3-branes on spaces with $R\times S^2\times S^3$ topology

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We study supergravity solutions representing D3-branes with transverse 6-space having $R\times S^2\times S^3$ topology. We consider regular and fractional D3-branes on a natural one-parameter extension of the standard Calabi-Yau metrics on singular and resolved conifolds. After imposing a $\mathbf{Z}_2$ identification on an angular coordinate these generalized ''6D conifolds'' are nonsingular spaces. The back reaction of D3-branes creates a curvature singularity that coincides with a horizon. In the presence of fractional D3-branes the solutions are similar to the original ones by Klebanov and Tseytlin and Pando Zayas and Tseytlin: the metric has a naked repulson-type singularity located behind the radius where the 5-form flux vanishes. The semiclassical behavior of the Wilson loop suggests that the corresponding gauge theory duals are confining.

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I. INTRODUCTION

One fruitful approach to generalize the original AdS/conformal field theory (CFT) correspondence [1] to more ''realistic'' gauge theories with less supersymmetries is based on considering D3-branes on conifold singularities [2–5]. To get non-conformal theories one may add [6,7] ''fractional'' [8] D3-branes.

Recently, exact supergravity solutions representing such configurations were constructed with the 6D space transverse to the D3-brane being a conifold [9], its deformation [10] and its resolution [11].

The deformed conifold [10] and resolved conifold [11] backgrounds are two different (deformation ϵ and resolution a) one-parameter generalizations of the conifold [9] one. The three solutions coincide for large values of the radial coordinate ρ, or in the UV in the language of gauge theory duals. However, the small-distance or IR behavior is different in each case. In particular, the conifold and the resolved conifold solutions have naked singularities at finite ρ, while (a special case of) the deformed conifold solution is regular [10].

In view of the interest of these solutions both from the supergravity and the gauge theory points of view (see [12–21] for some recent related work) it is important to study their various generalizations. That may help to clarify which of their features are truly universal, in particular regarding singularities and IR behavior.

Here we shall point out that there exists a very natural one-parameter extension of the three solutions of [9–11]. The key observation is that the most general Ricci-flat Kähler 6D metrics on the three conifolds [22,23] (with nontrivial dependence on radial direction only) contain [11,16] one extra parameter (called b below). This parameter was set equal to zero in the previous discussions of D3-branes on conifolds.[1] These metrics have the same $R\times S^2\times S^3$ topology as their $b=0$ limits, in particular, they again allow the introduction of fractional 3-branes (D5-branes wrapped over a 2-cycle). For $b\neq 0$ the conifold and resolved conifold metrics have regular curvature; it is possible to remove their bolt-type singularity by imposing a $\mathbf{Z}_2$ identification on the coordinate of the $U(1)$ fiber, thus obtaining non-singular 6D metrics. In contrast, the $b\neq 0$ generalization of the (regular) deformed conifold metric [22,23] has a curvature singularity at the origin.

As was shown in [16], for any zero or non-zero value of b these 6D metrics preserve the same fraction (1/4) of type II supersymmetry, so that the corresponding D3-brane solutions should also have [12,13] the same amount of supersymmetry as the solutions of [9–11].[2]

When $b\neq 0$, the ''standard'' (no 3-form flux) D3-brane on the conifold solution has the small ρ limit which is no longer $\mathrm{AdS}_5\times T^{1,1}$ as was in the standard $b=0$ case [4]. As a result, the conformal invariance is broken and b plays the role of a ''mass scale.'' An interesting feature of these solutions is that they have a curvature singularity coinciding with the horizon at $\rho=b$, like in the ''standard'' ($b=0$) resolved conifold case discussed in [11] (where the singularity and the horizon were located at $\rho=0$).

Including fractional D3-branes changes the situation radically. As in the conifold [9] and the resolved conifold [11] cases there is a naked singularity of a repulson type [37] located behind the ''zero-charge'' ($F_5=0$) locus. The gauge

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[1] For $b\neq 0$ the conifold metric is no longer that of a cone over $T^{1,1}$.

[2] Preservation of supersymmetry is obvious for the pure D3-brane solutions. For the fractional D3-branes, it was claimed in [14,15] that the resolved conifold solution of [11] breaks all supersymmetry.

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theory interpretation of these solutions should probably go along the lines of the discussion in [21]. It would be very interesting to understand the meaning of the parameter b on the gauge theory side.

Since the simplest (though non-supersymmetric [24]) Ricci flat 6D space with the same topology $R\times S^2\times S^3$ is the cone over $T^{1,0}=S^2\times S^3$, we shall, for comparison, consider also the corresponding D3-brane solution.

In Sec. II we shall present the explicit form of the Ricci-flat metrics with the $R\times S^2\times S^3$ topology referred to above. In Sec. III we shall construct the generalizations of the standard D3-brane solution [25,26] to the case when the transverse 6D spaces are the generalized ($b\neq 0$) standard, resolved and deformed conifolds.

The fractional D3-brane solutions which are the $b\neq 0$ analogues of the conifold and resolved conifold backgrounds of [9,11] will be found in Sec. IV. In Sec. V we shall study, following [27,28], the energy of a static fundamental string (with both ends at the ''boundary'' $\rho=\infty$) in these backgrounds and argue that the corresponding Wilson loop has confining (area law) behavior. This conclusion seems robust since the ''bent'' string does not reach the singular region.

II. RICCI-FLAT METRICS WITH TOPOLOGY $R\times S^2\times S^3$

One natural way to construct Ricci flat spaces of topology $R\times S^2\times S^3$ is to consider cones over Einstein spaces with topology $S^2\times S^3$. Examples of the latter are $T^{1,1}$ (supersymmetric) and $T^{1,0}$ (non-supersymmetric). In what follows we shall start with these simplest examples and consider their natural generalizations.

A. Cone over $S^3\times S^2$

$T^{1,0}$ is not only $S^3\times S^2$ topologically, but geometrically too. The resulting 6D Ricci-flat metric is (the radii of the two spheres are adjusted to make the whole space an Einstein one)

$$ds_6^2=d\rho^2+\rho^2\left[\frac{1}{8}(e_{\psi_1}^2+e_{\theta_1}^2+e_{\phi_1}^2)+\frac{1}{4}(e_{\theta_2}^2+e_{\phi_2}^2)\right], \tag{2.1}$$

where the vielbein is

$$e_{\theta_i}=d\theta_i\,,\quad e_{\phi_i}=\sin\theta_i d\phi_i\,,\quad e_{\psi_1}=d\psi_1+\cos\theta_1 d\phi_1\,. \tag{2.2}$$

B. Conifold

The standard conifold metric, i.e., the metric of the cone over $T^{1,1}=[SU(2)\times SU(2)]/U(1)$ is [22,29]

$$ds_6^2=d\rho^2+\rho^2\left[\frac{1}{9}e_{\psi}^2+\frac{1}{6}(e_{\theta_1}^2+e_{\phi_1}^2+e_{\theta_2}^2+e_{\phi_2}^2)\right], \tag{2.3}$$

where e_{θ_i} and e_{ϕ_i} are the same as in Eqs. (2.2) and

$$e_{\psi}=d\psi+\cos\theta_1 d\phi_1+\cos\theta_2 d\phi_2\,. \tag{2.4}$$

To get a more general class of Ricci flat metrics on the conifold let us recall some basic relations from [22,23,11,16] (where further details and notation may be found). The conifold can be described as a quadric in $\mathbf{C}^4$, $\Sigma_{i=1}^4 w_i^2=0$, or a solution of $\det\mathcal{W}=0$. In terms of the 2×2 matrix $\mathcal{W}$ and the potential K the Kähler metric on the conifold is [$r^2=\mathrm{tr}(\mathcal{W}^\dagger\mathcal{W})=\Sigma_{i=1}^4|w_i|^2$]

$$ds^2=K'\ \mathrm{tr}(d\mathcal{W}^\dagger d\mathcal{W})+K''|\mathrm{tr}(\mathcal{W}^\dagger d\mathcal{W})|^2,$$

$$(\cdots)'\equiv\frac{d}{dr^2}(\cdots). \tag{2.5}$$

The Ricci tensor for a Kähler metric is $R_{p\bar{q}}=-\partial_p\partial_{\bar{q}}\ln\det g$, where, for the metric in Eq. (2.5),

$$\det g=\frac{1}{|w_4|^2}(K')^2r^2(K'+r^2K''). \tag{2.6}$$

The Ricci-flatness condition implies

$$[(r^2K')^3]'=2r^2, \tag{2.7}$$

which is integrated to give

$$(r^2K')^3=r^4+c. \tag{2.8}$$

We shall assume that the constant c is non-negative to avoid a curvature singularity at finite r.

The conifold metric (2.5) then is[3]

$$ds^2=\frac{2}{3}(c+r^4)^{-2/3}\left(r^2dr^2+\frac{1}{4}r^4e_{\psi}^2\right)+\frac{1}{4}(c+r^4)^{1/3}(e_{\theta_1}^2+e_{\phi_1}^2+e_{\theta_2}^2+e_{\phi_2}^2). \tag{2.9}$$

For $c>0$, writing the metric in terms of $z=\frac{1}{2}r^2$ we get, near $r=0$,

$$(ds^2)_{r\to 0}=\frac{2}{3}c^{-2/3}(dz^2+z^2e_{\psi}^2)+\frac{1}{4}c^{1/3}(e_{\theta_1}^2+e_{\phi_1}^2+e_{\theta_2}^2+e_{\phi_2}^2). \tag{2.10}$$

Thus (i) near the apex ($r=0$) the 2-cycles stay finite, and (ii) it is possible to avoid the conical curvature singularity at $r=0$ by changing the range of ψ from the original one $[0,4\pi)$ to $[0,2\pi)$.

The generalized conifold with $b\neq 0$, Eq. (2.9), is the 6D analogue of the Eguchi-Hanson metric [30] where to avoid the singularity one is to change $S^3\to S^3/\mathbf{Z}_2=\mathbf{P}^3$. Here ψ is the coordinate of the fiber of the $U(1)$ bundle over $S^2\times S^2$. Taking $\psi\in[0,2\pi)$ one finds that for large r the space is now the cone over $T^{1,1}/\mathbf{Z}_2$; this is an example of asymptotically locally Euclidean metric.

To establish the analogy with the Eguchi-Hanson metric more explicitly let us define the constant $b\geqslant 0$ by

[3] The coordinate r here was denoted ρ in [16].

$$c\equiv\left(\frac{2}{3}b^2\right)^3, \tag{2.11}$$

and introduce the new radial coordinate ρ through the relation

$$\rho^6=\left[\frac{3}{2}r^2K'(r)\right]^3=\left(\frac{3}{2}\right)^3r^4+b^6. \tag{2.12}$$

Since $0\leqslant r<\infty$, the range of variation of ρ is $b\leqslant\rho<\infty$.

The metric (2.5),(2.9) then becomes

$$ds_6^2=\kappa^{-1}(\rho)d\rho^2+\frac{1}{9}\kappa(\rho)\rho^2e_\psi^2+\frac{1}{6}\rho^2(e_{\theta_1}^2+e_{\phi_1}^2+e_{\theta_2}^2+e_{\phi_2}^2), \tag{2.13}$$

where

$$\kappa(\rho)=1-\frac{b^6}{\rho^6}. \tag{2.14}$$

Note that $0\leqslant\kappa\leqslant1$. The analysis of this metric follows closely that of the Eguchi-Hanson metric in [30,31]. It is straightforward to establish that this Ricci-flat metric does not have any scalar curvature singularity; e.g., $R_{ijkl}R^{ijkl}=96\rho^{-16}(\rho^{12}+20b^{12})$ is finite for $0<b\leqslant\rho<\infty$, so that introducing the parameter b smoothens the curvature singularity of the original conifold metric. In that sense the b-generalized conifold introduced above may be called a ''regularized conifold'' (by analogy with resolved and deformed conifolds which also contain an extra parameters eliminating the curvature singularity).

The point $\rho=b$ is a removable bolt singularity, as can be seen by introducing the coordinate $u^2=\frac{1}{9}\rho^2\kappa(\rho)$ and considering the $\rho\to b$ limit.

C. Resolved conifold

Following [22,11,16], one finds the analogous one new parameter (b) extended metric on the resolved conifold [cf. Eq. (2.13)]

$$ds^2=\kappa^{-1}(\rho)d\rho^2+\frac{1}{9}\kappa(\rho)\rho^2e_\psi^2+\frac{1}{6}\rho^2(e_{\theta_1}^2+e_{\phi_1}^2)+\frac{1}{6}(\rho^2+6a^2)(e_{\theta_2}^2+e_{\phi_2}^2). \tag{2.15}$$

Here a is the resolution parameter and

$$\kappa(\rho)=\frac{1+\dfrac{9a^2}{\rho^2}-\dfrac{b^6}{\rho^6}}{1+\dfrac{6a^2}{\rho^2}}. \tag{2.16}$$

For $a=0$ this metric reduces to the above conifold metric (2.13),(2.14). For ρ much greater than any of the two length scales a and b we get the standard conifold metric (2.3).

The coordinate ρ and the original coordinate $0\leqslant r<\infty$ of the conifold (see [11,16]) are related according to [cf. Eq. (2.12)]

$$\rho^6+9a^2\rho^4=\left(\frac{3}{2}\right)^3r^4+b^6. \tag{2.17}$$

The range of ρ is thus $\rho_0\leqslant\rho<\infty$, where $\rho_0=\rho_0(a,b)\geqslant0$ corresponds to the apex $r=0$, i.e. is the solution of $\rho_0^6+9a^2\rho_0^4-b^6=0$ which is positive and becomes zero when $b=0$.[4] We shall assume again that $\psi\in[0,2\pi)$ to avoid the conical bolt singularity at $\rho=\rho_0$.

The curvature invariants for the metric (2.15) are regular [unless both a and b are zero when we get back to the original singular conifold metric (2.3)]. In particular,

$$R_{ijkl}R^{ijkl}=\frac{96}{\rho^{12}(\rho^2+6a^2)^6}[b^{12}(5184a^8+4320a^6\rho^2+1440a^4\rho^4+240a^2\rho^6+20\rho^8)+144b^6a^4\rho^{10}+\rho^{12}(6480a^8+2160a^6\rho^2+360a^4\rho^4+30a^2\rho^6+\rho^8)],$$

which is non-singular at $\rho\to\rho_0$ (for $a=0$ this expression reduces to the one conifold one given at the end of the previous subsection, and for $b=0$ it gives the curvature invariant for the resolved conifold metric of [11]).

To understand the short-distance $\rho\to\rho_0$ ($r\to0$) behavior of the $b\neq0$ resolved conifold metric we introduce as in Eq. (2.10) the coordinate $z=\frac{1}{2}r^2\to0$, thus obtaining

$$(ds_6^2)_{r\to0}=\frac{3}{8\rho_0^2(\rho_0^2+6a^2)}(dz^2+z^2e_\psi^2)+\frac{1}{6}\rho_0^2(e_{\theta_1}^2+e_{\phi_1}^2)+\frac{1}{6}(\rho_0^2+6a^2)(e_{\theta_2}^2+e_{\phi_2}^2). \tag{2.18}$$

For fixed θ_i and ϕ_i the metric is thus proportional to $dz^2+z^2d\psi^2$, so that to avoid a conical singularity we need again to apply a $\mathbf{Z}_2$ identification to $\psi\in[0,4\pi)$.[5]

The short-distance limit of the $b=0$ resolved conifold metric (which is to be considered separately as $\rho_0=0$ for $b=0$) is [11] ($\rho^2\approx(\sqrt{3/8a^2})r^2$)

$$(ds_6^2)_{\rho\to0}=\frac{2}{3}d\rho^2+\frac{1}{6}\rho^2(e_\psi^2+e_{\theta_1}^2+e_{\phi_1}^2)+\left(a^2+\frac{1}{6}\rho^2\right)(e_{\theta_2}^2+e_{\phi_2}^2). \tag{2.19}$$

[4]Explicitly, $\rho_0^2=9a^4\nu^{-1/3}+\nu^{1/3}-3a^2$, where $\nu=\frac{1}{2}[b^6-54a^6+b^6\sqrt{1-108a^6/b^6}]$ (see [22,11]).

[5]Note that in the $\rho\to\rho_0$ limit the metric is topologically the same as that of the generalized conifold (2.10), i.e. $\mathbf{R}^2\times S^2\times S^2$, but geometrically the two metrics are different [e.g., the radii of the two 2-spheres spanned by (θ_i,ϕ_i) are different].

This metric has regular curvature invariants, e.g., $(R_{ijkl}R^{ijkl})_{\rho\to 0}\to 40/3a^4$, illustrating that the parameter $a\neq 0$ indeed resolves the curvature singularity of the standard conifold.

D. Deformed conifold

The b-parameter family of metrics on the deformed conifold is found in a similar way [22,23,16,11]. Using the basis of [23,10] the metric can be written as ($g_5=e_\psi$)

$$ds_6^2=\kappa^{-1}(\rho)d\rho^2+\frac{1}{9}\kappa(\rho)\rho^2 e_\psi^2+\frac{1}{6}\rho^2\left[\sqrt{\frac{r^2-\epsilon^2}{r^2+\epsilon^2}}(g_1^2+g_2^2)+\sqrt{\frac{r^2+\epsilon^2}{r^2-\epsilon^2}}(g_3^2+g_4^2)\right], \tag{2.20}$$

where ρ is related to r according to [cf. Eqs. (2.12),(2.17)]

$$\rho^6=\left(\frac{3}{2}\right)^3 r^4\left[\sqrt{1-\frac{\epsilon^4}{r^4}}-\frac{\epsilon^4}{r^4}\ln\frac{r^2+\sqrt{r^4-\epsilon^4}}{\epsilon^2}\right]+b^6, \tag{2.21}$$

and [cf. Eqs. (2.14),(2.16)]

$$\kappa(\rho)=\left(\frac{3}{2}\right)^3\frac{r^4}{\rho^6}\left(1-\frac{\epsilon^4}{r^4}\right)=1-\frac{b^6}{\rho^6}+O(\epsilon). \tag{2.22}$$

Since $\epsilon\leqslant r\leqslant\infty$, we have $b\leqslant\rho<\infty$. This metric reduces to the generalized conifold metric (2.13) for $\epsilon\to 0$. For r greater than any of the two length scales b and ϵ, it becomes the standard conifold metric (2.3).

It is useful also to present the deformed conifold metric in terms of the radial coordinate τ used in [23,10,11]

$$r^2=\epsilon^2\cosh\tau,\quad \rho^6=\left(\frac{3}{2}\right)^3\epsilon^4\left[\frac{1}{2}\sinh(2\tau)-\tau\right]+b^6,\quad 0\leqslant\tau<\infty, \tag{2.23}$$

$$ds_6^2=\frac{1}{2}\mathcal{K}\left[(3\mathcal{K}^3)^{-1}\epsilon^4(d\tau^2+g_5^2)+\sinh^2\frac{\tau}{2}\ (g_1^2+g_2^2)+\cosh^2\frac{\tau}{2}\ (g_3^2+g_4^2)\right], \tag{2.24}$$

where

$$\mathcal{K}(\tau)=\frac{\{c+\frac{1}{2}\epsilon^4[\sinh(2\tau)-2\tau]\}^{1/3}}{\sinh\tau}=\frac{2}{3}\rho^2(\sinh\tau)^{-1},$$

$$c=\left(\frac{2}{3}\right)^3 b^6.$$

For large τ we again get the standard conifold metric, while for small values of τ we get (for $b\neq 0$)

$$(ds)^2_{\tau\to 0}=\frac{3\epsilon^4}{8b^4}\tau^2(d\tau^2+g_5^2)+\frac{b^2}{12}\tau(g_1^2+g_2^2)+\frac{b^2}{3}\tau^{-1}(g_3^2+g_4^2) \tag{2.25}$$

or, in terms of $\rho\to b$,

$$(ds_6^2)_{\rho\to b}=\kappa^{-1}(\rho)d\rho^2+\frac{1}{9}\kappa(\rho)\rho^2 e_\psi^2+\frac{1}{6}\rho^2\left[\left(\frac{\rho^6-b^6}{18\epsilon^4}\right)^{1/3}\times(g_1^2+g_2^2)+\left(\frac{\rho^6-b^6}{18\epsilon^4}\right)^{-1/3}(g_3^2+g_4^2)\right], \tag{2.26}$$

where for $\kappa=\kappa_{\rho\to b}=[(\frac{3}{2})^5\epsilon^4]^{1/3}(\rho^6-b^6)^{2/3}/\rho^6\to 0$. Note that the volume of the 2-cycle (g_1,g_2) shrinks to zero, while volume of the 3-cycle (g_3,g_4,g_5) stays constant $(=\epsilon^2/\sqrt{6})$. This metric has a curvature singularity at $\tau=0$ for any $b\neq 0$.

For comparsion, the small τ limit of the standard $b=0$ deformed conifold metric is [23,10]

$$(ds)^2_{\tau\to 0}=\left(\frac{\epsilon^4}{12}\right)^{1/3}\left[\frac{1}{2}(d\tau^2+g_5^2)+g_3^2+g_4^2+\frac{1}{4}\tau^2(g_1^2+g_2^2)\right],$$

and is regular at $\tau=0$. Indeed, computing the curvature invariant $R_{ijkl}R^{ijkl}$ for the general form of the metric (2.24) and then expanding it near $\tau=0$ (i.e. $r=\epsilon$) one finds a singular expression for $b\neq 0$,[6]

$$(R_{ijkl}R^{ijkl})_{\tau\to 0}=\frac{5120\,b^8}{9\epsilon^8}\frac{1}{\tau^8}\left[1-\frac{8}{15}\tau^2+\frac{3\epsilon^4}{4b^6}\tau^3+O(\tau^4)\right],$$

and a regular expression for $b=0$:

$$(R_{ijkl}R^{ijkl})_{\tau\to 0}=\frac{192\cdot(18)^{1/3}}{5\ \epsilon^{8/3}}\left[1-\frac{4}{5}\tau^2+\frac{202}{525}\tau^4+O(\tau^5)\right].$$

E. Remarks

The three one-parameter families of the Ricci-flat metrics presented above Eqs. (2.13), (2.15) and (2.20) are the most general solutions (with non-trivial dependence on the radial coordinate only) for Ricci-flat metrics on respective conifolds [11,16]. As was shown in [16], they define supersymmetric backgrounds of type II supergravity. This is in contrast to what happens in the case of the metric of a cone over $S^3\times S^2$, Eq. (2.1), which breaks all supersymmetries and thus may lead to unstable D3-brane solutions.

The analogy with the Eguchi-Hanson metric elucidates the geometrical meaning of the ''mass'' parameter b. From the perspective of dual gauge theories associated with D3-brane solutions on these generalized conifolds which will be

[6]Here ρ and b have canonical length l dimensions while $r\sim l^{3/2}$, $\epsilon\sim l^{3/2}$, $\tau\sim l^0$. Note that the leading term in the expression below is what one finds directly from the metric (2.25).

discussed below this parameter should play the role of an IR ''mass'' or ''confinement'' scale.

A feature of the conifold (2.13) and the resolved conifold (2.15) is that near the apex $(r=0)$ the respective metrics effectively ''factorize'' into an $\mathbf{R}^2\times S^2\times S^2$ part. This will have consequences for the structure of D3-brane solutions in the IR region.

We have seen that while the $b=0$ conifold had curvature singularity at the origin, its $b\neq 0$ generalization has regular curvature (and have no conical singularity after changing the period of the angle ψ). The resolved conifold metric depending on two parameters (a,b) is regular for all of their values except $a=b=0$. Curiously, this is different from the situation for the deformed conifold: while the standard $b=0$ deformed conifold metric was regular (with the parameter ϵ playing the role of the ''cutoff''), its $b\neq 0$ generalization has a curvature singularity at the origin $r=\epsilon$. This implies, in particular, that one is unlikely to find a direct $b\neq 0$ generalization of the regular fractional D3-brane on deformed conifold solution of [10] (which is thus a very special point in the parameter space of solutions). For that reason in what follows we shall mostly concentrate on the resolved conifold case.

Finally, let us note that the conifold solutions discussed above can be derived as special cases of the following ''interpolating'' ansatz for 6D metric (see [16]):

$$ds_6^2=e^{2z+3x}du^2+e^{z+x}[e^{g}(e_{\theta_1}^2+e_{\phi_1}^2)+e^{-g}(\tilde{\epsilon}_1^2+\tilde{\epsilon}_2^2)]+e^{-2z-x}e_\psi^2, \qquad (2.27)$$

where $\tilde{\epsilon}_1=\epsilon_1-a(u)e_{\theta_1}$, $\tilde{\epsilon}_2=\epsilon_2+a(u)e_{\phi_1}$, $\epsilon_1=\sin\psi\sin\theta_2 d\phi_2+\cos\psi d\theta_2$, $\epsilon_2=\cos\psi\sin\theta_2 d\phi_2-\sin\psi d\theta_2$, and $e_{\theta_1},e_{\phi_1},e_\psi$ are as in Eqs. (2.2),(2.4). Resolved and deformed conifold metrics are special cases of this ansatz [16] corresponding to $a=0$ and $a^2=1-e^{2g}$ respectively. The unknown functions x,z,g,a of the radial coordinate u representing Ricci-flat 6D spaces are subject to equations following from the following 1D action (plus the ''zero-energy'' constraint):

$$S_1=\int du[x'^2-z'^2-g'^2-e^{-2g}a'^2-V(x,z,g,a)], \qquad (2.28)$$

$$V=\frac{1}{2}e^{-2z}[e^{2g}+(a^2-1)^2e^{-2g}]-4e^{z+2x}\cosh g+a^2(e^{-z}-e^{2z+2x-g})^2. \qquad (2.29)$$

For the special cases corresponding to the resolved and the deformed conifolds one finds that this system admits a superpotential W [11], i.e. $V=-G^{ij}(\partial W/\partial q^i)\partial W/\partial q^j$, where $q^i=(x,z,g,a)$, $G_{ij}=(1,-1,-1,-e^{-2g})$, so that one gets a first-order system $q'^i=G^{ij}\partial W/\partial q^j$. For example, in the resolved conifold case $(a=0)$ $W=e^{-z}\cosh g+e^{2z+2x}$. It would be very interesting to find new solutions of the sytem (2.28) representing 6D Ricci-flat spaces that ''interpolate'' between the generalized resolved and deformed conifold metrics discussed above.

III. PURE D3-BRANE SOLUTIONS

A. General remarks

As is well known, given a Ricci flat 6D space with metric g_{mn}, one can construct the following generalization of the standard [25,26] 3-brane solution (see, e.g., [3,23,32]):

$$ds_{10}^2=h^{-1/2}(y)dx^\mu dx^\mu+h^{1/2}(y)g_{mn}(y)dy^m dy^n, \qquad (3.1)$$

$$F_5=(1+*)dh^{-1}\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3,$$

$$\Phi=\text{const}, \qquad (3.2)$$

where h is a harmonic function on the transverse 6D space,

$$\frac{1}{\sqrt{g}}\partial_m(\sqrt{g}g^{mn}\partial_n h)=0. \qquad (3.3)$$

In the case of a Ricci-flat cone with metric

$$g_{mn}(y)dy^m dy^n=d\rho^2+\rho^2\gamma_{ij}(z)dz^i dz^j, \qquad (3.4)$$

one can choose h to depend only on ρ, thus getting the single-center solution

$$h=h_0+\frac{L^4}{\rho^4}. \qquad (3.5)$$

Here and below we assume that $h_0\geqslant 0$. Then in the near-core region the space becomes $\mathrm{AdS}_5\times X^5$, where X^5 is the Einstein space which is the base of the cone with the metric γ_{ij}. Particular examples are provided by the standard cone metrics (2.1) and (2.3) where $X^5=T^{1,0}=S^2\times S^3$ and $X^5=T^{1,1}$.[7]

In the general case when the transverse 6-space is not a cone, one will not find the near-core geometry having an AdS_5 factor and thus the dual gauge theory will have broken conformal invariance.

B. Conifold case

Let us determine the harmonic function h that solves Eq. (3.3) for the generalized conifold metric (2.13) assuming $h=h(\rho)$. Using that $\sqrt{g_6}=1/108\rho^5\sin\theta_1\sin\theta_2$, $g^{\rho\rho}=1-b^6/\rho^6$ we get

$$h=h_0-\frac{2L^4}{b^4}\left[\frac{1}{6}\ln\frac{(\bar{\rho}^2-1)^3}{\bar{\rho}^6-1}+\frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\arctan\frac{2\bar{\rho}^2+1}{\sqrt{3}}\right)\right], \qquad (3.6)$$

$$\bar{\rho}\equiv\frac{\rho}{b}.$$

[7]Compactifications of type IIB supergrativity on such X^5 were discussed in [24] where it was pointed out that in the family of $T^{p,q}$ spaces only $T^{1,1}$ preserves supersymmetry.

For $\rho/b \gg 1$ we recover the standard metric of D3-branes on the conifold (3.5), i.e.

$$h(\rho \to \infty) = h_0 + \frac{L^4}{\rho^4}. \tag{3.7}$$

For small values of ρ we get $h \to \infty$:

$$h(\rho \to b) = -\frac{2L^4}{3b^4} \ln\left(\frac{\rho}{b} - 1\right) + h_1,$$

$$h_1 = h_0 + \frac{L^4}{9b^4}\left(\sqrt{3}\pi + 3\ln\frac{4}{3}\right). \tag{3.8}$$

Recalling from Eq. (3.1) that the 00-component of the metric is $h^{-1/2}$ we see that $\rho = b$ is a horizon. It is also a curvature singularity. Thus, starting with $\mathbf{R}^{1,3} \times M^6$ where M^6 is the generalized $b \neq 0$ conifold, the introduction of D3-branes creates a back reaction that transforms the previously nonsingular point $\rho = b$ into a curvature singularity coinciding with the horizon. In contrast, for $b = 0$ [4] the near-core geometry was regular—$AdS_5 \times T^{1,1}$.

This is similar to what was found for the D3-brane on a $b = 0$ resolved $(a \neq 0)$ conifold in [11]. Below we will see that this behavior extends also to the case of the resolved conifold with $b \neq 0$.

One of the generic features of D3-branes on the conifold and the resolved conifold is the logarithmic form of $h(\rho)$ at small distances. The origin of this lies in the fact (noted in Sec. II E) that for small ρ the transverse geometry becomes effectively 2 dimensional as far as the dependence on the radial coordinate is concerned, so that one finds that the harmonic function h has the ''7-brane-like'' $\log(\rho - \rho_0)$ structure.

C. Resolved conifold case

Starting with the resolved conifold metric (2.15) and solving Eq. (3.3) one finds the following expression for $h(\rho)$:

$$h = h_0 - \frac{2L^4}{\alpha b^4}\left[\frac{1}{6}\ln\frac{(\bar\rho^2 - \bar\rho_0^2)^3}{\bar\rho^6 + 3q^2\bar\rho^4 - 1} + \frac{\beta + q^2}{\lambda}\left(\frac{\pi}{2} - \arctan\frac{2\bar\rho^2 + \sigma}{\lambda}\right)\right], \tag{3.9}$$

where, as in Eq. (3.6) (ρ_0 is the same as in Sec. II C)

$$\bar\rho = \frac{\rho}{b}, \quad \bar\rho_0 = \frac{\rho_0}{b},$$

and for simplicity of presentation we have introduced the following constants depending on the ratio a/b:

$$q = \sqrt{3}\,\frac{a}{b}, \quad \beta = q^4\mu^{-1} + \mu - q^2,$$

$$\mu \equiv \left[\frac{1}{2}(1 - 2q^6 + \sqrt{1 - 4q^6})\right]^{1/3},$$

$$\alpha = \frac{1}{3}(2\beta^2 + 3q^2\beta + \beta^{-1}), \quad \sigma = \beta + 3q^2,$$

$$\lambda = \sqrt{4\beta^{-1} - \sigma^2}. \tag{3.10}$$

We have assumed that $1 - 4q^6 > 0$ (the opposite case is discussed below).

The behavior for $\rho \to \rho_0$ is the same as in Eqs. (3.8):

$$h(\rho \to \rho_0) = -\frac{2L^4}{3\alpha b^4}\ln(\bar\rho - \bar\rho_0) + O(1); \tag{3.11}$$

i.e., as in the previous case, at $\rho = \rho_0$ there is a horizon coinciding with curvature singularity.

Compared to Eq. (3.6) here the function h contains another (resolution) scale a represented by the parameter $q \in [0,\infty)$. Depending on the value of q, i.e. on the ratio of a and b, one may distinguish three regions: b dominated, intermediate, and a dominated. The metric defined by Eqs. (3.9), (3.10) is valid in the b-dominated region where $4q^6 < 1$, i.e., $b > 3^{1/2}2^{1/3}a$. For $q = 0$ $(a = 0)$ we get back to the conifold case (3.6) which should be viewed as the limiting case of the b-dominated resolved conifold solution (3.9). In the intermediate region, i.e. $4q^6 = 1$ [this corresponds to $\lambda = 0$ in Eqs. (3.9), (3.10)], we get

$$h = h_0 - \frac{2L^4}{81a^4}\left(\frac{9a^2}{\rho^2 + 6a^2} + \ln\frac{\rho^2 - 3a^2}{\rho^2 + 6a^2}\right), \tag{3.12}$$

where $\sqrt{3}a \leq \rho < \infty$. This metric also has a curvature singularity and the horizon at $\rho = \rho_0 = \sqrt{3}a$. In the a-dominated region $4q^6 > 1$, i.e. $a > 3^{-1/2}2^{-1/3}b$, we find

$$h = h_0 - \frac{2L^4}{9a^4\alpha}\left[\frac{1}{6}\ln\frac{(\bar\rho^2 - \bar\rho_0^2)^3}{\bar\rho^6 + \bar\rho^4 - q^{-6}} - \frac{\beta + 1}{\lambda}\ln\frac{2\bar\rho^2 + \sigma - \lambda}{2\bar\rho^2 + \sigma + \lambda}\right], \tag{3.13}$$

where[8]

$$\bar\rho = \frac{\rho}{\sqrt{3}a}, \quad \beta = \mu + \mu^{-1} - 1,$$

$$\mu \equiv e^{i\pi/3}[1 - (2q^6)^{-1} - iq^{-3}\sqrt{1 - (4q^6)^{-1}}]^{1/3},$$

$$\alpha = \frac{1}{3}[2\beta^2 + 3\beta + (q^6\beta)^{-1}], \quad \sigma = \beta + 3,$$

$$\lambda = \sqrt{\sigma^2 - 4(q^6\beta)^{-1}}. \tag{3.14}$$

[8]Note that since $|\mu| = 1$, the combination $\mu + \mu^{-1}$ is always real so that β, σ, λ, and α are always real and ρ_0 is the positive real root of the cubic equation.

This solution has the same generic logarithmic behavior near ρ_0, indicating the existence of a horizon and a singularity.

In the *a*-dominated expression (3.13) we are able to take the $q\to\infty$ $(b\to 0)$ limit to get back to the ''standard'' resolved conifold case. The function h in the metric of D3-branes on $b=0$ the resolved conifold found in [11] is

$$h_{b=0}=h_0+\frac{2L^4}{81a^4}\left[\frac{9a^2}{\rho^2}-\ln\left(1+\frac{9a^2}{\rho^2}\right)\right]. \quad (3.15)$$

Again, here $\rho=0$ is both the horizon and the singularity.

D. Deformed conifold case

In the deformed conifold case (2.20),(2.21) the harmonic function h in Eq. (3.1) is found to be

$$h=h_0-\frac{2^5}{3^3}L^4\int\frac{\rho d\rho}{r^4-\epsilon^4}, \quad (3.16)$$

where the explicit form of $r(\rho)$ in Eq. (2.21) is transcendental. For $r\gg\epsilon$ we have $r^4=(2\rho^2/3)^3$ and therefore recover the D3-brane on the conifold metric with $h=h_0+L^4/\rho^4$. For $r\to\epsilon$ $(\rho\to b)$ we get

$$ds_{10}^2=h^{-1/2}dx^\mu dx^\mu+h^{1/2}(ds_6^2)_{\rho\to b}, \quad (3.17)$$

where $(ds_6^2)_{\rho\to b}$ is given by Eq. (2.26) and

$$h=h_1-h_2\rho^2+O(\rho^4),\quad h_2=\frac{2^{8/3}L^4}{3^{5/3}b^2\epsilon^{4/3}}\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{4}{3};1\right), \quad (3.18)$$

and $h_1=h_0+O(L^4/b^4)$.[9] This $b\neq 0$ case is different from the $b=0$ deformed conifold case [10,11] where $\rho=0$ was a horizon. Here for $\rho\to b$ the space factorizes into $\mathbf{R}^{1,3}\times M^6$ where M^6 has $\rho=b$ as its curvature singularity.

IV. FRACTIONAL D3-BRANE SOLUTIONS

Let us now construct the $b\neq 0$ generalization of the fractional D3-brane on resolved conifold solution of [9,11], i.e. The extension of the D3 brane solution of the previous section to the case of additional (self-dual) 3-form flux. The resolved conifold solution includes the conifold one as a special $(a=0)$ case. The first-order system corresponding to this background was already obtained in [11]. It is straightforward also to construct a similar $b\neq 0$ generalization of the solution [10] in the deformed conifold case (see [11]), but we shall not discuss the details of this here.

For comparison, we shall start with a similar case of 3-branes on the cone over $S^2\times S^3$. This solution was previously discussed in [33] (see also [20]).

A. $S^2\times S^3$ cone case

The 3-brane ansatz for the metric with transverse part given by Eq. (2.1) is

$$ds^2=h^{-1/2}dx^\mu dx^\mu+h^{1/2}\left(d\rho^2+\frac{1}{8}\rho^2 d\Omega_3^2+\frac{1}{4}\rho^2 d\Omega_2^2\right), \quad (4.1)$$

and the natural ansatz for the form fields is similar to the one in the conifold [7,9] case:

$$B_2=f(\rho)e_{\theta_2}\wedge e_{\phi_2}\to H_3=f'(\rho)d\rho\wedge e_{\theta_2}\wedge e_{\phi_2},$$

$$F_3=Pe_\psi\wedge e_{\theta_1}\wedge e_{\phi_1},$$

$$F_5=\mathcal{F}+*\mathcal{F},\quad \mathcal{F}=\mathrm{K}(\rho)e_\psi\wedge e_{\theta_1}\wedge e_{\phi_1}\wedge e_{\theta_2}\wedge e_{\phi_2}. \quad (4.2)$$

The 10D duals of these fields are

$$*\mathcal{F}=\frac{2^{13/2}\mathrm{K}}{\rho^5h^2}d\rho\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3, \quad (4.3)$$

$$*F_3=\frac{2^{5/2}P}{\rho h}d\rho\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge e_{\theta_2}\wedge e_{\phi_2}, \quad (4.4)$$

$$*H_3=-\frac{\rho f'}{2^{5/2}h}dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge e_\psi\wedge e_{\theta_1}\wedge e_{\phi_1}. \quad (4.5)$$

We shall assume that the dilaton Φ is constant. Then the F_3 equation of motion $d(e^\Phi *F_3)=F_5\wedge H_3$ is satisfied automatically, and from the H_3 equation $d(e^{-\Phi}*H_3)=-F_5\wedge F_3$ one obtains the following equation ($e^\Phi=g_s$)

$$\left(\frac{f_1'\rho}{h}\right)'=\frac{2^9g_sP\mathrm{K}}{h^2\rho^5}. \quad (4.6)$$

The constant dilaton condition implies $H_3^2=e^{2\Phi}F_3^2$, i.e., using Eq. (2.13) we get[10] $\rho f'=2^{5/2}g_sP$. The Bianchi identity for the 5-form $d*F_5=dF_5=H_3\wedge F_3$ gives $\mathrm{K}'=Pf'$, i.e. $\mathrm{K}=Q+Pf$. The two linearly independent Einstein equations are a consequence of this system of first order differential equations. The solution is thus very similar to the original conifold one [9]:

$$f=2^{5/2}g_sP\ln\frac{\rho}{\rho_0},\quad \mathrm{K}=Q+2^{5/2}g_sP^2\ln\frac{\rho}{\rho_0},$$

$$h=h_0+\frac{2^{9/2}}{\rho^4}\left[Q+2^{5/2}g_sP^2\left(\ln\frac{\rho}{\rho_0}+\frac{1}{4}\right)\right]. \quad (4.7)$$

[9]Here ${}_2F_1(\frac{1}{3},\frac{2}{3};\frac{4}{3};1)=\Gamma(\frac{1}{3})\Gamma(\frac{4}{3})/\Gamma(\frac{2}{3})\approx 1.77$.

[10]As in [9], the axion equation is satisfied automatically since $H_3\cdot F_3=0$.

Note that as in [9], the complex 3-form $G_3=g_sF_3+iH_3$ is self-dual in the 6D sense. Like the conifold solution [9], this solution has a naked singularity located at $\rho=\rho_h$, $Q+2^{5/2}g_sP^2[\ln(\rho_h/\rho_0)+\frac{1}{4}]=0$, i.e. very close to the origin if the number of fractional D3-branes is small, $Q\gg g_sP^2$. In this case the singularity is behind the "zero charge" ($\mathrm{K}=0$) locus.

B. Resolved conifold case

The ansatz for the metric will be the same as in Eq. (3.1),

$$ds_{10}^2=h^{-1/2}(\rho)dx^\mu dx^\mu+h^{1/2}(\rho)ds_6^2, \tag{4.8}$$

where ds_6^2 will be the metric of the generalized $b\neq 0$ resolved conifold (2.15). Our ansatz for the Neveu-Schwarz–Neveu-Schwarz (NS-NS) 2-form will be as in [11], i.e. a natural generalization of the ansatz in [9] motivated by an asymmetry between the two S^2 parts of the resolved conifold metric:

$$B_2=f_1(\rho)e_{\theta_1}\wedge e_{\phi_1}+f_2(\rho)e_{\theta_2}\wedge e_{\phi_2},$$

$$H_3=dB_2=d\rho\wedge[f_1'(\rho)e_{\theta_1}\wedge e_{\phi_1}+f_2'(\rho)e_{\theta_2}\wedge e_{\phi_2}]. \tag{4.9}$$

The conifold case ($a=0$) corresponds [7,9] to $f_1=-f_2$. The forms F_3 and F_5 will also have the same structure as in [9,11] (we follow the notation of [11]):

$$F_3=Pe_\psi\wedge(e_{\theta_2}\wedge e_{\phi_2}-e_{\theta_1}\wedge e_{\phi_1}), \tag{4.10}$$

$$F_5=\mathcal{F}+*\mathcal{F},\quad \mathcal{F}=\mathrm{K}(\rho)e_\psi\wedge e_{\theta_1}\wedge e_{\phi_1}\wedge e_{\theta_2}\wedge e_{\phi_2}. \tag{4.11}$$

The rest of the discussion is essentially the same as in [11] with κ in the 6D metric now being dependent also on b according to Eq. (2.16).

Assuming that the dilaton Φ is constant, the F_3 equation of motion $d(e^{\Phi}*F_3)=F_5\wedge H_3$ is satisfied automatically, and from the H_3 equation $d(e^{-\Phi}*H_3)=-F_5\wedge F_3$ one obtains the following three equations ($e^{\Phi}=g_s$):

$$\left(\frac{f_1'\rho\kappa\Gamma}{h}\right)'=\frac{324g_sP\mathrm{K}}{h^2\rho^5\kappa\Gamma},\quad \left(\frac{f_2'\rho\kappa}{h\Gamma}\right)'=-\frac{324g_sP\mathrm{K}}{h^2\rho^5\kappa\Gamma}, \tag{4.12}$$

$$f_1'+\Gamma^{-2}f_2'=0,\quad \Gamma\equiv\frac{\rho^2+6a^2}{\rho^2}, \tag{4.13}$$

where Γ is the ratio of the squares of the radii of the two spheres in the resolved conifold metric (2.15). The constant dilaton condition implies $H_3^2=e^{2\Phi}F_3^2$, i.e.

$$f_1'^2+\Gamma^{-2}f_2'^2=\frac{9g_s^2P^2}{k^2\rho^2}(1+\Gamma^{-2}). \tag{4.14}$$

Combined with Eqs. (4.13) that gives

$$f_1'=\frac{3g_sP}{\rho\kappa\Gamma},\quad f_2'=-\frac{3g_sP\Gamma}{\rho\kappa}. \tag{4.15}$$

The Bianchi identity for the 5-form, $d*F_5=dF_5=H_3\wedge F_3$, implies

$$\mathrm{K}'=P(f_1'-f_2'),\quad \text{i.e.}\quad \mathrm{K}=Q+P(f_1-f_2). \tag{4.16}$$

As in [9,11], to determine the metric function $h(\rho)$ it is sufficient to consider the trace of the Einstein equations,[11] $R=-\frac{1}{2}\Delta h=\frac{1}{24}(e^{-\Phi}H_3^2+e^{\Phi}F_3^2)$, i.e.

$$h^{-3/2}\frac{1}{\sqrt{g}}\partial_\rho(\sqrt{g}g^{\rho\rho}\partial_\rho h)=-\frac{1}{12}(g_s^{-1}H_3^2+g_sF_3^2)=-\frac{1}{6}g_sF_3^2 \tag{4.17}$$

or

$$(\rho^5\kappa\Gamma h')'=-324g_sP^2\frac{(1+\Gamma^2)}{\rho\kappa\Gamma}. \tag{4.18}$$

Integrating this we get

$$h'=-\frac{108\mathrm{K}}{\rho^5\kappa\Gamma}. \tag{4.19}$$

Plugging in the function κ in Eq. (2.16) the system (4.15),(4.19) of first-order differential equations can be directly integrated.

Here we shall present the explicit form of the solution only in the $a=0$ limit, i.e., the b-generalized conifold case (2.14). For $a=0$ one finds $\Gamma=1$, implying $f_2=-f_1$, and with $\kappa=1-b^6/\rho^6$ we obtain

$$f_1=-f_2=\frac{1}{2}g_sP\ln(\bar{\rho}^6-1)+f_0,\quad \mathrm{K}=Q+g_sP^2\ln(\bar{\rho}^6-1), \tag{4.20}$$

$$\begin{aligned}h=h_0&-\frac{54Q}{b^4}\left[\frac{1}{6}\ln\frac{(\bar{\rho}^2-1)^3}{\bar{\rho}^6-1}+\frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\arctan\frac{2\bar{\rho}^2+1}{\sqrt{3}}\right)\right]\\&+\frac{27g_sP^2}{b^4(\bar{\rho}^6-1)^{2/3}}\left[\frac{3}{2}\,{}_3F_2\left(\frac{2}{3},\frac{2}{3},\frac{2}{3};\frac{5}{3},\frac{5}{3};-\frac{1}{\bar{\rho}^6-1}\right)\right.\\&\left.+{}_2F_1\left(\frac{2}{3},\frac{2}{3};\frac{5}{3};-\frac{1}{\bar{\rho}^6-1}\right)\ln(\bar{\rho}^6-1)\right],\end{aligned} \tag{4.21}$$

where $\bar{\rho}=\rho/b$ [cf. Eq. (3.6)] and ${}_pF_q$ is the hypergeometric function.

The large ρ behavior of h is

$$h(\rho\to\infty)=h_0+\frac{27}{\rho^4}\left[Q+6g_sP^2\left(\ln\frac{\rho}{b}+\frac{1}{4}\right)\right]; \tag{4.22}$$

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[11] More precisely, there are two linearly independent Einstein equations: one is the square of Eq. (4.13) and another, written above, can be expressed in terms of the first derivative of Eq. (4.13) using Eq. (4.16).

i.e., this solution has the same UV asymptotic as the $b=0$ conifold one of [9]. In the short-distance limit $\rho\to b$ limit we have, to the leading order,

$$h(\overline{\rho}\to 1)=h_0-\frac{18Q}{b^4}\ln(\overline{\rho}^2-1)-\frac{9g_sP^2}{b^4}\ln^2(\overline{\rho}^2-1). \tag{4.23}$$

At $\overline{\rho}_h\approx 1+e^{-2Q/(g_sP^2)}$ the solution has a naked singularity of a repulson type. The ''zero charge'' locus (K=0) is located at $\overline{\rho}_{\rm K}=1+e^{-Q/(g_sP^2)}$, i.e. $\overline{\rho}_{\rm K}>\overline{\rho}_h$.

We can thus conclude, based on the analysis in [9] ($b=a=0$), in [11] ($b=0,a\neq 0$) and here ($a=0,b\neq 0$), that generically fractional 3-branes on the conifold and resolved conifold have a repulson-type naked singularity which is located behind the ''zero-charge'' locus.

V. WILSON LOOP BEHAVIOR

Let us now investigate, following [27,28], the behavior of the Wilson loop corresponding to a ''quark-antiquark'' potential in the dual gauge theory. It is given by the exponential of the classical fundamental string action in these D3-brane backgrounds evaluated for a static configuration of open string ending on the probe D3-brane placed at the ''boundary'' $\rho=\infty$.

We will show that one gets an area law (confining) behavior for the ''pure'' D3-brane backgrounds of Sec. III, assuming that at least one of the scales of the transverse space is kept non-zero. This is different from what is found in the standard conifold case [4] where the near-core geometry has an AdS_5 factor and thus the potential is Coulombic as in [27,28] (in the single-center case as well as in the multicenter case [34]).

For simplicity, we shall consider only the D3-brane background with the resolved conifold as the transverse space. The corresponding metric (2.15),(3.9) depends on the two scale parameters b and a. Expressed in terms of $\overline{\rho}=\rho/b$ it depends only on their ratio $q=\sqrt{3}a/b$. It is sufficient to analyze the Wilson loop in the two limiting cases $q=0$ and $q=\infty$: (i) $a=0$, $b\neq 0$, i.e., the D3-brane on the generalized conifold (2.13),(3.6), and (ii) $a\neq=0$, $b=0$, i.e., the D3-brane on the ''standard'' resolved conifold (3.15). In both special cases the scale of the transverse space (b or a) determines the confinement scale. The behavior of the Wilson loop for general values of q will be similar, given that the behavior of h is generic. Let us emphasize that in contrast with other supergravity solutions dual to confining $\mathcal{N}=1$ gauge theories [36,10,18], this confinement behavior is found for the pure D3-brane background which does not have any non-trivial 3-form fluxes.

A. General setup

All examples we have discussed above have metrics of the type

$$ds^2=h^{-1/2}(\rho)(-dx_0^2+dx_kdx_k)+h^{1/2}(\rho)[\kappa^{-1}(\rho)d\rho^2+ds_5^2], \tag{5.1}$$

where ds_5^2 is the metric of the corresponding 5D compact space. The Nambu-Goto string action which determines the expression for the Wilson loop depends on this 10D metric G_{MN} as $\int d\tau d\sigma\sqrt{-\det(G_{MN}\partial_aX^M\partial_bX^N)}$. In the static gauge ($x_0=\tau$, $x_1\equiv x=\sigma$) and assuming that the string is stretched only in the radial direction—i.e., only the ρ coordinate depends on σ—we get[12]

$$S=T\int dx\sqrt{G_{00}G_{xx}+G_{00}G_{\rho\rho}(\partial_x\rho)^2}=T\int dx\sqrt{h^{-1}+\kappa^{-1}(\partial_x\rho)^2}. \tag{5.2}$$

Since the Lagrangian of this ''mechanical system'' does not depend explicitly on ''time'' x, we have a conserved quantity $h^{-1}/[\sqrt{h^{-1}+\kappa^{-1}(\partial_x\rho)^2}]$; i.e., the first integral is ($c_0=\mathrm{const}$)

$$dx=\frac{d\rho}{\sqrt{\kappa h^{-1}(h^{-1}/c_0^2-1)}}. \tag{5.3}$$

The energy of a static string configuration is thus

$$E=\frac{S}{T}=\int dx\sqrt{h^{-1}+\kappa^{-1}(\partial_x\rho)^2}=\int\frac{d\rho}{\sqrt{\kappa(1-c_0^2h)}}. \tag{5.4}$$

Following [27,28], the question about confinement is then reduced to finding the dependence of the energy E on the distance l between the string end points (between ''quark'' and the ''antiquark'').

B. Conifold case

For the generalized conifold metric with the scale b, Eqs. (2.13),(2.14), the function h of the D3-brane solution is given by Eq. (3.6). Introducing the new coordinate

$$y=\overline{\rho}^2=\frac{\rho^2}{b^2}, \tag{5.5}$$

and removing the asymptotically flat region (i.e. dropping h_0) we obtain the following relation for the quark-antiquark separation:

$$\frac{l}{2}=\frac{L^2}{\sqrt{2}b}\int_{y_*}^{\infty}dy\frac{y}{\sqrt{y^3-1}}\frac{f(y)}{\sqrt{f(y_*)-f(y)}}, \tag{5.6}$$

where y_* is the turning point and

[12] T is the time interval and the string tension is set equal to 1.

$$f(y)=\frac{1}{\sqrt{3}}\left(\arctan\frac{2y+1}{\sqrt{3}}-\frac{\pi}{2}\right)-\frac{1}{6}\ln\frac{(y-1)^3}{y^3-1},$$

$$f(y_*)=\frac{b^4}{2L^4c_0^2}. \tag{5.7}$$

Note that for any finite value of $f(y_*)$ one has $y_*>1$, meaning that the minimal surface does not reach $\rho=b$ which is the horizon and the curvature singularity. The energy of the string configuration is

$$E=\frac{b^3}{2^{3/2}L^2c_0}\int_{y_*}^{\infty}\frac{y\,dy}{\sqrt{y^3-1}}\frac{1}{\sqrt{f(y_*)-f(y)}}. \tag{5.8}$$

Evaluating the integrals as in [35], i.e. assuming that the main contribution comes from the region near y_*, we find the ''area law,'' i.e., the linear confinement behavior

$$E\approx\frac{c_0}{2}l. \tag{5.9}$$

C. Resolved conifold case

Let us first consider the ''standard'' $b=0$ version of the D3-brane solution on the resolved conifold [11], i.e. Eq. (3.15). Introducing the new coordinate

$$y=\frac{\rho^2}{9a^2} \tag{5.10}$$

and setting $h_0=0$ we get

$$h=\frac{2L^4}{81a^4}f(y),\quad f(y)\equiv y^{-1}-\ln(1+y^{-1}),\quad \kappa=\frac{y+1}{y+\frac{2}{3}}. \tag{5.11}$$

Then the analogue of Eq. (5.6) is

$$\frac{l}{2}=\frac{L^2}{3\sqrt{2}a}\int_{y_*}^{\infty}\frac{dy}{y^{1/2}}\sqrt{\frac{y+\frac{2}{3}}{y+1}}\frac{f(y)}{\sqrt{f(y_*)-f(y)}}, \tag{5.12}$$

where $f(y_*)=81a^4/2L^4c_0^2$. We have used that from the form of the denominator in the analogue of Eq. (5.3) [cf. Eq. (5.12)] it follows that there is a turning point for y; i.e., y changes from ∞ ($\rho=\infty$) to y_*. Note that $f(y)$ is a positive function and it increases monotonically from zero at $y=\infty$. Therefore, for any positive constant d_0 there is $y=y_*$ that solves $d_0^2=y^{-1}-\ln(1+y^{-1})$.

Similar behavior is found when we switch on the b parameter, i.e. start with κ and h given in Eqs. (2.16) and (3.9). Thus the minimal surface does not reach the curvature singularity located at ρ_0.[13] The expression for the energy is (for $b=0$)

$$E=\frac{27a^3}{2^{3/2}L^2c_0}\int_{y_*}^{\infty}\frac{dy}{y^{1/2}}\sqrt{\frac{y+\frac{2}{3}}{y+1}}\frac{1}{\sqrt{f(y_*)-f(y)}}. \tag{5.13}$$

Assuming that the main contribution comes from the region near y_* and expanding $f(y)\approx f(y_*)+f'(y_*)(y-y_*)$ we again get the area law behavior, i.e. the relation (5.9). An analogous result is found when one switches on the dependence of the background metric on the parameter b [37].

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[13]For $b=0$ one can estimate the value of y_* as follows. For small d_0 we expand $f(y)$ near zero and find $y_*=(\sqrt{2}d_0)^{-1}$, i.e. $\rho_*\approx L\sqrt{c_0}$. For large d_0, we expand $f(y)$ for large y to find $y_*=d_0^{-2}$, i.e. $\rho_*\approx\sqrt{2/L^2c_0/3a}$.

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