Effective string theory of vortices and Regge trajectories

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Starting from a field theory containing classical vortex solutions, we obtain an effective string theory of these vortices as a path integral over the two transverse degrees of freedom of the string. We carry out a semiclassical expansion of this effective theory, and use it to obtain corrections to Regge trajectories due to string fluctuations.

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I. INTRODUCTION

The goal of this paper is to derive an effective string theory of vortices beginning with a field theory containing classical vortex solutions. The Abelian Higgs model is an example of such a theory. Nielsen and Olesen [1] showed that this model has classical magnetic vortex solutions. These vortices are tubes of magnetic flux with constant energy per unit length.

The motivation for this work came from the dual superconductor picture of confinement [2–4]. In this picture, a dual Meissner effect confines electric color flux (Z_3 flux) to narrow tubes connecting quark-antiquark pairs. Calculations with explicit models of this type [5] have been compared both with experimental data and with Monte Carlo simulations of QCD [6]. To a good approximation, aside from a color factor, the dual Abelian Higgs model, coupling dual potentials to a scalar Higgs field carrying magnetic charge, can be used to describe the results of these calculations. However, these calculations neglect the effect of fluctuations in the shape of the flux tube on the $q\bar{q}$ interactions. We show in this paper that taking account of those fluctuations leads to an effective string theory of long distance QCD.

Well before the introduction of the idea of dual superconductivity, string models [7] had been used to understand the origin of Regge trajectories, and they have continued to be used to describe other features of hadron physics, such as the spectrum of hybrid mesons. In the dual superconductor picture, a string arises because the dual potentials couple to a quark-antiquark pair via a Dirac string whose ends are a source and sink of electric color flux. The effect of the string is to create a flux tube (or Abrikosov-Nielsen-Olesen vortex [8,1]) connecting the quark-antiquark pair. As the pair moves, this flux tube sweeps out a space time surface on which the dual Higgs field must vanish. This condition determines the location of the QCD string in the dual superconductor picture.

The effort to obtain an effective string theory for Abrikosov-Nielsen-Olesen (ANO) vortices has a long history, independent of any connection to QCD. Nambu [2] attached quarks to the ends of superconducting vortices, and found an expression for the classical action of the resulting ANO vortex in the singular London limit of infinite Higgs boson mass. He introduced a cutoff to render this action finite, and showed that it was proportional to the area of the world sheet (the Nambu-Goto action).

Förster [9] took into consideration the curvature of the world sheet. He showed that in the strong coupling limit, with the ratio of vector and scalar masses held fixed, the effects of curvature were unimportant, and the classical action for the vortex reduced to the Nambu-Goto action. This limit can be regarded as the long distance limit, since only zero mass excitations are left in the theory. Equivalently, since the flux tube radius vanishes in this limit, all physical distances, measured in units of the flux tube radius, are becoming large. All degrees of freedom except the transverse oscillations of the vortex are frozen out.

Gervais and Sakita [10] first considered the quantum theory of the vortices of the Abelian Higgs model in the same long distance limit. They used the results of Förster to define collective coordinates for the vortices, by means of which they constructed an effective vortex action. They also obtained a formal expression for the Feynman path integral of the Abelian Higgs model as an integration over vortex sheets. However, they were not able to write this expression as an integral over the physical degrees of freedom of the vortices.

Lüscher, Symanzik, and Weisz [11] considered the leading semiclassical corrections to the classical Nambu-Goto action due to transverse string fluctuations, and showed how to regulate the resulting divergences. They showed that for a string of length R with fixed ends, the leading semiclassical contribution to the heavy quark potential is $-\pi/12R$. In a second paper, Lüscher [12] showed that this result was unaffected by the addition of other terms to the effective string action.

Polchinski and Strominger [13] discussed the relation of the Abelian Higgs model to fundamental string theory, regarding the theory of ANO vortices as an effective string theory. They explained how existing string quantization methods were inappropriate for quantizing the vortices. To compensate for the anomalies [14] in these quantization methods, they introduced an additional term, the ''Polchinski Strominger term,'' into the effective vortex action.

Akhmedov, Chernodub, Polikarpov, and Zubkov [15] studied the quantum theory of ANO vortices in the London limit. In particular, they studied the transformation from field degrees of freedom to vortex degrees of freedom. They showed that the Jacobian of this transformation contained the ''Polchinski Strominger term'' as a factor. Although they, similar to Gervais and Sakita, did not obtain a complete expression for the path integral, this paper provided an important stimulus to our own work.

In the current paper, we simplify and extend work done in an earlier paper [16]. We begin with the path integral representation of a field theory having vortex solutions. It is an effective field theory describing phenomena at distances greater than the flux tube radius. We end up with an effective string theory of vortices in a form suitable for explicit calculations.

We apply this theory to calculate the energy E and angular momentum J of the fluctuations of a string bounded by the curve generated by the worldlines of a quark-antiquark pair separated by a fixed distance and rotating with fixed angular velocity. This gives the contribution of string fluctuations to the Regge trajectory $J(E^2)$, which we compare with the experimental ρ and ω trajectories.

II. OUTLINE

In Sec. III, we rewrite the path integral over field configurations of the Abelian Higgs model containing vortices as an integral over surfaces on which the Higgs field vanishes. This introduces a Jacobian due to the change from field variables to string variables (surfaces). This Jacobian is the key to determining the action of the effective string theory, and to defining the integral over all surfaces. We next use the formalism described in Sec. III to obtain an effective theory of ANO vortices. In Sec. IV, we show how the Jacobian divides into a field part and a string part. The two parts of the Jacobian play different roles in the effective theory. In Sec. V, we define an expression for the action of the effective string theory. All the dependence on the Abelian Higgs model is contained in the string action. We also obtain an expression for the path integral over vortices. In Sec. VI, we show how to express the integral over surfaces as an integral over the two physical degrees of freedom of the vortex, and obtain the final form of the effective string theory.

In the remaining sections we compute the leading semiclassical contribution to Regge trajectories due to the fluctuations of the string. We obtain an expression for the contribution of string fluctuations to the effective action in Sec. VII, and in Secs. VIII and IX describe how to regularize this expression, making use of the results of Lüscher, Symanzik, and Weisz [11]. In Sec. X we calculate the contribution of string fluctuations to the effective action for a straight, rotating string, and in Sec. XI obtain the resulting corrections to Regge trajectories.

III. THE TRANSFORMATION FROM FIELDS TO STRINGS

In this section we consider the Abelian Higgs model coupled via a Dirac string to a moving quark-antiquark pair. We transform the path integral over field configurations containing vortices to an integral over the surfaces $\tilde{x}^{\mu}$ determining the location of the vortices.

We denote the (dual) potentials by C_{μ} and the complex (monopole) Higgs field by ϕ. The dual coupling constant is $g=2\pi/e$, where e is the Yang-Mills coupling constant ($\alpha_s=e^2/4\pi$). The world lines of the quark and antiquark trajectories form the closed loop Γ (see Fig. 1). The moving quark-antiquark pair couples to the dual potentials C_{μ} via a Dirac string tensor $G^{S}_{\mu\nu}$, which is nonvanishing along some line L connecting the $q\bar{q}$ pair. As the pair moves, the line L sweeps out a world sheet $\tilde{x}^{\mu}(\xi)$ parametrized by coordinates ξ^a, $a=1,2$. The field ϕ vanishes on this world sheet

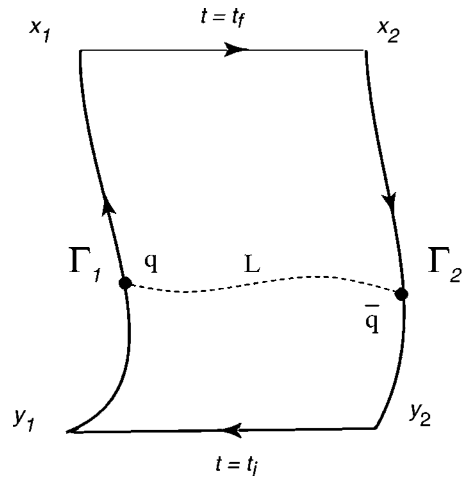

FIG. 1. The loop Γ.

$$\phi(x^{\mu})=0, \quad \text{at } x^{\mu}=\tilde{x}^{\mu}(\xi). \tag{3.1}$$

The corresponding Dirac string tensor $G^{S}_{\mu\nu}$ is given by

$$G^{S}_{\mu\nu}=-e\int d^2\xi\frac{1}{2}\epsilon^{ab}\epsilon_{\mu\nu\alpha\beta}\frac{\partial\tilde{x}^{\alpha}}{\partial\xi^{a}}\frac{\partial\tilde{x}^{\beta}}{\partial\xi^{b}}\delta^{(4)}[x^{\mu}-\tilde{x}^{\mu}(\xi)]. \tag{3.2}$$

The action S of a field configuration which has a vortex on the sheet $\tilde{x}^{\mu}(\xi)$ is

$$S=\frac{4}{3}\int d^4x\left[-\frac{1}{4}(G_{\mu\nu})^2-\frac{1}{2}|(\partial_{\mu}-igC_{\mu})\phi|^2 -\frac{\lambda}{4}(|\phi|^2-\phi_0^2)^2\right], \tag{3.3}$$

where the field strength $G_{\mu\nu}$ is given by

$$G_{\mu\nu}=\partial_{\mu}C_{\nu}-\partial_{\nu}C_{\mu}+G^{S}_{\mu\nu}. \tag{3.4}$$

The Higgs mechanism gives the vector particle (dual gluon) a mass $M_V=g\phi_0$ and the scalar particle a mass $M_S=\sqrt{2\lambda}\phi_0$, where ϕ_0 is the vacuum expectation value of the Higgs field. We have introduced the color factor $\frac{4}{3}$ in Eq. (3.4) because we are interested in using S as a model for long distance QCD. We consider S to be an effective action describing distances greater than the flux tube radius a.

The long distance $q\bar{q}$ interaction is determined by the Wilson loop $W[\Gamma]$,

$$W[\Gamma]=\int \mathcal{D}\phi^{*}\mathcal{D}\phi\mathcal{D}C^{\mu}e^{i(S[\phi,C]+S_{\mathrm{GF}})}, \tag{3.5}$$

where S_{GF} is a gauge fixing term. The functional integrals are cut off at the momentum scale $1/a$. The action (3.3) describes a field theory having classical vortex solutions. The functional integral (3.5) goes over all field configurations containing a vortex bounded by Γ.

Previous calculations [5] of $W[\Gamma]$ were carried out in the classical approximation (corresponding to a flat vortex sheet $\tilde{x}^{\mu}$), and showed that the Landau-Ginzburg parameter λ/g^2 is approximately equal to $\frac{1}{2}$. This corresponds to a superconductor on the border between type I and type II. In this situation, both particles have the same mass $M=M_V=M_S$, the string tension is $\sigma=\frac{4}{3}\pi\phi_0^2$, and the flux tube radius is $a=\sqrt{2}/M$.

To take into account the fluctuations of these vortices, we must evaluate $W[\Gamma]$ beyond the classical approximation. We carry out the functional integration (3.5) in two steps: (1) We fix the location of a vortex sheet $\tilde{x}^{\mu}$, and integrate only over field configurations for which $\phi(x^{\mu})$ vanishes on $\tilde{x}^{\mu}$. (2) We integrate over all possible vortex sheets. To implement this procedure, we introduce into the functional integral (3.5) the factor one, written in the form

$$1=J[\phi]\int \mathcal{D}\tilde{x}^{\mu}\,\delta\{\mathrm{Re}\,\phi[\tilde{x}^{\mu}(\xi)]\}\,\delta\{\mathrm{Im}\,\phi[\tilde{x}^{\mu}(\xi)]\}. \tag{3.6}$$

The integration $\mathcal{D}\tilde{x}^{\mu}$ is over the four functions $\tilde{x}^{\mu}(\xi)$. The functions $\tilde{x}^{\mu}(\xi)$ are a particular parametrization of the world sheet $\tilde{x}^{\mu}$.

The expression (3.6) implies that the string world sheet $\tilde{x}^{\mu}$, determined by the δ functions, is the surface of the zeros of the field ϕ. The factor $J[\phi]$ is a Jacobian, and is defined by Eq. (3.6). Inserting Eq. (3.6) into Eq. (3.5) puts the Wilson loop in the form

$$W[\Gamma]=\int \mathcal{D}\phi^*\mathcal{D}\phi\mathcal{D}C^{\mu}e^{i(S[\phi,C]+S_{\mathrm{GF}})}J[\phi]\times\int \mathcal{D}\tilde{x}^{\mu}\,\delta\{\mathrm{Re}\,\phi[\tilde{x}^{\mu}(\xi)]\}\,\delta\{\mathrm{Im}\,\phi[\tilde{x}^{\mu}(\xi)]\}. \tag{3.7}$$

We then reverse the order of the field integration and the string integration over surfaces $\tilde{x}^{\mu}(\xi)$,

$$W[\Gamma]=\int \mathcal{D}\tilde{x}^{\mu}\int \mathcal{D}\phi^*\mathcal{D}\phi\mathcal{D}C^{\mu}J[\phi]\,\delta\{\mathrm{Re}\,\phi[\tilde{x}^{\mu}(\xi)]\}\times\delta\{\mathrm{Im}\,\phi[\tilde{x}^{\mu}(\xi)]\}e^{i(S[\phi,C]+S_{\mathrm{GF}})}. \tag{3.8}$$

In Eq. (3.7), the δ functions fix $\tilde{x}^{\mu}$ to lie on the surface of the zeros of a given field ϕ, while in Eq. (3.8), they restrict the field ϕ to vanish on a given surface $\tilde{x}^{\mu}$. The integral over ϕ in Eq. (3.8) is therefore restricted to functions ϕ which vanish on $\tilde{x}^{\mu}$, in contrast to the integral over ϕ in Eq. (3.7), in which ϕ can be any function.

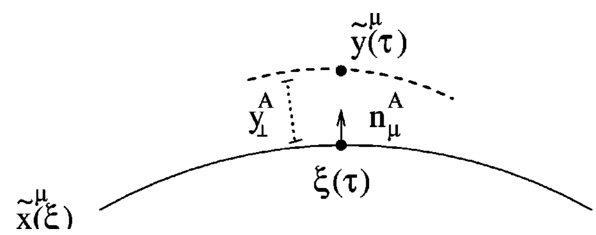

FIG. 2. World sheets and normal vectors.

IV. FACTORIZATION OF THE JACOBIAN

To evaluate $W[\Gamma]$ we divide $J[\phi]$ into two parts. The Jacobian $J[\phi]$ in Eq. (3.8) is evaluated for field configurations ϕ which vanish on a particular surface $\tilde{x}^{\mu}$. We make this explicit by writing Eq. (3.6) as

$$J[\phi,\tilde{x}^{\mu}]^{-1}=\int \mathcal{D}\tilde{y}^{\mu}\,\delta\{\mathrm{Re}\,\phi[\tilde{y}^{\mu}(\tau)]\}\,\delta\{\mathrm{Im}\,\phi[\tilde{y}^{\mu}(\tau)]\}, \tag{4.1}$$

where $\tilde{y}^{\mu}$ is some other string worldsheet, distinct from $\tilde{x}^{\mu}$. The evaluation of the Jacobian is the essential new ingredient in deriving $W[\Gamma]$.

The δ functions in Eq. (4.1) select surfaces $\tilde{y}^{\mu}(\tau)$ which lie in a neighborhood of the surface $\tilde{x}^{\mu}(\xi)$ of the zeros of ϕ. We separate $\tilde{y}^{\mu}(\tau)$ into components lying on the surface $\tilde{x}^{\mu}(\xi)$ and components lying along vectors $n_{\mu}^{A}(\xi)$ normal to $\tilde{x}^{\mu}(\xi)$ at the point ξ:

$$\tilde{y}^{\mu}(\tau)=\tilde{x}^{\mu}[\xi(\tau)]+y_{\perp}^{A}[\xi(\tau)]n_{\mu}^{A}[\xi(\tau)]. \tag{4.2}$$

The point $\tilde{x}^{\mu}[\xi(\tau)]$ is the point on the surface $\tilde{x}^{\mu}(\xi)$ lying closest to $\tilde{y}^{\mu}(\tau)$, and the magnitude of $y_{\perp}^{A}[\xi(\tau)]$ is the distance from $\tilde{y}^{\mu}(\tau)$ to $\tilde{x}^{\mu}[\xi(\tau)]$ (see Fig. 2).

We evaluate the Jacobian (4.1) by making the change of variables

$$\tilde{y}^{\mu}(\tau)\rightarrow[\xi(\tau),y_{\perp}^{A}(\xi)] \tag{4.3}$$

defined by Eq. (4.2). Although the δ functions in Eq. (4.1) force $y_{\perp}^{A}$ to vanish, the integrations over $y_{\perp}^{A}$ give a contribution to the Jacobian. Furthermore, this contribution depends on the field variable ϕ in a neighborhood of the surface. The integration over the reparametrizations $\xi(\tau)$ of the surface $\tilde{x}^{\mu}(\xi)$, on the other hand, depends upon the surface, but not on the fields. The change of variables (4.3) leads to a factorization of the Jacobian into a field contribution, and into a contribution depending only on the intrinsic properties of the world sheet $\tilde{x}^{\mu}(\xi)$.

We now exhibit the factorization of the Jacobian. Under the transformation (4.3), the integral over $\tilde{y}^{\mu}$ becomes

$$\mathcal{D}\tilde{y}^{\mu}=\mathrm{Det}_{\tau}\left[\epsilon^{\mu\nu\alpha\beta}\frac{1}{2}\epsilon^{ab}\frac{\partial\tilde{x}^{\mu}}{\partial\xi^{a}}\frac{\partial\tilde{x}^{\nu}}{\partial\xi^{b}}\frac{1}{2}\epsilon^{AB}n_{\alpha}^{A}n_{\beta}^{B}\right]\mathcal{D}y_{\perp}^{A}\mathcal{D}\xi$$
$$=\mathrm{Det}_{\tau}\left[\sqrt{-\frac{1}{2}\left(\epsilon^{ab}\frac{\partial\tilde{x}^{\mu}}{\partial\xi^{a}}\frac{\partial\tilde{x}^{\nu}}{\partial\xi^{b}}\right)^{2}\frac{1}{2}(\epsilon^{AB}n_{\alpha}^{A}n_{\beta}^{B})^{2}}\right]\mathcal{D}y_{\perp}^{A}\mathcal{D}\xi$$
$$=\mathrm{Det}_{\tau}[\sqrt{-g(\xi)}|_{\xi=\xi(\tau)}]\mathcal{D}y_{\perp}^{A}\mathcal{D}\xi, \tag{4.4}$$

where $\sqrt{-g}$ is the square root of the determinant of the induced metric

$$g_{ab}=\frac{\partial\tilde{x}^{\mu}}{\partial\xi^{a}}\frac{\partial\tilde{x}^{\mu}}{\partial\xi^{b}} \tag{4.5}$$

evaluated on the world sheet $\tilde{x}^{\mu}$. Appendix A gives a summary of our notation, and of the relations used to obtain Eq. (4.4).

The functional determinant in Eq. (4.4) is the product of its argument evaluated at all points τ on the sheet, in the same way that the integration over $\mathcal{D}\tilde{y}^{\mu}$ is a product of integrals at all points τ. Making the change of coordinates (4.2), (4.3) in the Jacobian (4.1) gives

$$J[\phi,\tilde{x}^{\mu}]^{-1}=\int\mathcal{D}\xi\mathcal{D}y_{\perp}^{A}\,\mathrm{Det}_{\tau}[\sqrt{-g}]\delta(\mathrm{Re}\,\phi\{\tilde{x}^{\mu}[\xi(\tau)]$$
$$+y_{\perp}^{A}[\xi(\tau)]n_{A}^{\mu}[\xi(\tau)]\})\delta(\mathrm{Im}\,\phi\{\tilde{x}^{\mu}[\xi(\tau)]$$
$$+y_{\perp}^{A}[\xi(\tau)]n_{A}^{\mu}[\xi(\tau)]\}). \tag{4.6}$$

Equation (4.6) has the form

$$J[\phi,\tilde{x}]^{-1}=\int\mathcal{D}\xi(\tau)\mathrm{Det}_{\tau}[\sqrt{-g}]J_{\perp}\{\phi,\tilde{x}^{\mu}[\xi(\tau)]\}^{-1}, \tag{4.7}$$

where

$$J_{\perp}\{\phi,\tilde{x}^{\mu}[\xi(\tau)]\}^{-1}=\int\mathcal{D}y_{\perp}^{A}\,\delta(\mathrm{Re}\,\phi\{\tilde{x}^{\mu}[\xi(\tau)]$$
$$+y_{\perp}^{A}[\xi(\tau)]n_{A}^{\mu}[\xi(\tau)]\})$$
$$\times\delta(\mathrm{Im}\,\phi\{\tilde{x}^{\mu}[\xi(\tau)]$$
$$+y_{\perp}^{A}[\xi(\tau)]n_{A}^{\mu}[\xi(\tau)]\}) \tag{4.8}$$

contains all the dependence on ϕ. Since $J_{\perp}$ is independent of the parametrization $\xi(\tau)$, the Jacobian factors into two parts

$$J[\phi,\tilde{x}]^{-1}=J_{\|}[\tilde{x}]^{-1}J_{\perp}[\phi,\tilde{x}]^{-1}, \tag{4.9}$$

where

$$J_{\|}[\tilde{x}]^{-1}=\int\mathcal{D}\xi\,\mathrm{Det}_{\tau}[\sqrt{-g}]. \tag{4.10}$$

The string part $J_{\|}$ of the Jacobian arises from the parametrization degrees of freedom. In the next section, we show that $J_{\perp}$ is the Faddeev-Popov determinant for the δ functions in Eq. (3.8). This allows us to define the action of the effective string theory. In the following section, we will use $J_{\|}$ to fix the reparametrization degrees of freedom.

V. THE STRING ACTION

Inserting the factorized form (4.9) of $J[\phi]$ into the expression (3.8) for $W[\Gamma]$ gives the Wilson Loop the form

$$W[\Gamma]=\int\mathcal{D}\tilde{x}^{\mu}J_{\|}[\tilde{x}]e^{iS_{\mathrm{eff}}}, \tag{5.1}$$

where the action S_{eff} of the effective string theory is given by

$$e^{iS_{\mathrm{eff}}[\tilde{x}^{\mu}(\xi)]}=\int\mathcal{D}\phi^{*}\mathcal{D}\phi\mathcal{D}C^{\mu}J_{\perp}[\phi]\delta\{\mathrm{Re}\,\phi[\tilde{x}^{\mu}(\xi)]\}$$
$$\times\delta\{\mathrm{Im}\,\phi[\tilde{x}^{\mu}(\xi)]\}e^{i(S+S_{\mathrm{GF}})}. \tag{5.2}$$

The string action (5.2) was obtained previously by Gervais and Sakita [10]. The novel feature of our result is the string integration measure of the Wilson loop (5.1).

The string action depends upon the field part $J_{\perp}$ of the Jacobian

$$J_{\perp}[\phi,\tilde{x}^{\mu}]^{-1}=\int\mathcal{D}y_{\perp}^{A}\,\delta[\mathrm{Re}\,\phi(\tilde{x}^{\mu}+y_{\perp}^{A}n_{\mu A})]$$
$$\times\delta[\mathrm{Im}\,\phi(\tilde{x}^{\mu}+y_{\perp}^{A}n_{\mu A})]. \tag{5.3}$$

The δ functions force $y_{\perp}^{A}$ to be zero, so we can expand their arguments in a power series in $y_{\perp}^{A}$,

$$\phi(y^{\mu})=\phi(\tilde{x}^{\mu})+y_{\perp}^{A}n_{A}^{\nu}\partial_{\nu}\phi(\tilde{x}^{\mu})+O(y_{\perp}^{2}). \tag{5.4}$$

The zeroth order term in Eq. (5.4) vanishes because $\tilde{x}^{\mu}$ is the surface of the zeros of ϕ. The integration (5.3) over $y_{\perp}^{A}$ gives the result

$$J_{\perp}[\phi,\tilde{x}^{\mu}]^{-1}=\mathrm{Det}_{\xi}^{-1}[\epsilon^{AB}n_{A}^{\mu}n_{B}^{\nu}(\partial_{\mu}\mathrm{Re}\,\phi)(\partial_{\nu}\mathrm{Im}\,\phi)|_{x^{\mu}=\tilde{x}^{\mu}}]. \tag{5.5}$$

The Jacobian $J_{\perp}$ is a Faddeev-Popov determinant, which we discuss in Appendix B.

Equation (5.2) gives the action $S_{\mathrm{eff}}(\tilde{x}^{\mu})$ of the effective string theory as an integral over field configurations which have a vortex fixed at $\tilde{x}^{\mu}$. Since the vortex theory (3.5) is an effective long distance theory, the path integral (3.5) for $W[\Gamma]$, written in terms of the fields of the Abelian Higgs model, is cutoff at a scale Λ which is on the order of the mass M of the dual gluon. Furthermore, the integration (5.1) over $\tilde{x}^{\mu}$ includes all the long distance fluctuations of the theory. Therefore, the path integral (5.2) contains neither short distance nor long distance fluctuations, and is determined by minimizing the field action $S[\tilde{x}^{\mu},\phi,C_{\mu}]$ for a fixed position of the vortex sheet

$$S_{\mathrm{eff}}[\tilde{x}^{\mu}]=S[\tilde{x}^{\mu},\phi^{\mathrm{class}},C_{\mu}^{\mathrm{class}}],\quad \phi^{\mathrm{class}}(\tilde{x}^{\mu})=0. \tag{5.6}$$

The fields ϕ^{class} and C_μ^{class} are the solutions of the classical equations of motion, subject to the boundary condition $\phi(\tilde{x}^\mu)=0$.

The action S_{eff} depends both on the distance R between the quarks, and the radius of curvature R_V of the vortex sheet bounded by Γ. In the long distance limit, when the length of the string R and its radius R_V are large compared to the thickness of the flux tube a, the string action (5.6) becomes the Nambu-Goto action S_{NG},

$$S_{\mathrm{NG}}=-\sigma\int d^2\xi\sqrt{-g}, \tag{5.7}$$

where σ is the classical string tension, determined from the solution of the Nielsen-Olesen equations for a straight, infinitely long string.

It is convenient to separate the action (5.6) into its perturbative and nonperturbative parts

$$S_{\mathrm{eff}}[\tilde{x}^\mu]=S[\tilde{x}^\mu,\phi^{\mathrm{class}},C_\mu^{\mathrm{class}}]=S^{\mathrm{Maxwell}}[\tilde{x}^\mu]+S^{\mathrm{NP}}[\tilde{x}^\mu], \tag{5.8}$$

where S^{Maxwell} is the action obtained by setting $\lambda=g=0$ in Eq. (3.3). The value of S^{Maxwell} depends only upon the boundary Γ, and is the usual electromagnetic interaction between charged particles

$$S^{\mathrm{Maxwell}}[\Gamma]=\frac{4}{3}\frac{e^2}{2}\oint dx^\mu\oint dx'^\mu\mathcal{D}_{\mu\nu}(x^\mu-x'^\mu), \tag{5.9}$$

where $\mathcal{D}_{\mu\nu}$ is the photon propagator.

To calculate the Wilson loop $W[\Gamma]$ from the effective string theory (5.1), we must also examine S_{eff} at smaller values of R and R_V, on the order of the string thickness a. We first consider the dependence of S_{eff} on R for a flat string, where $R_V\to\infty$. In this case, the curve Γ is a rectangle of length T in the time direction, and width R in the space direction. In the large T limit, the action S_{eff} reduces to the product of T and the potential $V^{\mathrm{class}}(R)$ previously used to fit the energy levels of heavy quark systems.

Evaluation of Eq. (5.8) for a flat sheet gives a corresponding decomposition of $V^{\mathrm{class}}(R)$,

$$V^{\mathrm{class}}(R)=V^{\mathrm{Coulomb}}(R)+V^{\mathrm{NP}}(R). \tag{5.10}$$

For small R,

$$V^{\mathrm{class}}(R)\xrightarrow[R\to 0]{}V^{\mathrm{Coulomb}}(R)=-\frac{4}{3}\frac{\alpha_s}{R}, \tag{5.11}$$

while Eq. (5.7) gives the large R behavior

$$V^{\mathrm{NP}}(R)\xrightarrow[R\to\infty]{}\sigma R. \tag{5.12}$$

Recent numerical studies [17] of the classical equations of motion for a flat sheet have shown that for a superconductor on the I-II border, the long distance behavior (5.12) of $V^{\mathrm{NP}}(R)$ persists to small values of R, even to values less than the string thickness a. Therefore, for a superconductor on the I-II border, $V^{\mathrm{class}}(R)$ is, to a good approximation, equal to the Cornell potential [18]

$$V^{\mathrm{class}}(R)\approx-\frac{4}{3}\frac{\alpha_s}{R}+\sigma R. \tag{5.13}$$

In other words, for a flat sheet,

$$S^{\mathrm{NP}}(R)\approx S_{\mathrm{NG}}. \tag{5.14}$$

Thus, for short straight strings the Nambu-Goto action remains a good approximation to the nonperturbative part of the classical action for a superconductor on the type I-II border.

Next, consider the nonperturbative contribution to the classical action for a long bent string. [The Maxwell action has the value (5.9) independent of the shape of the vortex.] The leading correction to the Nambu-Goto action when the string is bent is the curvature term

$$S_{\mathrm{curvature}}=-\beta\int d^2\xi\sqrt{-g}(\mathcal{K}_{ab}^A)^2, \tag{5.15}$$

where $\mathcal{K}_{ab}^A$ is the extrinsic curvature.

$$\mathcal{K}_{ab}^A=n_\mu^A\partial_a\partial_b x^\mu. \tag{5.16}$$

The magnitude of $\mathcal{K}_{ab}^A$ is of the order of $1/R_V$, so that $S_{\mathrm{curvature}}\sim(a^2/R_V^2)S_{\mathrm{NG}}$.

The calculation of the ''rigidity'' β determining the size of $S_{\mathrm{curvature}}$ has been considered by a number of authors [19], but the value of β for a superconductor on the I-II border was never calculated. We conjecture that the value of β is small, because de Vega and Schaposnik [20] have shown that the components of the stress tensor perpendicular to the axis of a straight Nielsen-Olesen flux tube vanish for a superconductor on the border between type I and type II. In other words, there are no ''bonds'' perpendicular to the field lines of a straight flux tube of infinite extent. When the flux tube is bent slightly, there are no perpendicular bonds to be stretched or compressed, and the change in the energy is just the string tension multiplied by the change in length. That is, the curvature term, which in a sense represents the attraction or repulsion between neighboring parts of the string, should vanish. A more formal argument can be made by regarding the borderline superconductor as the long distance limit of a theory where the forces between vortices become weak. Polyakov [21] has shown, using renormalization group methods, that β also vanishes in this limit.

Similar Heuristic arguments give a reason for the above mentioned result that the Nambu-Goto action is a good approximation for short, straight strings on the I-II border. The bending of the field lines as the quark-antiquark separation becomes smaller causes no additional changes in the energy.

We therefore take the action of the effective string theory to be equal to the sum of the Maxwell action (5.9) and the Nambu-Goto action (5.7):

$$S_{\rm eff}[\tilde{x}^\mu] = S^{\rm Maxwell}[\Gamma] - \sigma \int d^2\xi \sqrt{-g}. \tag{5.17}$$

We use Eq. (5.17) for the full range of string lengths R and radii of curvature R_V greater than the inverse of the mass M of the dual gluon, which is the cutoff of the effective string theory (5.1).

Equation (5.17) for $S_{\rm eff}[\tilde{x}^\mu]$ is the generalization of Eq. (5.13) to a general sheet. The first term, $S^{\rm Maxwell}[\Gamma]$, is just a boundary term, independent of the fluctuating string, and we take take $S_{\rm eff} = S_{\rm NG}$ for the calculations carried out in the rest of this paper. In the next section we show how to carry out the integration over $\tilde{x}^\mu$ in Eq. (5.1) by separating the degrees of freedom of the world sheet $\tilde{x}^\mu$ into two physical degrees of freedom and two reparametrization degrees of freedom. This treatment makes no use of Eq. (5.17), and is applicable to any effective string theory of vortices.

VI. EFFECTIVE THEORY OF TRANSVERSE STRING FLUCTUATIONS

We next show how to evaluate the integral over $\tilde{x}^\mu(\xi)$ in Eq. (5.1),

$$W[\Gamma] = \int \mathcal{D}\tilde{x}^\mu J_\parallel[\tilde{x}^\mu] e^{iS_{\rm eff}}. \tag{6.1}$$

The integration over $\tilde{x}^\mu(\xi)$ is the product of an integral over string world sheets and an integral over reparametrizations of the coordinates of the string. The Jacobian $J_\parallel$ is the inverse of the integration (4.10) over reparametrization degrees of freedom. In this section, we fix the parametrization of the string, and show that $J_\parallel$ cancels the integration over reparametrizations.

Any surface $\tilde{x}^\mu$ has only two physical degrees of freedom. The other two degrees of freedom represent the invariance of the surface under coordinate reparametrizations. We fix the coordinate reparametrization symmetry by choosing a particular ''representation'' x_p^μ of the surface, which depends on two functions $f^1(\xi), f^2(\xi)$,

$$x_p^\mu(\xi) = x_p^\mu[f^1(\xi), f^2(\xi), \xi]. \tag{6.2}$$

A particular example of a representation x_p^μ is obtained by expanding in transverse fluctuations $x_\perp^A$ about a fixed sheet $\overline{x_p^\mu}$,

$$x_p^\mu(\xi) = x_p^\mu[x_\perp^A(\xi), \xi] = \overline{x_p^\mu}(\xi) + x_\perp^A(\xi)\overline{n_A^\mu}(\xi). \tag{6.3}$$

The vectors $\overline{n_A^\mu}$ are orthogonal to the surface $\overline{x_p^\mu}$. In this example, f^1 and f^2 are the transverse coordinates $x_\perp^A$.

Any physical surface can be expressed in terms of x_p^μ by a suitable choice of f^1 and f^2. In particular, the world sheet $\tilde{x}^\mu(\xi)$ appearing in the integral (6.1) can be written in terms of a reparametrization $\tilde{\xi}(\xi)$ of the representation x_p^μ,

$$\tilde{x}^\mu(\xi) = x_p^\mu\{f^1[\tilde{\xi}(\xi)], f^2[\tilde{\xi}(\xi)], \tilde{\xi}(\xi)\}. \tag{6.4}$$

The four degrees of freedom in $\tilde{x}^\mu(\xi)$ are replaced by two physical degrees of freedom $f^1(\xi), f^2(\xi)$ and two reparametrization degrees of freedom $\tilde{\xi}(\xi)$.

We can write the integral over $\tilde{x}^\mu(\xi)$ in Eq. (6.1) in terms of integrals over $f^1(\xi)$, $f^2(\xi)$, and $\tilde{\xi}(\xi)$,

$$\mathcal{D}\tilde{x}^\mu = {\rm Det}\left[\epsilon_{\mu\nu\alpha\beta}\frac{1}{2}\epsilon^{ab}\frac{\partial\tilde{x}^\mu}{\partial\tilde{\xi}^a}\frac{\partial\tilde{x}^\nu}{\partial\tilde{\xi}^b}\frac{\partial\tilde{x}^\alpha}{\partial f^1}\frac{\partial\tilde{x}^\beta}{\partial f^2}\right]\mathcal{D}f^1\mathcal{D}f^2\mathcal{D}\tilde{\xi}. \tag{6.5}$$

Noting that the derivative of $\tilde{x}^\mu$ with respect to $\tilde{\xi}(\xi)$ is

$$\frac{\partial\tilde{x}^\mu}{\partial\tilde{\xi}^a} = \left[\frac{\partial x_p^\mu}{\partial f^1}\frac{\partial f^1}{\partial\xi^a} + \frac{\partial x_p^\mu}{\partial f^2}\frac{\partial f^2}{\partial\xi^a} + \frac{\partial x_p^\mu}{\partial\xi^a}\right]\Bigg|_{\xi=\tilde{\xi}}, \tag{6.6}$$

we can write

$$\begin{aligned}\mathcal{D}\tilde{x}^\mu &= {\rm Det}\left[\epsilon_{\mu\nu\alpha\beta}\frac{1}{2}\epsilon^{ab}\frac{\partial x_p^\alpha}{\partial\xi^a}\frac{\partial x_p^\beta}{\partial\xi^b}\frac{\partial x_p^\mu}{\partial f^1}\frac{\partial x_p^\nu}{\partial f^2}\Bigg|_{\xi=\tilde{\xi}}\right]\mathcal{D}f^1\mathcal{D}f^2\mathcal{D}\tilde{\xi}\\ &= {\rm Det}\left[\tilde{t}_{\mu\nu}\sqrt{-g^p}\frac{\partial x_p^\mu}{\partial f^1}\frac{\partial x_p^\nu}{\partial f^2}\Bigg|_{\xi=\tilde{\xi}}\right]\mathcal{D}f^1\mathcal{D}f^2\mathcal{D}\tilde{\xi},\end{aligned} \tag{6.7}$$

where

$$\tilde{t}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\frac{\epsilon^{ab}}{\sqrt{-g}}\frac{\partial\tilde{x}^\alpha}{\partial\xi^a}\frac{\partial\tilde{x}^\beta}{\partial\xi^b} \tag{6.8}$$

is the antisymmetric tensor normal to the world sheet and

$$g_{ab}^p = \frac{\partial x_p^\mu}{\partial\xi^a}\frac{\partial x_p^\mu}{\partial\xi^b} \tag{6.9}$$

is the induced metric of $x_p^\mu(\xi)$. The metric g_{ab}^p is related to the metric g_{ab} of $\tilde{x}^\mu(\xi)$,

$$g_{ab} = \frac{\partial\tilde{\xi}^c}{\partial\xi^a}\frac{\partial\tilde{\xi}^d}{\partial\xi^b}g_{cd}^p. \tag{6.10}$$

The induced metric g_{ab} of the original world sheet $\tilde{x}^\mu(\xi)$ does not appear in Eq. (6.7) because the determinant is independent of $\tilde{\xi}$. Only the induced metric g_{ab}^p of the world sheet $x_p^\mu(\xi)$ enters into the determinant.

With the parametrization (6.4) of $\tilde{x}^\mu$, the path integral (6.1) takes the form

$$W[\Gamma] = \int \mathcal{D}\tilde{\xi}\mathcal{D}f^1\mathcal{D}f^2{\rm Det}\left[\tilde{t}^{\mu\nu}\frac{\partial x_p^\mu}{\partial f^1}\frac{\partial x_p^\nu}{\partial f^2}\right]{\rm Det}[\sqrt{-g^p}]J_\parallel e^{iS_{\rm eff}}. \tag{6.11}$$

The action $S_{\rm eff}$ is parametrization independent, so it is independent of $\tilde{\xi}(\xi)$. The same is true for $J_\parallel$. Furthermore, $\tilde{t}_{\mu\nu}$ is parametrization independent, so that the product

$$\mathcal{D}f^1\mathcal{D}f^2\,\mathrm{Det}\left[\tilde{t}^{\mu\nu}\frac{\partial x_p^\mu}{\partial f^1}\frac{\partial x_p^\nu}{\partial f^2}\right] \tag{6.12}$$

is independent of $\tilde{\xi}(\xi)$. Therefore, this product, along with $J_\parallel$ and $e^{iS_{\rm eff}}$, can be brought outside the $\tilde{\xi}$ integral in Eq. (6.11). The path integral then takes the form

$$W[\Gamma]=\int\mathcal{D}f^1\mathcal{D}f^2\,\mathrm{Det}\left[\tilde{t}^{\mu\nu}\frac{\partial x_p^\mu}{\partial f^1}\frac{\partial x_p^\nu}{\partial f^2}\right]J_\parallel e^{iS_{\rm eff}} \times\int\mathcal{D}\tilde{\xi}\,\mathrm{Det}[\sqrt{-g^p}]. \tag{6.13}$$

The remaining integral over reparametrizations $\tilde{\xi}$ is equal to $J_\parallel^{-1}$, defined by Eq. (4.10), and is canceled by the explicit factor of $J_\parallel$ appearing in Eq. (6.13). This means we do not need to evaluate $J_\parallel$, and can avoid the complications inherent in evaluating the integral over reparametrizations of the string coordinates. The anomalies produced in string theory by evaluating this integral are not present, so we do not have a Polchinski-Strominger term in the theory. Equation (6.13) gives the final result for the Wilson loop

$$W[\Gamma]=\int\mathcal{D}f^1\mathcal{D}f^2\,\mathrm{Det}\left[\tilde{t}_{\mu\nu}\frac{\partial x_p^\mu}{\partial f^1}\frac{\partial x_p^\nu}{\partial f^2}\right]e^{iS_{\rm eff}}, \tag{6.14}$$

as an integration over two function $f^1(\xi)$ and $f^2(\xi)$, the physical degrees of freedom of the string.

The path integral (6.14) is invariant under reparametrizations of the string, and describes a two-dimensional field theory with two degrees of freedom, the two transverse oscillations of a two-dimensional sheet. The integration (6.14) goes over the normal fluctuations of the string world sheet. The components of f^1 and f^2 along the sheet are nonphysical. The determinant in Eq. (6.14) is a normalization factor for f^1 and f^2. This can be seen by applying the identity $\tilde{t}_{\mu\nu}=\epsilon^{AB}n_{\mu A}n_{\nu B}$ to the determinant

$$\mathrm{Det}\left[\tilde{t}_{\mu\nu}\frac{\partial x_p^\mu}{\partial f^1}\frac{\partial x_p^\nu}{\partial f^2}\right]=\mathrm{Det}\left[\epsilon^{AB}\left(n_{\mu A}\frac{\partial x_p^\mu}{\partial f^1}\right)\left(n_{\nu B}\frac{\partial x_p^\nu}{\partial f^2}\right)\right]. \tag{6.15}$$

The factors of $n_{\mu A}(\partial x_p^\mu/\partial f^i)$ determine the amount of the fluctuation f^i which is in a direction normal to the sheet.

Equation (6.14) is the string representation of any field theory containing classical vortex solutions. The expression (5.2) for $S_{\rm eff}$ had been obtained previously by Gervais and Sakita [10]; we are unaware of any previous derivation of the string representation (6.14) for the path integral. We now show how it provides a method for explicit calculations.

VII. THE SEMICLASSICAL APPROXIMATION

In this section we carry out the semiclassical expansion of $W[\Gamma]$ about a classical solution of the effective string theory, and find the leading contribution of string fluctuations to the effective action $-i\log W[\Gamma]$. The ends of the string follow the path Γ fixed by the prescribed trajectory of the quarks, and the fluctuations of the string are cutoff at the momentum scale M of the inverse string radius. As explained in Sec. V, we take the action $S_{\rm eff}$ of the effective string theory to be the Nambu-Goto action

$$S_{\rm eff}=-\sigma\int d^2\xi\sqrt{-g}, \tag{7.1}$$

and Eq. (6.14) becomes

$$W[\Gamma]=\int\mathcal{D}f^1(\xi)\mathcal{D}f^2(\xi)\mathrm{Det}\left[\tilde{t}^{\mu\nu}\frac{\partial x_p^\mu}{\partial f^1}\frac{\partial x_p^\nu}{\partial f^2}\right]e^{-i\sigma\int d^2\xi\sqrt{-g}}. \tag{7.2}$$

We expand Eq. (7.2) in small fluctuations f^i of $x_p^\mu[f^i,\xi]$ around a fixed sheet $\overline{x_p^\mu}(\xi)$, subject to the condition that the boundary of $\overline{x_p^\mu}$ lies on the curve Γ,

$$x_p^\mu(f^i,\xi)=\overline{x_p^\mu}(\xi)+f^i\left.\frac{\partial x_p^\mu}{\partial f^i}\right|_{f^i=0}+\frac{1}{2}f^if^j\left.\frac{\partial^2x_p^\mu}{\partial f^i\partial f^j}\right|_{f^i=0}+O(f^3), \tag{7.3}$$

where $\overline{x_p^\mu}(\xi)\equiv x_p^\mu(f^i=0,\xi)$ is the position of the string worldsheet when $f^1=f^2=0$. Expanding $\sqrt{-g}$ to quadratic order in small f^1 and f^2, we obtain

$$\begin{aligned}W[\Gamma]=\int\mathcal{D}f^i(\xi)\mathrm{Det}&\left[\frac{1}{2}\tilde{t}^{\mu\nu}\epsilon^{ij}\frac{\partial x_p^\mu}{\partial f^i}\left.\frac{\partial x_p^\nu}{\partial f^j}\right|_{f^i=0}\right]\\ &\times\exp\Bigg\{-i\sigma\int d^2\xi\sqrt{-\bar{g}}\\ &\times\Bigg[1+\bar{g}^{ab}\left.\frac{\partial x_p^\mu}{\partial\xi^a}\right|_{f^i=0}\frac{\partial}{\partial\xi^b}\left(\left.\frac{\partial x_p^\mu}{\partial f^i}\right|_{f^i=0}f^i\right)\\ &+\frac{1}{2}f^iG_{ij}^{-1}f^j\Bigg]\Bigg\},\end{aligned} \tag{7.4}$$

where $\bar{g}_{ab}$ is

$$\bar{g}_{ab}=\frac{\partial\overline{x_p^\mu}}{\partial\xi^a}\frac{\partial\overline{x_p^\mu}}{\partial\xi^b}, \tag{7.5}$$

the metric of the fixed world sheet $\overline{x_p^\mu}$, and

$$G_{ij}^{-1}=\frac{1}{\sqrt{-g}}\left.\frac{\partial^2\sqrt{-g}}{\partial f^i\partial f^j}\right|_{f^i=0}. \tag{7.6}$$

We choose $\overline{x_p^\mu}$ to be the surface which minimizes the action. Then $\overline{x_p^\mu}$ satisfies the ''classical equation of motion''

$$\left.\frac{\partial x_p^\mu}{\partial f^i}\right|_{f^i=0}(-\nabla^2)\overline{x_p^\mu}=0, \tag{7.7}$$

where the covariant Laplacian is

$$-\nabla^2=\frac{1}{\sqrt{-\bar{g}}}\frac{\partial}{\partial\xi^a}\bar{g}^{ab}\sqrt{-\bar{g}}\frac{\partial}{\partial\xi^b}. \tag{7.8}$$

Using the fact that the covariant derivative of the metric is zero, we show in Appendix A that

$$(\partial_a\bar{x}_p^\mu)(-\nabla^2\bar{x}_p^\mu)=0. \tag{7.9}$$

The vectors $\partial x_p^\mu/\partial f^i|_{f^i=0}$ and $\partial_a\bar{x}_p^\mu$ form a complete basis, so Eqs. (7.7) and (7.9) imply

$$-\nabla^2\bar{x}_p^\mu=0. \tag{7.10}$$

Evaluating the f^i integral in Eq. (7.4) gives

$$W[\Gamma]=e^{-i\sigma\int d^2\xi\sqrt{-\bar{g}}}\mathrm{Det}\left[\frac{1}{2}\tilde{t}_{\mu\nu}\epsilon^{ij}\frac{\partial x_p^\mu}{\partial f^i}\frac{\partial x_p^\nu}{\partial f^j}\bigg|_{f^i=0}\right]$$
$$\times\mathrm{Det}^{-1/2}[G_{ij}^{-1}]. \tag{7.11}$$

The inverse propagator G_{ij}^{-1} (7.6) can be shown to be

$$G_{ij}^{-1}=-\frac{\partial x_p^\mu}{\partial f^i}\bigg|_{f^i=0}\bar{n}_{\mu A}[-\nabla^2\delta_{AB}-\bar{\mathcal{K}}_{ab}^A\bar{\mathcal{K}}^{Bab}]\bar{n}_{\nu B}\frac{\partial x_p^\nu}{\partial f^j}\bigg|_{f^i=0}, \tag{7.12}$$

where $\bar{\mathcal{K}}_{ab}^A$ is the extrinsic curvature tensor of the sheet $\bar{x}_p^\mu$. A derivation of Eq. (7.12) is given in Appendix A of Ref. [11]. The $\bar{n}_{\mu A}$ are vectors normal to the world sheet $\bar{x}_p^\mu$. Equation (7.12) gives

$$\mathrm{Det}^{-1/2}[G_{ij}^{-1}]=\mathrm{Det}^{-1/2}[-\nabla^2\delta_{AB}-\bar{\mathcal{K}}_{ab}^A\bar{\mathcal{K}}^{Bab}]\mathrm{Det}^{-1}$$
$$\times\left[\frac{1}{2}\epsilon^{AB}\bar{n}_{\mu A}\bar{n}_{\nu B}\epsilon^{ij}\frac{\partial x_p^\mu}{\partial f^i}\frac{\partial x_p^\nu}{\partial f^j}\bigg|_{f^i=0}\right]. \tag{7.13}$$

From the identity (A15),

$$\tilde{t}_{\mu\nu}=\epsilon^{AB}\bar{n}_{\mu A}\bar{n}_{\nu B}, \tag{7.14}$$

we see that the first determinant in Eq. (7.11) and the second determinant in Eq. (7.13) cancel. The determinant appearing in Eq. (6.14) produces exactly the correct normalization for the Green's function. The functional integral (7.11) becomes

$$W[\Gamma]=e^{-i\sigma\int d^2\xi\sqrt{-\bar{g}}}\mathrm{Det}^{-1/2}[-\nabla^2\delta_{AB}-\bar{\mathcal{K}}_{ab}^A\bar{\mathcal{K}}^{Bab}]. \tag{7.15}$$

We note that Eq. (7.15) is independent of the factors of $n_\mu^A(\partial x_p^\mu/\partial f^i)|_{f^i=0}$ which appeared in the inverse propagator (7.12). These factors are the projections of the fluctuations f^i normal to the string world sheet. For small f^i, the world sheet x_p^μ is

$$x_p^\mu=\bar{x}_p^\mu+f_i\frac{\partial x_p^\mu}{\partial f^i}\bigg|_{f^i=0}+O(f^{i2}). \tag{7.16}$$

The perturbation of the world sheet in the direction $\bar{n}^{\mu A}$ is

$$n_\mu^A(x_p^\mu-\bar{x}_p^\mu)=n_\mu^A\frac{\partial\bar{x}_p^\mu}{\partial f^i}\bigg|_{f^i=0}f_i+O(f^{i2}). \tag{7.17}$$

The factors of $n_\mu^A(\partial x_p^\mu/\partial f^i)|_{f^i=0}$ in Eq. (7.12) project out the part of the fluctuations f^i perpendicular to the world sheet $\bar{x}_p^\mu$. Only normal fluctuations contribute to $W[\Gamma]$, since fluctuations along the world sheet are equivalent to a reparametrization of the sheet coordinates.

The effective action obtained from Eq. (7.15) is

$$-i\ln W[\Gamma]=S_{cl}+S_{\mathrm{fluc}}. \tag{7.18}$$

The first term in Eq. (7.18) is the Nambu-Goto action evaluated at the "classical" world sheet $\bar{x}_p^\mu(\xi)$,

$$S_{cl}=-\sigma\int d^2\xi\sqrt{-\bar{g}}. \tag{7.19}$$

The semiclassical correction S_{fluc} due to the transverse string fluctuations is

$$S_{\mathrm{fluc}}=\frac{i}{2}\mathrm{Tr}\ln[-\nabla^2\delta_{AB}-\bar{\mathcal{K}}_{ab}^A\bar{\mathcal{K}}^{Bab}]. \tag{7.20}$$

To summarize, we have integrated out the string fluctuations, and reduced the problem to the evaluation of the determinant in Eq. (7.15). This is a quantum mechanical scattering problem in the background of the solution of the classical equation (7.10), with appropriate boundary conditions. In the next section, we describe how to evaluate this determinant.

VIII. REGULARIZATION OF STRING INTEGRALS AND THE ROLE OF THE LÜSCHER TERM

The argument of the logarithm in Eq. (7.20) is the inverse propagator for fluctuations on the string. This inverse propagator can also be obtained by direct variation of the Nambu-Goto action with respect to any transverse coordinates $x_\perp^A$,

$$\frac{\partial^2 S}{\partial x_\perp^A\partial x_\perp^B}=-\nabla^2\delta_{AB}-\bar{\mathcal{K}}_{ab}^A\bar{\mathcal{K}}^{Bab}, \tag{8.1}$$

up to an overall normalization factor. In fact, the correction (7.20) to the effective action has already been studied by Lüscher, Symanzik, and Weisz (LSW) [11] in the case of a straight string with fixed ends. We describe their results, which we will use in evaluating Eq. (7.20). LSW used Pauli-Villars regularization to obtain a regulated form S_{reg} of the trace in Eq. (7.20),

$$S_{\mathrm{reg}}=-\int_0^\infty\frac{dt}{t}\left(1+\sum_j\epsilon_j e^{-t\mathcal{M}_j^2}\right)\mathrm{Tr}\,e^{-t(-\nabla^2\delta_{AB}-\bar{\mathcal{K}}_{ab}^A\bar{\mathcal{K}}^{Bab})}. \tag{8.2}$$

The $\mathcal{M}_j$ are the masses of the regulators, and the ϵ_j are suitably chosen coefficients. The Laplacian in Eq. (8.2) has been Wick rotated from Minkowski to Euclidean space.

The regulated quantity $S_{\rm reg}$ is separated into a divergent part $S_{\rm div}$ and a finite part $S_{\rm PV}$,

$$S_{\rm reg}=S_{\rm div}(\mathcal{M}_j,\epsilon_j)+S_{\rm PV}. \tag{8.3}$$

LSW evaluated the divergent part $S_{\rm div}(\mathcal{M}_j,\epsilon_j)$, and obtained terms which are quadratically, linearly, and logarithmically divergent in the cutoffs $\mathcal{M}_j$. The quadratic term is a renormalization of the string tension, the linear term is a renormalization of the quark masses, and the logarithmically divergent term is proportional to the integral over all space of the scalar curvature $\mathcal{R}$ of the string world sheet.

LSW also obtained a formal expression for the finite part $S_{\rm PV}$. They evaluated this expression only for the case of a straight string of length R with fixed ends, and calculated a correction $V_{\rm Lüscher}$ to the static potential

$$V_{\rm Lüscher}=-\lim_{T\to\infty}\frac{1}{T}S_{\rm PV}=-\frac{\pi}{12R}. \tag{8.4}$$

We are interested in calculating $S_{\rm fluc}$ for rotating quarks, so we must evaluate $S_{\rm reg}$ for a more general surface. We break Eq. (7.20) into two parts

$$S_{\rm reg}=i\,{\rm Tr}\ln[-\nabla^2]+\frac{i}{2}{\rm Tr}\ln\left[\frac{-\nabla^2\delta_{AB}-\overline{\mathcal{K}}^A_{ab}\overline{\mathcal{K}}^{Bab}}{-\nabla^2}\right]. \tag{8.5}$$

We will evaluate the first term in Eq. (8.5) by generalizing the calculation of LSW. We will calculate the second term directly.

The first term in Eq. (8.5),

$$S_1=i\,{\rm Tr}\ln[-\nabla^2], \tag{8.6}$$

involves the Laplacian in the curved background of the classical solution $\bar{x}_p^\mu$. In the flat case studied by LSW, the Laplacian is equal to $-\partial^2$,

$$-\partial^2=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}. \tag{8.7}$$

The coordinate t is the time in the lab frame, and r is a radial coordinate which takes the values $-R_1$ and R_2 at the two ends of the string. The length of the string is $R=R_1+R_2$.

To calculate S_1 we extend the calculation of LSW to more general coordinate systems. We make a coordinate transformation $\xi\to\xi'$ to conformal coordinates, where the transformed metric $g'_{ab}=\eta_{ab}e^{\varphi}$, $\eta_{ab}={\rm diag}(-1,1)$. This transformation puts the Laplacian in a form similar to the flat sheet Laplacian (8.7), and allows us to evaluate Eq. (8.6) by extending the calculation of LSW.

To see how this works, we express e^{iS_1} as a functional integral

$$e^{iS_1}=\int\mathcal{D}f_1\mathcal{D}f_2\exp\left\{-i\int d^2\xi\sqrt{-g}g^{ab}\frac{\partial f^i}{\partial\xi^a}\frac{\partial f^i}{\partial\xi^b}\right\}. \tag{8.8}$$

The transformation to conformal coordinates ξ' gives

$$e^{iS_1}=\int\mathcal{D}f_1\mathcal{D}f_2\exp\left\{-i\int d^2\xi'\,\eta^{ab}\frac{\partial f^i}{\partial\xi'^a}\frac{\partial f^i}{\partial\xi'^b}\right\}, \tag{8.9}$$

which is of the form of $S_{\rm reg}$ treated by LSW.

We will need S_1 in the limit of large T, and hence are only interested in strings whose metric is time independent. To determine which metrics are time independent, we must choose a coordinate system. We choose coordinates r and t, where t is the time in the lab frame, and r is orthogonal to t $(g_{rt}=0)$. This guarantees that t is the physical time. From now on we consider only metrics g_{ab} which are independent of t.

In Appendix C, we show that

$$S_1=S_{1,\rm div}+S_{1,\rm finite}, \tag{8.10}$$

where $S_{1,\rm div}$ contains quark mass and string tension renormalizations. The finite part of S_1 is

$$S_{1,\rm finite}=T\frac{\pi}{12R_p}, \tag{8.11}$$

where

$$R_p=\int_{-R_1}^{R_2}dr\sqrt{\frac{\overline{g}_{rr}(r)}{-\overline{g}_{tt}(r)}}. \tag{8.12}$$

The results (8.11) and (8.12) are valid for any orthogonal coordinate system with a time independent metric.

We show in Appendix C that R_p is equal to the classical energy of the string divided by the string tension σ. We call R_p the ''proper length'' of the string. For a flat metric, where $\overline{g}_{rr}=-\overline{g}_{tt}=1$, the proper length R_p of the string reduces to the distance R between its end points.

IX. CORRECTION TO THE EFFECTIVE ACTION FOR A CURVED SHEET

In the previous section, we evaluated the finite part of S_1 for a sheet with a time independent metric using the results of LSW. In this section we evaluate the second term in Eq. (8.5),

$$S_2\equiv i\frac{T}{2}{\rm Tr}\ln\left[\frac{-\nabla^2\delta_{AB}-\overline{\mathcal{K}}^A_{ab}\overline{\mathcal{K}}^{Bab}}{-\nabla^2}\right]. \tag{9.1}$$

The trace in Eq. (9.1) is over functions of r and t.

We first make the coordinate transformation $r\to x$,

$$\frac{dx}{dr}=\sqrt{\frac{\overline{g}_{rr}(r)}{-\overline{g}_{tt}(r)}},\quad x|_{r=0}=0. \tag{9.2}$$

The coordinate x runs from $-X_1$ to X_2,

$$X_1=\int_{-R_1}^{0} dr\sqrt{\frac{\overline{g}_{rr}(r)}{-\overline{g}_{tt}(r)}},$$

$$X_2=\int_{0}^{R_2} dr\sqrt{\frac{\overline{g}_{rr}(r)}{-\overline{g}_{tt}(r)}}, \tag{9.3}$$

and $X_1+X_2=R_p$. In Appendix C, we show that the metric in the system (x,t) is conformal $(\overline{g}_{xx}=-\overline{g}_{tt},\overline{g}_{xt}=0)$. In this coordinate system, the inverse propagator for string fluctuations is

$$-\nabla^2\delta_{AB}-\overline{\mathcal{K}}^A_{ab}\overline{\mathcal{K}}^{Bab}=\frac{1}{\sqrt{-\overline{g}}}\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\delta_{AB}-\overline{\mathcal{K}}^A_{ab}\overline{\mathcal{K}}^{Bab}. \tag{9.4}$$

The string has infinite extent in time, and the curvature $\overline{\mathcal{K}}^A_{ab}\overline{\mathcal{K}}^{Bab}$ is independent of t, so we can take the Fourier transform with respect to the time coordinate. We express the trace in (9.1) over functions of t and r as an integral over a frequency ν and a trace over functions of a single variable x,

$$S_2=\frac{T}{2}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\mathrm{Tr}_x\ln\left[\frac{\left(\nu^2-\frac{\partial^2}{\partial x^2}\right)\delta_{AB}-\sqrt{-\overline{g}}\overline{\mathcal{K}}^A_{ab}\overline{\mathcal{K}}^{Bab}}{\nu^2-\frac{\partial^2}{\partial x^2}}\right]. \tag{9.5}$$

In going from Eqs. (9.1) to (9.5), we have also carried out the Wick rotation $\nu\to -i\nu$. The integration over ν gives

$$S_2=\frac{T}{2}\left[\mathrm{Tr}_x\sqrt{-\frac{\partial^2}{\partial x^2}\delta_{AB}-\sqrt{-\overline{g}}\overline{\mathcal{K}}^A_{ab}\overline{\mathcal{K}}^{Bab}}-\mathrm{Tr}\sqrt{-\frac{\partial^2}{\partial x^2}\delta_{AB}}\right]. \tag{9.6}$$

Equation (9.6) expresses S_2 as the trace of the difference of two operators. The first has the form of a Hamiltonian for a relativistic particle in the local potential $\sqrt{-\overline{g}}\overline{\mathcal{K}}^A_{ab}\overline{\mathcal{K}}^{Bab}$. The second operator has the form of a free Hamiltonian. The square roots enter because we are working with relativistic degrees of freedom.

The terms S_1 and S_2 are proportional to the time T, but are otherwise time independent. We define the ''effective Lagrangian'' of the string to be the effective action divided by the time T. The sum of Eqs. (8.11) and (9.6) gives the effective Lagrangian L_{fluc} determining the contribution of the string fluctuations to $W[\Gamma]$,

$$\begin{aligned}L_{\text{fluc}}&\equiv\lim_{T\to\infty}\frac{1}{T}(S_{1,\text{finite}}+S_2)\\&=\frac{\pi}{12R_p}+\frac{1}{2}\left(\mathrm{Tr}\sqrt{-\frac{\partial^2}{\partial x^2}\delta_{AB}-\sqrt{-\overline{g}}\overline{\mathcal{K}}^A_{ab}\overline{\mathcal{K}}^{Bab}}-\mathrm{Tr}\sqrt{-\frac{\partial^2}{\partial x^2}\delta_{AB}}\right).\end{aligned} \tag{9.7}$$

In the next section, we will evaluate L_{fluc} for a string of length R rotating with angular velocity ω.

In Appendix D, we show that, for a general sheet, S_2 is logarithmically divergent. We show that its divergent part is given by

$$S_{2,\text{div}}=\frac{T}{4}\sum_{n=1}^{MR_p/\pi}\frac{1}{\pi n}\int_{-X_1}^{X_2}dx\sqrt{-\overline{g}}\mathcal{R}, \tag{9.8}$$

where $\mathcal{R}$ is the scalar curvature,

$$\mathcal{R}=(\mathcal{K}^{Aa}_{a})^2-(\mathcal{K}^{A}_{ab})^2. \tag{9.9}$$

This result agrees, in the large time limit, with the logarithmically divergent term in the cutoff dependent part of the effective string action (C10) found by LSW.

X. EFFECTIVE LAGRANGIAN FOR ROTATING STRING

We now evaluate the effective Lagrangian of a string with boundary Γ generated by a quark-antiquark pair separated at fixed distances R_1 and R_2 from the origin, and rotating with angular velocity ω. This Lagrangian has two parts, the classical string Lagrangian and the contribution (9.7) of string fluctuations.

We evaluate the classical string Lagrangian first. The solution to the classical equations of motion (7.10) yields the classical, straight rotating string

$$\overline{x}^\mu(r,t)=t\,\hat{\mathbf{e}}_0^\mu+r\cos(\omega t)\hat{\mathbf{e}}_1^\mu l r\sin(\omega t)\hat{\mathbf{e}}_2^\mu. \tag{10.1}$$

The coordinate r is chosen so that the velocity of the string is zero when $r=0$. The coordinate r runs from $-R_1$ to R_2, and t runs from $-\infty$ to ∞. The vectors $\hat{\mathbf{e}}_1^\mu$ and $\hat{\mathbf{e}}_2^\mu$ are two orthogonal unit vectors in the plane of rotation, and $\hat{\mathbf{e}}_0^\mu$ is a unit vector in the time direction. The classical Lagrangian L_{cl}^{string} obtained from Eq. (7.19) is

$$\begin{aligned}L_{cl}^{\text{string}}&\equiv-\lim_{T\to\infty}\frac{1}{T}\sigma\int d^2\xi\sqrt{-\overline{g}}\\&=-\sigma\int_{-R_1}^{R_2}dr\sqrt{1-r^2\omega^2}\\&=-\sigma\sum_{i=1}^{2}\frac{R_i}{2}\left(\frac{\arcsin(\omega R_i)}{\omega R_i}+\sqrt{1-R_i^2\omega^2}\right).\end{aligned} \tag{10.2}$$

Next, we calculate the contribution L_{fluc} (9.7) due to string fluctuations. The metric of the sheet (10.1) is

$$\overline{g}_{tt} = -1 + r^2\overline{\omega}^2, \quad \overline{g}_{rr} = 1, \quad \overline{g}_{rt} = 0. \tag{10.3}$$

This metric is independent of t, and $\overline{g}_{rt}=0$. We make the transformation (9.2) from coordinates r and t to coordinates x and t, and find

$$x = \frac{1}{\omega}\arcsin \omega r. \tag{10.4}$$

The coordinate x runs from $-X_1$ to X_2, where

$$X_i = \frac{\arcsin(\omega R_i)}{\omega}, \tag{10.5}$$

and the proper length R_p of the string is

$$R_p = X_1 + X_2 = \sum_i \frac{\arcsin(\omega R_i)}{\omega}. \tag{10.6}$$

Using this coordinate system , we evaluate L_{fluc} in Appendix E for the case of equal quark masses, $R_1 = R_2 = R/2$:

$$L_{\text{fluc}} = \frac{v}{\arcsin v}\frac{\pi}{12R} - \frac{2v}{\pi R} \times \left[v\gamma \ln\left(\frac{MR}{2(\gamma^2-1)}\right) + v\gamma - \frac{\pi}{2}\right] - \frac{v^2\gamma}{\pi R} f(v), \tag{10.7}$$

where

$$v \equiv \frac{R}{2}\omega, \quad \gamma = \frac{1}{\sqrt{1-v^2}}, \tag{10.8}$$

and the function $f(v)$ is

$$f(v) = \int_0^\infty ds \ln\left[\frac{s^2 + 2s\coth(2sv\gamma\arcsin v) + 1}{(s+1)^2}\right]. \tag{10.9}$$

Equation (10.7) becomes the Lüscher term in the zero velocity limit.

We are interested in the large R limit, where the quark velocity is close to the speed of light. For v close to one, Eq. (10.7) becomes

$$L_{\text{fluc}} = -\frac{2}{\pi R}\gamma\left[\ln\left(\frac{MR}{2\gamma^2}\right) + 1\right] + \frac{7}{6R} + O\left(\frac{\ln\gamma}{\gamma R}\right). \tag{10.10}$$

Furthermore, for the semiclassical expansion to be valid, the theory must be weakly coupled. That is, L_{fluc} must be less than L_{cl}^{string} (10.2). For large R,

$$L_{cl}^{\text{string}} = -\frac{\pi}{4}\sigma R - \frac{\pi}{8}\sigma R\gamma^{-2} + \frac{1}{6}\sigma R\gamma^{-3} + O(\gamma^{-4}\sigma R). \tag{10.11}$$

The semiclassical expansion is valid, since, as we will see, R grows as γ^2 in the $v \to 1$ limit. In this case, the long distance limit where the effective theory is applicable is automatically the region of weak coupling.

XI. REGGE TRAJECTORIES

We calculate classical Regge trajectories for equal mass quarks by adding a quark mass term to the string Lagrangian L_{cl}^{string},

$$L_{cl} = L_{cl}^{\text{string}} - 2m\sqrt{1-v^2} = -\sigma\frac{R}{2}\left(\frac{\arcsin v}{v} + \gamma^{-1}\right) - 2m\gamma^{-1}. \tag{11.1}$$

We have used Eq. (10.2) with $R_1 = R_2 = R/2$. The quark velocity is $v = \omega R/2$, and $\gamma = 1/\sqrt{1-v^2}$ is the quark boost factor. The Lagrangian (11.1) is a function of R and ω,

$$L_{cl} = L_{cl}(R,\omega). \tag{11.2}$$

The angular momentum of the meson is obtained by varying the Lagrangian with respect to the angular velocity

$$J = \frac{\partial L_{cl}}{\partial\omega} = \sigma\frac{R^2}{4v}\left(\frac{\arcsin v}{v} - \gamma^{-1}\right) + mRv\gamma^{-1}. \tag{11.3}$$

The meson energy is given by the Hamiltonian

$$E = \omega\frac{\partial L_{cl}}{\partial\omega} - L_{cl} = \sigma R\frac{\arcsin v}{v} + 2m\gamma. \tag{11.4}$$

The classical equation of motion

$$\frac{\partial L}{\partial R} = 0 \tag{11.5}$$

for the quarks determines R as a function of ω,

$$\sigma\frac{R}{2} = m(\gamma^2 - 1). \tag{11.6}$$

Equation (11.6) shows that R is proportional to γ^2 for large γ. Expanding Eqs. (11.3) and (11.4) in the large R limit, where the quark velocity v goes to one, yields the result

$$\frac{J}{E_{cl}^2} = \frac{1}{2\pi\sigma}\left(1 - \frac{8}{3\pi}\gamma^{-3} + O(\gamma^{-5})\right). \tag{11.7}$$

The first term in Eq. (11.7) is the classical formula for the slope of a Regge trajectory. The second term is the leading correction for nonzero classical quark mass, where $\gamma^{-1} = \sqrt{1-v^2} \neq 0$.

We now calculate the correction to the energy obtained by considering L_{fluc} a small perturbation to the classical Lagrangian L_{cl}. The Lagrangian

$$L(\omega) = L_{cl}(\omega) + L_{\text{fluc}}(\omega), \tag{11.8}$$

depends on only one degree of freedom, the rotation angle θ, through its time derivative $\omega=\dot{\theta}$. To first order in $L_{\rm fluc}$, the correction to the energy is minus the correction to the Lagrangian [22]

$$E(J)=[E_{cl}(\omega)-L_{\rm fluc}(\omega)]|_{\omega=\omega(J)}, \tag{11.9}$$

where ω is given as a function of J through the classical relation (11.3).

The correction (11.9) to the energy of the meson gives a correction to the slope (11.7) of a Regge trajectory

$$\frac{J}{E^2}=\frac{J}{E_{cl}^2}\frac{E_{cl}^2}{E^2}\simeq\frac{J}{E_{cl}^2}\left(1+2\frac{L_{\rm fluc}}{E}\right). \tag{11.10}$$

Using Eq. (11.7) for J/E_{cl}^2 and Eq. (10.10) for $L_{\rm fluc}$, we obtain

$$\frac{J}{E^2}=\frac{1}{2\pi\sigma}-\frac{2}{\pi^2\sigma RE}\gamma\left[\ln\left(\frac{MR}{2\gamma^2}\right)+1\right] -\frac{4}{3\pi^2\sigma}\gamma^{-3}+\frac{7}{6\pi\sigma RE}+O\left(\gamma^{-5},\frac{1}{RE\gamma}\right). \tag{11.11}$$

We write R and γ as functions of E using the definition (11.4) of E and the classical equation of motion (11.6). Because R and γ only appear in the small correction terms in the result (11.11), we only need their leading order dependence on E,

$$R\simeq\frac{2E}{\pi\sigma},\quad \gamma\simeq\sqrt{\frac{E}{\pi m}}. \tag{11.12}$$

Substituting Eq. (11.12) in Eq. (11.11) gives

$$J=\frac{E^2}{2\pi\sigma}-\sqrt{\frac{E}{\pi^3 m}}\left[\ln\left(\frac{Mm}{\sigma}\right)+1\right] -\frac{4}{3\sigma}\sqrt{\frac{m^3E}{\pi}}+\frac{7}{12}+O(E^{-1/2}). \tag{11.13}$$

The leading term is the classical Regge formula. The next term is the leading correction due to string fluctuations. The third term is a nonzero quark mass correction. The fourth term is another correction due to string fluctuations.

Equation (11.13) gives a meson Regge trajectory $J(E^2)$. We used values $\frac{4}{3}\alpha_s=0.25$ and $\sigma=(455\ \text{MeV})^2$ obtained from the Cornell fits of heavy quark potentials [18]. This gives $M=g\phi_0=\sqrt{3\sigma/4\alpha_s}=910$ MeV. The only other parameter is the quark mass m. In Fig. 3, we plot J versus the square of the energy (11.9) for quark masses of 30, 100, and 300 MeV. For comparison, we also plot the classical formula $J=E^2/2\pi\sigma$. The points [23] plotted on the graph are the ρ^1, A^2, ρ^3, A^4, ρ^5, and A^6 mesons. The crosses are the ω^1, f^2, ω^3, f^4, and f^6. We have added one to the value of the angular momentum J in Fig. 3 to account for the contribution of the spin of the quarks.

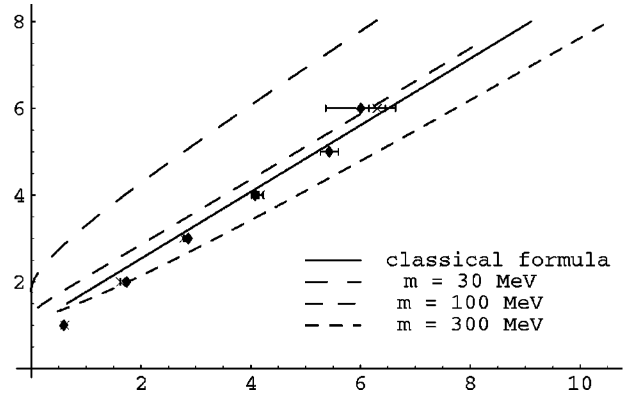

FIG. 3. J versus E^2.

We have chosen a range of values for the quark masses in Fig. 3 in order to give a qualitative picture of the dependence of the Regge trajectory on the quark mass. Since Eq. (11.13) does not include the contribution of quark fluctuations to the Regge trajectory, this formula is incomplete. We are now in the process of including the quark degrees of freedom in the functional integral (6.14). The boundary Γ of the sheet $\tilde{x}^\mu$ becomes dynamical, and couples to the string fluctuations. It is clear that a calculation of the contribution of these degrees of freedom is essential to understanding why the classical formula for Regge trajectories works so well.

XII. SUMMARY AND CONCLUSIONS

The primary results of the paper are Eqs. (6.14) and (11.13). We have expressed the path integral $W[\Gamma]$ (3.5) of a renormalizable quantum field theory having classical vortex solutions as the path integral formulation of an effective string theory of vortices (6.14). This theory describes the two transverse fluctuations of the vortex at scales larger than the inverse mass of the lightest particle in the field theory. Our method is applicable to any field theory containing vortex solutions.

Using the string representation of $W[\Gamma]$, we carried out a semiclassical expansion of the effective action $-i\log W[\Gamma]$ about a classical solution of the effective string theory. We calculated the contribution of these string fluctuations explicitly for the case where the world line Γ is generated by the trajectory of a quark-antiquark pair separated by a distance R. We are now calculating the contribution to the effective action $-i\log W[\Gamma]$ due to the quantum fluctuations of the boundary.

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APPENDIX A: NOTATION AND THE CURVATURE OF THE VORTEX

We describe the string world sheet by the function $\tilde{x}^\mu(\xi)$ of the coordinates ξ. The physics of the vortex should be independent of the coordinate system we choose, so we require the theory to be invariant under a reparametrization of the coordinates $\xi \to \tilde{\xi}(\xi)$. The tangent vectors to the vortex world sheet are defined by taking derivatives of $\tilde{x}^\mu(\xi)$,

$$t_a^\mu(\xi) \equiv \partial_a \tilde{x}^\mu(\xi), \tag{A1}$$

where $\partial_a = \partial/\partial\xi^a$ is a partial derivative with respect to one of the vortex coordinates. The induced metric on the world sheet $\tilde{x}^\mu(\xi)$ is

$$g_{ab} \equiv t_a^\mu t_{\mu b}. \tag{A2}$$

It is also convenient to define the square root of the determinant of the metric

$$\sqrt{-g} = \sqrt{-\tfrac{1}{2}\epsilon^{ab}\epsilon^{cd} g_{ac} g_{bd}}. \tag{A3}$$

We use the t_a^μ to define an antisymmetric tensor which describes the orientation of the string world sheet

$$t^{\mu\nu} \equiv \frac{\epsilon^{ab}}{\sqrt{-g}} t_a^\mu t_b^\nu. \tag{A4}$$

This quantity was defined by Polyakov [21]. It is the projection of the antisymmetric tensor ϵ^{ab} into the space of four-dimensional tensors. This tensor defines the orientation of the two-dimensional vortex world sheet in four space. The quantity (A4) is also independent of the coordinate parametrization of the world sheet $\tilde{x}^\mu(\xi)$.

We now describe the curvature of the vortex world sheet. We do this by taking covariant derivatives of the tangent vectors. The covariant derivative of t_a^μ is

$$\nabla_b t_a^\mu = \partial_b t_\mu^a - \Gamma_{ab}^c t_c^\mu, \tag{A5}$$

where the Γ_{ab}^c are Christoffel symbols

$$\Gamma_{ab}^c = \tfrac{1}{2} g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}). \tag{A6}$$

The covariant derivatives of the tangent vectors are orthogonal to the world sheet,

$$t_{\mu c}\nabla_a t_b^\mu = t_{\mu c}(\partial_a t_b^\mu - \Gamma_{ab}^d t_d^\mu) = 0. \tag{A7}$$

This identity is derived using the definition (A6) of the Christoffel symbols, the definition (A2) of the metric, and the relationship between derivatives of different t_a^μ,

$$\partial_a t_b^\mu = \partial_b t_a^\mu = \partial_a \partial_b \tilde{x}^\mu. \tag{A8}$$

The covariant derivatives of the tangent vectors are normal to the string world sheet. We therefore define a basis of normal vectors n_A^μ, which satisfy the conditions

$$n_{\mu A} t_a^\mu = 0, \quad n_{\mu A} n_B^\mu = \delta_{AB}. \tag{A9}$$

The n_A^μ are an orthonormal basis for the vectors normal to the world sheet. Equation (A7) implies that

$$\nabla_a t_b^\mu = n_A^\mu \mathcal{K}_{ab}^A \tag{A10}$$

for some tensor $\mathcal{K}_{ab}^A$. The tensor $\mathcal{K}_{ab}^A$ is called the extrinsic curvature tensor of the string world sheet. With the definition (A10), the curvature tensor $\mathcal{K}_{ab}^A$ is

$$\mathcal{K}_{ab}^A \equiv n_\mu^A \nabla_a t_b^\mu = n_\mu^A \partial_a \partial_b x^\mu. \tag{A11}$$

It is symmetric in the indices a and b due to the relationship (A8) between derivatives of tangent vectors.

The extrinsic curvature of the string world sheet can also be described using derivatives of the normal vectors. The orthogonality of the $t_{\mu a}$ and the n_A^μ implies

$$t_{\mu b}\partial_a n_A^\mu = -\mathcal{K}_{ab}^A. \tag{A12}$$

Therefore, the derivatives of the normal vectors can be written as

$$\partial_a n_A^\mu = -t^{\mu b}\mathcal{K}_{ab}^A + n_B^\mu \mathcal{A}_a^{AB}. \tag{A13}$$

The tensor $\mathcal{A}_a^{AB}$ is called the torsion, and it describes the twisting of the basis of normal vectors as we move along the world sheet. The torsion depends on our choice of the n_A^μ, so we will choose them so that the torsion is zero. This is done by requiring that the n_A^μ satisfy the differential equation

$$\partial_a n_A^\mu = n_A^\nu(\nabla_a t_{\nu b}) t^{\mu b}. \tag{A14}$$

Equation (A14) is equivalent to the statement $\mathcal{A}_b^{AB} = 0$. It is consistent with the conditions (A9) which define the normal vectors. As long as the normal vectors have an orthonormal basis at one point, Eq. (A14) guarantees they will be orthonormal in a neighborhood of that point. Therefore, it is always possible to find a local, orthonormal, torsion free basis for the normal vectors.

There is one additional property of the normal vectors we will use. The antisymmetric combination of the normal vectors is (with proper ordering) equal to the dual of the world sheet orientation tensor $t^{\mu\nu}$ (A4)

$$\epsilon^{AB} n_A^\mu n_B^\nu = \tilde{t}^{\mu\nu}, \tag{A15}$$

where

$$\tilde{t}^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta} t_{\alpha\beta}. \tag{A16}$$

The relationship (A15) can be understood by noting that any antisymmetric tensor is of the form

$$A^{\mu\nu}(\xi) = T(\xi) t^{\mu\nu} + N(\xi)\epsilon^{AB} n_A^\mu n_B^\nu + M^{Aa}(n_A^\mu t_a^\nu - n_A^\nu t_a^\mu). \tag{A17}$$

The tensor $\tilde{t}^{\mu\nu}$ is orthogonal to the t_a^μ, so it must be proportional to $\epsilon^{AB} n_A^\mu n_B^\nu$. Squaring both of these tensors gives

$$(\tilde{t}^{\mu\nu})^2 = (\epsilon^{AB} n_A^\mu n_B^\nu)^2 = 2. \tag{A18}$$

Therefore, $\tilde{t}^{\mu\nu}$ and $\epsilon^{AB}n_A^\mu n_B^\nu$ are equal up to an overall sign, which is fixed by choosing an appropriate ordering for the normal vectors.

APPENDIX B: DISCUSSION OF $J_\perp$

The Jacobian $J_\perp$ (5.5) is a Faddeev-Popov determinant, because fixing the position of the string in the field integrals is analogous to fixing a gauge in a gauge theory. In the string action, we fix the degrees of freedom which generate the transformation

$$\tilde{x}^\mu(\xi)\rightarrow\tilde{x}^\mu(\xi)+\delta x_\perp^A(\xi)n_{\mu A}(\xi) \tag{B1}$$

which displaces the vortex.

The Jacobian $J_\perp$ is analogous to the Faddeev-Popov determinant in a gauge theory. In a gauge theory, where the δ function fixes the symmetry generated by the transformation

$$A_\mu\rightarrow UA_\mu U^{-1}+U\partial_\mu U^{-1}, \tag{B2}$$

the Faddeev-Popov determinant appears as a normalization for the δ function [24]

$$Z_{\text{gauge}}=\int \mathcal{D}A^\mu\,\delta[F(A^\mu)]\Delta_{\text{FP}}e^{-S}, \tag{B3}$$

where

$$\Delta_{\text{FP}}^{-1}=\int \mathcal{D}U\,\delta[F(A^\mu)]. \tag{B4}$$

The Wilson loop (B3) is analogous to our Eq. (5.2) for the effective action. The determinant Δ_{FP} is analogous to $J_\perp$.

In the gauge theory, the Faddeev-Popov method is used to remove nonphysical degrees of freedom from the problem. The δ function is inserted in the path integral part to fix the fields in some particular gauge. This creates an integral over all gauges which appears as a normalization factor, and is removed. The δ function in Eq. (5.2), on the other hand, fixes the position of the vortex sheet, which is a physical degree of freedom.

APPENDIX C: EVALUATION OF S_1

We want to evaluate the term

$$S_1=i\,\mathrm{Tr}\ln[-\nabla^2], \tag{C1}$$

in the effective action for a general string world sheet. We work in coordinates r and t, such that t is the time in the lab frame, and r is orthogonal to t ($g_{rt}=0$). In these coordinates, the functional integral (8.8) for e^{iS_1} takes the form

$$e^{iS_1}=\int \mathcal{D}f_1\mathcal{D}f_2\exp\left\{-i\int dt\int_{-R_1}^{R_2}dr\sqrt{-g_{tt}g_{rr}}\times\left[g^{tt}\left(\frac{\partial f^i}{\partial t}\right)^2+g^{rr}\left(\frac{\partial f^i}{\partial r}\right)^2\right]\right\}. \tag{C2}$$

We consider the case where the metric is independent of t, and we make the coordinate transformation $r\rightarrow x$ defined by

$$\frac{dx}{dr}=\sqrt{\frac{g_{rr}}{-g_{tt}}},\quad x|_{r=0}=0. \tag{C3}$$

The coordinate x runs from $-X_1$ to X_2,

$$X_1=\int_{-R_1}^{0}dr\sqrt{\frac{g_{rr}}{-g_{tt}}},\qquad X_2=\int_0^{R_2}dr\sqrt{\frac{g_{rr}}{-g_{tt}}}. \tag{C4}$$

In these coordinates, the length of the string is $X_1+X_2=R_p$, the proper length of the string

$$R_p=\int_{-R_1}^{R_2}dr\sqrt{\frac{g_{rr}}{-g_{tt}}}. \tag{C5}$$

In the coordinate system (x,t), the metric is conformal

$$g_{xx}=\left(\frac{dx}{dr}\right)^{-2}g_{rr}=-g_{tt},\qquad g_{xt}=\left(\frac{dx}{dr}\right)^{-1}g_{rt}=0, \tag{C6}$$

and

$$e^{iS_1}=\int \mathcal{D}f_1\mathcal{D}f_2\exp\left\{\int dt\int_{-X_1}^{X_2}dx\left[-\left(\frac{\partial f^i}{\partial t}\right)^2+\left(\frac{\partial f^i}{\partial x}\right)^2\right]\right\}. \tag{C7}$$

We evaluate Eq. (C7) in a manner analogous to our treatment of S_2 in Sec. IX. We Fourier transform in both space and time, introducing variables ν and $k_n=\pi n/R_p$. This transformation puts the action in Eq. (C7) in a diagonal form. Doing the f_1 and f_2 integrations gives

$$S_1=\sum_{\nu,n}\ln\left[\nu^2+\left(\frac{\pi n}{R_p}\right)^2\right], \tag{C8}$$

where we have Wick rotated $\nu\rightarrow -i\nu$ to avoid the poles at $\nu=\pm\pi n/R_p$.

The length R_p is just the classical string energy E_{cl} divided by the string tension σ, since E_{cl} is

$$E_{cl}=-\sigma\frac{1}{T}\int d^2\xi\frac{\partial}{\partial\dot{x}^0}\sqrt{-g}=\sigma\frac{1}{T}\int dt\int_{-R_1}^{R_2}dr\sqrt{\frac{g_{rr}}{-g_{tt}}}=\sigma R_p. \tag{C9}$$

The quantity $R_p = E_{cl}/\sigma$ is the length of the string measured in local comoving coordinates, which are at rest with respect to the string. This is different from the string length R in the laboratory frame.

We will regulate S_1 using the results of LSW. Their result for $S_{\rm reg}$ is the following:

$$S_{\rm reg} = -\frac{d-2}{4\pi}A(\mathcal{C})\sum_j \epsilon_j \mathcal{M}_j^2 \ln \mathcal{M}_j^2 - \frac{d-2}{4}L(\mathcal{C})\sum_j \epsilon_j \mathcal{M}_j + \left(\frac{d-2}{6} - \frac{1}{4\pi}\int d^2\xi\sqrt{-g}\mathcal{R}\right)\sum_j \epsilon_j \ln \mathcal{M}_j^2 + S_{\rm PV}, \tag{C10}$$

where d is the number of dimensions, $A(\mathcal{C})$ is the area of the string world sheet, $L(\mathcal{C})$ is the length of its boundary, and $\mathcal{R}$ is the scalar curvature of the sheet. The $\mathcal{M}_j$ are regulator masses, and the ϵ_j are appropriate coefficients. The final term, $S_{\rm PV}$, is finite in the limit where the $\mathcal{M}_j \to \infty$.

LSW evaluate the finite term $S_{\rm PV}$ only for a straight string of length R with fixed ends. In this case, the area of the sheet $A(\mathcal{C}) = RT$, the length of the boundary $L(\mathcal{C}) = 2T$, and the curvature of the sheet is zero. They then obtained the explicit contribution to the heavy quark potential,

$$-\lim_{T\to\infty}\frac{1}{T}S_{\rm reg} = \frac{1}{2\pi}R\sum_j \epsilon_j \mathcal{M}_j^2 \ln \mathcal{M}_j^2 + \sum_j \epsilon_j \mathcal{M}_j - \frac{\pi}{12R}. \tag{C11}$$

The first term in Eq. (C11) renormalizes the string tension. The second renormalizes the quark mass. The third is the well known Lüscher term in the heavy quark potential.

Since the extrinsic curvature vanishes for a flat sheet, we can identify the result (C11) with our expression (C8) for S_1, with R_p replaced by R:

$$-\lim_{T\to\infty}\frac{1}{T}\sum_{\nu,n}\ln\left[\nu^2 + \left(\frac{\pi n}{R}\right)^2\right] = \frac{1}{2\pi}R\sum_j \epsilon_j \mathcal{M}_j^2 \ln \mathcal{M}_j^2 + \sum_j \epsilon_j \mathcal{M}_j - \frac{\pi}{12R}. \tag{C12}$$

Equation (C12) tells us how to regulate S_1. Replacing R by R_p in Eq. (C12) gives the regulated form of S_1:

$$-\lim_{T\to\infty}\frac{1}{T}S_1 = \frac{1}{2\pi}R_p\sum_j \epsilon_j \mathcal{M}_j^2 \ln \mathcal{M}_j + \sum_j \epsilon_j \mathcal{M}_j - \frac{\pi}{12R_p}. \tag{C13}$$

The first term in Eq. (C13) is still a string tension renormalization, since both the string tension contribution to the energy (C9) and the first term in Eq. (C13) are proportional to R_p. The second term in Eq. (C13) is, as before, a renormalization of the quark mass. The finite part of the contribution of S_1 to the action is

$$S_1|_{\text{finite part}} = T\frac{\pi}{12R_p}. \tag{C14}$$

This is the result stated in Sec. VIII. The result (C14) is the Lüscher term, with the distance R between the quarks replaced by the proper length R_p of the string. Our result is Eq. (C5), the derivation of R_p.

APPENDIX D: CUTOFF DEPENDENCE OF S_2

In this appendix, we show that the divergent part of S_2 for a general sheet is proportional to the integral of the scalar curvature $\mathcal{R}$. This agrees with the logarithmic divergence (C10) derived by LSW by other means. To obtain the divergent part of S_2, we carry out a Fourier transform with respect to the variable x. For functions defined on the interval $-X_1 < x < X_2$, the δ function can be expressed as a sum of sines

$$\delta(x-x') = \frac{2}{R_p}\sum_{n=1}^{\infty}\sin[k_n(x+X_1)]\sin[k_n(x'+X_1)], \tag{D1}$$

where $k_n = \pi n/R_p$. The Fourier transform of an operator of the form $-\partial^2/\partial x^2 + U(x)$ can then be written

$$\left\langle k_m\left| -\frac{\partial^2}{\partial x^2} + U(x)\right| k_n\right\rangle = \frac{2}{R_p}\int_{-X_1}^{X_2} dx\,\sin[k_m(x+X_1)]\left(-\frac{\partial^2}{\partial x^2} + U(x)\right)\sin[k_n(x+X_1)]$$
$$= k_n^2\delta_{n,m} + \frac{2}{R_p}\int_{-X_1}^{X_2} dx\,\sin[k_m(x+X_1)]\sin[k_n(x+X_1)]U(x). \tag{D2}$$

Using the formula (D2) to evaluate Eq. (9.6) gives

$$S_2 = \frac{T}{2}\mathrm{Tr}_n\left[\sqrt{k_n^2\delta_{n,m}\delta_{AB} - \frac{2}{R_p}\int_{-X_1}^{X_2} dx\,\sin[k_m(x+X_1)]\sin[k_n(x+X_1)]\sqrt{-g}\bar{\mathcal{K}}^A_{ab}\bar{\mathcal{K}}^{Bab}} - k_n\delta_{n,m}\delta_{AB}\right]. \tag{D3}$$

The trace is over indices A,B which run from 1 to 2, and indices n,m which run from 1 to ∞. The trace is cutoff at $k_n = M$, the mass of the vector particle in the original field theory.

We expand Eq. (D3) for large k_n and obtain the cutoff dependent part of S_2,

$$S_2 = -\frac{T}{4}\sum_{n=1}^{MR_p/\pi}\frac{1}{R_p k_n}\int_{-X_1}^{X_2} dx\,\sqrt{-g}(\overline{\mathcal{K}}^A_{ab})^2 + \text{finite}. \tag{D4}$$

The term $(\overline{\mathcal{K}}^A_{ab})^2$ is equal to minus the scalar curvature $\mathcal{R}$,

$$\mathcal{R} = (\overline{\mathcal{K}}^{Aa}_a)^2 - (\overline{\mathcal{K}}^A_{ab})^2, \tag{D5}$$

since the equation of motion (7.10) implies $\overline{\mathcal{K}}^{Aa}_a = 0$. The cutoff dependent part of S_2 is therefore

$$S_2 = \frac{T}{4}\sum_{n=1}^{MR_p/\pi}\frac{1}{\pi n}\int_{-X_1}^{X_2} dx\,\sqrt{-g}\mathcal{R} + \text{finite}. \tag{D6}$$

Equation (D6) agrees with the result of LSW for the leading semiclassical logarithmic divergence.

APPENDIX E: EVALUATION OF S_2

We want to evaluate S_2,

$$S_2 = \frac{T}{2}\left(\mathrm{Tr}\sqrt{-\frac{\partial^2}{\partial x^2}\delta_{AB} - \sqrt{-g}\mathcal{K}^A_{ab}\mathcal{K}^{Bab}} - \mathrm{Tr}\sqrt{-\frac{\partial^2}{\partial x^2}\delta_{AB}}\right) \tag{E1}$$

for the fluctuations about a straight string of length R rotating with angular velocity ω. To evaluate this, we must determine the value of the extrinsic curvature $\mathcal{K}^A_{ab}$. The definition of the extrinsic curvature is

$$\mathcal{K}^A_{ab} = n^A_\mu \partial_a \partial_b x^\mu. \tag{E2}$$

The string x^μ is

$$x^\mu(x,t) = t\hat{\mathbf{e}}^\mu_0 + \frac{1}{\omega}\sin(\omega x)[\cos(\omega t)\hat{\mathbf{e}}^\mu_1 + \sin(\omega t)\hat{\mathbf{e}}^\mu_2]. \tag{E3}$$

The $\hat{\mathbf{e}}^\mu_i$ are a basis of orthonormal unit vectors in Minkowski space. The n^μ_A are a basis for the vectors normal to x^μ. We choose the basis

$$n^\mu_1 = \hat{\mathbf{e}}^\mu_3,$$

$$n^\mu_2 = \tan(\omega x)\hat{\mathbf{e}}^\mu_0 + \sec(\omega x)[-\sin(\omega t)\hat{\mathbf{e}}^\mu_1 + \cos(\omega t)\hat{\mathbf{e}}^\mu_2]. \tag{E4}$$

With this choice for the n^μ_A, $\mathcal{K}^1_{ab}$ is zero, because the $\hat{\mathbf{e}}^\mu_3$ component of x^μ is zero. The only nonzero component of $\mathcal{K}^A_{ab}\mathcal{K}^{Bab}$ is

$$\sqrt{-g}(\mathcal{K}^2_{ab})^2 = -2\omega^2\sec^2\omega x. \tag{E5}$$

Now that we know what $\mathcal{K}^A_{ab}$ is, we can evaluate Eq. (E1). Inserting Eq. (E5) into Eq. (E1) gives

$$S_2 = \frac{T}{2}\left(\mathrm{Tr}\sqrt{-\frac{\partial^2}{\partial x^2} + 2\omega^2\sec^2\omega x} - \mathrm{Tr}\sqrt{-\frac{\partial^2}{\partial x^2}}\right). \tag{E6}$$

The traces in Eq. (E6) are defined as sums over the eigenvalues of the given operators. Replacing the traces with explicit sums gives

$$S_2 = \frac{T}{2}\sum_{n=1}^{\Lambda R_p/\pi}\left(\sqrt{\lambda_n} - \frac{\pi n}{R_p}\right), \tag{E7}$$

where

$$R_p = \frac{\arcsin v}{v}R. \tag{E8}$$

The eigenvalues λ_n are determined by the eigenfunction equation

$$\left(-\frac{\partial^2}{\partial x^2} + 2\omega^2\sec^2\omega x\right)\psi_n(x) = \lambda_n\psi_n(x), \tag{E9}$$

with the boundary conditions $\psi_n(\pm R_p/2) = 0$. The difference between the traces in Eq. (E6) is logarithmically dependent on the cutoff Λ (the mass of the dual gluon).

Equation (E9) has the form of the Schrödinger equation, with the potential $2\omega^2\sec^2\omega x$. This potential is an analytic continuation of the potential $2\omega^2\,\mathrm{sech}^2\omega x$, whose eigenfunctions can be expressed terms of hypergeometric functions [25]. Using this result, we find the eigenfunctions

$$\psi_n(x) = \begin{cases} \sqrt{\lambda_n}\cos(\sqrt{\lambda_n}x) + \omega\tan\omega x\sin(\sqrt{\lambda_n}x) & \text{for } n \text{ odd},\\ \sqrt{\lambda_n}\sin(\sqrt{\lambda_n}x) - \omega\tan\omega x\cos(\sqrt{\lambda_n}x) & \text{for } n \text{ even}. \end{cases} \tag{E10}$$

The eigenvalues λ_n are

$$\lambda_n = \left(\frac{(\pi n + 2\alpha_n)v}{R\arcsin v}\right)^2, \tag{E11}$$

where α_n satisfies the transcendental equation

$$\frac{\frac{\pi}{2}n + \alpha_n}{\arcsin v} = \frac{v}{\sqrt{1-v^2}}\cot\alpha_n, \tag{E12}$$

and $0<\alpha_n<\pi/2$. There is no $n=0$ eigenvalue, despite the fact that $\alpha_0=\arcsin v$ satisfies Eq. (E12), because the corresponding eigenvalue $\lambda_0=\omega^2$ makes ψ_n zero everywhere.

We will carry out the sum (E7),

$$S_2=\frac{T}{2}\sum_{n=1}^{\Lambda R_p/\pi}\left(\sqrt{\lambda_n}-\frac{\pi n}{R_p}\right), \tag{E13}$$

by converting it to a contour integral. We will find a function $F_\lambda(z)$ which has zeros whenever $z=\pm\sqrt{\lambda_n}$. We will find another function $F_{R_p}(z)$ which has zeros whenever $z=\pm\pi n/R_p$. We will then define a function $F_{\rm int}(z)$,

$$F_{\rm int}(z)=\frac{d\ln F_\lambda(z)}{dz}-\frac{d\ln F_{R_p}(z)}{dz}. \tag{E14}$$

The function $F_{\rm int}(z)$ has poles of residue 1 when $z=\pm\sqrt{\lambda_n}$ and poles of residue -1 when $z=\pm\pi n/R_p$. We then rewrite the sum (E13) as a contour integral,

$$S_2=\frac{T}{4\pi i}\int dz z F_{\rm int}(z). \tag{E15}$$

The contour of the integral (E15) lies along the imaginary axis, and on a semicircle at $|z|=\Lambda$ with the real part of z positive.

To write S_2 as the contour integral (E15), we need to find the functions $F_\lambda(z)$ and $F_{R_p}(z)$. The function $F_{R_p}(z)$ is

$$F_{R_p}(z)=\sin(R_p z), \tag{E16}$$

which is zero for $z=\pm\pi n/R_p$. We find $F_\lambda(z)$ by recalling that the eigenfunctions (E10) vanish at $x=\pm R_p/2$. Therefore,

$$F_{\rm odd}(z)=z\cos\left(\frac{R_p}{2}z\right)+\omega\tan\left(\frac{R_p}{2}\omega\right)\sin\left(\frac{R_p}{2}z\right), \tag{E17}$$

has zeros at $z=\sqrt{\lambda_n}$ for n odd, and

$$F_{\rm even}(z)=z\sin\left(\frac{R_p}{2}z\right)-\omega\tan\left(\frac{R_p}{2}\omega\right)\cos\left(\frac{R_p}{2}z\right), \tag{E18}$$

has zeros at $z=\sqrt{\lambda_n}$ for n even. Thus,

$$\begin{aligned}F_\lambda(z)&=\frac{F_{\rm odd}(z)F_{\rm even}(z)}{z^2-\omega^2}\\&=\frac{1}{2}\,\frac{z^2\sin(R_pz)-2\omega z\tan\left(\frac{R_p}{2}\omega\right)\cos(R_pz)-\omega^2\tan^2\left(\frac{R_p}{2}\omega\right)\sin(R_pz)}{z^2-\omega^2}.\end{aligned} \tag{E19}$$

The factor $(z^2-\omega^2)^{-1}$ removes nonphysical zeros which appear because $F_{\rm even}(\pm\omega)=0$. These zeros correspond to the $n=0$ "eigenfunction" which is zero everywhere for $\lambda_0=\omega^2$. The function $F_{\rm int}(z)$ is

$$F_{\rm int}(z)=\frac{d}{dz}\ln\left[\frac{z^2-2\omega z\tan\left(\frac{R_p}{2}\omega\right)\cot(R_pz)-\omega^2\tan^2\left(\frac{R_p}{2}\omega\right)}{z^2-\omega^2}\right]. \tag{E20}$$

Inserting Eq. (E20) in Eq. (E15) and integrating by parts gives

$$S_2=-\frac{T}{4\pi i}\int dz\ln\left[\frac{z^2-2\omega z\tan\left(\frac{R_p}{2}\omega\right)\cot(R_pz)-\omega^2\tan^2\left(\frac{R_p}{2}\omega\right)}{z^2-\omega^2}\right]. \tag{E21}$$

Now, instead of having poles at $z=\sqrt{\lambda_n}$ and $z=\pi n/R_p$, the integrand has branch points at these points. The branch cuts run from $\sqrt{\lambda_n}$ to $\pi n/R_p$ for all n along the real axis. There is one branch cut for each value of n. Since the contour either includes both $\sqrt{\lambda_n}$ and $\pi n/R_p$ or excludes both these points, none of these branch cuts cross the contour of integration. There are no branch points at $z=\pm\omega$, since both the numerator and denominator vanish there. None of the branch cuts crosses the contour, so the contour is still closed, and the integration by parts does not produce a boundary term.

The contour of the integral (E21) lies on the imaginary axis and a semicircle passing through positive real infinity. We rewrite Eq. (E21) as two integrals over the different pieces of the contour. For large values of the cutoff Λ, the action S_2 is

$$S_2=-\frac{T}{4\pi}\int_{-\Lambda}^{\Lambda}dy\,\ln\left[\frac{y^2+2\omega y\tan\left(\frac{R_p}{2}\omega\right)\coth(R_p y)+\omega^2\tan^2\left(\frac{R_p}{2}\omega\right)}{y^2+\omega^2}\right]$$
$$-T\frac{1}{4\pi}\int_{-\pi/2}^{\pi/2}d\theta\Lambda e^{i\theta}\left[-2\frac{\omega}{\Lambda}e^{-i\theta}\tan\left(\frac{R_p}{2}\omega\right)\cot(R_p\Lambda e^{i\theta})+O(\Lambda^{-2})\right]. \quad \text{(E22)}$$

For large Λ, $\cot(R_p\Lambda e^{i\theta})$ is proportional to the sign of θ, so the θ integral vanishes. The y integral is symmetric under $y\to -y$. Changing variables to $s=(y/\omega)\cot(\omega R_p/2)$ gives

$$S_2=-T\frac{\omega}{2\pi}\tan\left(\frac{R_p}{2}\omega\right)\int_0^{(\Lambda/\omega)\cot(\omega R_p/2)}ds\,\ln\left[\frac{s^2+2s\coth\left[R_p\omega\tan\left(\frac{R_p}{2}\omega\right)s\right]+1}{s^2+\cot^2\left(\frac{R_p}{2}\omega\right)}\right]. \quad \text{(E23)}$$

The numerator in Eq. (E23) is approximately $(s+1)^2$ for large values of s. We use this fact to extract the divergent part of Eq. (E23), getting

$$S_2=-T\frac{2v}{\pi R}\left[v\gamma\ln\left(\frac{\Lambda R}{2v^2\gamma}\right)+v\gamma-\frac{\pi}{2}\right]-T\frac{v^2\gamma}{\pi R}f(v). \quad \text{(E24)}$$

We have replaced R_p with its definition

$$R_p=\frac{2}{\omega}\arcsin v. \quad \text{(E25)}$$

$\gamma=(1-v^2)^{-1/2}$ is the quark boost factor. The function $f(v)$ contains the rest of the integral

$$f(v)\equiv\int_0^{\infty}ds\,\ln\left[\frac{s^2+2s\coth(2sv\gamma\arcsin v)+1}{(s+1)^2}\right]. \quad \text{(E26)}$$

For $v\to 1$, the asymptotic value of $f(v)$ is

$$f(v)\simeq\frac{1}{6\gamma^2}. \quad \text{(E27)}$$

The cutoff Λ used in Eq. (E24) is the cutoff in the x coordinate. We must express Λ in terms of the cutoff M for the r coordinate, which measures physical distance. The cutoffs Λ and M are related by the equation $\Lambda\,\delta x=M\,\delta r$, or

$$\Lambda=M\frac{dr}{dx}=M\gamma^{-1}. \quad \text{(E28)}$$

Inserting Eq. (E28) into Eq. (E24) gives

$$S_2=-T\frac{2v}{\pi R}\left[v\gamma\ln\left(\frac{MR}{2(\gamma^2-1)}\right)+v\gamma-\frac{\pi}{2}\right]-T\frac{v^2\gamma}{\pi R}f(v). \quad \text{(E29)}$$

Using the classical equation of motion (11.6) then gives

$$S_2=-T\frac{2v}{\pi R}\left[v\gamma\ln\left(\frac{Mm}{\sigma}\right)+v\gamma-\frac{\pi}{2}\right]-T\frac{v^2\gamma}{\pi R}f(v). \quad \text{(E30)}$$

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