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Gauging the SU(2) Skyrme model

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In this paper the SU(2) Skyrme model will be reformulated as a gauge theory and the hidden symmetry will be investigated and explored in the energy spectrum computation. To this end we propose a constraint conversion scheme, based on the symplectic framework with the introduction of Wess-Zumino terms in an unambiguous way. It is a positive feature not present in the Batalin-Fradkin-Fradkina-Tyutin constraint conversion. Dirac's procedure for the first-class constraints is employed to quantize this gauge-invariant nonlinear system and the energy spectrum is computed. The result shows the power of the symplectic gauge-invariant formalism when compared with other constraint conversion procedures present in the literature.

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I. INTRODUCTION

We unveil the hidden symmetry of the SU(2) Skyrme model [1] lying in the original phase space. It is a conception not yet investigated. This hidden symmetry will be investigated using the symplectic gauge-invariant formalism. This technique, developed by us in this paper, reformulates noninvariant models as gauge-invariant theories.

The SU(2) Skyrme model is an effective theory that describes the weakly interacting mesons in the chiral limit resulting from the more fundamental theory for strong interactions (QCD) in the limit when the number of colors N_c is taken very large. The collective semiclassical approach [2,3] leads to the isospin quantum corrections to baryon properties. This process reduces the SU(2) Skyrme model to that of a nonrelativistic particle constrained over a sphere, a well-known second-class problem [4,5].

The quantization of nonlinear constrained systems is a serious physical question that has been intensively studied over some decades by many authors [6–9]. However, some problems remain. For example, in the light of the Dirac Hamiltonian formalism [10], these models have field-dependent brackets identified as quantum commutators. As established by quantum mechanics, the quantum operators must be symmetrized adopting an ordering scheme. Since there are different acceptable prescriptions to construct a Hermitian operator, some of them may lead to different physical values, characterizing an operator ordering ambiguity.

Recently, an alternative approach, based on the reformulation of a nonlinear model as a gauge-invariant theory [11–14], has been explored and some success has been achieved. In these papers, Wess-Zumino (WZ) variables were introduced in the theory, as suggested by Faddeev [15], following different constraint-conversion methods [16,17].

In pioneer papers, two of us developed the reduced-SU(2) Skyrme model as a gauge-invariant theory using the Batalin-Fradkin-Fradkina-Tyutin (BFFT) formalism [18,19]. These works inspired many authors [20–22] to investigate the gauge-invariant version for the Skyrme model using different procedures. In these gauge-invariant formalisms, based on the Dirac's framework, the second-class constraints were converted into first-class ones with the introduction of the WZ variables. This process is affected by an ambiguity problem as shown in Ref. [13]. To overcome this kind of problem, we propose to use gauge-invariant formalisms that eliminate this arbitrariness. For example, the gauge-unfixing Hamiltonian formalism [23,24]. This formalism considers half (in the case of bosonic system) of total second-class constraints as gauge fixing terms while the remaining ones form a subset that satisfies a first-class algebra. However, this scheme is restrained to treat systems with even numbers of second-class constraints. In view of this, it is imperative to propose a new approach to carry out the gauge-invariant reformulation, namely, the symplectic gauge-invariant formalism. It is one of the main goals of this paper.

To prove that the symplectic gauge-invariant formalism does not change the physical contents originally present in the second-class reduced-SU(2) Skyrme model, the energy spectrum will be explicitly computed. The result shows that this model may be described, in the same phase-space coordinates, by both gauge invariant and noninvariant descriptions.

To make this paper self-consistent, it was organized as follows. In Sec. II, we shall review the semiclassical expansion of the Skyrme's collective rotational mode. Reduction to a nonlinear quantum mechanical model depending explicitly on the time-dependent collective variables satisfying a spherical constraint is performed. In Sec. III, the symplectic gauge-invariant formalism will be systematized, emphasizing the main steps and advantages. In Sec. IV, we shall disclose the hidden symmetry for the reduced-SU(2) Skyrme model. To this end, this model will be reformulated as a gauge theory via symplectic gauge-invariant method and the infinitesimal gauge transformation will be computed. In Sec. V, the gauge invariant system will be quantized employing Dirac's first-class procedure, and the energy spectrum will be computed. In the Appendix, an alternative approach based on the gauge-unfixing Hamiltonian method [23,24] is shown to lead to canonically equivalent results. The last section is reserved to discuss the physical meaning of our findings together with our final comments and conclusions.

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II. THE REDUCED-SU(2) SKYRME MODEL

The Skyrme model describes baryons and their interactions through soliton solution of the nonlinear sigma model-type Lagrangian given by

$$L=\int d^3x\left[-\frac{F_\pi^2}{16}Tr\,(\partial_\mu U\,\partial^u U^+)+\frac{1}{32e^2}Tr[U^+\,\partial_\mu U,U^+\,\partial_\nu U]^2\right], \tag{1}$$

where F_π is the pion decay constant, e is a dimensionless parameter, and U is a SU(2) matrix. The collective semiclassical expansion [2] is performed just substituting $U(x^\mu)$ by $U(x^\mu)=A(t)U(r)A^+(t)$ in (1), where A is a SU(2) matrix, we obtain

$$L=-M+\lambda Tr[\partial_0 A\,\partial_0 A^{-1}], \tag{2}$$

where

$$M=\frac{F_\pi}{e}I_1 \tag{3}$$

and

$$\lambda=\frac{1}{e^3F_\pi}I_2 \tag{4}$$

are the soliton mass and the moment of inertia, respectively, and I_1,I_2 are adimensional values depending on the classical solution of the model. A is a SU(2) matrix that can be written as $A=a_0+i a\cdot\tau$, where τ_i are the Pauli matrices, and satisfies the constraint relation

$$T_1=a_ia_i-1\approx 0,\quad i=0,1,2,3. \tag{5}$$

Then, the Lagrangian (2) can be read as a function of the a_i as

$$L=-M+2\lambda\dot{a}_i\dot{a}_i. \tag{6}$$

Calculating the canonical momenta

$$\pi_i=\frac{\partial L}{\partial\dot{a}_i}=4\lambda\dot{a}_i, \tag{7}$$

and using the Legendre transformation, the canonical Hamiltonian is computed as

$$H_c=\pi_i\dot{a}_i-L=M+2\lambda\dot{a}_i\dot{a}_i=M+\frac{1}{8\lambda}\sum_{i=0}^{3}\pi_i\pi_i. \tag{8}$$

A typical polynomial wave function, $1/N(l)(a_1+ia_2)^l=|\text{polynomial}\rangle$, is an eigenvector of the Hamiltonian (8). This wave function is also eigenvector of the spin and isospin operators, written in [3] as $J_k=\frac{1}{2}(a_0\pi_k-a_k\pi_0-\epsilon_{klm}a_l\pi_m)$ and $I_k=\frac{1}{2}(a_k\pi_0-a_0\pi_k-\epsilon_{klm}a_l\pi_m)$.

Constructing the total Hamiltonian and imposing that the constraint has no time evolution [10], we get a new constraint

$$T_2=a_i\pi_i\approx 0. \tag{9}$$

We observe that no further constraints are generated via this iterative procedure because T_1 and T_2 are second-class constraints. The matrix elements of their Poisson brackets read

$$\Delta_{\alpha\beta}=\{T_\alpha,T_\beta\}=-2\epsilon_{\alpha\beta}a_ia_i,\quad \alpha,\beta=1,2, \tag{10}$$

where $\epsilon_{\alpha\beta}$ is the antisymmetric tensor normalized as $\epsilon_{12}=-\epsilon^{12}=-1$.

III. SYMPLECTIC GAUGE-INVARIANT FORMALISM

In the literature there are several schemes to reformulate noninvariant models as gauge theories. Recently, some constraint-conversion formalisms, based on Dirac's method [10], were developed following Faddeev's idea of phase-space extension with the introduction of auxiliary variables [15]. Among them, the BFFT [16] and the iterative [17] methods were powerful enough to be successfully applied to a great number of important physical models. Although these techniques share the same conceptual basis [15] and follow Dirac's framework [10], these constraint-conversion methods were implemented following different directions. Historically, both BFFT and the iterative methods were applied to deal with linear systems such as chiral gauge theories [17,25] in order to eliminate the gauge anomaly that hampers the quantization process. In spite of the great success achieved by these methods, they have an ambiguity problem [13]. This problem naturally arises when the second-class constraint is converted into a first-class one with the introduction of WZ variables. Due to this, the constraint conversion process may become a hard task [13]. In this section, we reformulate noninvariant systems as gauge theories using a technique that is not affected by this ambiguity problem. This technique follows Faddeev's suggestion [15] and is set up on a contemporary framework to handle noninvariant model, namely, the symplectic formalism [26,27].

In order to systematize the symplectic gauge-invariant formalism, we consider a general noninvariant mechanical model whose dynamics is governed by a Lagrangian $\mathcal{L}(a_i,\dot{a}_i,t)$ (with $i=1,2,\dots,N$), where a_i and $\dot{a}_i$ are the space and velocity variables, respectively. Notice that this model does not lead to lost generality or physical content. Following the symplectic method the Lagrangian is written in its first-order form as

$$\mathcal{L}^{(0)}=A_\alpha^{(0)}\dot{\xi}_\alpha^{(0)}-V^{(0)}, \tag{11}$$

where $\xi_\alpha^{(0)}(a_i,p_i)$ (with $\alpha=1,2,\dots,2N$) are the symplectic variables, $A_\alpha^{(0)}$ are the one-form canonical momenta, (0) indicates that it is the zeroth-iterative Lagrangian and $V^{(0)}$ is the symplectic potential. Then the symplectic tensor, defined as

$$f^{(0)}_{\alpha\beta}=\frac{\partial A^{(0)}_{\beta}}{\partial \xi^{(0)}_{\alpha}}-\frac{\partial A^{(0)}_{\alpha}}{\partial \xi^{(0)}_{\beta}} \tag{12}$$

is computed. Since this symplectic matrix is singular, it has a zero-mode ($\nu^{(0)}$) that generates a new constraint when contracted with the gradient of potential: namely,

$$\Omega^{(0)}=\nu^{(0)}_{\alpha}\frac{\partial V^{(0)}}{\partial \xi^{(0)}_{\alpha}}. \tag{13}$$

Through a Lagrange multiplier η, this constraint is introduced into the zeroth-iterative Lagrangian (11), generating the next one,

$$\begin{aligned}\mathcal{L}^{(1)}&=A^{(0)}_{\alpha}\dot{\xi}^{(0)}_{\alpha}-V^{(0)}+\dot{\eta}\Omega^{(0)},\\&=A^{(1)}_{\alpha}\dot{\xi}^{(1)}_{\alpha}-V^{(1)},\end{aligned} \tag{14}$$

where

$$\begin{aligned}V^{(1)}&=V^{(0)}\big|_{\Omega^{(0)}=0},\\ \xi^{(1)}_{\alpha}&=(\xi^{(0)}_{\alpha},\eta),\\ A^{(1)}_{\alpha}&=A^{(0)}_{\alpha}+\eta\frac{\partial\Omega^{(0)}}{\partial\xi^{(0)}_{\alpha}}.\end{aligned} \tag{15}$$

The first-iterative symplectic tensor is computed as

$$f^{(1)}_{\alpha\beta}=\frac{\partial A^{(1)}_{\beta}}{\partial \xi^{(1)}_{\alpha}}-\frac{\partial A^{(1)}_{\alpha}}{\partial \xi^{(1)}_{\beta}}. \tag{16}$$

Since this tensor is nonsingular, the iterative process stops and Dirac's brackets among the phase-space variables are obtained from the inverse matrix $(f^{(1)}_{\alpha\beta})^{-1}$. On the contrary, the tensor is singular and a new constraint arises, indicating that the iterative process goes on.

After this brief review, the symplectic gauge-invariant formalism will be systematized. It starts with the introduction of an extra term dependent on the original and WZ variable, $G(a_i,p_i,\theta)$, into the first-order Lagrangian. This extra term, expands as

$$G(a_i,p_i,\theta)=\sum_{n=0}^{\infty}\mathcal{G}^{(n)}(a_i,p_i,\theta), \tag{17}$$

where $\mathcal{G}^{(n)}(a_i,p_i,\theta)$ is a term of order n in θ, satisfies the boundary condition

$$G(a_i,p_i,\theta=0)=\mathcal{G}^{(n=0)}(a_i,p_i,\theta=0)=0. \tag{18}$$

The symplectic variables were extended to also contain the WZ variable $\tilde{\xi}^{(1)}_{\tilde{\alpha}}=(\xi^{(0)}_{\alpha},\eta,\theta)$ (with $\tilde{\alpha}=1,2,\ldots,2N+2$) and the first-iterative symplectic potential becomes

$$\tilde{V}^{(1)}_{(n)}(a_i,p_i,\theta)=V^{(1)}(a_i,p_i)-\sum_{n=0}^{\infty}\mathcal{G}^{(n)}(a_i,p_i,\theta). \tag{19}$$

For $n=0$, we have

$$\tilde{V}^{(1)}_{(n=0)}(a_i,p_i,\theta)=V^{(1)}(a_i,p_i). \tag{20}$$

Subsequently, we impose that the symplectic tensor ($f^{(1)}$) is a singular matrix with the corresponding zero-mode

$$\tilde{\nu}^{(1)}_{\tilde{\alpha}}=\begin{pmatrix}\nu^{(1)}_{\alpha} & 1\end{pmatrix}, \tag{21}$$

as the generator of gauge symmetry. Due to this, the correction terms $\mathcal{G}^{(n)}(a_i,p_i,\theta)$ in order of θ can be explicitly computed. Contracting the zero-mode ($\tilde{\nu}^{(1)}_{\tilde{\alpha}}$) with the gradient of potential $\tilde{V}^{(1)}_{(n)}(a_i,p_i,\eta,\theta)$ and imposing that no more constraint is generated, a general differential equation is obtained, that reads as

$$\tilde{\nu}^{(1)}_{\tilde{\alpha}}\frac{\partial\tilde{V}^{(1)}_{(n)}(a_i,p_i,\theta)}{\partial\tilde{\xi}^{(1)}_{\tilde{\alpha}}}=0, \tag{22}$$

$$\nu^{(1)}_{\alpha}\frac{\partial V^{(1)}(a_i,p_i)}{\partial\xi^{(1)}_{\alpha}}-\sum_{n=0}^{\infty}\frac{\partial\mathcal{G}^{(n)}(a_i,p_i,\theta)}{\partial\theta}=0,$$

which allows us to compute all correction terms in order of θ. For linear correction term, we have

$$\nu^{(1)}_{\alpha}\frac{\partial V^{(1)}_{(n=0)}(a_i,p_i)}{\partial\xi^{(1)}_{\alpha}}-\frac{\partial\mathcal{G}^{(n=1)}(a_i,p_i,\theta)}{\partial\theta}=0. \tag{23}$$

For quadratic correction term, we get

$$\tilde{\nu}^{(1)}_{\tilde{\alpha}}\frac{\partial V^{(1)}_{(n=1)}(a_i,p_i,\theta)}{\partial\tilde{\xi}^{(1)}_{\tilde{\alpha}}}-\frac{\partial\mathcal{G}^{(n=2)}(a_i,p_i,\theta)}{\partial\theta}=0. \tag{24}$$

From these equations, a recursive equation for $n\geqslant 1$ is proposed as

$$\tilde{\nu}^{(1)}_{\tilde{\alpha}}\frac{\partial V^{(1)}_{(n-1)}(a_i,p_i,\theta)}{\partial\tilde{\xi}^{(1)}_{\tilde{\alpha}}}-\frac{\partial\mathcal{G}^{(n)}(a_i,p_i,\theta)}{\partial\theta}=0, \tag{25}$$

which allows us to compute each correction term in order of θ. This iterative process is successively repeated until Eq. (22) becomes identically null, consequently, the extra term $G(a_i,p_i,\theta)$ is obtained explicitly. Then, the gauge-invariant Hamiltonian, identified as being the symplectic potential, is obtained as

$$\tilde{\mathcal{H}}(a_i,p_i,\theta)=V^{(1)}_{(n)}(a_i,p_i,\theta)=V^{(1)}(a_i,p_i)+G(a_i,p_i,\theta), \tag{26}$$

and the zero-mode $\tilde{\nu}^{(1)}_{\tilde{\alpha}}$ is identified as being the generator of an infinitesimal gauge transformation, given by

$$\delta\tilde{\xi}_{\tilde{\alpha}}=\varepsilon\,\tilde{\nu}^{(1)}_{\tilde{\alpha}}, \tag{27}$$

where ε is an infinitesimal time-dependent parameter.

In the next section, we reformulate the SU(2) Skyrme model as a gauge theory that recently has been intensively studied in the literature from many points of view [9,18–20,22], using the symplectic gauge-invariant process.

IV. EMBEDDING THE SU(2) SKYRME MODEL

In this section, the hidden symmetry of the reduced SU(2) Skyrme model will be disclosed enlarging the phase space with the introduction of the Wess-Zumino variable via symplectic gauge-invariant formalism. To put this work in a correct perspective, we first apply the symplectic method to the original second-class model that allows us to show the second-class nature of the model and also to obtain the usual Dirac's brackets. Later, we unveil the hidden gauge symmetry of the model.

In order to implement the symplectic method, the original second-order Lagrangian in the velocity, given in Eq. (6), is reduced to a first-order form, namely,

$$L^{(0)}=\pi_i\dot{a}_i-M-\frac{1}{8\lambda}\pi_i\pi_i+\eta(a_ia_i-1), \tag{28}$$

where the index (0) indicates the zeroth-iterative Lagrangian, and the Lagrange multiplier (η) enforces the spherical constraint (5) into the theory. Then the symplectic tensor, defined as

$$f_{\alpha\beta}=\frac{\partial A_\beta}{\partial\xi^\alpha}-\frac{\partial A_\alpha}{\partial\xi^\beta}, \tag{29}$$

must be computed. The zeroth-iterative symplectic variables are $\xi_\alpha^{(0)}=(a_j,\pi_j,\eta)$ and the corresponding one-form canonical momenta are given by

$$A_{a_i}^{(0)}=\pi_i,$$
$$A_{\pi_i}^{(0)}=A_\lambda^{(0)}=0. \tag{30}$$

Then, the zeroth-iterative symplectic tensor is

$$f^{(0)}=\begin{pmatrix}0 & -\delta_{ij} & 0\\ \delta_{ij} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{31}$$

This matrix is obviously singular, thus, it has a zero-mode

$$v^{(0)}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ 1\end{pmatrix}, \tag{32}$$

which generates the following constraint:

$$\Omega_1=v_\alpha^{(0)}\frac{\partial V^{(0)}}{\partial A_\alpha^{(0)}},$$
$$=a_ia_i-1, \tag{33}$$

where the zeroth-iterative potential $V^{(0)}$ is given as

$$V^{(0)}=M+\frac{1}{8\lambda}\pi_i\pi_i-\eta(a_ia_i-1). \tag{34}$$

Bringing back the constraint Ω_1 into the canonical sector of the first-order Lagrangian by means of a Lagrange multiplier ρ, we get the first-iterative Lagrangian $L^{(1)}$, namely,

$$L^{(1)}=\pi_i\dot{a}_i+(a_ia_i-1)\dot{\rho}-M-\frac{1}{8\lambda}\pi_i\pi_i, \tag{35}$$

where $\eta\to\dot{\rho}$. Therefore, the symplectic variables become $\xi_\alpha^{(1)}=(a_j,\pi_j,\rho)$ with the following one-form canonical momenta:

$$A_{a_i}^{(1)}=\pi_i,$$
$$A_{\pi_i}^{(1)}=0, \tag{36}$$
$$A_\rho^{(1)}=a_ia_i-1.$$

The corresponding matrix $f^{(1)}$ is

$$f^{(1)}=\begin{pmatrix}0 & -\delta_{ij} & 2a_i\\ \delta_{ij} & 0 & 0\\ -2a_i & 0 & 0\end{pmatrix}, \tag{37}$$

which is singular. The corresponding zero-mode is

$$v^{(1)}=\begin{pmatrix}\mathbf{0}\\ a_i\\ 1/2\end{pmatrix}, \tag{38}$$

which generates the following constraint:

$$\Omega_2=v_\alpha^{(1)}\frac{\partial V^{(1)}}{\partial A_\alpha^{(1)}}=a_i\pi_i\approx 0, \tag{39}$$

where

$$V^{(1)}=+M+\frac{1}{8\lambda}\pi_i\pi_i. \tag{40}$$

The twice-iterated Lagrangian, obtained after including the constraint (39) into the Lagrangian (35) through a Lagrange multiplier ζ, reads

$$L^{(2)}=\pi_i\dot{a}_i+(a_ia_i-1)\dot{\rho}+a_i\pi_i\dot{\zeta}-V^{(2)}, \tag{41}$$

with $V^{(2)}=V^{(1)}$. The enlarged symplectic variables are $\xi_\alpha^{(2)}=(a_j,\pi_j,\rho,\zeta)$. The new one-form canonical momenta are

$$A_{a_i}^{(2)}=\pi_i,$$
$$A_{\pi_i}^{(2)}=0,$$
$$A_\rho^{(2)}=a_ia_i-1,$$
$$A_\zeta^{(2)}=a_i\pi_i,$$

and the corresponding matrix $f^{(2)}$ is

$$f^{(2)}=\begin{pmatrix} 0 & -\delta_{ij} & 2a_i & \pi_i \\ \delta_{ij} & 0 & 0 & a_i \\ -2a_i & 0 & 0 & 0 \\ -\pi_i & -a_i & 0 & 0 \end{pmatrix}, \tag{42}$$

which is a nonsingular matrix. The inverse of $f^{(2)}$ gives the usual Dirac brackets among the physical variables obtained in a straightforward calculation. This means that the SU(2) Skyrme model is not a gauge-invariant theory.

At this stage we are ready to implement our proposal. In order to disclose the hidden symmetry present on the reduced-SU(2) Skyrme model via symplectic gauge-invariant formalism, the original phase space will be extended with the introduction of an extra function G depending on the original phase-space variables and the WZ variable θ, defined as

$$G(a_i,\pi_i,\theta)=\sum_{n=0}^{\infty}\mathcal{G}^{(n)}, \tag{43}$$

which satisfies the boundary condition

$$G(a_i,\pi_i,\theta=0)=\mathcal{G}^{(0)}=0. \tag{44}$$

Introducing the new term G into the Lagrangian (35), we have

$$\tilde{L}^{(1)}=\pi_i\dot{a}_i+(a_ia_i-1)\dot{\rho}-M-\frac{1}{8\lambda}\pi_i\pi_i+G(a_i,\pi_i,\theta). \tag{45}$$

The enlarged symplectic variables are $\tilde{\xi}_{\alpha}^{(1)}=(a_j,\pi_j,\rho,\theta)$ with the following one-form canonical momenta

$$\begin{aligned}\tilde{A}_{a_i}^{(1)}&=\pi_i,\\ \tilde{A}_{\pi_i}^{(1)}&=0,\\ \tilde{A}_{\rho}^{(1)}&=a_ia_i-1,\\ \tilde{A}_{\theta}^{(1)}&=0.\end{aligned} \tag{46}$$

Then, we compute the matrix $\tilde{f}^{(1)}$ as

$$\tilde{f}^{(1)}=\begin{pmatrix} 0 & -\delta_{ij} & 2a_i & 0 \\ \delta_{ij} & 0 & 0 & 0 \\ -2a_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{47}$$

which is obviously singular. Consequently, it has the following zero-mode

$$v^{(1)}=\begin{pmatrix} \mathbf{0} \\ a_i \\ 1/2 \\ 1 \end{pmatrix}. \tag{48}$$

Imposing that no more constraint is generated by this zero mode ($v^{(1)}$), the first-order correction term in θ, $\mathcal{G}^{(1)}$, is determined after an integration process, namely,

$$\mathcal{G}^{(1)}(a_i,\pi_i,\theta)=\frac{1}{4\lambda}(a_i\pi_i)\theta. \tag{49}$$

Bringing back this expression into Eq. (45), the new Lagrangian is obtained as

$$\tilde{L}^{(1)}=\pi_i\dot{a}_i+(a_ia_i-1)\dot{\rho}-M-\frac{1}{8\lambda}\pi_i\pi_i+\frac{1}{4\lambda}(a_i\pi_i)\theta, \tag{50}$$

which is not yet a gauge-invariant Lagrangian because the zero-mode $v^{(1)}$ still generates a new constraint that reads as

$$v_{\alpha}^{(1)}\frac{\partial V^{(1)}}{\partial\xi_{\alpha}}=+\frac{1}{4\lambda}a_i^2\theta. \tag{51}$$

It indicates that it is necessary to obtain the remaining correction terms $\mathcal{G}^{(n)}$ in order of θ. It is achieved by just imposing that the zero-mode does not generate more constraint. It allows us to determine the second-order correction term $\mathcal{G}^{(2)}$ given by

$$\begin{aligned}v_{\alpha}^{(1)}\frac{\partial V^{(1)}}{\partial\xi_{\alpha}}&=\frac{1}{4\lambda}a_i^2\theta+\frac{\partial\mathcal{G}^{(2)}}{\partial\theta}=0,\\ \mathcal{G}^{(2)}&=-\frac{1}{8\lambda}a_i^2\theta.\end{aligned} \tag{52}$$

Bringing this result into the first-order Lagrangian (50), we obtain

$$\tilde{L}^{(1)}=\pi_i\dot{a}_i+(a_ia_i-1)\dot{\rho}-M-\frac{1}{8\lambda}\pi_i\pi_i+\frac{1}{4\lambda}(a_i\pi_i)\theta-\frac{1}{8\lambda}a_i^2\theta. \tag{53}$$

The zero-mode $v^{(1)}$ does not produce a new constraint, consequently, the model has a symmetry and it is the generator of an infinitesimal gauge transformation. Due to this, all correction terms $\mathcal{G}^{(n)}$ with $n\geqslant 3$ are null.

At this moment, we are interested to recover the invariant second-order Lagrangian from its first-order form given in Eq. (53). To this end, the canonical momenta must be eliminated from the Lagrangian (53). From the equation of motion for π_i, the canonical momenta are computed as

$$\pi_i=4\lambda\dot{a}_i+a_i\theta. \tag{54}$$

Inserting this result into the first-order Lagrangian, given by

$$\tilde{L}^{(0)}=\pi_i\dot{a}_i+(a_ia_i-1)\eta-M-\frac{1}{8\lambda}\pi_i\pi_i+\frac{1}{4\lambda}(a_i\pi_i)\theta -\frac{1}{8\lambda}a_i^2\theta^2, \tag{55}$$

the second-order Lagrangian is obtained as

$$\tilde{L}=-M+2\lambda\dot{a}_i^2+(a_i\dot{a}_i)\theta+(a_ia_i-1)\eta, \tag{56}$$

with the corresponding gauge-invariant Hamiltonian

$$\tilde{H}=M+\frac{1}{8\lambda}\pi_i\pi_i-\frac{1}{4\lambda}(a_i\pi_i)\theta+\frac{1}{8\lambda}a_i^2\theta^2-\eta(a_ia_i-1). \tag{57}$$

By construction, both Lagrangian (56) and Hamiltonian (57) are gauge invariant.

To make this work self-consistent the infinitesimal gauge transformation will be determined using the symplectic method. To this end, we start with the first-order Lagrangian (53) in terms of the symplectic variables $\tilde{\xi}_\alpha^{(1)}=(a_j,\pi_j,\rho,\theta)$, that generates the singular symplectic matrix (47) with the zero-mode (48). This zero-mode is identified as being the generator of the infinitesimal gauge transformation $\delta\tilde{\xi}_\alpha^{(1)}=\varepsilon v^{(1)}$, given by

$$\delta a_i=0,$$

$$\delta\pi_i=\varepsilon a_i,$$

$$\delta\eta=\frac{1}{2}\dot{\varepsilon}, \quad (\eta\to\dot{\rho}) \tag{58}$$

$$\delta\theta=\varepsilon. \tag{59}$$

Note that both Hamiltonian and Lagrangian are invariant under this transformation. Similar results were also obtained in the literature using different methods based on Dirac's constraint idea [11,12,18–21]. However, these methods are affected by some ambiguity problems that naturally arise when it is necessary to obtain the second-class constraints and then determine how they will be converted to first-class ones. It occurs when the phase space is extended with the introduction of the WZ variables. In our procedure, this problem does not arise, consequently, the arbitrariety disappears. This completes one of the main goals of this paper.

Henceforth we are interested to disclose the hidden symmetry of the reduced-SU(2) Skyrme model and obtain both Hamiltonian and Lagrangian in terms of the original coordinates (a_i,π_i). To this end, we will obtain the set of constraints of the invariant model described by the Lagrangian (56) and Hamiltonian (57). Indeed, the model has two constraint chains, namely,

$$\phi_1=\pi_\eta,$$

$$\phi_2=a_ia_i-1, \tag{60}$$

and

$$\varphi_1=\pi_\theta,$$

$$\varphi_2=a_i\pi_i-a_i^2\theta, \tag{61}$$

where π_θ is the canonical momentum conjugated to the WZ variable θ. The Dirac matrix is singular, however, there are nonvanishing Poisson brackets among some constraints, indicating that there are both second-class and first-class constraints. It is solved by splitting up the second-class constraints from the first-class ones through the constraints combination. The set of first-class constraints is

$$\chi_1=\pi_\eta,$$

$$\chi_2=a_ia_i-1-2\pi_\theta, \tag{62}$$

while the set of second-class constraints is

$$\chi_1=\pi_\theta,$$

$$\chi_1=a_i\pi_i-a_i^2\theta. \tag{63}$$

Since the second-class constraints are assumed in a strong way, and using the Maskawa-Nakajima theorem [28], Dirac's brackets are worked out as

$$\{a_i,a_i\}=0,$$

$$\{a_i,p_i\}=\delta_{ij},$$

$$\{p_i,p_i\}=0, \tag{64}$$

as well as the Hamiltonian,

$$\tilde{H}=M+\frac{1}{8\lambda}\pi_i\pi_i-\frac{1}{8\lambda}\frac{(a_i\pi_i)^2}{a_ia_i}-\eta(a_ia_i-1)$$

$$=M+\frac{1}{8\lambda}\pi_iM_{ij}\pi_j-\eta(a_ia_i-1), \tag{65}$$

where

$$M_{ij}=\delta_{ij}-\frac{a_ia_j}{a_k^2} \tag{66}$$

is a singular matrix. We can show that $\tilde{H}$, Eq. (65), satisfies the first-class property

$$\{T_1,\tilde{H}\}=0. \tag{67}$$

Due to this the first-class constraint (T_1) is the generator of the gauge symmetry. The infinitesimal gauge transformation is computed as

$$\delta a_i=\varepsilon\{a_i,T_1\}=0,$$

$$\delta\pi_i=\varepsilon\{\pi_i,T_1\}=\varepsilon a_i, \tag{68}$$

where ε is an infinitesimal time-dependent parameter. It is easy to verify that the Hamiltonian (65) is invariant under these transformations because a_i are eigenvectors of the phase space metric (M_{ij}) with eigenvalue null. It reproduces the result discussed in the Appendix using the gauge-unfixing Hamiltonian formalism.

V. THE SPECTRUM OF THE HAMILTONIAN

In this section, we will derive the SU(2) Skyrmion energy levels. Normally, these results were employed to obtain the baryon static properties [2,3]. In this first-class theory the quantization is performed following Dirac's first-class prescription [10] by just imposing that the physical wave functions are annihilated by the first-class operator constraint that reads as

$$\xi|\psi\rangle_{phys}=0. \tag{69}$$

The physical states that satisfy (69) are

$$|\psi\rangle_{phys}=\frac{1}{V}\,\delta(a_ia_i-1)\,|\text{polynomial}\rangle, \tag{70}$$

where V is the normalization factor and $|\text{polynomial}\rangle=1/N(l)(a_1+ia_2)^l$. The corresponding quantum Hamiltonian is

$$\tilde{H}-M+\frac{1}{8\lambda}\pi_iM_{ij}\pi_j-\eta(a_ia_i-1). \tag{71}$$

Thus, in order to obtain the spectrum of the theory, we take the scalar product, ${}_{phys}\langle\psi|\tilde{H}|\psi\rangle_{phys}$, which is the mean value of the first-class Hamiltonian. We begin calculating the scalar product, given by

$${}_{phys}\langle\psi|\tilde{H}|\psi\rangle_{phys}=\langle\text{polynomial}|\ \frac{1}{V^2}\int da_i\ \delta(a_ia_i-1)\,\tilde{H}\times\delta(a_ia_i-1)|\text{polynomial}\rangle. \tag{72}$$

Integrating over a_i, we obtain

$${}_{phys}\langle\psi|\tilde{H}|\psi\rangle_{phys}=\langle\text{polynomial}|M+\frac{1}{8\lambda}[\pi_i\pi_i-(a_i\pi_i)^2]|\text{polynomial}\rangle. \tag{73}$$

Here we would like to comment that the regularization of delta function squared $\delta(a_ia_i-1)^2$ is performed using the delta relation, $(2\pi)^2\delta(0)=\lim_{k\to 0}\int d^2x\, e^{ik\cdot x}=\int d^2x=V$. Then, we use the parameter V as the normalization factor. The Hamiltonian operator inside the kets, Eq. (73), can be rewritten as

$${}_{phys}\langle\psi|\tilde{H}|\psi\rangle_{phys}=\langle\text{polynomial}|M+\frac{1}{8\lambda}[p_k\cdot p_k]|\text{polynomial}\rangle, \tag{74}$$

where $p_k=\pi_k-a_k(a_j\pi_j)$. The operator π_k describes a free particle and their representations on the collective coordinates a_k are

$$\pi_k=-i\frac{\partial}{\partial a_k}. \tag{75}$$

The algebraic expression of p_k lead to ordering problems in the first-class Hamiltonian operator $\tilde{H}$. We adopt the well-known Weyl ordering prescription [29] to symmetrize the p_k expression, and consequently $\tilde{H}$. We count all possible random orders of π_i and a_k. Then, the symmetrized expressions for p_k are

$$\begin{aligned}[p_k]_{sym}&=\frac{1}{6i}(6\partial_k-a_ka_i\partial_i-a_k\partial_ia_i-a_ia_k\partial_i\\&\quad-a_i\partial_ia_k-\partial_ia_ka_i-\partial_ia_ia_k)\\&=\frac{1}{i}\left(\partial_k-a_ka_i\partial_i-\frac{5}{2}a_k\right),\end{aligned} \tag{76}$$

leading to the symmetrized first-class Hamiltonian operator

$$[\tilde{H}]_{sym}=M+\frac{1}{8\lambda}\left[-\partial_j\partial_j+\frac{1}{2}\left(OpOp+2Op+\frac{5}{4}\right)\right], \tag{77}$$

where Op is defined as $Op\equiv a_i\partial_i$. Putting the expression (77) in the mean value (74) we obtain the energy levels as

$$E_l={}_{phys}\langle\psi|\tilde{H}|\psi\rangle_{phys}=M+\frac{1}{8\lambda}\left[l(l+2)+\frac{5}{4}\right]. \tag{78}$$

We would like to comment that the last expression, Eq. (78), matches the result obtained in Ref. [9], where the SU(2) Skyrme model was quantized via second-class Dirac's method. It becomes an interesting point since this extra term plays an important role in the energy Skyrmion spectrum [20]. It can be shown by just observing in Eq. (78) that the value of the soliton mass (M) Eq. (3), and the inertia moment (λ) Eq. (4) are determined using the nucleon (l=1) and the delta (l=2) masses as input parameters. Consequently, the values of F_π, e, and the remaining phenomenological results can be predicted. Then, it is clear that an extra term, resulting from a second- or first-class quantization scheme together with a symmetrization procedure can modify the spectrum and, therefore, the physical values predicted by the Skyrme model. In the context of the non-Abelian and Abelian BFFT formalisms (used by two of us in early papers [18,19]) the extra constant term in the energy formula Eq. (78) does not match that obtained in the second-class formalism [9].

VI. FINAL DISCUSSIONS

In this paper, we propose a gauge-invariant formalism that is not affected by an ambiguity problem related to the introduction of the WZ variables. This formalism was systematized and applied on the reduced-SU(2) Skyrme model. The hidden symmetry living in the original phase space was investigated which is an unexpected result for a second-class system. Afterward, this invariant model was quantized employing Dirac's first-class procedure. Using the Weyl ordering prescription to symmetrize the operators, we obtained exactly the same energy spectrum when compared with the reduced second-class Skyrme model. It is an important feature that does not occur when the BFFT method is used [18–20]. We believe that the arbitrary algebra in the extended model, induced by the introduction of the Wess-Zumino variables, leads to the discrepancy between the first- and the second-class Skyrmion energy spectrum. In view of this, different constraint conversion schemes introduce distinct modifications in the energy spectrum [18–20] and, consequently, change the phenomenological results, as discussed in Sec. V.

Our results prove that the SU(2) Skyrme model has also a gauge invariant description [on the original phase-space coordinates (a_i, p_i)] dynamically equivalent to the usual second-class treatment. It seems important since our scheme does not affect the baryon phenomenology initially predicted by the second-class model, in opposition to another gauge-invariant formalism [18–22]. Thus, the symplectic gauge-invariant formalism leads to a more elegant and simplified first-class Hamiltonian structure than the Abelian and non-Abelian BFFT cases.

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APPENDIX

1. The gauge-unfixing formalism for the reduced SU(2) Skyrme model

The main idea of the gauge unfixing procedure is to consider half of the total second-class constraints as gauge fixing terms while the remaining ones are the gauge symmetry generators [24,30]. Here, the gauge-unfixing Hamiltonian formalism will be applied to the reduced-SU(2) Skyrme model reviewed in Sec. II. We start redefining the constraint $T_1 = a_i a_i - 1$ as

$$\xi = C^{-1} T_1, \tag{A1}$$

where C is

$$C = \{T_1, T_2\} = 2a_i a_i = 2. \tag{A2}$$

Then the total Hamiltonian is written as

$$H = M + \frac{1}{8\lambda}\pi_i\pi_i + \eta_1\xi + \eta_2 T_2, \tag{A3}$$

where η_1 and η_2 are the Lagrange multipliers that enforces the constraints ξ and T_2 into the Hamiltonian. Imposing that the constraints ξ and ψ are conserved on time, the Lagrange multipliers are obtained as

$$\eta_1 = \frac{1}{4\lambda}\pi_i\pi_i, \tag{A4}$$

$$\eta_2 = -\frac{1}{4\lambda} a_i\pi_i. \tag{A5}$$

Substituting Eq. (A4) and Eq. (A5) in the total Hamiltonian given in Eq. (A3), we get

$$H = M + \frac{1}{8\lambda}\pi_i\pi_i - \frac{1}{4\lambda}(a_i\pi_i)^2. \tag{A6}$$

Then, we are ready to derive the gauge-invariant Hamiltonian using the formula [24] given by

$$\tilde{H} = H - \psi\{\xi, H\} + \frac{1}{2!}\psi^2\{\xi,\{\xi,H\}\} - \frac{1}{3!}\psi^3\{\xi,\{\xi,\{\xi,H\}\}\} + \cdots. \tag{A7}$$

The right-hand terms $\{\xi,H\}$ and $\{\xi,\{\xi,H\}\}$ are computed,

$$\{\xi,H\} = -\frac{1}{4\lambda}a_i\pi_i, \tag{A8}$$

$$\{\xi,\{\xi,H\}\} = -\frac{1}{4\lambda}. \tag{A9}$$

From Eq. (A9) we note that the terms in (A7), $\{\xi,\{\xi,\{\xi,H\}\}\}$, and the remaining higher orders are zero. Then, the invariant Hamiltonian reads

$$\tilde{H} = M + \frac{1}{8\lambda}\pi_i\pi_i - \frac{1}{8\lambda}(a_i\pi_i)^2, \\ = M + \frac{1}{8\lambda}\pi_i\bar{M}_{ij}\pi_j, \tag{A10}$$

where

$$\bar{M}_{ij} = \delta_{ij} - a_i a_j \tag{A11}$$

is a singular matrix. We can show that $\tilde{H}$, Eq. (A10), satisfies the first-class property

$$\{\xi, \tilde{H}\} = 0. \tag{A12}$$

Due to this the first-class constraint (ξ) is the generator of the gauge symmetry. The infinitesimal gauge transformations are computed as

$$\delta a_i = \varepsilon\{a_i, \xi\} = 0, \\ \delta\pi_i = \varepsilon\{\pi_i, \xi\} = \varepsilon a_i, \tag{A13}$$

where ε is an infinitesimal time-dependent parameter. It is easy to verify that the Hamiltonian (A10) is invariant under these transformations because a_i are eigenvectors of the phase space metric ($\bar{M}_{ij}$) with eigenvalues null.

To complete this section, we would like to remark that the algebraic expression for the first-class Hamiltonian, Eq. (A10), is more simple than obtained via the Abelian and non-Abelian BFFT formalism as shown by two of us in Ref. [18,19]. In the context of Abelian formalism [18], the first-class Hamiltonian has a geometrical series form, while in the non-Abelian formalism [19,20] the first-class Hamiltonian has a finite number of terms, but this algebraic formula is large.

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