=-0.060$
$\sin^2(2\theta_{12})=0.856$	$\sin^2(2\theta_{23})=0.968$	$\sin^2(2\theta_{13})=0.014$

TABLE XIX. List of the $U(1)_{\bar X}$ charges that determine the family structure of the Yukawa and neutrino Majorana matrices in the SMA model.

	X_{F_1}	X_{F_2}	X_{F_3}	$X_{F_1^c}$	$X_{F_2^c}$	$X_{F_3^c}$	X_h	X_H	$X_{\bar H}$
$U(1)_{\bar X}$	2	0	0	3	2	0	0	0	0

TABLE XX. Approximate structure of the Yukawa and neutrino Majorana matrices in the SMA model resulting from Eqs. (E5),(E6) when the numerical effect of the Clebsch and of the c,C parameters is neglected.

$$\lambda_u(M_X)\sim\begin{pmatrix} \delta^3\epsilon^5 & \delta^2\epsilon^4 & \delta\epsilon^2 \\ \delta^3\epsilon^3 & \delta^2\epsilon^2 & 0 \\ \delta^3\epsilon^3 & \delta^2\epsilon^2 & 1 \end{pmatrix}\sim\begin{pmatrix} \lambda^8 & \lambda^6 & \lambda^3 \\ \lambda^6 & \lambda^4 & 0 \\ \lambda^6 & \lambda^4 & 1 \end{pmatrix}$$

$$\lambda_d(M_X)\sim\begin{pmatrix} \delta\epsilon^5 & \delta\epsilon^4 & \delta\epsilon^2 \\ \delta\epsilon^3 & \delta\epsilon^2 & \delta^2 \\ \delta\epsilon^3 & \delta^2\epsilon^2 & 1 \end{pmatrix}\sim\begin{pmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^4 & \lambda^3 & \lambda^2 \\ \lambda^4 & \lambda^4 & 1 \end{pmatrix}$$

$$\lambda_e(M_X)\sim\begin{pmatrix} \delta\epsilon^5 & \delta\epsilon^4 & \delta\epsilon^2 \\ \delta\epsilon^3 & \delta\epsilon^2 & \delta^2 \\ \delta\epsilon^3 & \delta\epsilon^2 & 1 \end{pmatrix}\sim\begin{pmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^4 & \lambda^3 & \lambda^2 \\ \lambda^4 & \lambda^3 & 1 \end{pmatrix}$$

$$\lambda_\nu(M_X)\sim\begin{pmatrix} \delta^3\epsilon^5 & \delta^2\epsilon^4 & \delta\epsilon^2 \\ \delta^3\epsilon^3 & \delta^2\epsilon^2 & \delta \\ \delta^3\epsilon^3 & \delta\epsilon^2 & 1 \end{pmatrix}\sim\begin{pmatrix} \lambda^8 & \lambda^6 & \lambda^3 \\ \lambda^6 & \lambda^4 & \lambda \\ \lambda^6 & \lambda^3 & 1 \end{pmatrix}$$

$$M_{RR}(M_X)\sim\begin{pmatrix} \epsilon^6 & \epsilon^5 & \epsilon^3 \\ \epsilon^5 & \epsilon^4 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}\sim\begin{pmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}$$

anomaly. Although the SMA region is disfavored by the latest results from the Super-Kamiokande experiment, the SMA solution is not completely ruled out.[30] The SMA solution data indicate [49]

$$\text{SMA}: \quad \sin^2(2\theta_{12})\sim1.6\times10^{-3}, \quad \Delta m^2_{12}\sim5\times10^{-6}\ \text{eV}^2. \tag{E1}$$

In analogy with the LMA model we start by recalling that the Yukawa and the neutrino Majorana matrices in the SMA model can be decomposed into a ''vertical'' δ component and a ''horizontal'' ϵ component given by

$$(\lambda_f)_{AB}\sim(\lambda^\delta)_{AB}(\lambda^\epsilon)_{AB}, \quad (M_{RR})_{AB}\sim(M^\delta_{RR})_{AB}(M^\epsilon_{RR})_{AB}. \tag{E2}$$

The $U(1)_{\bar X}$ charges of the SMA given in Table XIX fix the ''horizontal'' structure of λ^ϵ and M^ϵ_{RR} in the SMA model to be

$$\lambda^\epsilon\sim\begin{pmatrix} \epsilon^5 & \epsilon^4 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1 \\ \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}, \quad M^\epsilon_{RR}\sim\begin{pmatrix} \epsilon^6 & \epsilon^5 & \epsilon^3 \\ \epsilon^5 & \epsilon^4 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}. \tag{E3}$$

On the other hand, the ''vertical'' structure of λ^δ and M^δ_{RR} is given by the following operator matrices:

[30] Statistically, the SMA solution can still describe the neutrino data with a probability of 34% [49].

TABLE XXI. Numerical values of the order-1 c,C parameters that parametrize the Yukawa and neutrino Majorana matrices of Eqs. (E5),(E6) in the SMA model.

$$c=\begin{pmatrix} -1.403 & -1.656 & 1.000 \\ 1.000 & -1.563 & 1.000 \\ 1.000 & 1.000 & 0.682 \end{pmatrix}$$

$$c'=\begin{pmatrix} 0.000 & 1.854 & 1.000 \\ 0.000 & -0.807 & 0.702 \\ 0.000 & 1.000 & 0.000 \end{pmatrix}$$

$$c''=\begin{pmatrix} -1.131 & 0.000 & 0.000 \\ 1.000 & 0.000 & 0.000 \\ 1.000 & 0.000 & 0.000 \end{pmatrix}$$

$$C=\begin{pmatrix} 1.069 & 0.533 & 0.799 \\ 0.533 & 1.054 & 0.753 \\ 0.799 & 0.753 & 1.000 \end{pmatrix}$$

$$\lambda^{\delta}\sim\begin{pmatrix} \mathcal{O}^G+\mathcal{O}''^K & \mathcal{O}^R+\mathcal{O}'^O & \mathcal{O}^H+\mathcal{O}'^a \\ \mathcal{O}^G+\mathcal{O}''^V & \mathcal{O}^W+\mathcal{O}'^H & \mathcal{O}^I+\mathcal{O}'^W \\ \mathcal{O}^M+\mathcal{O}''^V & \mathcal{O}^g+\mathcal{O}'^T & 1 \end{pmatrix}, \quad M^{\delta}_{RR}\sim\mathbf{1}. \tag{E4}$$

As a result of Eqs. (E3),(E4) the Yukawa and the neutrino Majorana matrices in Eq. (E2) can be written as

TABLE XXII. Numerical values for the entries of the Yukawa $(\lambda_u,\lambda_d,\lambda_e,\lambda_\nu)$ and neutrino Majorana (M_{RR}) matrices at the unification scale M_X in the SMA model (M_{RR} is given in GeV mass units).

$$\lambda_u(M_X)=\begin{pmatrix} -9.935\times10^{-6} & 1.880\times10^{-4} & 1.183\times10^{-2} \\ 1.603\times10^{-4} & -1.511\times10^{-3} & 0.000 \\ 1.603\times10^{-4} & 1.325\times10^{-3} & 0.682 \end{pmatrix}$$

$$\lambda_d(M_X)=\begin{pmatrix} -1.423\times10^{-4} & -1.365\times10^{-3} & 7.572\times10^{-3} \\ 2.095\times10^{-3} & -1.128\times10^{-2} & 2.149\times10^{-2} \\ 3.313\times10^{-3} & 6.626\times10^{-4} & 0.682 \end{pmatrix}$$

$$\lambda_e(M_X)=\begin{pmatrix} -2.846\times10^{-4} & -1.024\times10^{-3} & 8.518\times10^{-3} \\ 4.190\times10^{-3} & 3.006\times10^{-2} & -6.447\times10^{-2} \\ 3.313\times10^{-3} & 1.572\times10^{-2} & 0.683 \end{pmatrix}$$

$$\lambda_\nu(M_X)=\begin{pmatrix} -7.451\times10^{-6} & 3.760\times10^{-4} & 1.704\times10^{-2} \\ 1.603\times10^{-4} & -3.023\times10^{-3} & 0.440 \\ 1.603\times10^{-4} & 1.638\times10^{-2} & 0.682 \end{pmatrix}$$

$$M_{RR}(M_X)=\begin{pmatrix} 1.177\times10^{11} & 2.664\times10^{11} & 8.261\times10^{12} \\ 2.664\times10^{11} & 2.398\times10^{12} & 3.541\times10^{13} \\ 8.261\times10^{12} & 3.541\times10^{13} & 9.708\times10^{14} \end{pmatrix}$$

TABLE XXIII. Numerical values for the entries of the Yukawa $(\lambda_u,\lambda_d,\lambda_e,\lambda_\nu)$ and neutrino Majorana (M_{RR}) matrices at the $Q=M_Z$ scale in the SMA model (M_{RR} is given in GeV mass units).

$$\lambda_u(M_Z)=\begin{pmatrix} -2.323\times10^{-5} & 3.837\times10^{-4} & 1.644\times10^{-2} \\ 3.409\times10^{-4} & -3.228\times10^{-3} & -3.588\times10^{-3} \\ 2.125\times10^{-4} & 1.767\times10^{-3} & 0.907 \end{pmatrix}$$

$$\lambda_d(M_Z)=\begin{pmatrix} -3.155\times10^{-4} & -2.734\times10^{-3} & 8.864\times10^{-3} \\ 4.130\times10^{-3} & -2.256\times10^{-2} & 3.088\times10^{-2} \\ 4.162\times10^{-3} & 1.053\times10^{-3} & 0.864 \end{pmatrix}$$

$$\lambda_e(M_Z)=\begin{pmatrix} -2.846\times10^{-4} & -1.032\times10^{-3} & 6.511\times10^{-3} \\ 4.090\times10^{-3} & 2.923\times10^{-2} & -5.322\times10^{-2} \\ 2.619\times10^{-3} & 1.257\times10^{-2} & 0.526 \end{pmatrix}$$

$$\lambda_\nu(M_Z)=\begin{pmatrix} -7.251\times10^{-6} & 3.290\times10^{-4} & 1.481\times10^{-2} \\ 1.464\times10^{-4} & -2.961\times10^{-3} & 0.399 \\ 1.341\times10^{-4} & 1.387\times10^{-2} & 0.572 \end{pmatrix}$$

$$M_{RR}(M_Z)=\begin{pmatrix} 1.176\times10^{11} & 2.629\times10^{11} & 8.034\times10^{12} \\ 2.629\times10^{11} & 2.370\times10^{12} & 3.409\times10^{13} \\ 8.034\times10^{12} & 3.409\times10^{13} & 9.197\times10^{14} \end{pmatrix}$$

TABLE XXIV. Predicted values for the left-handed neutrino mass matrix $m_{LL}(M_Z)$ in units of $m_{LL}(M_Z)_{33}=2.893\times10^{-3}$ eV and for the MNS neutrino mixing matrix $V^{MNS}(M_Z)$ in the SMA model.

$$\frac{m_{LL}(M_Z)}{m_{LL}(M_Z)_{33}}=\begin{pmatrix} 7.530\times10^{-4} & 2.284\times10^{-2} & 2.741\times10^{-2} \\ 2.284\times10^{-2} & 8.151\times10^{-1} & 8.239\times10^{-1} \\ 2.741\times10^{-2} & 8.239\times10^{-1} & 1 \end{pmatrix}$$

$$V^{MNS}(M_Z)=\begin{pmatrix} 0.9990 & 0.0192 & 0.0407 \\ -0.0430 & 0.6753 & 0.7363 \\ -0.0133 & -0.7373 & 0.6754 \end{pmatrix}$$

TABLE XXV. Predicted values for the left-handed neutrino masses $m_{\nu_{1,2,3}}$ and squared mass splittings $\Delta m^2_{12}=|m^2_{\nu_2}-m^2_{\nu_1}|$, $\Delta m^2_{23}=|m^2_{\nu_3}-m^2_{\nu_2}|$ in the SMA model.

$m_{\nu_1}=3.07\times10^{-8}$ eV $m_{\nu_2}=2.27\times10^{-3}$ eV $m_{\nu_3}=5.03\times10^{-2}$ eV

$\Delta m^2_{12}=5.15\times10^{-6}$ eV^2 $\Delta m^2_{23}=2.52\times10^{-3}$ eV^2

TABLE XXVI. Running of the neutrino mixing angles at the unification $Q=M_X$, the right-handed tau neutrino mass $Q=M_{\nu_3}$ and Z boson mass $Q=M_Z$ energy scales in the SMA model.

$Q=M_X \sim 3\times 10^{16}$ GeV		
$\sin\theta_{12}=2.17\times 10^{-2}$	$\sin\theta_{23}=0.703$	$\sin\theta_{13}=3.87\times 10^{-2}$
$\sin^2(2\theta_{12})=1.87\times 10^{-3}$	$\sin^2(2\theta_{23})=1.000$	$\sin^2(2\theta_{13})=5.97\times 10^{-3}$
$Q=M_{\nu_3}\sim 9\times 10^{14}$ GeV		
$\sin\theta_{12}=2.12\times 10^{-2}$	$\sin\theta_{23}=0.713$	$\sin\theta_{13}=3.94\times 10^{-2}$
$\sin^2(2\theta_{12})=1.80\times 10^{-3}$	$\sin^2(2\theta_{23})=1.000$	$\sin^2(2\theta_{13})=6.19\times 10^{-3}$
$Q=M_Z$		
$\sin\theta_{12}=1.92\times 10^{-2}$	$\sin\theta_{23}=0.737$	$\sin\theta_{13}=4.07\times 10^{-2}$
$\sin^2(2\theta_{12})=1.48\times 10^{-3}$	$\sin^2(2\theta_{23})=0.993$	$\sin^2(2\theta_{13})=6.60\times 10^{-3}$

$$\lambda_f(M_X)=\begin{pmatrix} x_f^G c_{11}\delta\epsilon^5 & x_f^R c_{12}\delta\epsilon^4 & x_f^H c_{13}\delta\epsilon^2 \\ x_f^G c_{21}\delta\epsilon^3 & x_f^W c_{22}\delta\epsilon^2 & x_f^I c_{23}\delta \\ x_f^M c_{31}\delta\epsilon^3 & x_f^g c_{32}\delta\epsilon^2 & c_{33} \end{pmatrix} \tag{E5}$$

$$+\begin{pmatrix} 0 & x_f^O c'_{12}\delta^2\epsilon^4 & x_f^a c'_{13}\delta^2\epsilon^2 \\ 0 & x_f^H c'_{22}\delta^2\epsilon^2 & x_f^W c'_{23}\delta^2 \\ 0 & x_f^T c'_{32}\delta^2\epsilon^2 & 0 \end{pmatrix}$$

$$+\begin{pmatrix} x_f^K c''_{11}\delta^3\epsilon^5 & 0 & 0 \\ x_f^V c''_{21}\delta^3\epsilon^3 & 0 & 0 \\ x_f^V c''_{31}\delta^3\epsilon^3 & 0 & 0 \end{pmatrix}$$

TABLE XXVII. Predictions for the running $\overline{\rm MS}$ masses of the down (m_d) and strange (m_s) quarks at $Q=1$ GeV and $Q=M_s$ respectively in the SMA model (M_s indicates the strange pole quark mass).

$m_d(1\text{ GeV})=7.6$ MeV	$m_s(M_s)=193$ MeV

$$\frac{M_{RR}(M_X)}{M_{RR}(M_X)_{33}}=\begin{pmatrix} C_{11}\epsilon^6 & C_{12}\epsilon^5 & C_{13}\epsilon^3 \\ C_{21}\epsilon^5 & C_{22}\epsilon^4 & C_{23}\epsilon^2 \\ C_{31}\epsilon^3 & C_{32}\epsilon^2 & C_{33} \end{pmatrix}. \tag{E6}$$

In the rest of this appendix we apply the same systematic approach used in the main part of the paper for the LMA solution to the SMA model. The approximate structure of the effective matrices resulting from Eqs. (E5),(E6) is given in Table XX. In Table XXI we give the values of the c,C parameters appearing in Eqs. (E5),(E6). In Table XXII we present the exact numerical values of the Yukawa and Majorana matrices at the unification scale and in Table XXIII the values of the same matrices at the M_Z scale. In Table XXIV we present the predicted values for the left-handed neutrino mass and for the MNS mixing matrices. The predictions for masses of the physical neutrinos in the SMA model are listed in Table XXV and in Table XXVI we give the predictions for the neutrino mixing angles at several energy scales. Finally, in Table XXVII we show the predictions for the masses of the down and strange quarks in the SMA model.

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