Numerical study of the scaling properties of $SU(2)$ lattice gauge theory in Palumbo noncompact regularization

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(Received 18 September 2000; published 4 April 2001)

In the framework of a noncompact lattice regularization of non-Abelian gauge theories, we look, in the $SU(2)$ case, for the scaling window through the analysis of the ratio of two masses of hadronic states. In the two-dimensional parameter space of the theory we find the region where the ratio is constant and equal to the one in the Wilson regularization. In the scaling region we calculate the lattice spacing, finding it at least 20% larger than in the Wilson case; therefore, the simulated physical volume is larger.

DOI: 10.1103/PhysRevD.63.094501 PACS number(s): 11.15.Ha, 12.38.Gc

I. INTRODUCTION

Lattice regularization is the most effective method, if not the only one, to study the behavior of quantum field theories outside the limits of application of perturbative expansions. One of the paradigms of this approach, namely the Wilson regularization of gauge theories [1], has been widely used from the beginning, and for a large period of time has been thought, apart from a few exploratory studies of alternatives, essentially as the only way, especially for non-Abelian theories. The Wilson regularization implies the use of gauge group elements as dynamical (link) variables, instead of fields in the algebra of the gauge group as in the continuum. Therefore, it is called compact regularization because the links take values in a compact space, the manifold of the gauge group. In the continuum limit one expects that the gauge fields will pass through an effective decompactification restoring the properties of the continuum physics. Instead, the naive discretization of gauge theories using the usual field representation of the continuum formulation, replacing derivatives with finite differences and a flat measure for the gauge fields, leads to a theory where gauge invariance is explicitly broken at finite lattice spacing.

The possible spurious effects of compactification have been investigated in the past, leading to the conclusion that the main features of non-Abelian gauge theories, e.g., confinement and spontaneous breaking of chiral symmetry, do not depend on the compactification of dynamical variables. Nevertheless, one can wonder if there are alternatives to this approach, which are the possible advantages of a formulation of a gauge theory on the lattice where the dynamical variables stay noncompact from the beginning.

Some schemes of noncompact regularization of lattice gauge theories have been proposed in the last ten years; we will concentrate our attention on a particular one, the Palumbo regularization [2], and we will determine in a nonperturbative way the scaling region and the lattice spacing in physical units. This regularization has already been studied and used in the past both in the Lagrangian [3–5] and Hamiltonian [6] formulations. Numerical results [5] obtained in the Lagrangian formulation show a discrepancy with respect to the perturbative expansion [4], while there is a full agreement between the latter and the calculations in the Hamiltonian framework [6]; a reanalysis of this slightly controversial situation is an additional (minor) motivation for the present work.

Besides being an alternative to Wilson regularization, Palumbo regularization is interesting because of its relation with the tadpole improvement technique used to obtain improvement of compact lattice actions (see, for example, Ref. [7], and references therein). As will be shown in the following, in this regularization, because of the use of noncompact fields as dynamic variables, the tadpoles are resummed from the very beginning in some auxiliary fields which decouple in the continuum limit.

In the present study, in some sense an exploratory one, we will use $SU(2)$ as a gauge group mainly for the significant simplification we get in the numerical procedure with respect to the (more interesting) $SU(3)$ case; but there are no *a priori* obstructions in repeating the whole procedure we will depict in the following using the noncompact regularization with $SU(N)$ as gauge group [8]. Moreover, in the $SU(2)$ case there are, as said before, other results in the literature useful for comparison.

We will proceed as follows: after the introduction of the more important features of the Palumbo regularization, we will explain our scheme for identifying the scaling region using a ratio of two hadron masses. The next step will be the definition of the scheme used in the numerical simulations. The results will be presented in the last section of the paper, together with a comparison with other analytical as well as numerical results obtained in the same framework in the past.

II. THE PALUMBO NONCOMPACT REGULARIZATION

The noncompact regularization we used is fully explained in Refs. [3,4], but in order to keep the paper self-contained

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we recall here its main features. The exact gauge invariance at finite lattice spacing is obtained by using a covariant derivative $\mathcal{D}_\mu(x)$ which, under a gauge transformation $g \epsilon SU(N)$, transforms according to the equation

$$\mathcal{D}'_\mu(x) = g(x)\mathcal{D}_\mu(x) g^\dagger(x + a\hat{\mu}), \tag{1}$$

where $\mathcal{D}$ is an element of $GL(M,\mathcal{C})$, M is the dimension of the matrix representation, and a is the lattice spacing. The covariant derivative can then be written as a function of a field in the Lie algebra[1] of $SU(N)$ plus some auxiliary fields (see also [8]), whose transformation equations can be worked out straightforwardly. In the case of the $SU(2)$ gauge group, the covariant derivative in the fundamental representation depends on only one auxiliary field W_μ

$$\mathcal{D}_\mu(x) = \left(\frac{1}{a} - W_\mu\right) I + i\mathcal{A}_\mu, \tag{2}$$

$$\mathcal{A}_\mu = A^a_\mu T_a, \tag{3}$$

where I is the (2×2) identity matrix and the T_a are the generators of the gauge group

$$[T_a, T_b] = i\varepsilon^c_{ab} T_c, \quad \{T_a, T_b\} = \frac{1}{2}\delta^a_b. \tag{4}$$

The strength tensor is defined in analogy to the continuum, and the same holds for the Yang-Mills Lagrangian density:

$$\mathcal{F}_{\mu\nu}(x) = -i[\mathcal{D}_\mu(x)\mathcal{D}_\nu(x + a\hat{\mu}) - \mathcal{D}_\nu(x)\mathcal{D}_\mu(x + a\hat{\nu})], \tag{5}$$

$$\mathcal{L}_{YM}(x) = \frac{\beta}{4}\sum_{\nu>\mu} \mathrm{Tr}\,\mathcal{F}_{\mu\nu}(x)\mathcal{F}^\dagger_{\mu\nu}(x). \tag{6}$$

The lattice theory is defined with a flat measure for the fields in $\mathcal{D}_\mu(x)$, as in the continuum.

This is a regularization of the Yang-Mills theory if, in the continuum limit, the auxiliary field W_μ is decoupled. This can be achieved at the quantum level by introducing a potential which gives a divergent mass to this field. The potential is constructed using the gauge invariant quantity t_μ

$$t_\mu(x) I = \mathcal{D}_\mu(x)\mathcal{D}^\dagger_\mu(x) - \frac{1}{a^2} I = \left[\frac{1}{4}\mathcal{A}^2_\mu(x) + W^2_\mu(x) - \frac{2}{a} W_\mu\right] I; \tag{7}$$

therefore, the basic noncompact Lagrangian is obtained by adding to $\mathcal{L}_{YM}$ the potential

$$\mathcal{L}_c(x) = \beta_c \sum_\mu t^2_\mu(x), \quad \beta_c > 0. \tag{8}$$

If in the continuum limit

$$\beta_c \simeq (a\Sigma)^{2-\varepsilon}, \quad \varepsilon > 0, \tag{9}$$

where Σ is a parameter with the dimension of a mass, the auxiliary field has a mass of the order of $a^{-\varepsilon/2}\Sigma^{1-\varepsilon/2}$. As in Ref. [4] we confine ourselves to the case $\varepsilon = 2$.

The Wilson regularization can be obtained by eliminating the auxiliary field by imposing the constraint

$$t_\mu = 0. \tag{10}$$

This condition produces a compactification of the covariant derivative which becomes

$$\mathcal{D}_\mu = \frac{1}{a} U_\mu, \tag{11}$$

where $U_\mu \epsilon SU(2)$ can be identified with the Wilson link variable. The imposition of the constraint (10) is equivalent to taking the limit $\beta_c \to \infty$. This can be made explicit by introducing a polar representation for the covariant derivative [4]; we notice that the Jacobian for the change of variables provides the Haar measure for the link variables $U_\mu(x)$.

We stress that the coupling β_c is not an irrelevant one because it is necessary to render the lattice theory as a regularization of the Yang-Mills gauge theory.

In Ref. [4], the properties of the regularization were studied using a perturbative approach and adopting a polar representation for the covariant derivative. We checked that the results obtained in numerical simulations do not depend (as they must be) on the parametrization used for the covariant derivative. It is worth noticing that the use of a Cartesian parametrization for the covariant derivative makes evident one advantage of this noncompact regularization, with respect to the compact ones. The number of vertices in perturbative calculations stays finite independently from the order considered.

Some irrelevant terms were introduced in the action to make the perturbative analysis of the regularization easier. They are constructed from the gauge-invariant quantity t_μ, and cancel some contributions originated from $\mathcal{L}_{YM}$ in Eq. (6) which depend only on the field t_μ, so that the auxiliary field does not propagate at tree level. By an appropriate choice of the couplings of the irrelevant terms the lattice theory can be defined in terms of only two parameters, β, γ, with

$$\gamma^2 = 2\left(\beta_c + \frac{3}{4}\beta\right). \tag{12}$$

The condition $\beta_c > 0$ corresponds to

$$\gamma > \sqrt{\frac{3}{2}\beta}. \tag{13}$$

[1]As shown in Ref. [3] it transforms like a continuum gauge field, except for lattice artifacts that vanish in the continuum limit.

To render possible a comparison with the previous perturbative (see, also, Ref. [6]) and numerical [5] calculations we used the same Lagrangian of Ref. [4], explicitly

$$\mathcal{L}=\mathcal{L}_{YM}+\frac{1}{8}\beta a^2\sum_{\nu>\mu}(\nabla_\mu t_\nu-\nabla_\nu t_\mu)^2+\frac{1}{2}\gamma^2\sum_\mu t_\mu^2, \tag{14}$$

where

$$\nabla_\mu f(x)=\frac{1}{a}[f(x+a\hat{\mu})-f(x)]. \tag{15}$$

III. NONPERTURBATIVE DETERMINATION OF THE SCALING PROPERTIES OF THE REGULARIZATION

The goal of our work is to answer the question: What about the convergence to continuum in this regularization? In this form the question is misleading, because removing the regularization has different meanings depending on the context considered. In fact, in the one-coupling-constant lattice regularizations of Yang-Mills theory, like Wilson's, the evolution of the bare coupling $g^2=2N/\beta$ is obtained from the equation

$$a\frac{dg(a)}{da}=-\beta(g)=b_0g^3(a)+b_1g^5(a)+O(g^7), \tag{16}$$

where b_0,b_1 are the universal one and two loop coefficients of the beta function expansion. In a $SU(N)$ pure gauge lattice theory they are

$$b_0=\frac{11}{3}\frac{N}{16\pi^2};\quad b_1=\frac{34}{3}\left(\frac{N}{16\pi^2}\right)^2. \tag{17}$$

In perturbative calculations, to take the continuum limit means to evaluate the limit for $a\to 0$ of the renormalized quantities calculated.

In a numerical simulation the situation is very different because the calculated quantities are not in analytical form; moreover, the lattice spacing is always finite. Therefore, to take the continuum limit in this case means to determine a region, in the parameters space of the lattice theory used, where the results (at finite lattice spacing) are as close as possible to their (experimental) continuum limit. Usually this region is characterized by a value as large as possible of the correlation length of the system. There it is also possible to study the asymptotic scaling of the calculated quantities, i.e., in which measure they follow the perturbative scaling equations. It is important to notice here that the size of the scaling violations depends on the quantity considered. As is well known, the ratio of two quantities with the same physical dimension (for example, two masses) is constant (if not for scaling violations) in a scaling region that is usually larger than the asymptotic scaling window. In presence of irrelevant couplings, the scaling of the physical quantities should not depend on their values.

In Palumbo noncompact regularization there are two coupling constants, as explained before, whose evolution as functions of the lattice spacing can be determined from equations analogous to Eq. (16). We stress that γ is not an irrelevant coupling (in the common meaning of the term), as explained above. In Ref. [4], the authors define the quantity Λ_{NC}, which in the sequel we call a noncompact scale parameter, using the expression

$$\Lambda_{NC}^2a^2=\left(1+\frac{b_1^2}{b_0^3}g^2\right)\exp\left\{-\frac{1}{b_0g^2}-\frac{b_1}{b_0^2}\ln(b_0g^2)\right\}, \tag{18}$$

which is exactly the same function obtained in the case of Wilson regularization, except for the fact that Λ_{NC} is a function of $g^2=2N/\beta$ and γ. Such a function can be evaluated in perturbation theory, solving the evolution equations for the two coupling constants. Actually, in Ref. [4], the authors obtained

$$\Lambda_{NC}=\Lambda_W\exp\left\{-\frac{c}{g^2\gamma^2}\right\}, \tag{19}$$

where Λ_W is the scale parameter for Wilson regularization and

$$c=\frac{12}{11}\pi^2\times 0.88323. \tag{20}$$

In the limit of vanishing lattice spacing we have $g\to 0$, where Eq. (19) would then imply $\Lambda_{NC}\to 0$, i.e., an inconsistency of the regularization if not for an appropriate evolution of γ. On general grounds, given Eq. (19) for Λ_{NC} as a function of the bare couplings, to have $\Lambda_{NC}\neq 0$ in the continuum limit it is necessary that

$$\lim_{a\to 0}g^2\gamma^2=\kappa\neq 0. \tag{21}$$

Actually, in Ref. [4], the authors obtained to one loop order

$$\gamma=\frac{\gamma_1}{g^2}+\gamma_2, \tag{22}$$

where γ_1 is an arbitrary constant and γ_2 is be determined by a higher loop calculation. Then in the limit $a\to 0$,

$$\lim_{a\to 0}\Lambda_{NC}=\Lambda_W. \tag{23}$$

Such results mean that in perturbative calculations with Palumbo regularization it is possible to get the continuum limit with a scale parameter equal to the Wilson one.

In numerical simulations the situation is different, as explained above. On general grounds we expect that there will exist a scaling region in the plane $(\beta,1/\gamma)$[2] and that, because of the finite lattice spacing, the properties of convergence to

[2]We use as a natural variable $1/\gamma$ because the Wilson limit can be identified with the $1/\gamma=0$ line.

the continuum of the regularization vary in this region. For example, we could have, in scaling conditions, different physical values for the lattice spacing (therefore, different physical volumes) as a function of the value of the bare parameters β,$1/\gamma$. One possible strategy to determine the scaling properties of the regularization (which we followed) goes through the following steps. The ratio of two quantities with the same physical dimension has to be calculated as a function of the bare parameters of the regularization, e.g., on a regular grid in the (β,$1/\gamma$) plane. These values can be fitted so as to obtain a surface continuously varying in the (β,$1/\gamma$) plane. The scaling region for the noncompact regularization is therefore the region in the (β,$1/\gamma$) plane where the ratio considered agrees with the same quantity evaluated in Wilson lattice theory, which corresponds to the $1/\gamma=0$ line. The comparison with Wilson regularization (or with any other regularization for which perturbative relations are available to determine the asymptotic scaling region) is mandatory only in the case of a nonphysical theory, like in pure $SU(2)$ gauge theory. In fact, in the $SU(3)$ case we can fix the physical value of the ratio by using experimental data, therefore, in a fully independent way from perturbative calculations.

In summary, the only two necessary ingredients are the ratio calculated using the noncompact regularization and the perturbative scaling equations for Wilson regularization to determine the physical value of the ratio, which is the value it assumes in the asymptotic scaling region for Wilson regularization. We stress that this technique was well known in principle (see, for example, Ref. [9]) and has been used in the past, although in a different form, in Ref. [10]. We chose to use the ratio of two particle masses due to the following considerations. Other pure gluonic observables, like the plaquette, have significant perturbative contributions which may obscure the nonperturbative features of the regularization. As for the glueball masses, usually smearing techniques are used to obtain clearer signals, but this implies the introduction of additional parameters besides the two peculiar to the Palumbo regularization. As for the string tension, in this preliminary work, we preferred to use ratios of quantities with the same physical dimensions to avoid spurious effects. Lastly we notice that the particle masses have a well-defined physical meaning and depend in a fundamental way on the nonperturbative properties of the theory.

It is a crucial condition for the above-depicted scheme to be valid, since the only dimensionful quantity of the theory is the renormalization group scaling parameter; this is true if we work in the chiral limit. Otherwise, we would have another dimensionful quantity (the quark mass). As is well known, it is extremely difficult (and numerically very expensive) to perform simulations or to measure masses directly in the chiral limit; to stay within the limits of this work, we have chosen to evaluate the mass spectrum at four finite quark mass values and then extrapolate to the chiral limit. To have better control on this extrapolation we have used Kogut-Susskind fermions, where chiral limits can be easily defined and reached, with good accuracy, by means of a linear extrapolation in the bare quark mass. The same considerations led us to work in the quenched approximation; moreover, one of our (minor) scopes is to compare with the analytical calculations in Ref. [4] that were carried out for the pure gauge theory.

IV. DETAILS ON THE NUMERICAL APPROACH

As follows from the previous considerations, the action for $SU(2)$ lattice gauge theory using the Palumbo regularization contains two parameters (β and γ); the role of γ is to assure the decoupling of the auxiliary field in the continuum limit. Notice that, in practice, the action can assume different specific forms depending on which class of irrelevant terms we decide to include; different choices of the set of irrelevant terms can lead to an action easier to use in numerical or analytical calculations, or to an action more complex, but with a better approach to the continuum limit. We will not address this issue, but instead choose to work with the lattice action in Eq. (14).

We have explicitly checked that the term

$$\frac{1}{8}\beta a^2 \sum_{\nu>\mu} (\nabla_\mu t_\nu - \nabla_\nu t_\mu)^2 \tag{24}$$

is actually irrelevant, in the sense that the results for the hadron masses do not depend in a sensible way on its inclusion or exclusion in the action.

The action (14) can usefully be thought of as an action for a gauge field living in the GL(2) group; therefore, we decided to use a Cartesian representation that includes both the physical fields and the auxiliary one. Other choices are possible; in particular, we recall the polar representation used in Ref. [4]. Starting from the action (14) we have written a generic Metropolis+Overrelaxation code; the overrelaxation part of the procedure applies only to the $SU(2)$ part of the GL(2) fields, i.e., it amounts to a microcanonical rotation in the $SU(2)$ subgroup leaving untouched the determinant of the GL(2) matrix. CPU time and memory requirement for the computation is essentially the same needed for a corresponding Monte Carlo with the Wilson regularization.

Looking at the action (14), it can be easily understood that the parameter γ is bound to be larger than a minimum value [Eq. (13)], as discussed in Ref. [4].

The other limit, i.e., $\gamma\rightarrow\infty$, reproduces the Wilson regularization in the sense that the determinant of GL(2) gauge fields is constrained to be one and we recover the usual compact $SU(2)$ gauge action.

In order to have a better readability of the results, in particular, in the region of large γ where the Wilson results have to be recovered, we decided to work on a (quasi-) regular grid in the β,$1/\gamma$ parameter space (in this space the $1/\gamma=0$ line is the Wilson theory). We have chosen to work with $2.0\leqslant\beta\leqslant2.7$ and $0.0<1/\gamma\leqslant0.2$, using 12 values of β and 20 values of $1/\gamma$; we have also included in our analysis the results obtained for the Wilson regularization (the $1/\gamma=0$ line in the following).

As said before, we are interested mainly in the masses of the (lighter) hadron states; in $SU(2)$ gauge theory we have four states made from two quarks, namely a scalar (σ), a pseudoscalar (π), a vector (ρ), and a pseudovector (A_1).

Moreover, due to the use of Kogut-Susskind fermions, we can have a signal in the correlation function both from nonoscillating and oscillating channels; we will refer to them as $+$ and $-$ states, respectively.

We would like to stress here that we had 240 different simulations to carry out in order to complete our program; faced with our limited computing resources, and taking into account the exploratory character of this work, we confined ourselves to small lattices, namely a $6^3\times 12$ one.

The actual scheme we have followed is that for every point in the $\beta,1/\gamma$ grid we have thermalized a starting configuration, then measured the hadron propagator for one of the four values of the quark mass (namely $m_q=0.15$, 0.20, 0.25, and 0.30) on a configuration separated by 150 circles of combined Metropolis+Overrelaxation sweeps from the previous one used to measure the observables, for a total of 180 hadron propagators per mass value.

We notice that we have estimated the integrated autocorrelation time in a representative point inside the scaling window, finding a value around 40; therefore, we used an ensemble of well-decorrelated configurations.

From the averaged propagators we have extracted the mass of hadron states (m^+ and m^-) fitting with the following function:

$$P(\tau)=A(e^{-\tau m^+}+e^{-(N_t-\tau)m^+}) +B(-1)^\tau(e^{-\tau m^-}+e^{-(N_t-\tau)m^-}). \tag{25}$$

At the end we obtain, for each $\beta,1/\gamma$ point, the mass of seven hadron states (for the pseudoscalar channel the oscillating state is not observable), each one evaluated at four values of the quark mass.

Among these seven masses we notice a well-defined pattern. The pion mass is affected by the smallest statistical error (well below 1%), but, this particle being a Goldstone boson, it vanishes in the chiral limit, and then cannot be used for the determination of the scaling window. The other three particles, namely ρ^+, A_1^-, and σ^+, have small statistical errors (around or less than 1%); finally, the other three states are worst defined being affected by large statistical errors, and then useless for our purposes.

Restricting ourselves to the three non-Goldstone states with small errors, a linear extrapolation to the chiral ($m_q\to 0$) limit gives us the dataset for the analysis explained in the following section.

Aside from the determination of the hadron propagators, we have used the configuration generated to also measure local observables as the plaquette and its specific heat.

The simulations have been fully performed on small systems such as Unix workstations and Linux PCs at L.N.F., L.N.G.S., and the University of Perugia.

V. RESULTS

Let us proceed to show our results. As said in the previous section, we have evaluated the ratio of the masses of A_1^- and ρ^+. The motivation for this choice comes from the observation that the errors for these two masses are smaller; nevertheless, we have repeated the whole analysis also using the mass of the σ^+ particle, finding similar results.

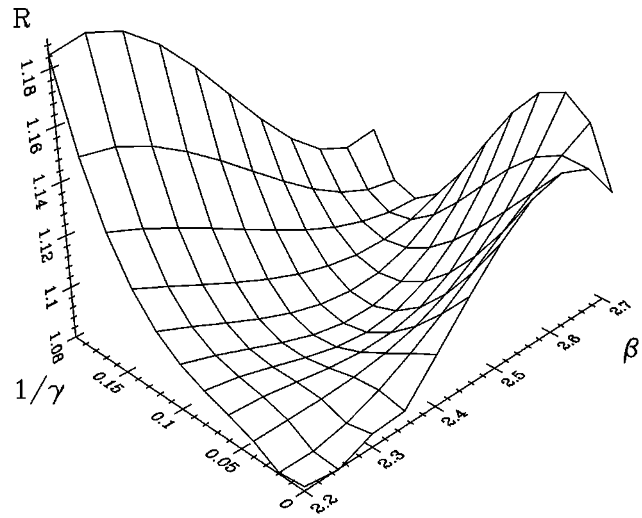

FIG. 1. $R(\beta,1/\gamma)$ fitting surface.

The mass ratio R has been obtained on a grid of points in β and $1/\gamma$ and then a regular surface has been reconstructed using a bipolynomial spline fitting procedure. We postpone a discussion on the errors to the end of this section.

In Fig. 1 we report the fitting surface $R(\beta,1/\gamma)$. Looking at the figure we can recognize some important features; first of all, the existence of an (almost) flat region (valley) that originates from the $1/\gamma=0$ line (Wilson results), and propagates towards larger values of β for increasing $1/\gamma$. The flat region in the Wilson limit coincides with the usual scaling region for $SU(2)$ pure gauge lattice theory for these (intermediate) lattice sizes [11].

We have checked that, in this region, the good asymptotic scaling can be obtained only in a narrow interval near $\beta=2.3$. Therefore, we tentatively identify the valley as the scaling window (although not the asymptotic scaling region) for the noncompact regularization. In order to make this observation more precise, we report in Fig. 2 the curves of constant R in the plane $\beta,1/\gamma$.

Including in the analysis the data of the single particle

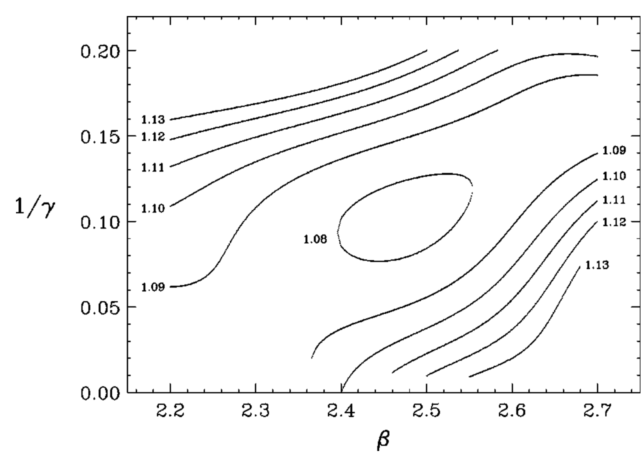

FIG. 2. Lines of constant R in the $\beta,1/\gamma$ plane.

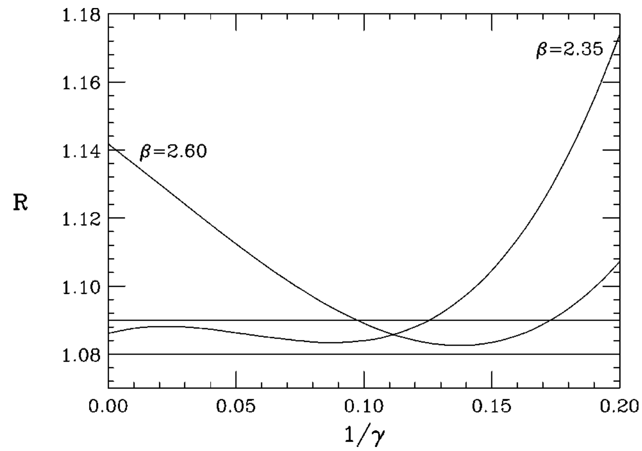

FIG. 3. $R(\beta,1/\gamma)$ for $\beta=2.35$ and $\beta=2.60$ vs $1/\gamma$; the horizontal lines limit the scaling region.

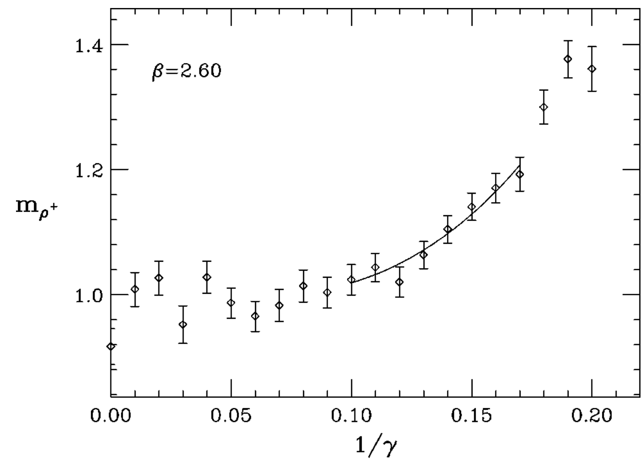

FIG. 5. Mass of the ρ^+ particle for $\beta=2.60$ vs $1/\gamma$.

masses in the scaling region, we can make our statement more definitive. Looking at Fig. 2, we can identify two different zones where a more detailed analysis allows us to clarify the properties of the region we proposed as the scaling region. If we define the scaling region as limited from the $R=1.09$ level, we can say that $1/\gamma<0.12$ and $0.10<1/\gamma<0.17$ are the scaling regions for, respectively, $\beta=2.35$ and $\beta=2.60$. Consider the lines $\beta=2.35$ and $\beta=2.60$. In the first case, the Wilson point $(1/\gamma=0)$ is inside the scaling region, whereas in the second case only a segment $1/\gamma_1<1/\gamma<1/\gamma_2$ is. In Fig. 3, we can see the behavior of R (from the fitting surface) along these two lines.

In Figs. 4 and 5 we report the results for the ρ^+ mass in two cases ($\beta=2.35$ for Fig. 4 and $\beta=2.60$ for Fig. 5). In these figures we have reported the raw data for the mass.

We are allowed, now, to compare our numerical results with the perturbative analysis in Ref. [4]; following this analysis we expect that the lattice spacing, and then the lattice mass inside the scaling region, follows a behavior like

$$a(\beta,\gamma)\propto \exp\left\{-\frac{12\pi^2}{11}0.2208\frac{\beta}{\gamma^2}\right\}, \qquad (26)$$

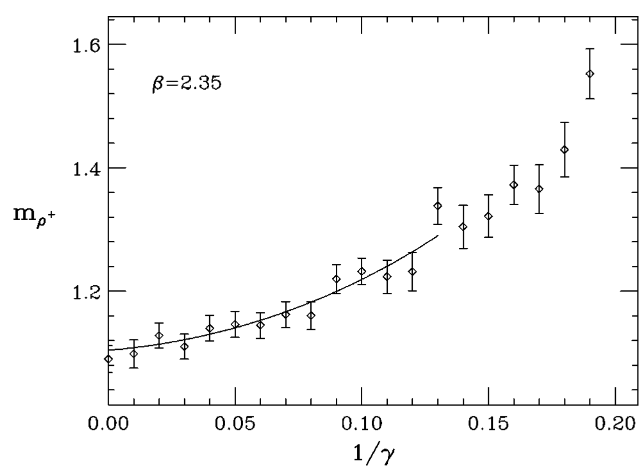

FIG. 4. Mass of the ρ^+ particle for $\beta=2.35$ vs $1/\gamma$.

where the numerical coefficients result from a one loop calculation.

Coming back to Figs. 4 and 5 we can see, superimposed to the data in the scaling regions, a fit with the exponential of a second order polynomial. Notice that in the $\beta=2.35$ case we have not included the Wilson point in the fitted data; the Wilson result is in good agreement with the extrapolation from $1/\gamma>0$ data; in the other case we expect this kind of extrapolation to be meaningless. For $\beta=2.35$ we can try a direct comparison with the perturbative results, keeping in mind that we are working on a small lattice and, in any case, we have not performed a detailed estimation of systematic effects largely outside the scope of this work. Following Ref. [4], we expect the coefficient of the linear term in $1/\gamma$ to be zero and that of the quadratic one to be 5.56. From our fit we get for the former a value compatible with zero (0.3 ± 0.8) and for the latter 7 ± 2. We do not claim an agreement with the perturbative formula, but in any case we have, at this level, no trace of the large discrepancies found in Ref. [5].

Finally, in Fig. 5, we note that the decreasing trend for decreasing $1/\gamma$, which can be clearly seen in the scaling region, ceases in correspondence of the lower limit of the scaling region itself. We have checked that this behavior is present also in the data for the other particles and values of β where, as in the case of $\beta=2.6$, the Wilson limit lies outside the scaling window. Again, the behavior of the mass, and hence the lattice spacing, is well reproduced by an exponential of a polynomial in $1/\gamma$.

Fortified by these results, we can now proceed to a final check on our scaling window; we expect that the scaling window contains the value of the parameters in which the specific heat has a maximum, signaling a large correlation length. In Fig. 6, we can see the constant R lines as in Fig. 2 (continuous lines) with the superimposed position, in the $(\beta,1/\gamma)$ plane, of the peaks of the plaquette specific heat (dashed line). In this figure, four lines of constant plaquette (dotted lines) are also reported. We can see a substantial correspondence between the fluxes in the parameter space as identified by different operators. This scenario is the one expected for an honest theory in the scaling regime. We therefore are confident to have correctly identified the scal-

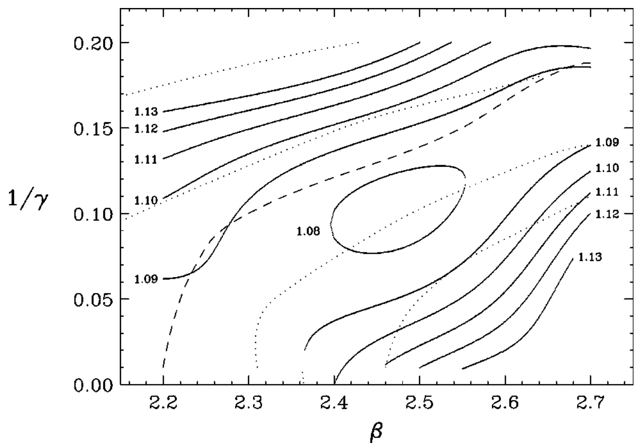

FIG. 6. Lines of constant R (continuous line), constant plaquette (dotted line), and position of the peak of specific heat (dashed line) in the β,$1/\gamma$ plane.

ing window for noncompact regularization. We stress that by using this scheme, we can leave aside any perturbative calculation in the noncompact regularization.

Any point inside the scaling window is a good one for approximating the behavior of continuum theory, but actual results can be different. In particular, the value of the lattice spacing, and then the lattice volume, varies from point to point. As we have seen before for the noncompact formulation, this value can be larger than that obtained using the Wilson formulation. In order to make this assertion less qualitative, we present in Fig. 7 the behavior of the lattice spacing, as extracted from the ρ^+ mass, along the center of the valley. We can clearly see that the lattice spacing becomes larger and larger the more we depart from the Wilson case $1/\gamma=0$; with the lattice size used in this work, we can access to regions where the lattice spacing is around 20% larger than the Wilson regularization. This improvement can be made larger if we move towards larger β and $1/\gamma$, but with a narrower scaling valley and nearer to the instability regime [see Eq. (13)]. Again these results are in substantial agreement with the prediction of the perturbative calculation

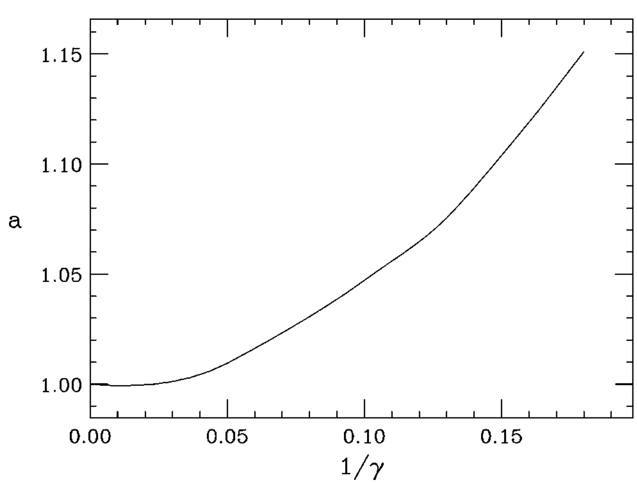

FIG. 7. Ratio of noncompact and Wilson lattice spacing along a curve lying on the bottom of the scaling valley.

in Refs. [4,6] and do not show any sign of the large deviations claimed in Ref. [5]; we remind, however, that these deviations have been obtained using a different approach and looking at different operators.

We believe that a complete knowledge of the entire scaling region, as well as the corresponding gain in terms of lattice spacing, is out of the scope of this work; it is more interesting to address this point in a more realistic simulation, using larger lattices in the $SU(3)$ case. In the present paper, we have devoted our attention more to the development of a scheme for addressing the problem, than to give definitive and quantitative answers to questions about the (improving) potentiality of this regularization.

Errors

The main ingredient of our analysis is the ratio R. This quantity is obtained, starting from the raw propagators, by means of a complex procedure amounting essentially to three levels of fitting: fitting the correlator to extract the mass, fitting the masses to extract the chiral limit, and eventually, the final fit of $R(\beta,1/\gamma)$ to get a smooth surface. It is extremely hard to trace the propagation of statistical errors from the raw data to the final surface. Nevertheless, it is mandatory to have at least a rough idea of the effects of the statistical fluctuations on our procedure.

To this end we have chosen to proceed in this way: we have divided our statistical ensemble of 180 independent measures of the correlator (for each β, $1/\gamma$, and quark mass) in two independent subsets of 90; we have then repeated the whole procedure [from the fitting to Eq. (25) to the construction of the smooth surface for R] for the two sets independently. We have then computed the root mean square (rms) of the deviation between the two surfaces on a regular grid of $O(100)$ points P_i, placed in the core of the $(\beta,1/\gamma)$ region to avoid edge effects. From this procedure we find a rms of 1% which we assume to approximate the error on the R surface; we have checked the independence of this evaluation on the number of points P_i used to compute the average of the deviation. This result helped us to establish the criterion $(R\leqslant 1.09)$ used to define the scaling region (see Fig. 3) and made us more confident on the robustness of the emerging scenario.

VI. CONCLUSIONS

We have studied the approach to the continuum of $SU(2)$ lattice gauge theory within a noncompact regularization scheme with two dimensionless parameters. We have determined the scaling window using a nonperturbative approach, defined through the use of the ratio of the masses of two hadronic states. We have found a clear scaling window that, stemming from one of the Wilson regularizations, moves towards larger β and $1/\gamma$.

Inside this region, we have determined the lattice spacing, finding it increasing with increasing distance from the Wilson line (in the parameter space), and in a way compatible with the expectations based on perturbative analytical calculations without the large discrepancies observed in Ref. [5]. Our determination of the scaling region is corroborated by the observation of the behavior of other quantities, i.e., the specific heat, linked to the correlation length.

All this work has been carried out in small lattices; therefore we cannot give a more sound quantitative estimation of the effects observed. In any case, the use of this noncompact regularization leads to clear advantages in terms of simulated physical volumes.

This analysis needs to be improved with the use of larger lattices and other observables more suitable for the accurate determination of the lattice spacing, possibly for the physically interesting case of $SU(3)$ theory [8]. It would also be of interest to study the effects of unquenching on the advantages for the physical volume in the scaling region in Palumbo regularization.

ACKNOWLEDGMENTS

We thank F. Palumbo and S. Caracciolo for invaluable discussions and suggestions during every stage of this work. We thank M. P. Lombardo for providing us the program for Dirac matrix inversion and propagator calculation.

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