Approximate decoherence of histories and 't Hooft's deterministic quantum theory

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In the decoherent histories approach to quantum theory, sets of histories are said to be decoherent when the decoherence functional, measuring interference between pairs of histories, is exactly diagonal. In realistic situations, however, only approximate diagonality is ever achieved, raising the question of what approximate decoherence actually means and how it is related to exact decoherence. This paper explores the possibility that an exactly decoherent set of histories may be constructed from an approximate set by small distortions of the operators characterizing the histories. In particular, for the case of histories of positions and momenta, this is achieved by doubling the set of operators and then finding, among this enlarged set, new position and momentum operators that commute, and so decohere exactly, and which are ''close'' to the original operators. Two derivations are given: one in terms of the decoherence functional, the second in terms of Wigner functions. The enlarged, exactly decoherent theory has the same classical dynamics as the original one, and coincides with the so-called deterministic quantum theories of the type recently studied by 't Hooft. These results suggest that the comparison of standard and deterministic quantum theories may provide an alternative method of characterizing emergent classicality. A side product is the surprising result that histories of momenta in the quantum Brownian motion model (for the free particle in the high-temperature limit) are exactly decoherent.

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I. INTRODUCTION

How close to classical mechanics can quantum mechanics be? One of the main aims of the decoherent histories approach is to demonstrate the emergence of classical mechanics as an effective theory, starting from the assumption that quantum mechanics is the exact underlying theory [1–5]. In such studies, the effective classical theory almost always emerges in an approximate way, rarely exact. The main reason for this is that decoherence, the destruction of quantum interference, is almost always approximate. What does approximate decoherence mean? What is the nature of the histories that approximately decoherent histories are an approximation to?

The aim of this paper is to explore the idea that approximate decoherence of histories can be turned into exact decoherence by suitable ''small'' modifications of the operators characterizing the histories. In particular, histories characterized by fixed values of coordinates and momenta x,p are rendered exactly decoherent by replacing x,p with new coordinates and momenta X,P, which commute. This replacement, we show, is a valid approximation provided that the original histories are approximately decoherent. The new theory in terms of the commuting variables X,P has the same form as the so-called deterministic quantum theories of the type recently studied by 't Hooft in the context of quantum gravity [6].

To set up the problem in more detail, we briefly review the decoherent histories approach [1–4,7,8]. In the decoherent histories approach to quantum theory, probabilities are assigned to histories of a closed system via the formula

$$p(\alpha_1,\alpha_2,\ldots,\alpha_n) = \mathrm{Tr}[P_{\alpha_n}(t_n)\ldots P_{\alpha_1}(t_1)\rho P_{\alpha_1}(t_1)\ldots P_{\alpha_n}(t_n)]. \quad (1.1)$$

The projection operators P_α characterized the different alternatives describing the histories at each moment of time. The projectors satisfy

$$\sum_\alpha P_\alpha = 1, \quad P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha \quad (1.2)$$

and the projectors appearing in Eq. (1.1) are, in the Heisenberg picture,

$$P_{\alpha_k}(t_k) = e^{iH(t_k-t_0)} P_{\alpha_k} e^{-iH(t_k-t_0)} \quad (1.3)$$

Probabilities can be assigned to histories, if and only if, all histories in the set obey the condition of consistency, which is that

$$\mathrm{Re}\, D(\underline{\alpha},\underline{\alpha}') = 0 \quad (1.4)$$

for $\underline{\alpha} \neq \underline{\alpha}'$. Here $\underline{\alpha}$ denotes the string $\alpha_1,\ldots,\alpha_n$ and $D(\underline{\alpha},\underline{\alpha}')$ is the decoherence functional

$$D(\underline{\alpha},\underline{\alpha}') = \mathrm{Tr}[P_{\alpha_n}(t_n)\ldots P_{\alpha_1}(t_1)\rho P_{\alpha_1'}(t_1)\ldots P_{\alpha_n'}(t_n)]. \quad (1.5)$$

Loosely speaking, the decoherence functional measures the amount of interference between pairs of histories. It is observed in numerous examples involving physical mechanisms for decoherence that the imaginary part of the decoherence functional often also vanishes when the real part vanishes, and it is therefore of interest to consider the stronger condition of decoherence,

$$D(\underline{\alpha},\underline{\alpha}') = 0 \quad (1.6)$$

for $\underline{\alpha} \neq \underline{\alpha}'$. This condition may be shown to be related to the existence of record projectors, which may be added to the very end of the string of projectors that are perfectly correlated with the earlier alternatives $\alpha_1,\ldots,\alpha_n$, and are related to the physical process of information storage [4,9].

In its application to physical interesting situations, therefore, one of the first aims of the approach is to find out how the decoherence condition (1.6) may come to be satisfied. This is often accomplished, for example, by coupling the system of interest to an environment and then tracing out the environment. Or more generally, by some kind of coarse-graining procedure. However, as indicated earlier, it is almost universally observed in such situations that the condition (1.6) is only satisfied approximately, not exactly. The degree to which this condition is satisfied can be exceptionally good, by any standards (see Refs. [10,11], for example), but it is still nevertheless approximate. Although to work with approximate decoherence seems very reasonable physically, from a more rigorous point of view it leaves a gray area in the formalism, since it is not clear what the approximately decoherent histories are an approximation to, if anything [12]. It would be highly desirable to find a more controlled way of moving between approximate and exact decoherence.

As stated above, we shall show that there is a closely related theory that is exactly decoherent and which, under certain circumstances, approximately coincides in its predictions with the approximately decoherent theory.

We start with the observation that the generic lack of decoherence of histories is due to the fact that operators at different times generally do not commute. In the case of histories characterized by projections onto positions, positions at different times can be completely expressed in terms of $\hat{p}$ and $\hat{x}$ at the initial time, so the nondecoherence is due to noncommutativity of the basic canonical pair. Histories characterized by operators that do commute at different times are exactly decoherent, as may be seen from Eq. (1.6). (Histories of conserved quantities are important examples of this type [13].)

We now recall a very old result due to von Neumann, concerning the noncommuting pair, $\hat{p},\hat{x}$. Von Neumann showed that it is possible to find a new pair of operators, $\hat{p}'$, $\hat{x}'$, say, which do commute, and which are in some sense ''close'' to the original pair [14]. The key issue is then to explain what is meant by ''close.'' This is obviously a rather subtle issue. Every interesting quantum effect can be traced back to noncommuting operators, so clearly there will be many situations in which this replacement is a very poor approximation. The point, of course, is that the measure of closeness depends on the context. We are primarily interested in situations that are almost classical anyway, and in that case there is a chance that such an approximation may be good.

This suggests the following approach to approximate decoherence. We start with a decoherence functional that is approximately diagonal. We replace the operators with commuting operators, thereby achieving exact diagonality. The degree of closeness is then measured by the amount that the probabilities for the histories change on replacing the original operators with the commuting operators. We expect this change to be small when the original set of histories are approximately decoherent. Of course, *any* set of histories can be made exactly decoherent in this way. The point, however, is that we expect only histories that are approximately decoherent in the first place will undergo a small change in their probabilities through this procedure. Sets of histories that are not, by any reasonable standard, close to being decoherent, will suffer a large change in their probabilities.

The von Neumann method above is one way of obtaining a commuting set of operators, and there are probably many ways of achieving similar results. Here, we will use a different method, which is perhaps easier and more physically insightful, but is also perhaps more radical in that involves changing the fundamental theory one is quantizing. Suppose we start with a noncommuting canonical pair, $\hat{p},\hat{x}$, for a single particle in one dimension, so

$$[\hat{x},\hat{p}]=i\hbar. \tag{1.7}$$

Denote this system *A*, and now adjoin to it an auxiliary system, denoted *B*, identical to *A*, with canonical pair, $\hat{k},\hat{y}$, and consider the variables

$$\hat{X}=\hat{x}+\hat{y},\quad \hat{Q}=\frac{1}{2}(\hat{x}-\hat{y}),\quad \hat{K}=\frac{1}{2}(\hat{p}+\hat{k}),\quad \hat{P}=\hat{p}-\hat{k}. \tag{1.8}$$

We now have the commutation relations

$$[\hat{Q},\hat{P}]=i\hbar,\quad [\hat{X},\hat{K}]=i\hbar. \tag{1.9}$$

All other commutators are zero, and in particular, we note that

$$[\hat{X},\hat{P}]=0. \tag{1.10}$$

Classically, we could set $y=0=k$ identically, so $X=x$ and $P=p$. Quantum mechanically, we cannot do this, but we can see how close we can get. Suppose we put system *B* in a minimum uncertainty state with $\langle\hat{y}\rangle=0=\langle\hat{k}\rangle$. Then

$$\langle\hat{X}\rangle=\langle\hat{x}\rangle,\quad \langle\hat{P}\rangle=\langle\hat{p}\rangle \tag{1.11}$$

but the higher moments of $\hat{y}$ and $\hat{k}$ are nonzero. This indicates that the pair $\hat{p},\hat{x}$ are equal to the commuting pair $\hat{P},\hat{X}$ up to ''quantum fluctuations.'' More precisely, a measure of the degree of closeness is indicated by the relations

$$\langle(\hat{X}-\hat{x})^2\rangle\langle(\hat{P}-\hat{p})^2\rangle=\langle\hat{y}^2\rangle\langle\hat{k}^2\rangle=\frac{\hbar^2}{4}, \tag{1.12}$$

The issue is then to determine to what extent and under what conditions these fluctuations are significant. Clearly they will be significant when quantum-mechanical effects are important, but it is reasonable suppose that they won't be significant close to the classical regime.

To use this scheme in the decoherent histories approach it is useful to write down an action for the extended system, so that we can use path integrals. Recall that what we ultimately need to get decoherence of position histories is that positions at different times need to commute. We therefore require that the noncommuting operators $\hat{x}_t$ and $\hat{x}$ at different times are distorted into commuting operators $\hat{X}_t, \hat{X}$, which will guarantee exact decoherence. Since, in any reasonable dynamics, $\hat{X}_t$ is a function of $\dot{\hat{X}}$ and $\hat{X}$, the relationship between velocities and momenta (so far unspecified) must be such that $[\dot{\hat{X}}, \hat{X}]=0$. With the standard action, we would have $K=m\dot{X}$, since K is defined to be the conjugate to X, but this clearly will not work since $[\hat{K}, \hat{X}]\neq 0$. We must instead arrange that $P=m\dot{X}$. It is easily seen that this is achieved using that action

$$S=\int dt\left[\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\dot{y}^2\right]=\int dt\, m\dot{X}\dot{Q} \tag{1.13}$$

(in the free particle case). The classical solution for X in the free particle case is

$$X_t=X+t\dot{X}=X+\frac{Pt}{m}, \tag{1.14}$$

On quantization, this implies that $[\hat{X}_t, \hat{X}]=0$ as required.

The action for the new variables X and Q has the form of the action for the deterministic quantum theory (DQT) discussed by 't Hooft [6]. For the more general case of a particle in a potential, this action is

$$S=\int dt[m\dot{Q}\dot{X}-QV'(X)], \tag{1.15}$$

This produces the classical equations of motion

$$m\ddot{X}+V'(X)=0 \tag{1.16}$$

and therefore gives the same classical dynamics as the usual action. But the quantum theory will generally be quite different, since there are twice as many variables. Furthermore, there is a price to pay in that the Hamiltonian for this theory is unbounded below, although there is some chance that this problem may be rectified by fixing the quantum state of the auxiliary system B. Nevertheless, this theory does have properties to recommend it for the purposes of this paper; it is exactly decoherent, and its classical dynamics coincides with the dynamics of the original theory.

The work of 't Hooft concerns the possibility that the deterministic quantum theory is a new fundamental theory, replacing the standard one [6]. The reproduction of quantizationlike effects (in particular, discrete spectra) is argued to arise from dissipative effects in the underlying classical theory [6,15], making use of the fact that the basic action (1.15) is readily modified to include dissipation at a fundamental level,

$$S=\int dt[m\dot{Q}\dot{X}-2m\gamma Q\dot{X}-QV'(X)]. \tag{1.17}$$

The present paper is not primarily concerned with promoting this point of view, but rather, with finding what sort of mathematical statements one can make about the relationship between approximate and exact decoherence. The results do, however, contribute to 't Hooft's program, in that they show in detail how the predictions of standard quantum theory and deterministic quantum theory become indistinguishable as the classical regime is approached.

The results of this paper are basically simple and in some ways almost obvious: DQT reproduces classical predictions exactly, and standard quantum theory reproduces classical predictions approximately when approximate decoherence holds, hence it is no surprise that the two theories approximately coincide. The main task of this paper, however, is to show in detail exactly how this works out.

In Sec. II we discuss the quantization of systems described by the action (1.15). We show that histories of X are exactly decoherent and that the predictions of the theory may be arranged to coincide *exactly* with those of the classical theory.

In Sec. III, we discuss the standard picture of approximate decoherence of histories of a simple linear system, with decoherence provided by coupling to a thermal environment.

The main result of this paper is contained in Sec. IV, where we repeat the analysis of Sec. III but with the addition of an identical auxiliary system with the wrong sign action. We verify that histories of $X=x+y$ are exactly decoherent, as in Sec. II, but here complicated by the presence of an environment. Most importantly, the environment ensures that the exactly decoherent deterministic theory makes predictions that are indeed very close to the predictions of the standard theory with approximate decoherence.

In Sec. V, we give an alternative account of the results of Sec. IV, working with the Wigner function rather than the decoherent histories approach. We show that the Wigner function of the DQT is a good approximation to the Wigner function of the standard quantum theory approach if there is an environment present. The role of the environment in both Secs. IV and V is seen to be, through its fluctuations, to smear out the positions and momenta so that the distinction between x,p and X,P becomes insignificant.

In Sec. VI, we consider a different issue related to the general theme of exact decoherence. This is the observation that there is, in fact, an exactly decoherent set of histories already buried in the standard approach, in the much-studied quantum Brownian motion model. Namely, histories of momenta in this model are exactly decoherent (for the free particle with a high-temperature environment). This is a different sort of exact decoherence, since it is related to total momentum conservation of the system coupled to the environment, but it does not seem to have been noticed previously.

In Sec. VII we briefly consider the question of how the scheme may extend to quantum systems not described by a simple canonical pair obeying Eq. (1.7). We summarize and conclude in Sec. VIII.

II. DETERMINISTIC QUANTUM THEORIES

We now consider the quantization of the DQT described by the action (1.15). The Hamiltonian is

$$H=\frac{1}{m}PK+QV'(X), \tag{2.1}$$

where recall we have the fundamental commutation relations

$$[\hat{Q},\hat{P}]=i\hbar, \quad [\hat{X},\hat{K}]=i\hbar. \tag{2.2}$$

Since $[\hat{X},\hat{P}]=0$, we may quantize using a representation in which the wave functions depend on X and P, $\tilde{\Psi}=\tilde{\Psi}(X,P)$. [Note one could instead work with the commuting pair $\hat{Q},\hat{K}$ and work in a representation in which $\Psi=\Psi(Q,K)$.] We therefore make the replacements,

$$\hat{Q}=i\hbar\frac{\partial}{\partial P}, \quad \hat{K}=-i\hbar\frac{\partial}{\partial X}. \tag{2.3}$$

Hence the Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\tilde{\Psi}(X,P,t)=\left(-\frac{i\hbar}{m}P\frac{\partial}{\partial X}+i\hbar V'(X)\frac{\partial}{\partial P}\right)\tilde{\Psi}(X,P,t). \tag{2.4}$$

The factors of i and $\hbar$ drop out, giving the Schödinger equation a totally classical form:

$$\frac{\partial}{\partial t}\tilde{\Psi}(X,P,t)=\left(-\frac{P}{m}\frac{\partial}{\partial X}+V'(X)\frac{\partial}{\partial P}\right)\tilde{\Psi}(X,P,t). \tag{2.5}$$

This is a classical Liouville equation (although note that the wave function is not necessarily real). The solution is

$$\tilde{\Psi}(X,P,t)=\tilde{\Psi}(X_{-t},P_{-t},0), \tag{2.6}$$

where X_{-t},P_{-t} are the (backwards evolved) classical solutions with initial data X,P.

We now see why the quantum theory of this system may be called deterministic. First of all, since $[\hat{X},\hat{P}]=0$, we may choose initial states that are arbitrarily concentrated in both P and X. Secondly, there is no wave-packet spreading in the dynamics (2.6), and the states therefore remain arbitrarily peaked in P and X. There is therefore no obstruction to assigning definite values to X and P for all times. There is also no possibility of interference because interference arises from wave-packet spreading. Because of these properties, the predictions of this quantum theory may be arranged to *exactly* coincide with the classical theory. Much of the above has already been noted by 't Hooft [6].

In the decoherent histories approach, these features ensure that the histories of fixed X are exactly decoherent, not surprisingly. We briefly sketch the proof of this using a path-integral representation of the decoherence functional. It is

$$D(\alpha,\alpha')=\int_{\alpha}\mathcal{D}X(t)\int_{\alpha'}\mathcal{D}X'(t)\int\mathcal{D}Q(t)\mathcal{D}Q'(t)\times\exp\left(\frac{i}{\hbar}S[X,Q]-\frac{i}{\hbar}S[X',Q']\right)\times\Psi_0(X_0,Q_0)\Psi_0^*(X_0',Q_0'). \tag{2.7}$$

The sum is over pairs of paths $X(t),Q(t)$ and $X'(t),Q'(t)$, where $X(t),X'(t)$ are constrained to pass through a series of gates denoted by α,α' (described in more detail in Sec. III), and $Q(t),Q'(t)$ are unrestricted. The paths meet at the final point $t=t_f$, hence

$$X_f=X_f', \quad Q_f=Q_f', \tag{2.8}$$

After an integration by parts, the action (1.15) may be written

$$S[X,Q]=-\int dt\,Q[m\ddot{X}+V'(X)]+mQ_f\dot{X}_f-mQ_0\dot{X}_0 \tag{2.9}$$

and similarly,

$$S[X',Q']=-\int dt\,Q'[m\ddot{X}'+V'(X')]+mQ_f\dot{X}_f'-mQ_0'\dot{X}_0', \tag{2.10}$$

where the final conditions (2.8) have been used. Now consider the functional integral over Q. In a time-slicing definition of this path integral, we may split the functional integral into an integral of the initial values Q_0,Q_0', the final value $Q_f=Q_f'$, and the values on the interior slices. The $Q(t)$ and $Q'(t)$ in the integrands in Eqs. (2.9) and (2.10), sit on the interior slices only, and integrating them out pulls down delta functions on the equations of motion. Furthermore, the integral over $Q_f=Q_f'$ pulls down a delta function $\delta(\dot{X}_f-\dot{X}_f')$. Hence, we obtain

$$D(\alpha,\alpha')=\int_{\alpha}\mathcal{D}X(t)\int_{\alpha'}\mathcal{D}X'(t)\int dQ_0dQ_0'\delta[m\ddot{X}-V'(X)]\times\delta[m\ddot{X}'-V'(X')]\delta(\dot{X}_f'-\dot{X}_f)\times\exp\left(\frac{im}{\hbar}(Q_0'\dot{X}_0'-Q_0\dot{X}_0)\right)\times\Psi_0(X_0,Q_0)\Psi_0^*(X_0',Q_0'). \tag{2.11}$$

Because of the delta functions on the equations of motion, the sums over paths $X(t)$ and $X'(t)$ take contributions only from histories satisfying the classical equations of motion. But we also have the final condition $X_f=X_f'$, together with the delta function in Eq. (2.11), which ensures that $\dot{X}_f=\dot{X}_f'$. Therefore $X(t)$ and $X'(t)$ satisfy the same second-order equation and the same final conditions. It follows that $X(t)=X'(t)$ in this path integral and therefore there is exact decoherence.

The integral over Q_0 and Q_0' performs a Fourier transformation of the initial wave function to the representation $\tilde{\Psi}(X,P)$ used earlier,

$$\tilde{\Psi}(X,P)=\int dQ\, e^{-(i/\hbar)PQ}\Psi(X,Q), \tag{2.12}$$

and we find that the probabilities for the histories are given by

$$p(\alpha)=\int_{\alpha}\mathcal{D}X(t)\,\delta[m\ddot{X}+V'(X)]|\tilde{\Psi}(X_0,m\dot{X}_0)|^2, \tag{2.13}$$

This is precisely the expected result for a classical deterministic theory with probability for initial conditions given by $|\tilde{\Psi}(X,P)|^2$.

Finally, it is of interest to compare the initial phase-space distribution $|\tilde{\Psi}(X,P)|^2$ with the Wigner function, which often crops up in this sort of decoherence functional calculation [4,16]. The Wigner function is defined in terms of the wave function $\Psi(X,Q)$ by [17]

$$\begin{aligned}W(K,X,P,Q)=&\frac{1}{(2\pi\hbar)^2}\int d\xi_1 d\xi_2\\&\times\exp[-(i/\hbar)K\xi_1-(i/\hbar)P\xi_2]\\&\times\Psi\left(X+\frac{1}{2}\xi_1,Q+\frac{1}{2}\xi_2\right)\\&\times\Psi^*\left(X-\frac{1}{2}\xi_1,Q-\frac{1}{2}\xi_2\right).\end{aligned} \tag{2.14}$$

Inserting the expression for $\Psi(X,Q)$ in terms of its Fourier transform $\tilde{\Psi}(X,P)$ [the inverse of Eq. (2.12)], it is easily shown that the reduced Wigner function $\tilde{W}(X,P)$ is

$$\tilde{W}(X,P)=\int dK\,dQ\,W(K,X,P,Q)=|\tilde{\Psi}(X,P)|^2, \tag{2.15}$$

which is the intuitively expected result.

III. APPROXIMATE DECOHERENCE IN THE STANDARD PICTURE

We now briefly review the approximate decoherence of position histories in standard quantum theory (SQT). We consider a single particle in a potential $V(x)$ linearly coupled to a large environment of harmonic oscillators in an initial thermal state with temperature T_A. The action for this system is

$$\begin{aligned}S[x,q_n]=&\int dt\left[\frac{1}{2}m\dot{x}^2-V(x)\right]\\&+\sum_n\int dt\left[\frac{1}{2}m_n\dot{q}_n^2-\frac{1}{2}m_n\omega_n^2q_n^2-c_nq_nx\right]\end{aligned} \tag{3.1}$$

and the Hamiltonian is

$$H=\frac{p^2}{2m}+V(x)+\sum_n\left[\frac{p_n^2}{2m}+\frac{1}{2}m_n\omega_n^2q_n^2+c_nq_nx\right]. \tag{3.2}$$

This model, the quantum Brownian motion model, has been considered many times elsewhere [18–20], especially in the context of decoherence [21] (see also the older related work Ref. [22]). We will describe it only in outline, quoting required results where necessary.

After tracing out the environment variables, the decoherence functional is

$$\begin{aligned}D(\underset{\sim}{\alpha},\underset{\sim}{\alpha}')=&\int\mathcal{D}x(t)\mathcal{D}x'(t)\prod_{k=1}^{n}\Upsilon[x(t_k)-\overline{x_k}]\Upsilon[x'(t_k)-\overline{x_k'}]\\&\times\exp\left(\frac{i}{\hbar}\int_0^{\tau}dt\left[\frac{1}{2}m\dot{x}^2-V(x)\right.\right.\\&\left.\left.-\frac{1}{2}m\dot{x}'^2+V(x')\right]\right)\\&\times F[x(t),x'(t)]\rho_A(x_0,x_0').\end{aligned} \tag{3.3}$$

Here, we use $\underset{\sim}{\alpha}$ to denote the string $\overline{x_1},\overline{x_2},\ldots\overline{x_n}$. The window functions Υ restrict the paths to pass through gates of width Δ centerd about points $\overline{x_1},\overline{x_2},\ldots$ at times $t_1,t_2\ldots t_n$ in a total time interval $[0,\tau]$. The only leftover of the environment is the influence functional

$$F[x(t),x'(t)]=\exp\left(\frac{i}{\hbar}W[x(t),x'(t)]\right), \tag{3.4}$$

where $W[x(t),x'(t)]$ is the Feynman-Vernon influence functional phase

$$\begin{aligned}W[x(t),x'(t)]=&-\int_0^{\tau}dt\int_0^{t}ds[x(t)-x'(t)]\eta(t-s)\\&\times[x(s)+x'(s)]\\&+i\int_0^{\tau}dt\int_0^{t}ds[x(t)-x'(t)]\\&\times\nu(t-s)[x(s)-x'(s)].\end{aligned} \tag{3.5}$$

Full details of the kernels η and ν may be found elsewhere [18,19,23,24]. They are in general nonlocal in time, but simplify enormously in the Fokker-Planck limit (high temperature and a continuum of oscillators with a high-frequency cutoff) in which,

$$W=-\int_0^\tau dt\, m\gamma(x-x')(\dot{x}+\dot{x}')-\int_0^\tau dt\ \delta\omega^2(x^2-x'^2)$$
$$+\frac{2M\gamma kT_A}{\hbar}i\int_0^\tau dt(x-x')^2. \tag{3.6}$$

In what follows, to make the exposition clearer, we will work entirely in this limit. [It is readily verified that the following calculations can be carried out with the fully general form (3.5), but the expressions are much more cumbersome.]

From Eq. (3.6) one can see that the real part of $W[x(t),x'(t)]$ contributes a dissipative part to the effective equations of motion, and also a renormalization $\delta\omega^2$ to the frequency. We shall assume that the latter has been absorbed into the potential $V(x)$. The imaginary part produces the decoherence, since it suppresses differing values of x and x'. Since the projectors coarse grain the paths into regions of size Δ, distinct histories have $|x-x'|$ greater than Δ. The condition for approximate decoherence is therefore loosely given by

$$2m\gamma kT_A\tau\Delta^2\gg\hbar^2, \tag{3.7}$$

and hence is satisfied for sufficiently large temperature. The imaginary part of $W[x(t),x'(t)]$ also produces fluctuations about the effective classical equations of motion.

Given approximate decoherence, we may take the probabilities for histories to be given, to a good approximation, by the diagonal elements of the decoherence functional. The resulting expression is most easily evaluated using the sum and difference coordinates,

$$\xi=x-x',\quad u=\frac{1}{2}(x+x') \tag{3.8}$$

and we obtain for the probabilities for histories,

$$p(\underline{\alpha})=\int \mathcal{D}u(t)\mathcal{D}\xi(t)\prod_{k=1}^{n}\Upsilon\left[u(t_k)+\frac{1}{2}\xi(t_k)-\overline{x_k}\right]$$
$$\times\Upsilon\left[u(t_k)-\frac{1}{2}\xi(t_k)-\overline{x_k}\right]$$
$$\times\exp\left\{\frac{i}{\hbar}\int dt\left[m\dot{u}\dot{\xi}-2m\gamma\dot{u}\xi-V\left(u+\frac{1}{2}\xi\right)\right.\right.$$
$$\left.\left.+V\left(u-\frac{1}{2}\xi\right)\right]\right\}\exp\left(-\frac{2m\gamma kT_A}{\hbar^2}\int dt\xi^2\right)$$
$$\times\rho_A\left(u_0+\frac{1}{2}\xi_0,u_0-\frac{1}{2}\xi_0\right), \tag{3.9}$$

Consider the functional integral over ξ. It is Gaussian except for the appearance of ξ in the window functions Υ and in the potential V. However, the contribution from ξ is very tightly concentrated around $\xi=0$. We there expect to be able to drop the ξ terms in the window functions, in comparison to u, and also to use a small ξ approximation in the potential

$$V\left(u+\frac{1}{2}\xi\right)-V\left(u-\frac{1}{2}\xi\right)=\xi V'(u)+\frac{1}{24}\xi^3V'''(u)+\cdots. \tag{3.10}$$

Dropping the order ξ^3 term (shown here only for comparison with later results), the integral in the imaginary part of the exponential may be integrated by parts yielding

$$-\int dt\ \xi[m\ddot{u}+2m\gamma\dot{u}+V'(u)]-\dot{u}_0\xi_0, \tag{3.11}$$

where we have used the fact that $x=x'$ at the final time so $\xi_f=0$. In a skeletonized version of the path integral, the integrand in Eq. (3.11) does not involve ξ_0, only the values of ξ on the internal time slices. The integral over ξ_0 with the boundary term from Eq. (3.11) therefore effectively performs the Wigner transformation of the initial density matrix,

$$W(p,u_0)=\frac{1}{2\pi\hbar}\int d\xi_0\, e^{-(i/\hbar)p\xi_0}\rho_A\left(u_0+\frac{1}{2}\xi_0,u_0-\frac{1}{2}\xi_0\right). \tag{3.12}$$

And carrying out the ξ integration on the internal times slices as well, we therefore obtain

$$p(\underline{\alpha})=\int \mathcal{D}u(t)\prod_{k=1}^{n}\Upsilon[u(t_k)-\overline{x_k}]$$
$$\times\exp\left(-\frac{1}{8m\gamma kT_A}\int dt[m\ddot{u}+2m\gamma\dot{u}+V'(u)]^2\right)$$
$$\times W(m\dot{u}_0,u_0). \tag{3.13}$$

This is the desired result, a simple expression for the probability for histories of positions. It is peaked about classical evolution with dissipation, with thermal fluctuations about that motion, and with the initial data weighted by the Wigner function of the initial state. (The Wigner function is not always positive, but a closer analysis of this sort of expression [16] reveals that the Wigner function is effectively smeared in such a way that it is positive.)

Equation (3.13) was derived under essentially one approximation: that the contribution from paths with large values of $\xi=x-x'$ could be neglected. This meant first, that the approximate decoherence could be taken as essentially exact. Secondly, that we could drop the ξ terms in the window functions in Eq. (3.9) and the higher powers of ξ in the expansion of the potential (3.10), so that we could carry out the ξ integration.

IV. COMPARISON WITH THE EXACTLY DECOHERENT DETERMINISTIC QUANTUM THEORY

The formula (3.13) bears a close resemblance to Eq. (2.13), the probabilities for histories in the exactly decoherent DQT. There are, however, three differences. First, Eq. (3.13) has dissipation in the equations of motion but Eq. (2.13) does not; this is easily fixed by the trivial generalization of Eq. (2.13) to the case of the dissipative action (1.17). Second, Eq. (2.13) has a delta-function peak about the equation of motion, while Eq. (3.13) has only a Gaussian peak, due to the thermal fluctuations. This Gaussian peak becomes sharper as the mass of the particle increases. Moreover, the difference between the two types of peaks will not be noticed if the width of the projections in Eq. (3.13) are much greater than the width of the Gaussian, Third, Eq. (3.13) has a (not necessarily positive) Wigner function weighting its initial conditions, while Eq. (2.13) has a positive weight function. But given that the fluctuations tend to smear W so as to be positive anyway (as will be discussed at greater length below), for a wide variety of initial states it ought to be possible to choose an initial state in Eq. (2.13) to give essentially the same results as Eq. (3.13).

Of the above differences, the most important one is the delta function versus the Gaussian peak. We therefore conclude that as long as the particle is sufficiently massive to substantially resist the effects of thermal fluctuations, the exactly decoherent DQT of Sec. II approximately reproduces the probabilities of the approximately decoherent histories of standard quantum theory described above. This is our first result on the closeness of DQT and standard quantum theory.

The above result applies, however, only to the case when the mass of the particle is sufficiently large to resist thermal fluctuations. It does not apply to the case where there is approximate decoherence but the fluctuations about classical deterministic behavior are not small, as in the case of small mass. The most general effective theories emerging from an underlying quantum theory are classical stochastic theories, perhaps with large fluctuations. We therefore need to generalize our comparison of DQT and standard quantum theory to this case, and this turns out to be somewhat more complicated. It requires comparing the quantum Brownian motion model of Sec. III to a DQT including an environment to provide fluctuations.

We have seen for a simple linear system with action $S[x]$, a closely related DQT may be constructed using the action $S=S[x]-S[y]$ and by focusing on the variable $X=x+y$. The coupling to an environment, as in Eq. (3.1), requires a reconsideration of the question of how to construct the related DQT. On the basis of what we have seen so far—that the DQT is obtained by doubling what we already have—it seems natural to double up both the system and the environment. While this in fact turns out to be correct, one might wonder whether it would be possible to obtain exact decoherence by the simpler procedure of doubling the system alone. As we shall see, however, the dissipative terms induced by the environmental interactions prevent this from working properly. We therefore do indeed need to double both system and environment.

One can imagine a number of different ways of proceeding at this point. For example, one could extend the analysis of Sec. II to include coupling to a thermal environment and then repeat the steps leading to Eq. (2.13). This would, however, involve getting into unnecessary detail about the environment dynamics and initial state. We will instead stay as close as possible to the calculation of Sec. III, in which all the environment dynamics are concisely summarized in the influence functional.

Consider therefore the same calculation as in Sec. III but with both system and environment doubled up. For simplicity, we first concentrate on the case of a linear system with $V(x)=(1/2)m\omega^2x^2$. We therefore consider system A with coordinates x coupled to its environment with temperature T_A, as before, with the auxiliary system B and its environment, with temperature T_B (which, we shall see, does not have to be the same as T_A). Following the general scheme, we consider histories specified by fixed values of $X=x+y$. After tracing out both environments, the decoherence functional is

$$D(\underline{\alpha},\underline{\alpha}')=\int \mathcal{D}x(t)\mathcal{D}x'(t)\mathcal{D}y(t)\mathcal{D}y'(t)\prod_{k=1}^{n} \Upsilon[x(t_k)+y(t_k)-\overline{x_k}]\Upsilon[x'(t_k)+y'(t_k)-\overline{x_k'}]$$
$$\times\exp\left(\frac{i}{\hbar}\int dt\left[\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2-\frac{1}{2}m\dot{x}'^2+\frac{1}{2}m\omega^2x'^2\right]\right)$$
$$\times\exp\left(\frac{i}{\hbar}\int dt\left[-\frac{1}{2}m\dot{y}^2+\frac{1}{2}m\omega^2y^2+\frac{1}{2}m\dot{y}'^2-\frac{1}{2}m\omega^2y'^2\right]\right)$$
$$\times F_A[x(t),x'(t)]F_B^*[y(t),y'(t)]\rho_A(x_0,x_0')\rho_B(y_0,y_0') \qquad (4.1)$$

(where the effect of the wrong sign action in the auxiliary system B effectively gives the complex conjugate of the influence functional). We will confirm that this is exactly decoherent and compute the probabilities for histories. Note that Eq. (4.1) and the corresponding approximately decoherent expression (3.3) are almost identical: if the projections in Eq. (4.1) were onto values of x,x', rather than $X=x+y$ and $X'=x'+y'$, then all the $y,y.'$ terms could be entirely integrated out yielding Eq. (3.3).

The path integral is most easily evaluated by changing variables from (x,y) to (X,y) (and similarly for the primed variables).

In these coordinates, and writing out the influence functional explicitly, it reads

$$D(\underset{\sim}{\alpha},\underset{\sim}{\alpha}')=\int \mathcal{D}X(t)\mathcal{D}X'(t)\mathcal{D}y(t)\mathcal{D}y'(t)\prod_{k=1}^{n}\Upsilon[X(t_k)-\overline{x_k}]\Upsilon[X'(t_k)-\overline{x_k'}]$$
$$\times\exp\left(\frac{i}{\hbar}\int dt\left[\frac{1}{2}m\dot{X}^2-\frac{1}{2}m\omega^2X^2-\frac{1}{2}m\dot{X}'^2+\frac{1}{2}m\omega^2X'^2\right]\right)\exp\left(\frac{i}{\hbar}\int dt[-m\dot{y}\dot{X}+m\omega^2yX+m\dot{y}'\dot{X}'-m\omega^2y'X']\right)$$
$$\times\exp\left(\frac{i}{\hbar}\int dt[-m\gamma(X-X')(\dot{X}+\dot{X}')+m\gamma(y-y')(\dot{X}+\dot{X}')+m\gamma(\dot{y}+\dot{y}')(X-X')]\right)$$
$$\times\exp\left(-\frac{2m\gamma kT_A}{\hbar^2}\int dt(X-X'-y+y')^2-\frac{2m\gamma kT_B}{\hbar^2}\int dt(y-y')^2\right)\rho_A(X_0-y_0,X_0'-y_0')\rho_B(y_0,y_0'). \qquad (4.2)$$

Recall that in Sec. II, exact decoherence in the deterministic model was obtained as a result of the action being linear in one of the variables, hence yielding a delta function on integration. In this case, note that the exponent is linear in the variable $y+y'$. So introduce new coordinates

$$Y=\frac{1}{2}(y+y'),\quad v=y-y' \qquad (4.3)$$

and note that

$$yX-y'X'=Y(X-X')+\frac{1}{2}v(X+X'). \qquad (4.4)$$

The y terms in the second and third exponential in Eq. (4.2) therefore become

$$\int dt[-m\dot{y}\dot{X}+m\omega^2yX+m\dot{y}'\dot{X}'-m\omega^2y'X'+m\gamma(y-y')$$
$$\times(\dot{X}+\dot{X}')+m\gamma(\dot{y}+\dot{y}')(X-X')]$$
$$=\int dt\Big[-m\dot{Y}(\dot{X}-\dot{X}')-\frac{1}{2}m\dot{v}(\dot{X}+\dot{X}')$$
$$+m\omega^2Y(X-X')+\frac{1}{2}m\omega^2v(X+X')$$
$$+m\gamma v(\dot{X}+\dot{X}')+2m\gamma\dot{Y}(X-X')\Big]. \qquad (4.5)$$

As advertized, the exponential in the path integral is now entirely linear in Y, and, after an integration by parts in Eq. (4.5), Y may be integrated out on the interior slices to produce a delta function on configurations satisfying the equation

$$\ddot{X}-\ddot{X}'-2\gamma(\dot{X}-\dot{X}')+\omega^2(X-X')=0. \qquad (4.6)$$

This is the antidamped dissipative equation for $X-X'$, but this does not matter since it is not the effective equation of motion (derived below). The integration by parts in Eq. (4.5) also produces the boundary terms

$$-[mY(\dot{X}-\dot{X}')+2m\gamma Y(X-X')]_0^{\tau}. \qquad (4.7)$$

As in Sec. II, the integration over Y_f produces a delta function enforcing $\dot{X}_f=\dot{X}_f'$, and since we also have $X_f=X_f'$, the solution to Eq. (4.6) is therefore $X(t)=X'(t)$ identically, for all t. We therefore have exact decoherence, as expected. It follows that the other boundary terms in Eq. (4.7) vanish (since they are proportional to $X-X'$).

We may now compute the probabilities for histories. With $X(t)=X'(t)$ throughout, we now have

$$p(\underset{\sim}{\alpha})=\int \mathcal{D}X(t)\mathcal{D}v(t)dy_0\,dy_0\prod_{k=1}^{n}\Upsilon[X(t_k)-\overline{x_k}]$$
$$\times\exp\left(\frac{i}{\hbar}\int dt[-m\dot{v}\dot{X}+2m\gamma v\dot{X}+m\omega^2vX]\right)$$
$$\times\exp\left(-\frac{2m\gamma k(T_A+T_B)}{\hbar^2}\int dtv^2\right)$$
$$\times\rho_A(X_0-y_0,X_0-y_0')\rho_B(y_0,y_0'). \qquad (4.8)$$

The v integral may now be carried out, and, noting that there is also a boundary term coming from the integration by parts of the $m\dot{v}\dot{X}$ term, we get

$$p(\underset{\sim}{\alpha})=\int \mathcal{D}X(t)\prod_{k=1}^{n}\Upsilon[X(t_k)-\overline{x_k}]$$
$$\times\exp\left(-\frac{m}{8\gamma kT'}\int dt[\ddot{X}+2\gamma\dot{X}+\omega^2X]^2\right)$$
$$\times\int dy_0\,dy_0'\exp[(i/\hbar)m(y_0-y_0')\dot{X}_0]$$
$$\times\rho_A(X_0-y_0,X_0-y_0')\rho_B(y_0,y_0'). \qquad (4.9)$$

where $T'=T_A+T_B$. Now consider the last part of this expression, the y_0,y_0' integral involving the initial state. We now choose ρ_B to be the ground state of the harmonic oscillator. If we also let $y_0 \to -y_0$, $y_0' \to -y_0'$, followed by the transformation, $y_0 \to y_0 - X_0$ and $y_0' \to y_0' - X_0$, then this integral becomes

$$\int dy_0\, dy_0' \exp\left(-\frac{i}{\hbar} m(y_0 - y_0')\dot{X}_0 - \frac{(y_0 - X_0)^2}{4\sigma^2} - \frac{(y_0' - X_0)^2}{4\sigma^2} \right) \rho_A(y_0, y_0'). \tag{4.10}$$

This is clearly just the average of the initial state ρ_A in a coherent state $|p,q\rangle$ with $p = m\dot{X}_0, q = X_0$. Hence the final expression is

$$p(\underset{\sim}{\alpha}) = \int \mathcal{D}X(t) \prod_{k=1}^{n} \Upsilon[X(t_k) - \overline{x_k}] \times \exp\left(-\frac{m}{8\gamma k T'} \int dt [\ddot{X} + 2\gamma \dot{X} + \omega^2 X]^2 \right) \times \langle m\dot{X}_0, X_0 | \rho_A | m\dot{X}_0, X_0 \rangle. \tag{4.11}$$

This is the desired result: the probability for histories of X for the exactly decoherent deterministic theory with an environment.

The main issue now is to compare this result with Eq. (3.13) derived using standard quantum theory under the conditions of approximate decoherence. Equation (4.11) is clearly a much better approximation to Eq. (3.13) than Eq. (2.13) was. Equation (4.11) has the desired dissipation term [although here it comes from the environment, and not from the action (1.17)]. Most importantly it has thermal fluctuations. The temperature in Eq. (4.11) is $T' = T_A + T_B$, versus a temperature T_A in Eq. (3.13), but this difference is clearly negligible if we choose $T_B \ll T_A$.

The only significant difference between Eqs. (4.11) and (3.13) is the appearance of the explicitly positive weight on initial data, $\langle p,q|\rho_A|p,q\rangle$, in Eq. (4.11), versus the Wigner function $W(p,q)$ in Eq. (3.13). The two objects are, however, close. $\langle p,q|\rho_A|p,q\rangle$ is readily shown to be equal to the Wigner function of ρ_A but smeared over an $\hbar$-sized region of phase space. Moreover, the subsequent evolution of the system renders the difference between these two objects negligible, since the thermal fluctuations produce a smearing in phase space that becomes much greater than $\hbar$ on a very short time scale [25,26]. The probabilities of the DQT and the approximately decoherent standard quantum theory are therefore very close.

The physical picture is as follows. We have proposed switching from noncommuting operators $\hat{x},\hat{p}$ to commuting ones $\hat{X},\hat{P}$ differing from the original ones by ''quantum fluctuations.'' The key point is that in the presence of the environment, the system also suffers thermal fluctuations that are typically much larger than the quantum fluctuations in $\hat{X} - \hat{x}$ and $\hat{P} - \hat{p}$. The difference between the two sets of operators is therefore negligible, and we may reasonably consider the two theories as ''close.''

We now consider some finer points of this derivation. Consider first the issue of why we need to include the environment of the auxiliary system B. As stated, this has to do with the dissipative term. The question is what would happen if we drop the environment of B? It is easy to see that dropping the fluctuation term for B's environment does no harm. In fact it improves things, since it is the same as setting $T_B = 0$, so we no longer need the condition $T_A \gg T_B$. On the other hand, dropping the dissipative terms for B is equivalent to including a term proportional to

$$m\gamma(y - y')(\dot{y} + \dot{y}') = 2m\gamma v \dot{Y} \tag{4.12}$$

in the exponent in Eq. (4.2). On carrying out the integral over Y, this produces a term proportional to v on the right-hand side of Eq. (4.6). The key point now is that the solution to this equation is no longer $X(t) = X'(t)$ identically. Therefore exact decoherence is destroyed. Hence the presence of the dissipative term is required.

A possible difficulty of having to include a second environment is that its effects may become significant at low temperatures. We have concentrated here on the high-temperature regime, but in standard quantum theory there is some decoherence at low temperatures, including zero temperature (although this does not seem to have been very extensively studied in the literature [9,27]). [At low temperatures note also that the fully nonlocal form of the influence functional (3.5) must be used.] In this regime it becomes less obvious that the DQT is close to the predictions of SQT.

At all temperatures, standard quantum theory, after approximate decoherence, is approximately equivalent to a classical but stochastic theory [described by Eq. (3.13) for example], consisting of deterministic evolution according to classical equations of motion with dissipation, with thermal fluctuations about that motion. This description is still good even if the fluctuations are not small. The DQT also leads to a description in terms of fluctuations about deterministic evolution, but the presence of *two* environments means that the fluctuations are not the same in general as the fluctuations in the SQT case—they are larger, as evidence by the presence of the temperature $T_A + T_B$ in Eq. (4.11). They are approximately the same if $T_A \gg T_B$, but they will be different if both T_A and T_B are the same order of magnitude. Hence, SQT and the DQT are generally not approximately the same in their predictions for low-temperature environments, since the fluctuations in the DQT case are significantly larger.

At least, that is the conclusion on the basis of the approach of this section, involving doubled environments. It does not rule out the possibility that another type of DQT might approximately reproduce the predictions of standard quantum theory at low temperatures. Indeed, if the mass of the particle is very large, Eq. (2.13) with a dissipative term will do the job moderately well (as discussed at the beginning of this section). Still, the analysis of this paper leaves space for a more thorough discussion of the connection between DQT and SQT in the low-temperature regime.

Finally, consider the case of nonlinear systems. When a more general potential is present, we need to replace the potential terms in $S[x] - S[y]$ with $(x-y)V'(x+y)$ [to coincide with the action (1.15)]. This means of course that the systems A and B are now coupled whereas previously they were not. It is readily shown that the analysis goes through in a very similar way with a term $V'(X)$ in the final result (4.11) in place of $m\omega^2 X$. General potentials are fully treated in the alternative formulation in the next section.

V. A WIGNER FUNCTION FORMULATION

We have examined the relationship between SQT and DQT by comparing the probabilities for histories of the two theories, when SQT is approximately decoherent. This still leaves, however, a certain amount of vagueness in a statement about the relationship between approximate and exact decoherence, since the probabilities from SQT are still only approximately defined due to imperfect decoherence. A perhaps more precise way of comparing the predictions of standard quantum theory with the deterministic one is to compare the density operator of standard quantum theory with the reduced density operator of DQT after the extra variables (K,Q, etc.) have been traced out. This we now do. We will in fact work with the Wigner function [17], rather than the density operator, but this is essentially the same since they are related by a simple Fourier transform.

The system A plus its environment has Hamiltonian (3.2) and is described by a Wigner function $W(p,x,p_n,q_n)$ obeying the equation

$$\frac{\partial W}{\partial t}=\{H,W\}+DW, \tag{5.1}$$

where $\{\}$ is the usual Poisson bracket and D is an operator acting on phase space,

$$D=\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n}}\frac{1}{(2n+1)!}\frac{d^{2n+1}V(x)}{dx^{2n+1}}\frac{\partial^{2n+1}}{\partial p^{2n+1}}. \tag{5.2}$$

Explicitly,

$$\frac{\partial W}{\partial t}=-\frac{p}{m}\frac{\partial W}{\partial x}+V'(x)\frac{\partial W}{\partial p}+DW+\sum_n\left[-\frac{p_n}{m_n}\frac{\partial W}{\partial q_n}\right.$$
$$\left.+m_n\omega_n^2 q_n\frac{\partial W}{\partial p_n}+c_n x\frac{\partial W}{\partial p_n}+c_n q_n\frac{\partial W}{\partial p}\right]. \tag{5.3}$$

This equation describes the exact dynamics of the system A coupled to its environment. Assuming a factored initial state between system and environment, and with a thermal initial state with temperature T_A for the environment, the environment coordinates may be traced out, and an equation for the reduced Wigner function for the system only $\overline{W}(x,p)$ may be derived. This is in general a non-Markovian equation, whose explicit form is only readily obtained for linear systems [24,28]. But in the Fokker-Planck limit (used in the previous section) it has the form

$$\frac{\partial \overline{W}}{\partial t}=-\frac{p}{m}\frac{\partial \overline{W}}{\partial x}+V'(x)\frac{\partial \overline{W}}{\partial p}+2\frac{\partial}{\partial p}(p\overline{W})$$
$$+2m\gamma kT_A\frac{\partial^2\overline{W}}{\partial p^2}+D\overline{W}. \tag{5.4}$$

It is well-known that the diffusion term spreads out the Wigner function so that the higher derivative terms $D\overline{W}$ may be neglected [29,30]. Furthermore, the Wigner function also becomes positive after a very short time [25]. It may therefore be regarded, approximately, as a classical phase-space distribution function. This is the usual account of the approximate emergence of classical behavior using the Wigner function or density operator, paralleling the discussion of Sec. III.

We now compare this to the Wigner function description of the deterministic quantum theory, which we know to be exactly decoherent, paralleling the derivation of Sec. IV. The action for the deterministic theory coupled to an environment is

$$S=\int dt[m\dot{Q}\dot{X}-QV'(X)]+\sum_n\int dt[m_n\dot{Q}_n\dot{X}_n$$
$$-m_n\omega_n^2 Q_nX_n-c_nQ_nX-c_nX_nQ], \tag{5.5}$$

where the coordinates are related to the coordinates x,y, etc., by

$$X=x+y,\quad Q=\frac{1}{2}(x-y),\quad X_n=q_n+\tilde{q}_n,$$
$$Q_n=\frac{1}{2}(q_n-\tilde{q}_n). \tag{5.6}$$

In the linear case, the action (5.5) is of the form

$$S=S[x,q_n]-S[y,\tilde{q}_n]. \tag{5.7}$$

The Hamiltonian is

$$H=\frac{1}{m}PK+QV'(X)+\sum_n\left[\frac{1}{m_n}P_nK_n+m_n\omega_n^2Q_nX_n\right.$$
$$\left.+c_nQ_nX+c_nX_nQ\right], \tag{5.8}$$

where P_n,K_n are the momenta conjugate to Q_n,X_n respectively. The Wigner function for this system $W=W(K,X,P,Q,K_n,X_n,P_n,Q_n)$ obeys the evolution equation

$$\frac{\partial W}{\partial t} = -\frac{K}{m}\frac{\partial W}{\partial Q} - \frac{P}{m}\frac{\partial W}{\partial X} + V'(X)\frac{\partial W}{\partial P} + QV''(X)\frac{\partial W}{\partial K} + \tilde{D}W + \sum_n \left[-\frac{K_n}{m_n}\frac{\partial W}{\partial Q_n} - \frac{P_n}{m_n}\frac{\partial W}{\partial X_n} + m_n\omega_n^2 X_n \frac{\partial W}{\partial P_n} + m_n \omega_n^2 Q_n \frac{\partial W}{\partial K_n}\right] + \sum_n c_n \left[X\frac{\partial W}{\partial P_n} + Q\frac{\partial W}{\partial K_n} + X_n \frac{\partial W}{\partial P} + Q_n \frac{\partial W}{\partial K}\right]. \tag{5.9}$$

Here, $\tilde{D}$ is a modified phase-space operator, appropriate to the fact that the potential is $QV'(X)$; hence,

$$\tilde{D} = Q\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{1}{(2n+1)!} \frac{d^{2n+2}V(X)}{dX^{2n+2}} \frac{\partial^{2n+1}}{\partial K^{2n+1}}. \tag{5.10}$$

This is the exact quantum dynamics of the deterministic quantum system coupled to an environment. It is exactly decoherent in terms of histories specified by fixed values of P, X, P_n, and X_n. It is subject to the initial conditions that, in terms of the original systems A,B, and their environments, the initial state completely factors:

$$W = W_A(p,x)W_B(k,y)W_{A\mathcal{E}}(p_n,q_n)W_{B\mathcal{E}}(\tilde{p}_n,\tilde{q}_n). \tag{5.11}$$

As in the previous section, the auxiliary system B is chosen to be in a minimum uncertainty state. The environments of A and B are chosen to be in thermal states, but with $T_A \gg T_B$.

To compare with the standard quantum theory results (5.3) and (5.4), we integrate out the variables, K, Q, K_n, and Q_n. Tracing the Wigner equation [to derive Eqs. (5.4) from (5.3), for example] is usually a nontrivial operation [28]. However, the fact that we not tracing out canonical pairs appears to make it essentially trivial, and it is easily seen that the resulting Wigner function $\tilde{W}(X,P,X_n,P_n)$ obeys the evolution equation

$$\frac{\partial \tilde{W}}{\partial t} = -\frac{P}{m}\frac{\partial \tilde{W}}{\partial X} + V'(X)\frac{\partial \tilde{W}}{\partial P} + \sum_n \left[-\frac{P_n}{m_n}\frac{\partial \tilde{W}}{\partial X_n} + m_n\omega_n^2 X_n \frac{\partial \tilde{W}}{\partial P_n} + c_n X\frac{\partial \tilde{W}}{\partial P_n} + c_n X_n \frac{\partial \tilde{W}}{\partial P}\right], \tag{5.12}$$

The evolution equations (5.12) and (5.3) are the same, except for the term DW in Eq. (5.3) [where note that the analogous term in Eq. (5.9) dropped out when K was integrated over]. In the absence of the environmental terms, the presence of DW would substantially modify the dynamics in Eq. (5.3) in comparison to Eq. (5.12). However, as stated, after tracing out the environment in Eq. (5.3) to yield Eq. (5.4), the diffusive effects induced in the evolution of W make the contribution of this term negligible. Moreover, tracing out the environment in the DQT of Eq. (5.12) leads to an equation of the form (5.4) without the term DW, and with the temperature T_A replaced by $T_A + T_B$. The two evolution equations are therefore approximately the same for $T_B \ll T_A$. We may therefore say the following: the dynamics described by Eqs. (5.3) and (5.12) will be essentially identical with respect to coarse grainings, asking questions only about the variables X and P. [We have phrased the statement in this way, in terms of Eqs. (5.12) and (5.3), rather than Eq. (5.4) since the former are exact equations whereas Eq. (5.4) holds only in the Fokker-Planck limit.]

Given identical dynamics, the comparison of the two systems then reduces to comparison of the initial states. In the SQT result, Eq. (5.4), the initial state is the Wigner function $W_A(p,x)$. In the corresponding DQT equation [Eq. (5.12) with environment traced out], by contrast, the initial state is the reduced Wigner function,

$$\tilde{W}(P,X) = \int dQ\, dK\, W(K,X,P,Q) = \int dQ\, dK\, W_A(p,x) W_B(k,y). \tag{5.13}$$

This is written most usefully by changing variables from X,P,Q,K to X,P,y,k, where, from Eq. (5.6), we have

$$Q = \frac{1}{2}X - y, \quad K = P + k, \quad p = P + k, \quad x = X - y. \tag{5.14}$$

It follows that

$$\tilde{W}(P,X) = \int dy\, dk\, W_A(P+k, X-y) W_B(k,y). \tag{5.15}$$

Since W_B is a minimum uncertainty state, this is a Wigner function smeared over an $\hbar$-size region of phase space, as in Eq. (4.11) (and is positive). We are therefore now comparing the smeared Wigner function $\tilde{W}(P,X)$, which solves the environment-traced version of Eq. (5.12) to the Wigner function of the SQT, $W_A(p,x)$. These will generally be different, but as stated in Sec. IV, the environment comes to the rescue—under evolution according to an equation of the form (4.4), thermal fluctuations rapidly overtake the quantum ones, and the difference between the smeared and unsmeared Wigner functions is negligible.

We therefore have an independent proof of the approximate equivalence of SQT and the DQT under the conditions of approximate decoherence.

VI. EXACT DECOHERENCE OF MOMENTA IN THE QUANTUM BROWNIAN MOTION MODEL OF SQT

We now produce an example of a situation in standard quantum theory, which does in fact exhibit exact decoherence, without having to resort to the DQT of the previous sections. The example is histories of momenta in the quantum Brownian motion model, for a free particle in the Fokker-Planck limit. It is in some ways a curious and pathological example, but it does not appear to have been noticed before, and is perhaps of interest in relation to the discussions of the previous sections.

We first consider the form of the decoherence functional for a system-environment model with, for simplicity, projectors at two moments of time. It is

$$D(\alpha_1,\alpha_2|\alpha_1',\alpha_2)=\mathrm{Tr}(P_{\alpha_2}K_0^t[P_{\alpha_1}\rho P_{\alpha_1'}]). \tag{6.1}$$

Here, the environment has been traced out, so the projectors and the trace refer to the system only. The evolution operator K_0^t refers to reduced system dynamics described by the Master equation whose Wigner transform is Eq. (4.4), that is, its solution is

$$\rho_t=K_0^t[\rho_0], \tag{6.2}$$

It is also useful to introduce a backwards time evolution operator $\tilde{K}_0^t$, defined by

$$\mathrm{Tr}(AK_0^t[\rho_0])=\mathrm{Tr}(\tilde{K}_0^t[A]\rho_0) \tag{6.3}$$

(this is not the inverse of K_0^t since the evolution is not unitary). In terms of it, the decoherence functional may be written,

$$D(\alpha_1,\alpha_2|\alpha_1',\alpha_2)=\mathrm{Tr}(\tilde{K}_0^t[P_{\alpha_2}]P_{\alpha_1}\rho P_{\alpha_1'}). \tag{6.4}$$

Backwards evolution may also be described by a Master equation whose Wigner transform is similar to the usual one [Eq. (4.4)], but the unitary and dissipative terms have the opposite sign [we consider only the case $V(x)=0$ here]. The decoherence term produces the same effect in either direction in time.

By way of a digression, from Eq. (6.4) we can see why decoherence of position histories is produced by essentially the same mechanism that diagonalizes the density matrix; the projector P_{α_2} starts out diagonal in x and remains approximately diagonal in x under evolution by $\tilde{K}_0^t$, hence when acted on by position projectors $P_{\alpha_1},P_{\alpha_1'}$ it gives approximate diagonality of the decoherence functional.

After these preliminaries, we turn to the case in which the projectors in Eq. (6.4) are onto ranges of momenta. We shall show that diagonality in p is exactly preserved by $\tilde{K}_0^t$, for the case of the free particle coupled to an environment in the Fokker-Planck limit. To see this, consider first the Wigner representation of the Master equation in this case. It is

$$\frac{\partial W}{\partial t}=-\frac{p}{m}\frac{\partial W}{\partial x}+2\gamma\frac{\partial}{\partial p}(pW)+2m\gamma kT\frac{\partial^2 W}{\partial p^2}. \tag{6.5}$$

The important property of this equation is the now following: if W is a solution to this equation, with initial condition W_0, then $\partial W/\partial x$ is also a solution, with initial condition $\partial W_0/\partial x$. Translated back into density operator language, this means that if ρ_t is a solution to the Master equation with initial condition ρ_0, then $[\rho_t,\hat{p}]$ is also a solution with initial condition $[\rho_0,\hat{p}]$, so

$$[\rho_t,\hat{p}]=K_0^t[[\rho_0,\hat{p}]]. \tag{6.6}$$

This may also be written

$$[K_0^t[\rho_0],\hat{p}]=K_0^t[[\rho_0,\hat{p}]] \tag{6.7}$$

or better,

$$e^{ia\hat{p}}K_0^t[\rho_0]e^{-ia\hat{p}}=K_0^t[e^{ia\hat{p}}\rho_0 e^{-ia\hat{p}}] \tag{6.8}$$

for any real constant a.

Now suppose that $[\rho_0,\hat{p}]=0$, which is equivalent to the statement that ρ_0 is diagonal in p. Then it follows that $[\rho_t,\hat{p}]=0$ for all t. This means that the evolution operator $\tilde{K}_0^t$ preserves diagonality in momenta. It follows immediately from this that the decoherence functional (6.4) with projectors onto momenta will be *exactly* diagonal.

Equation (6.8) shows that the exact decoherence of momenta comes from a translational invariance visible in the path-integral representation of K_0^t [essentially Eq. (3.3) without the projectors, with zero potential, and in the Fokker-Planck limit]; it is invariant under $x\to x+a, y\to y+a$. Furthermore it is broken by the frequency renormalization term in Eq. (3.6), but we have here assumed that the renormalized frequency is set to zero, along with the potential. This is all rather unnatural, and for this reason this property is an unphysical feature perhaps only of pedagogical value. It ultimately traces back to the conservation of momentum of the entire system [as long as the system environment coupling is of the form $(x-q_n)^2$ in Eq. (3.1)].

The equivalent Langevin description also gives some insight. The momenta, in this description, obeys the equation

$$\dot{p}+\gamma p=\eta(t),$$

where $\eta(t)$ is the usual Gaussian white noise. The important point is that this equation is first order, so p_t is a function of p, but not of $\dot{p}$, so we expect in the quantum theory that $[\hat{p}_t,\hat{p}]=0$, and therefore their histories will be exactly decoherent.

On the other hand, while the density matrix (and indeed any other evolving operator) will remain exactly diagonal in momenta, the distribution of momenta $\rho(p,p)$ will generally spread. We therefore have the perhaps surprising situation of a quantity that suffers fluctuations but is still exactly decoherent. The free particle without an environment is clearly exactly decoherent in momentum. Furthermore the distribution of momentum does not spread for the free particle. On coupling to an environment in such a way that the total (system plus environment) momentum is conserved, one might expect to get only approximate decoherence of the system momentum, since system momentum alone is no longer exactly conserved. The surprise is that in a certain regime of this model (the Fokker-Planck limit), the decoherence of momentum *remains exact*, the environment making its mark only on the momentum fluctuations, which now do spread. This emphasises the fact that the evolution of $[\hat{p},\rho]$ (which controls decoherence) and the evolution of $\rho(p,p)$ or $(\Delta p)^2$ (which controls fluctuations) can really be quite different.

As stated, this example is in many ways a curiosity, but it illustrates some interesting points. And in the hunt for theories, which are exactly decoherent, it is surely worth noting the places in which it was already lying under our noses!

VII. A GENERAL APPROACH?

We now turn to the question of how the construction described may be extended to quantum systems, which are not described by a single simple canonical pair satisfying Eq. (1.7), but instead by a more complicated algebra. Spin systems, for example, are not described by Eq. (1.7). While we do not have a comprehensive answer to this, the following is an indication of how one might proceed.

Suppose we have a quantum theory described by, a set of operators $A_k, k=1,2\ldots$ obeying a closed algebra, where $[A_k, A_j]\neq 0$ in general. (The case described so far has $A_1 = p, A_2 = x, A_3 = 1$.) The equations of motion are

$$\dot{A}_k = i[H, A_k] = f_k(A_1, A_2, \ldots) \tag{7.1}$$

for some Hamiltonian $H = H(A_1, A_2, \ldots)$, and the above relation defines the function f_k. Suppose we consider the decoherence functional for histories specified by fixed values of A_k. Since A_k at different times will generally not commute, the histories will generally not be decoherent.

Now consider a second theory described by a set of commuting operators B_k, with canonical momenta P_k. Suppose that at the classical level, they have the Poisson bracket relations,

$$\{B_k, B_j\} = 0, \quad \{B_k, P_j\} = \delta_{kj}, \quad \{P_k, P_j\} = 0. \tag{7.2}$$

Now define the Hamiltonian to be

$$\mathcal{H} = \sum_k P_k f_k(B_1, B_2 \ldots), \tag{7.3}$$

where f_k is the function defined in Eq. (7.1). Then the classical equations of motion for B_k are

$$\dot{B}_k = \{B_k, \mathcal{H}\} = f_k(B_1, B_2, \ldots), \tag{7.4}$$

On quantization (and with attention to operator ordering), we thus obtain a set of commuting operators B_k, which obey the same equations of motion as the original set of operators A_k. This means that histories of fixed B_k will be exactly decoherent. Furthermore, in the expression for the probabilities for histories (1.1), the probabilities for histories of A_k and B_k will be almost the same function of the operators, differing in the form of the initial state, and in the fact that the trace in the case of the B_k operators is over a Hilbert space twice as large. Of course, these differences may be substantial so this does not prove anything in terms of the closeness of the two theories, but the above shows that the question of the dynamics is straightforward. A more detailed description of the relationship between A_k and B_k is required for further analysis, and this is perhaps best carried out with specific examples. This will be pursued elsewhere.

VIII. SUMMARY AND DISCUSSION

A. Summary

We have shown in a variety of ways that approximate decoherence of histories of a system with canonical pair p,x may be turned into exact decoherence by doubling the Hilbert space and switching to the classically equivalent variables $P = p - k, X = x + y$, where the auxiliary variables k,y are in a minimum uncertainty state. Any nondecoherent set of histories may be made decoherent in this way, but the point is that the change in the probabilities (or the Wigner function) is small for histories that are already approximately decoherent. The role of the environment in this scheme is that, by giving the original system thermal fluctuations, it provides a kind of ''smoke screen'' rendering the shift from p,x to P,X undetectable.

B. An alternative approach to emergent classicality?

The approach described here might be regarded as giving an alternative approach to emergent classicality. Standard demonstrations of approximate classicality involve comparing the predictions of classical and quantum mechanics in a given situation. Although this comparison is often clear intuitively, at a more fundamental level the issue is perhaps clouded by the fact that classical and quantum mechanics are theories of different types: how can one measure the ''distance'' between them? Here, however, in considering deterministic quantum theories we are essentially writing down a quantum theory whose predictions are exactly the same as a given classical theory. To check for emergent classicality we then compare standard quantum theory with the deterministic quantum theory. Since the theories are the same type of thing—quantum theories—it is clearer how they may be compared. One may compare the density operators predicted by the two theories, for example.

Although this conceptual advantage is admittedly minimal, there could also be a practical advantage. The decoherence functional is in general rather complicated to calculate, in comparison to Wigner functions and density operators, to a degree that presents problems in some areas of interest (such as the study of histories of hydrodynamic variables [31]). The results of this paper suggest that a test for approximate decoherence of histories consists quite simply of comparing the Wigner functions (or density operators) of standard quantum theory and a suitably chosen deterministic quantum theory.

C. Other approaches to approximate decoherence

There are undoubtedly many other ways of investigating the connection between approximate and exact decoherence, and it would certainly be of interest to explore these. Here, we have adopted the device of doubling the set of dynamical variables, and employed a fundamentally different action. It would be of particular interest to see whether one could avoid this in a simple way. For example, the commuting position and momentum operators of von Neumann, described in the Introduction, appear to hold the possibility of moving from noncommuting to commuting operators without having to change the underlying dynamics or the number of dynamical variables.

One of the difficulties of the present scheme is that the Hamiltonian of the auxiliary system has wrong sign, leading to the possibility of negative energies, although it is not clear that this undesirable feature must arise in this context. For example, exactly conserved quantities are exactly decoherent, and also the model of Sec. VI gave exact decoherence without negative energies.

An alternative scheme, similar to the present one, which avoids negative energies, is to add an identical auxiliary system with the correct sign for the Hamiltonian, but work with complex canonical variable [32]. So we define

$$\hat{X}=\hat{x}+i\hat{y}, \quad \hat{P}=\hat{p}+i\hat{k}, \tag{8.1}$$

which clearly satisfy $[\hat{X},\hat{P}]=0$. The total Hamiltonian for a linear system is then

$$H=\frac{1}{2m}\hat{P}^{\dagger}\hat{P}+\frac{1}{2}m\omega^{2}\hat{X}^{\dagger}\hat{X}, \tag{8.2}$$

which is positive. The difficulty with this approach (although not obviously unsurmountable) is that now one is faced with the issue of interpreting position and momentum operators with an imaginary part.

These examples and their problems cause one to wonder whether all attempts to distort approximate decoherence into exact decoherence in a reasonably general way (i.e., not just for special initial states) will encounter features that are difficult to accept. We might expect difficulties because we are in a sense trying to rewrite quantum theory in essentially classical terms, and this is well-known to lead to problems. Another example of this is the Wigner function representation, which gives a deterministic evolution equation close to the classical one for a phase-space distribution function, but it is not always positive and so cannot be directly interpreted as a true probability distribution. Then there is the Bohm theory approach to quantum theory, which gives a direct interpretation of the wave function in terms of trajectories, but is explicitly nonlocal. The extent to which quantum theory cannot be interpreted in classical terms is elegantly summarized in the Bell inequalities (and other related results). This raises the question of whether inequalities of the Bell type have something to say about the degree to which decoherence may be made exact. These and other issues will be explored elsewhere.

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