Spin and dualization of $SU(5)$ dyons

Tanmay Vachaspati
Physics Department, Case Western Reserve University, Cleveland, Ohio 44106-7079

Danièle A. Steer*
DAMTP, CMS, Wilberforce Road, Cambridge CB3 OWA, United Kingdom
(Received 25 October 2000; published 19 March 2001)

Motivated by the dual standard model, we study the angular momentum spectrum of stable $SU(5)$ dyons that can be transformed into purely electric states by a suitable duality rotation, i.e., dyons that are dualizable. The problem reduces to solving a Diophantine equation for the holomorphic charges in each topological sector, but the solutions also have to satisfy certain constraints. We show that these equations can be solved and sets of dualizable, half-integer spin $SU(5)$ dyons can be found, each of which corresponds to a single family of the standard model fermions. We then find two predictions of the dual standard model. First, the family of dualizable, half-integer spin dyons is accompanied by a set of dualizable, integer-spin partner states. Secondly, the dyon corresponding to the electron must necessarily contain nontrivial color internal structure. In addition, we provide other general results regarding the spectrum of dualizable dyons and their novel properties, and extend the stability analysis of $SU(5)$ monopoles used in the dual standard model so far to discuss the stability of the half-integer spin dyons.

DOI: 10.1103/PhysRevD.63.085008 PACS number(s): 14.80.Hv, 12.60.−i

I. INTRODUCTION

The idea that particles may be viewed as solitons can be traced back to Skyrme [1] who introduced what is now called the Skyrme model in which a classical solution (''Skyrmion'') represents the proton. The model has proved useful in the discussion of the properties of light nuclei even though it is known that the Skyrmion does not have the constituent structure of the proton.

The recent attempts to build a dual standard model are along the lines that Skyrme developed—that is, to find a model that admits soliton analogues of the known fundamental particles. Partial success in this direction was achieved in the discovery that the topological charges of the stable magnetic monopoles in an $SU(5)$ field theory are in one to one correspondence with the electric charges of one family of fermions of the standard model [2,3]. A possible scheme to obtain three families of identically charged magnetic monopoles was outlined in Ref. [4], though at the expense of considerably complicating the group structure of the model.

So far, a substantial shortcoming of the model (summarized in Sec. II) has been that the monopoles emerging from the $SU(5)$ field theory are all bosonic while the standard model particles are known to be fermionic. The issue of spin and handedness of the solitons was discussed in Ref. [3] though not resolved in the $SU(5)$ context. The basis for the discussion was the discovery of ''spin from isospin'' [5–7] in which dyons can have half-integer spin even in a purely bosonic particle theory. The possibility for handedness was discussed in the context of a θ term in the action and the result that dyons can carry fractional electric charges proportional to θ [8]. Also needed was the angular momentum of dyons in presence of a θ term [9].

The success of the spin from isospin phenomenon for the dual standard model depends on the existence of half-integer spin states for *all* the dyons that ultimately correspond to the standard model particles. In the case of the 't Hooft–Polyakov [10,11] monopole, spin from isospin can provide half-integer spin to the fundamental monopole but not to the monopole with twice the topological winding. In contrast, in the $SU(5)$ case it has been shown that *all* the stable monopoles can be provided with electric charges to make them into half-integer spin dyons [12]. However, the particles that we observe are not ostensibly dyons. Hence it is important to show that all the half-integer spin dyons which arise in the $SU(5)$ field theory and which will be identified with standard model fermions can be transformed by a duality rotation into purely electric charges. This is the aim of the present paper.

Here we shall take the approach that the known standard model particles are purely electric (in contrast with Refs. [13,14] in which these particles have both electric and magnetic charge). We would then like to know whether the spin 1/2 dyonic states of the $SU(5)$ model—that are in one-one correspondence with the standard model particles—can all be dualized into purely electric charges. In trying to answer this question, we strictly need to consider duality rotations for gauge fields transforming in representations of the unbroken symmetry group $H=[SU(3)\times SU(2)\times U(1)]/Z_6$. Such non-Abelian duality transformations are not fully understood yet. Our approach (Sec. III) will be to *assume* that independent duality rotations can be applied to the field strengths in the directions of the four commuting generators of H. In other words, we assume that the transformation

$$E_i^a+iB_i^a\rightarrow e^{i\phi_a}(E_i^a+iB_i^a),\quad a=0,8,3,1 \tag{1}$$

leaves the equations of motion invariant, where the ϕ_a are

*Present address: Département de Physique Théorique, Université de Genève, 24 Quai Ernest Ansermet, 1211 Genève 4, Switzerland. Email address: daniele.steer@physics.unige.ch

independent phase angles for the generators λ_3 and λ_8 of $SU(3)$, τ_3 of $SU(2)$ and Y of $U(1)$:

$$\lambda_3=\frac{1}{2}\,\mathrm{diag}(1,-1,0,0,0), \tag{2}$$

$$\lambda_8=\frac{1}{2\sqrt{3}}\,\mathrm{diag}(1,1,-2,0,0), \tag{3}$$

$$\tau_3=\frac{1}{2}\,\mathrm{diag}(0,0,0,1,-1), \tag{4}$$

$$Y=\frac{1}{2\sqrt{15}}\,\mathrm{diag}(2,2,2,-3,-3). \tag{5}$$

The corresponding magnetic charges m_a and electric charges q_a transform as

$$(q_a+im_a)\rightarrow e^{i\phi_a}(q_a+im_a). \tag{6}$$

It should be emphasised that these transformations are a conjecture: we cannot rigorously justify them since the non-Abelian equations of motion explicitly involve the gauge fields and not just the field strengths (in contrast with Maxwell's equations). However, the equations of motion are indeed invariant under the transformation if only the commuting gauge field components (Cartan subalgebra) are non-vanishing. Also note that the Hamiltonian and the Euclidean action remain invariant under the transformation in Eq. (1). Furthermore, if $SU(5)$ were broken down to $U(1)^4$, each of the gauge field components labeled by the index $a=0,8,3,1$ would be Abelian and then the Abelian duality transformations would correspond exactly to Eq. (1).

While adopting the transformation in Eq. (1) as a ''working hypothesis'' for the duality rotation, we also discuss the case when this may not be true. If, for example, we restrict ourselves to $\phi_0=\phi_8$, we can show that the half-integer spin states in the even winding topological sectors must necessarily carry both magnetic and electric $SU(3)$ charge.

In the case when the ϕ_a are independent, we find an infinite set of dualizable, half-integer $SU(5)$ dyon states that are in one to one correspondence with the standard model particles. To arrive at this conclusion we need to solve constrained quadratic Diophantine equations that can be definite or indefinite. Such equations have been considered at least since 600 A.D. by Bhaskara and Brahmagupta and techniques to solve them can be found in number theory text books (e.g. [15]). We shall describe some of the equations and their solutions in Appendix B.

The infinity of solutions is unlikely to be of any direct physical relevance. The reason is that we are interested only in the lowest energy state in any given topological sector since, presumably, the higher energy states are unstable to decay into the lowest energy state. However, the energy of a dyon is not known at strong coupling—which is the relevant regime for making contact with the standard model—and so there is no sure way of determining the lowest energy states. The best that we can do at present is to assume a Bogomol'nyi-Prasad-Sommerfield (BPS) form for the energy [16,17] (see also the monopole reviews in Ref. [18]) in which the energy of a dyon is proportional to the magnitude of its charge:

$$E_{BPS}\propto\sqrt{q_a^2+m_a^2} \tag{7}$$

where a sum over the index a is understood. This form of the energy does not apply to the dual standard model where the monopoles may be close to being BPS but are not exactly BPS [2], and neither does it apply to the standard model particles. The purpose of considering Eq. (7) is simply that it enables us to find the lowest energy dyons in the weak coupling, near BPS limit.

If we assume Eq. (7) for the energy, then for any given value of the magnetic charge m^a, it would pick out the state $q^a=0$ as the state with the lowest energy. These purely magnetic states would have zero spin (see below). The situation is more interesting when we include a θ term in the $SU(5)$ action because then the electric charge contains a contribution from the θ term [8]. In that case, the lowest energy state can indeed have half integer spin. The hope then would be that for a certain value of θ, of the phases ϕ_a, and of the coupling constant g, one would obtain a complete family of spin half dyons which would be the lowest energy states. However, we show that this hope is not realized due to the monopole with topological winding $n=6$. In this topological class, the state with the lowest BPS energy necessarily has integer spin.

Ideally we would like to work with the energy of a dyon at strong coupling and then determine the lightest states for given parameters. This would require understanding the quantum properties of magnetic monopoles—a subject that has been under intense research over the last two decades. Remarkable progress has been achieved in the understanding of monopoles at strong coupling in the supersymmetric case [19] but several tantalizing issues remain open especially in the non-supersymmetric setting (e.g. [20]). An issue that is central to particle-soliton duality is the group representation in which the monopoles transform when they are considered as particles. Goddard-Nuyts-Olive conjectured that monopoles transform in a representation of a dual symmetry group [21]. Bais and Schroers [22,23] find that a richer structure is applicable to non-Abelian monopoles, since they carry ''holomorphic'' charges in addition to a topological charge. (This will be important to us in Sec. VI.) In the $SU(5)$ model, Lepora has provided strong evidence that the monopoles transform in the fundamental representation of the dual symmetry group $[SU(3)\times SU(2)\times U(1)]$ based on the transformation properties of the monopoles under rigid gauge transformations [24]. This evidence seems to support the concept of a dual standard model. Further support comes from Lepora's calculation of the value of the weak mixing angle θ_w in the context of the $SU(5)$ dual standard model [25]. Lepora finds $\sin^2\theta_w=0.22$ which is in good agreement with experiment at a few GeV. However the relevance of the few GeV scale to the dual standard model has not yet been investigated. Naively it seems that this should be the scale at which the monopole-like structure of elementary particles becomes relevant. Then it is possible that phenomenological considerations already impose strong constraints on the idea of the dual standard model. It would be very interesting to pursue this idea further.

II. REVIEW OF DUAL STANDARD MODEL

Consider the symmetry breaking

$$G=SU(5)\to H=[SU(3)\times SU(2)\times U(1)]/Z_6. \quad (8)$$

The magnetic monopoles in this symmetry breaking are labeled by their $SU(3)$, $SU(2)$ and $U(1)$ magnetic charges,

$$M=(m_0,m_8,m_3,m_1)=\left(0,\frac{n_8}{\sqrt{3}g},\frac{n_3}{2g},\frac{-1}{2g}\sqrt{\frac{5}{3}}n_1\right) \quad (9)$$

where

$$n_8=n+3k, \quad n_3=n+2l, \quad n_1=n. \quad (10)$$

Here, k and l are arbitrary integers since the λ_8 [of $SU(3)$] and τ_3 [of $SU(2)$] charges are only defined modulo 3 and 2 respectively.

The topological sector is only determined by the integer n_1 which gives the topological winding number $[\Pi_2(G/H)=Z]$. The integers n_8 and n_3 are related to the ''holomorphic'' charges which are discussed in Refs. [22,23,26] and which are not topological. In [26], Murray derived constraints that, in the BPS limit, the sum of the topological and holomorphic charges has to be greater than or equal to zero. The holomorphic charges are the diagonal entries of the magnetic charge matrix which in this $SU(5)$ case is

$$2\mathbf{M}=2g[m_0\lambda_3+m_8\lambda_8+m_3\tau_3+m_1Y]$$
$$=\mathrm{diag}\left(\frac{n_8-n_1}{3},\frac{n_8-n_1}{3},\frac{-2n_8-n_1}{3},\frac{n_3+n_1}{2},\frac{-n_3+n_1}{2}\right). \quad (11)$$

Murray's constraints [26] are then that the first three entries of the charge matrix must be non-negative and the last two entries must be greater than or equal to minus the topological charge (our n_1). For $n_1\leqslant 0$ this leads to

$$-n_1\geqslant 2k\geqslant 0, \quad -n_1\geqslant l\geqslant 0. \quad (12)$$

(For positive values of n_1, these inequalities would be reversed.)

As we shall see below, the integer k is crucially important in determining the spin of a dyon: there are values of k that violate the constraints but which give rise to angular momentum that cannot be achieved by states satisfying the constraints. Since Murray's constraints are only valid in the BPS limit in any case, we will assume them provided there is no state that violates them and which has a different value (integer versus half-integer) of the angular momentum.

A stability analysis of the non-BPS monopoles in any topological sector shows that only the $\pm n=1,2,3,4,6$ monopoles are stable. [This result assumes a range of parameters in the $SU(5)$ potential [3].] A comparison with the standard model particles shows that these monopoles are in one to one correspondence as depicted in Table I. On dualization of the $SU(5)$ model, we expect these magnetic monopoles to correspond to electrically charged particles, while the electrically charged scalar and vector particles of the $SU(5)$ model should correspond to very massive magnetically charged states.

TABLE I. The quantum numbers (n_8, n_3 and n_1) on stable $SU(5)$ monopoles are shown and these correspond to the $SU(3)$, $SU(2)$ and $U(1)$ charges on the corresponding standard model fermions shown in the right-most column.

n	$n_8/3$	$n_3/2$	$n_1/6$	
$+1$	1/3	1/2	1/6	$(u,d)_L$
-2	1/3	0	$-1/3$	d_R
-3	0	1/2	$-1/2$	$(\nu,e)_L$
$+4$	1/3	0	2/3	u_R
-6	0	0	-1	e_R

Non-trivial spin of the $SU(5)$ monopoles is provided by considering electrically charged bound states on the monopoles. If a scalar field transforming in the fundamental representation of $SU(5)$ is included in the model, quanta of this field will provide such electrically charged states—this is the ''spin from isospin'' idea [5–7] which was extended to $SU(5)$ monopoles [12,27]. Thus, as in those papers, we now add to the original theory a scalar field transforming in the fundamental representation of $SU(5)$. [The existence of such bound states will depend on the details of the $SU(5)$ potential. Here, as in [5,6,12], we will simply assume that the bound states exist.] To determine whether the spin is integer or half-integer, one needs to calculate the angular momentum in the gauge fields of two dyons of charges $(q_a^{(1)},m_a^{(1)})$ and $(q_a^{(2)},m_a^{(2)})$. It is given by the Zwanziger formula [28,29] applied to each of the charges

$$J=-\sum_a (q_a^{(1)}m_a^{(2)}-q_a^{(2)}m_a^{(1)}) \quad (13)$$

where the index a runs over 0,8,3,1 and labels the two $SU(3)$ charges, one $SU(2)$ charge and the hypercharge. The m_a have been defined in Eq. (9) and the q_a are the electric charges present in the state under consideration. Note that the expression for the angular momentum is invariant under the duality rotation in Eq. (6). In the applications below, we will only need to consider the case when dyon 1 is purely electric and dyon 2 is purely magnetic. Then the index labeling the dyons can be dropped and we can write [27]

$$J=-\sum_a q_a m_a. \quad (14)$$

The fundamental scalar field of $SU(5)$ has five components and we can consider dyonic states with any number of quanta of these five components. Let us label the components by the index h, then the four different electric charges on a single quanta of each of the five components can be written as

$$e_0^h=\frac{g}{2}\begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix},\quad e_8^h=\frac{g}{2\sqrt{3}}\begin{pmatrix}1\\1\\-2\\0\\0\end{pmatrix},$$

$$e_3^h=\frac{g}{2}\begin{pmatrix}0\\0\\0\\1\\-1\end{pmatrix},\quad e_1^h=\frac{g}{2\sqrt{15}}\begin{pmatrix}2\\2\\2\\-3\\-3\end{pmatrix}. \tag{15}$$

(These assignments are obtained by considering the corresponding Noether charges.) To clarify the meaning of these charge assignments consider the example in which we have one quanta of the first component ($h=1$) of the fundamental scalar field. This quanta will have $q_0=g/2$, $q_8=g/2\sqrt{3}$, $q_3=0$ and $q_1=g/\sqrt{15}$. Similarly we can work out the charges on any of the other four ($h=2,3,4,5$) scalar field components. If we now consider N_h quanta of the component h, then the total electric charge is

$$Q=(q_0,q_8,q_3,q_1), \tag{16}$$

with

$$q_0=\frac{g}{2}(N_1-N_2) \tag{17}$$

$$q_8=\frac{g}{2\sqrt{3}}(N_1+N_2-2N_3) \tag{18}$$

$$q_3=\frac{g}{2}(N_4-N_5) \tag{19}$$

$$q_1=\frac{g}{2\sqrt{15}}[2(N_1+N_2+N_3)-3(N_4+N_5)]. \tag{20}$$

Let us now define

$$M_0\equiv-(N_1-N_2) \tag{21}$$

$$M_8\equiv-(N_1+N_2-2N_3) \tag{22}$$

$$M_3\equiv-(N_4-N_5) \tag{23}$$

$$M_1\equiv-3(N_4+N_5)+2(N_1+N_2+N_3). \tag{24}$$

Since the N_h are integers, so are the M_a. Solving the above equations gives the N_h in terms of the M_a:

$$N_2=N_1+M_0 \tag{25}$$

$$N_3=N_1+\frac{M_8+M_0}{2} \tag{26}$$

$$N_4=N_1+\frac{M_0-M_3}{2}+\frac{M_8-M_1}{6} \tag{27}$$

$$N_5=N_1+\frac{M_0+M_3}{2}+\frac{M_8-M_1}{6}. \tag{28}$$

Now since the N_h are integers, we have the following two constraints on the M_a:

$$\frac{M_8+M_0}{2}=\text{integer} \tag{29}$$

$$\frac{M_0-M_3}{2}+\frac{M_8-M_1}{6}=\text{integer}. \tag{30}$$

The second constraint can be combined with the first to put it in a more useful form:

$$\frac{M_8}{3}+\frac{M_3}{2}+\frac{M_1}{6}=\text{integer}. \tag{31}$$

Then the angular momentum from Eq. (14) with Eqs. (16) and (9) is found to be

$$J=\frac{1}{2}\left[\frac{M_8n_8}{3}+\frac{M_3n_3}{2}+\frac{M_1n_1}{6}\right]. \tag{32}$$

In Ref. [12] it was shown that we can have $J=1/2$ for every value of n for suitable values of the electric charges M_a (which will be different on the different monopoles). Note that J is only the angular momentum in the long range gauge fields and does not contain other possible contributions such as orbital angular momentum and spin of the gauge particles. These extra contributions can only change the angular momentum by an integer and cannot change a half-integer angular momentum state to one that has integer angular momentum (or vice versa).

Next let us consider the addition of an $SU(5)$ θ term. In terms of the gauge fields corresponding to the diagonal generators, the additional piece of the Lagrangian is

$$L_\theta=\kappa[G^3_{\mu\nu}\tilde{G}^{\mu\nu 3}+G^8_{\mu\nu}\tilde{G}^{\mu\nu 8}+W^3_{\mu\nu}\tilde{W}^{\mu\nu 3}+Y_{\mu\nu}\tilde{Y}^{\mu\nu}] \tag{33}$$

where

$$\kappa=\frac{g^2\theta}{16\pi^2}. \tag{34}$$

The addition of such a term does not alter the expression for the angular momentum of the dyons given in Eq. (32) but it does affect the values of the electric charges in Eqs. (17)–(20). [In the case of $SU(2)$ monopoles, the effect of a θ term on the electric charge has been discussed in Ref. [8] and on

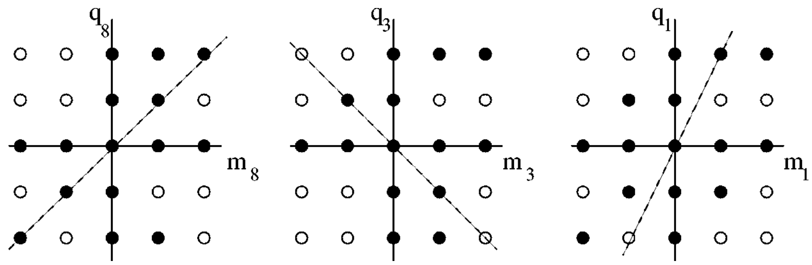

FIG. 1. The $SU(5)$ dyons have four magnetic and four electric charges and so they can be depicted as points in an eight dimensional space. (Only three planes in this eight dimensional space have been shown in the diagram.) On each of these planes, we depict bosonic states by filled circles and fermionic states by unfilled circles. The question we would like to address is whether, by an independent rotation in each of these four planes, a complete family of half-integer spin dyons can be made to lie in the four dimensional electric plane.

the angular momentum of dyons in Ref. [9].] The new expressions for the electric charges on the dyons are

$$q_0 = -\frac{g}{2} M_0 \tag{35}$$

$$q_8 = -\frac{g}{2\sqrt{3}} M_8 + \frac{g n_8}{\sqrt{3}} \frac{\theta}{2\pi} \tag{36}$$

$$q_3 = -\frac{g}{2} M_3 + \frac{g n_3}{2} \frac{\theta}{2\pi} \tag{37}$$

$$q_1 = \frac{g}{2\sqrt{15}} M_1 - \frac{g n_1}{2} \sqrt{\frac{5}{3}} \frac{\theta}{2\pi}. \tag{38}$$

It is straightforward to check that a shift of θ by 2π can be compensated for by shifts of the M_a that satisfy Eqs. (29) and (31), thus verifying that the spectrum of states is invariant under $\theta \to \theta + 2\pi j$ for any integer j.

We now want to know if a complete set (i.e. all topological sectors occurring in Table I) of the half-integer spin dyons can be made to be purely electric by performing a suitable duality rotation (see Fig. 1).

III. DUALIZATION OF HALF-INTEGER SPIN DYONS

The duality rotation phase angles ϕ_a [see Eq. (1)] required to make a dyon into a purely electric object are given by the inverse tangent of the ratios of its magnetic and electric charges. Therefore,

$$\tan\phi_0 = 0 \quad (\text{if} \quad M_0 \neq 0) \tag{39}$$

$$\tan\phi_8 = -\frac{2}{g^2} \frac{1}{M_8/n_8 - 2\theta/2\pi} \tag{40}$$

$$\tan\phi_3 = -\frac{1}{g^2} \frac{1}{M_3/n_3 - \theta/2\pi} \tag{41}$$

$$\tan\phi_1 = -\frac{5}{g^2} \frac{1}{M_1/n_1 - 5\theta/2\pi}. \tag{42}$$

Note that the M_a are integers and denote the electric charges on the dyons and hence can depend on the winding number n. Also the integers n_a clearly depend on n. For the dyons to be dualizable, we want that the duality phase angles be independent of n. Hence we require that α_a be independent of n where

$$M_8 = n_8 \alpha_8 \tag{43}$$

$$M_3 = n_3 \alpha_3 \tag{44}$$

$$M_1 = n_1 \alpha_1. \tag{45}$$

The α_a are independent of n and hence by considering the dyon with $n_1 = 1$ we find that α_1 must be an integer. The constraint in Eq. (12) shows that we must also take $n_8 = 1$ and $n_3 = 1$ for $n_1 = 1$ and so all the α_a must be taken to be integers.[1] Furthermore, there is a constraint that the α_a must satisfy, coming from the constraint Eq. (31) when combined with Eq. (10) and setting $n = 1$:

$$\frac{\alpha_8}{3} + \frac{\alpha_3}{2} + \frac{\alpha_1}{6} = \text{integer}. \tag{46}$$

In terms of the α_a, the angular momentum (14) is given by

$$2J_n = \left[\frac{\alpha_8}{3} + \frac{\alpha_3}{2} + \frac{\alpha_1}{6}\right] n^2 + 2[n(\alpha_8 k_n + \alpha_3 l_n) + \alpha_3 l_n^2] + 3\alpha_8 k_n^2 \tag{47}$$

where we have added a subscript n to emphasize the n-dependence of J. For the whole family of dyons ($\pm n = 1,2,3,4,6$) to have half-integer spin, we need the right-hand side of Eq. (47) to be odd for each member. First consider the $n = 2$ monopole. The first term on the right-hand side is clearly even in this case. The second term is also even since the α_a are integers. So $2J_2$ is odd if and only if $3\alpha_8 k_2^2$ is odd. Now suppose that α_8 and k_2 are chosen so that $3\alpha_8 k_2^2$ is odd. Then all the other dyons in the dualizable family will

[1]Following the discussion after Eq. (12), if we relax the constraint to allow $n_8 = -2$ for $n_1 = 1$, half-integer values of α_8 could still yield integer values of M_8. However, in Appendix A we show that half-integer values of α_8 cannot yield a family of spin half dyons and so we will restrict our discussion to integer α_8.

have half-integer spin if we set $k_n = \pm k_2$ when the first term on the right-hand side of Eq. (47) is even, and $k_n = 0$ when this term is odd.

Two explicit examples satisfying the constraint in Eq. (46) are

$$\alpha_8 = 1, \quad \alpha_3 = -1, \quad \alpha_1 = 1, \tag{48}$$

$$\alpha_8 = 1, \quad \alpha_3 = 0, \quad \alpha_1 = 4. \tag{49}$$

For the first example, the first term on the right-hand side of Eq. (47) vanishes and therefore $2J_n$ is odd provided $k_n^2 = \text{odd}$ for all n.[2] Hence a whole family of dyons has half-integer spin and is dualizable. In fact, there are an infinite number of solutions (α_a) that have this property. This can be seen by noting that a shift of each of the α_a by any fixed even integer also leads to a solution that satisfies the constraints and preserves the half-integer angular momentum.

The dualizable $2J_n = 1$ dyon states for a fixed set of α_a correspond to solutions of the Diophantine equation:

$$2\alpha_8 n_8^2 + 3\alpha_3 n_3^2 = 6 - \alpha_1 n_1^2. \tag{50}$$

In Appendix B we show that for the α_a in Eq. (48) there are an infinite number of dualizable dyonic states in every topological sector that have half-integer spin. This conclusion is expected to be valid whenever some of the α_a's differ in their signs, leading to indefinite (hyperbolic) Diophantine equations. If all the α_a have the same sign, we expect there to be a finite set (possibly empty) of solutions. In view of the constraints in Eq. (12) the infinite set of states is not of physical interest. In addition we only expect the lightest of the states for any given winding and angular momentum to be stable.

IV. GENERAL RESULTS

(1) *There are infinitely many solutions to the constraints leading to a dualizable family of half-integer spin dyons.*

This has been shown above in the paragraph following Eq. (49).

(2) *Each member of the family of dualizable half-integer spin dyons has an integer spin partner that is also dualizable.*

To see this conclusion, note that if for a certain n one has $2J_n = \text{odd}$, then the state with $k_n \to k_n \pm 1$ has [Eq. (47)]

$$2J_n \to (2J_n)' = 2J_n + \text{even integer} + 3\alpha_8. \tag{51}$$

[2] This is an illustration of the discussion following Eq. (12). For $n=1$ the constraint in Eq. (12) only allows $k_1 = 0$. However, $k_1 = 0$ gives a dyon with integer spin, while the state $k_1 = 1$ violates the constraint but gives half-integer spin. Since the spin of the dyons is not taken into account in deriving the constraints [26] we assume that the $k_1 = 1$ state is admissible.

However, $3\alpha_8$ has to be odd since $2J_2 = \text{even} + 3\alpha_8 k_2^2$ and this has to be odd for the $n=2$ monopole to have half-integer spin. Hence the state with the shifted value of k_n has integer spin.

Hence the dual standard model predicts bosonic partners of all the standard model fermions. Unlike in the case of supersymmetry, the masses of the partners do not have to be degenerate.

(3) *The $n=6$ dyon with the least BPS energy has integer spin.*

The energy of a BPS dyon is given by Eq. (7),

$$E_{BPS} = c\sqrt{q_a^2 + m_a^2}$$

where c is a proportionality constant. The state with the lowest energy is the one with the smallest electric and magnetic charge. For the $n=6$ dyon, this is the state with $n_8 = 0 = n_3$ since then, both the electric and magnetic charges in the $SU(3)$ and $SU(2)$ sectors vanish. Now using Eq. (32) together with Eq. (45) we see that this state has integer spin.

A general statement of this kind cannot be made for dyons with other windings since they necessarily have non-vanishing $SU(3)$ and/or $SU(2)$ magnetic charge. However it is not difficult to determine which spin state among the dualizable dyons has the least BPS energy. First note that dualizability implies $q_a \propto m_a \propto n_a$. [This relation does not hold for $a=0$ where we have $q_0 = -gM_0/2$ and M_0 is constrained by Eq. (29).] Therefore, for fixed values of the α_a, θ and for small values of g (when the electric charge contributions are subdominant), the least BPS energy state is one that has the minimum values of n_8^2 and n_3^2. For $A \equiv \alpha_8/3 + \alpha_3/2 + \alpha_1/6 = \text{odd}$, this ensures that the $n = 1, -2, 4$ states with half-integer spin have lower BPS energy than the corresponding integer spin states. However, for the $n=-3$ half-integer spin state to have lower energy than the integer spin state in the case of small g, we need $A = \text{even}$ because only then the $n_8 = 0$ $(k_3 = 1)$ state has half-integer spin.

It is worthwhile pointing out the role of the θ term in these considerations. The lowest BPS energy states for a non-zero θ angle will occur for non-zero values of the α_a. If θ were zero, the states with the least energy would be those with vanishing electric charges (since $\alpha_a = 0$ would minimize the BPS energy) and hence, with zero spin.

(4) *The $n = 2,4,6$ half-integer spin dualizable dyons carry λ_3 electric charge i.e., $M_0 \neq 0$.*

To see this, note that Eq. (31) implies that $3M_3 + M_1$ is even. Therefore both M_3 and M_1 are even or both are odd. For the even n dyons, $M_1 = \alpha_1 n_1$ is even. Hence M_3 is also even for even n. Now from the angular momentum formula Eq. (32) and the relations in Eq. (10) we get

$$2J_n = \left[\frac{M_8}{3} + \frac{M_3}{2} + \frac{M_1}{6}\right] n + M_8 k_n + M_3 l_n. \tag{52}$$

Therefore, taking Eq. (31) into account, we see that $2J_n$ is even for even n if M_8 is even. Hence to obtain an odd value for $2J_n$ (i.e. half-integer spin), we must necessarily set M_8 to be odd. Next we use the constraint in Eq. (29) which shows that M_0 has to be odd and, in particular, has to be non-zero. Therefore these half-integer states necessarily carry λ_3 electric charge.

A consequence of this conclusion is that the two $SU(3)$ duality rotation phase angles ϕ_0 and ϕ_8 cannot be equal. If non-Abelian duality rotations can only be applied with $\phi_0=\phi_8$ then the dual standard model would only work provided the particles transforming non-trivially under $SU(3)$ carry magnetic charge.

(5) *The $n=6$ half-integer spin dualizable dyon must have $n_8\neq 0$.*

Inserting $n=6$ in Eq. (47) shows that we must necessarily have $k_6=$odd to get half-integer spin. Therefore $n_8=n+3k_6=3(2+k_6)$ is necessarily non-vanishing and the $n=6$ half-integer spin state carries $SU(3)$ gluonic structure.

Similarly if α_8 and $\alpha_8/3+\alpha_3/2+\alpha_1/6$ are odd integers, then k_3 has to be even for the $n=3$ monopole to have half-integer spin. Then $n_8\neq 0$ and this monopole also carries gluonic structure.

V. STABILITY OF HALF-INTEGER SPIN DYONS

The monopoles in any topological sector have two decay channels. First, the monopoles can emit scalar and vector particles and change their values of k and l. Secondly, a monopole can fragment into two monopoles of smaller magnetic charge. We have to show that neither of these instabilities apply to the states that we would like to interpret as standard model particles.

The first instability will not apply to the lowest lying half-integer spin state in any given topological sector and so we need only concern ourselves with the second instability.

Next we show that the dyons with topological winding $n>6$ are all unstable to fragmentation into dyons with $n=6$ and something else.

Let us denote the dyonic states by their magnetic and electric charges as follows:

$$|n_8,n_3,n_1;M_8,M_3,M_1\rangle.$$

Then we want to show that the decay process

$$\begin{aligned}&|n_8,n_3,n_1;M_8,M_3,M_1\rangle\\&\quad\to|n_8,n_3,n_1-6;M_8-p_8,M_3-p_3,M_1-p_1\rangle\\&\qquad+|0,0,6;p_8,p_3,p_1\rangle\end{aligned} \tag{53}$$

is energetically favorable. The two states on the right-hand side interact by the $U(1)$ magnetic interactions and we know that this is repulsive. The electric interactions are small compared to the magnetic interactions at weak coupling by a factor g^4 and so we ignore them for the present. (Later we will check that the decay would proceed even with the electric interactions taken into account.) Hence it is clear that this decay process is energetically favorable. What is not so clear is if the process is allowed by angular momentum conservation. [The magnetic and electric charges are conserved in Eq. (53).] This is what we will now check.

The angular momentum of the states on the right-hand side can be written as [Eq. (32)]

$$2J_{rhs}=2J_{lhs}+2p_3-M_1-\frac{p_8n_8}{3}-\frac{p_3n_3}{2}-\frac{p_1n_1}{6} \tag{54}$$

up to the addition of an integer (which may be carried off in orbital angular momentum etc.). For angular momentum conservation—meaning that half-integer initial angular momentum should go to half-integer final angular momentum and similarly for integer angular momentum—we therefore need

$$-M_1-\frac{p_8n_8}{3}-\frac{p_3n_3}{2}-\frac{p_1n_1}{6}=\text{even integer}. \tag{55}$$

A solution is simply given by $p_8=0=p_3$, $p_1=6\alpha_1$ because then the left-hand side is even. With these values of the p_a, the electric interactions are also purely $U(1)$ and repulsive. This shows that the decay process is not forbidden by angular momentum conservation and hence can occur for purely energetic reasons which we know favor it.

A similar stability analysis goes through for the $n=5$ dyon. Consider the decay process

$$\begin{aligned}|n_8,n_3,5;M_8,M_3,M_1\rangle&\to|0,n_3,3;p_8,M_3-p_3,p_1\rangle\\&\quad+|n_8,0,2;M_8-p_8,p_3,M_1-p_1\rangle.\end{aligned} \tag{56}$$

This is energetically favored since the two dyons on the right-hand side interact only via the $U(1)$ magnetic interaction which is repulsive. Next we need to check if the decay is allowed by angular momentum conservation.

Using the formula for the angular momentum [Eq. (32)], we find

$$2J_{rhs}=2J_{lhs}-\left[\frac{n_8p_8}{3}+\frac{n_3p_3}{2}-\frac{p_1}{6}+\frac{M_1}{2}\right]. \tag{57}$$

So the decay will be allowed provided

$$\frac{n_8p_8}{3}+\frac{n_3p_3}{2}-\frac{p_1}{6}+\frac{M_1}{2}=\text{even integer}. \tag{58}$$

This is clearly so if we choose $p_8=0=p_3$ and $p_1=3\alpha_1$. (Recall that $M_1=5\alpha_1$ for the initial state to be dualizable.) With this choice of p_a, once again the electric interactions are purely $U(1)$ and repulsive. Hence the $n=5$ dyon is unstable.

This tells us that the dyons with $|n|=5$ and $|n|\geqslant 7$ are unstable, exactly as found for the monopoles in [2,30]. The $\pm n=1,2,3,4,6$ dyons will still be stable because the fragmentation is completely governed (in the weak coupling limit) by the magnetic interactions [2,30]. Therefore the spectrum of stable half-integer spin dyons also agrees with the standard model fermions.

VI. DISCUSSION

Our general results can be found in Sec. IV. The main conclusion is that it is possible to find a family of dyons each member of which has half-integer spin and the family as a whole can be dualized into purely electric states (subject to the discussion of duality rotations given in the Introduction). In addition, there are two new features that have emerged and that may be considered as predictions of the dual standard model. The first is that each of the half-integer spin dyons has a bosonic partner. In the dualized theory, these states would appear as bosonic partners of the known standard model fermions. Since the bosonic partners are not due to an imposed symmetry (e.g. supersymmetry), there is no reason to expect them to be degenerate in mass with their fermionic partners. The second new feature is that some of the half-integer spin dyons (in particular the $n=6$ dyon) may have non-vanishing values of n_8 and n_3 even though the minimum allowed values of these quantum numbers may be zero. For example, in the $n=6$ case, the minimum values are $n_8=0=n_3$, yet to get half-integer spin it is necessary to have $n_8\neq 0$ (see Sec. IV). Since these monopoles with $n_8=n+3k_n\neq 0$ carry the same topological charge as the monopole with winding number n but with $n_8=0$, they too must transform in the fundamental representation of the dual symmetry group [24]. However, the value of k_n is another charge associated with the monopole (related to the ''holomorphic'' charge in [22]) and must correspond to a new property of the particle obtained after dualization.

How is the holomorphic charge manifested in the context of the dual standard model? The holomorphic charge seems to label an internal degree of freedom of the dualized dyons and, according to Bais and Schroers [22], manifests itself as a magnetic dipole moment of the dyons i.e. an electric dipole moment of the particles. Then, for example, the $n=-6$, spin half dyon necessarily has $n_8\neq 0$ which means that it must have non-trivial $SU(3)$ internal structure even though it transforms as an $SU(3)$ singlet. The resolution to this apparent paradox is that the particles in the context of the dual standard model are composite objects and hence they can have internal $SU(3)$ structure in spite of having trivial $SU(3)$ long range interactions (as in the case of the proton). The novelty here is that the $n=-6$ dyon under discussion supposedly corresponds to the electron, implying that the electron must carry non-trivial internal $SU(3)$ structure.

ACKNOWLEDGMENTS

We wish to thank Matthias Blau, Nathan Lepora, Brian Steer, Mark Trodden, Erick Weinberg and Serge Winitzki for discussions. We are especially grateful to Bernd Schroers for help with the constraints derived from Ref. [26] and to David Singer for number theoretic help. T.V. was supported by the DOE. DAS was supported by PPARC of the UK and also the Flora Stone Mather Association at Case Western Reserve University.

APPENDIX A

Consider the possibility that α_8 is a half-integer. In this case, for M_8 to be an integer, n_8 should be an even integer. Then, for even n, all of n_8, n_3 and n_1 are even. Therefore we write $n_a=2\tilde{n}_a$ where $\tilde{n}_a$ are integers and insert into the equation for the angular momentum [Eq. (32)] to find

$$2J_n=2\left[\frac{1}{3}M_8\tilde{n}_8+M_3\tilde{n}_3+\frac{1}{3}M_1\tilde{n}_1\right]. \tag{A1}$$

Hence $2J_n$ is even and half-integer spin solutions do not exist. Therefore half-integer values of α_8 cannot yield a family of half-integer spin dyons.

APPENDIX B

Here we show that there are an infinite number of dyon states with $J=1/2$ for the choice of α_a in Eq. (48) (for example). This is not directly relevant to us because of Eq. (12) and further physical constraints. However it is still an interesting exercise.

To see the infinity of solutions, rewrite the angular momentum constraint [Eq. (32) with (43), (44), (45) and (48)] as

$$2n_8^2-3n_3^2=6-n_1^2. \tag{B1}$$

For the fundamental monopole ($n_1=1$), the problem then is to find all solutions to the equation

$$2p^2-3q^2=5 \tag{B2}$$

where p and q are integers.

This is a standard problem in number theory and is related to Pell's equation (for example, [15]) The idea of the construction is that given *one* solution to the equation

$$ap^2-bq^2=c \tag{B3}$$

where a, b and c are integers, and if there exists a non-trivial solution (l,m) to the equation

$$l^2-abm^2=1, \tag{B4}$$

then an infinite set of solutions can be generated. (The trivial solutions are $l^2=1$, $m=0$.) The construction uses the solution to the first equation, call it p_0,q_0, and the solution to the second equation, call it l,m, to determine another solution:

$$p=lp_0+bmq_0,\quad q=amp_0+lq_0. \tag{B5}$$

So this gives a relatively easy way to check if there are an infinite number of solutions and to generate them. Indeed for the unit winding monopole, one can check that this method generates an infinite number of spin 1/2 states. For the higher winding monopoles, we only need find one spin 1/2 solution [described below Eq. (49)] and that guarantees an infinite number since the secondary equation does not care about the value of c and this is the only place where the topological winding of the monopole (n_1) enters.

In our case we have another restriction on the solutions p and q since we require $p=n+3k$ and $q=n+2l$ where k and l are integers. However, it is easy to check that the construction still generates an infinite sequence of solutions. For the $n=1,2,4$ cases, every alternate member of the sequence described above has the desired form. For the $n=3,6$ cases, every member has the desired form.

[1] T. H. R. Skyrme, Proc. R. Soc. London **A247**, 260 (1958); **A262**, 237 (1961).
[2] T. Vachaspati, Phys. Rev. Lett. **76**, 188 (1996).
[3] H. Liu and T. Vachaspati, Phys. Rev. D **56**, 1300 (1997).
[4] H. Liu, G.D. Starkman, and T. Vachaspati, Phys. Rev. Lett. **78**, 1223 (1997).
[5] R. Jackiw and C. Rebbi, Phys. Rev. Lett. **36**, 1116 (1976).
[6] P. Hasenfratz and G. 't Hooft, Phys. Rev. Lett. **36**, 1119 (1976).
[7] A. S. Goldhaber, Phys. Rev. Lett. **36**, 1122 (1976).
[8] E. Witten, Phys. Lett. **86B**, 283 (1979).
[9] F. Wilczek, Phys. Rev. Lett. **48**, 1146 (1982).
[10] G. 't Hooft, Nucl. Phys. **B79**, 276 (1974).
[11] A. M. Polyakov, Pis'ma Zh. Éksp. Teor. Fiz. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)].
[12] T. Vachaspati, Phys. Lett. B **427**, 323 (1998).
[13] H. M. Chan and S. T. Tsou, Phys. Rev. D **57**, 2507 (1998).
[14] H. M. Chan, J. Bordes, and S. T. Tsou, Int. J. Mod. Phys. A **14**, 2173 (1999).
[15] D. Redmound, *Number Theory* (Dekker, New York, 1996).
[16] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975).
[17] E. B. Bogomolny, Yad. Fiz. **24**, 861 (1976) [Sov. J. Nucl. Phys. **24**, 449 (1976)].
[18] P. Goddard and D. I. Olive, Rep. Prog. Phys. **41**, 1357 (1978); S. Coleman, in *The Unity of Fundamental Interactions*, edited by A. Zichichi (Plenum, London, 1983); J. Preskill, in *Architecture of Fundamental Interactions at Short Distance*, edited by P. Ramond and R. Stora (Elsevier, New York, 1987).
[19] For a review, see K. Intriligator and N. Seiberg, Nucl. Phys. B (Proc. Suppl.) **45BC**, 1 (1996).
[20] A. Goldhaber, Phys. Rep. **315**, 83 (1999).
[21] P. Goddard, J. Nuyts, and D. I. Olive, Nucl. Phys. **B125**, 1 (1977).
[22] F. A. Bais and B. J. Schroers, Nucl. Phys. **B512**, 250 (1998).
[23] F. A. Bais and B. J. Schroers, Nucl. Phys. **B535**, 197 (1998).
[24] N. Lepora, hep-th/0002163.
[25] N. Lepora, J. High Energy Phys. **02**, 037 (2000).
[26] M. Murray, Commun. Math. Phys. **125**, 661 (1989).
[27] J. D. Lykken and A. Strominger, Phys. Rev. Lett. **44**, 1175 (1980).
[28] D. Zwanziger, Phys. Rev. **176**, 1480 (1968); **176**, 1489 (1968).
[29] J. Schwinger, Phys. Rev. **144**, 1087 (1966); **173**, 1536 (1968).
[30] C. Gardner and J. Harvey, Phys. Rev. Lett. **52**, 879 (1984).