Low-energy effective theories of quantum spin and quantum link models

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Quantum spin and quantum link models provide an unconventional regularization of field theory in which classical fields arise via dimensional reduction of discrete variables. This *D*-theory regularization leads to the same continuum theories as the conventional approach. We show this by deriving the low-energy effective Lagrangians of *D*-theory models using coherent state path integral techniques. We illustrate our method for the $(2+1)$D Heisenberg quantum spin model which is the *D*-theory regularization of the 2D $O(3)$ model. Similarly, we prove that in the continuum limit a $(2+1)$D quantum spin model with $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$ symmetry is equivalent to the 2D principal chiral model. Finally, we show that $(4+1)$D $SU(N)$ quantum link models reduce to ordinary 4D Yang-Mills theory.

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I. INTRODUCTION

In the conventional approach to quantum field theory the fundamental degrees of freedom are continuous classical fields. To regularize the ultraviolet divergences beyond perturbation theory it is natural to introduce a space-time lattice. For example, in Wilson's lattice gauge theory the gluons are represented by classical $SU(N)$ parallel transporter matrices. Similarly, the fundamental degrees of freedom of a lattice $O(3)$ model are classical unit vectors. In this paper we use an alternative approach to field theory—*D*-theory—in which classical fields are replaced by discrete variables (quantum spins or quantum links) that undergo dimensional reduction [1–4].

For example, at zero temperature the $O(3)$ symmetries of both the $(2+1)$D ferromagnetic and antiferromagnetic Heisenberg quantum spin models break spontaneously, giving rise to massless Goldstone bosons—the so-called magnons or spin waves. These magnons are collective excitations of many quantum spins and are effectively described by a continuous classical field. It is remarkable that these continuous degrees of freedom emerge from a microscopic theory of purely discrete quantum spins. The low-energy effective theory of magnons is an $O(3)$ model in $(2+1)$ dimensions. At small, non-zero temperature, and hence at finite extent β of the Euclidean time dimension, the correlation length ξ of the Goldstone bosons is large compared to β and hence the system undergoes dimensional reduction to the 2D $O(3)$ model. In this case, the Coleman-Hohenberg-Mermin-Wagner theorem [5] implies that ξ must become finite and that the magnons pick up a nonperturbatively generated mass gap $m=1/\xi$. As a consequence of asymptotic freedom of the 2D $O(3)$ model, ξ is exponentially large in β, $\xi \sim \exp(2\pi\rho_s\beta)$, where ρ_s is the spin stiffness of the underlying quantum spin system [6–8]. Hence, $\rho_s\beta=1/g^2$ plays the role of the coupling constant of the dimensionally reduced theory. The continuum limit of that theory is reached by varying the extent β of the extra dimension, not by adjusting a bare coupling constant.

Dimensional reduction of discrete variables is not limited to the quantum Heisenberg model. In fact, it is a generic phenomenon that gives rise to the *D*-theory formulation of field theory. For example, as we show in this paper, the 2D principal chiral model that is traditionally formulated in terms of continuous classical $U(N)$ matrix fields can also be expressed as a system of generalized quantum spins in $(2+1)$ dimensions (cf. [9]). The components of these quantum spins are generators of an $SU(2N)$ algebra. Again, the discrete variables undergo dimensional reduction to 2D if the $(2+1)$D system has massless Goldstone bosons. We identify appropriate representations of the $SU(2N)$ algebra for which this is indeed the case.

Gauge theories can be formulated in terms of discrete quantum links which are gauge covariant generalizations of quantum spins. A quantum link is an $N\times N$ parallel transporter matrix whose elements are generators of $SU(2N)$. The dimensional reduction of quantum link models works differently from the case of quantum spins. While in the spin case an infinite correlation length arises as a result of the spontaneous breakdown of a global symmetry, for gauge theories spontaneous symmetry breaking leads to a massive Higgs phase with a finite correlation length. Moreover, confined phases in non-Abelian gauge theories also have finite correlation lengths and therefore do not lead to dimensional reduction. However, gauge theories in five dimensions can exist in a non-Abelian Coulomb phase with massless unconfined gluons [2,10]. The massless gluons of a $(4+1)$D quantum link model are collective excitations of many quantum links, just as magnons are collective excitations of quantum spins. If a $(4+1)$D quantum link model exists in a Coulomb phase for an infinite extent β of the fifth dimension, it will undergo dimensional reduction to an ordinary 4D gauge theory once β becomes finite [1]. This is a consequence of the confinement hypothesis, which is the gauge analog of the Coleman-Hohenberg-Mermin-Wagner theorem of the spin case. In particular, a gluon cannot remain massless when β becomes finite because it then effectively lives in a 4D world and hence should be confined. The corresponding finite correlation length ξ is related to the glueball mass $m=1/\xi$. As before, asymptotic freedom of 4D non-Abelian gauge theories implies that ξ is exponentially large in β, $\xi \sim \exp(8\pi^2\beta/11e^2N)$, where e is the dimensionful gauge coupling of the underlying $(4+1)$D quantum link model. Hence, the role of the coupling constant of the dimensionally reduced theory is played by $\beta/e^2=1/g^2$. So again, the continuum limit of the theory is reached by varying the extent β of the extra dimension, not by adjusting a bare coupling constant.

As just explained, taking the continuum limit of a quantum link model requires a fifth dimension in order to obtain a large gluonic correlation length. It is then very natural to make use of the fifth dimension to include quarks as domain wall fermions. In particular, Shamir's variant [11,12] of Kaplan's original proposal [13] provides a suitable realization of full quantum link QCD. Again, the correlation length of the quarks is controlled by the extent of the fifth dimension and is exponentially large in β. Consequently, one reaches both the chiral and the continuum limit by sending β to infinity. This requires no fine-tuning of bare coupling constants, which makes *D*-theory an attractive alternative to the traditional approach to field theory [3].

Models with discrete degrees of freedom have been studied before. Generalizations of antiferromagnetic quantum spin models were discussed by Read and Sachdev in [14] for the case of an $SU(N)$ symmetry group. In [15], Radjbar-Daemi, Salam and Strathdee considered discrete spin systems with a general symmetry group and showed how the continuum limit of such theories corresponds to sigma-model-type field theories. They considered both ferro- and antiferromagnetic cases. In a follow up to that paper, they also investigated the renormalization group flow of a particular continuum theory on the manifold $SU(3)/[U(1)\times U(1)]$ [16]. A quantum link model with a $U(1)$ gauge symmetry was first constructed by Horn in [17]. Orland and Rohrlich introduced an $SU(2)$ quantum link model [18]. In the present context, it was realized in [1] that models with discrete variables can give rise to ordinary field theories, including QCD [3], via dimensional reduction. A detailed analysis of how the physics of conventional lattice gauge theory with $U(1)$ gauge group is reproduced by the $U(1)$ quantum link model is given in [19].

A key issue in the previous discussion is the existence of massless Goldstone bosons in quantum spin models, and of massless Coulombic gauge bosons in quantum link models. In the $(2+1)$D spin 1/2 antiferromagnetic quantum Heisenberg model it was unclear for some time if the $O(3)$ symmetry is spontaneously broken. For larger spin representations, however, one can prove analytically that this is indeed the case. By now, detailed numerical simulations have shown that spontaneous symmetry breaking also occurs for spin 1/2 [20,21]. For the $(2+1)$D $SU(N)_L\times SU(N)_R\times U(1)_{L=R}$ symmetric quantum spin model constructed in this paper it is *a priori* not clear if spontaneous symmetry breaking occurs, and thus if massless Goldstone bosons exist. Here we show that for a sufficiently large representation of the embedding algebra $SU(2N)$ with a rectangular Young tableau, this model becomes a Wilson-type lattice principal chiral model in 3D. Such a model is known to exist in a phase of spontaneously broken symmetry [22] and hence we conclude that at finite extent β of the third dimension it undergoes dimensional reduction to the usual 2D $SU(N)_L\times SU(N)_R\times U(1)_{L=R}$ principal chiral model. Similarly, we show that for a sufficiently large representation of $SU(2N)$ the low-energy behavior of a $(4+1)$D quantum link model is that of a Wilson-type lattice gauge theory in 5 dimensions. From numerical simulations, we know that such a model can exist in a non-Abelian Coulomb phase [2,10] and hence undergoes dimensional reduction to a 4D $SU(N)$ Yang-Mills theory once the extent of the fifth dimension becomes finite. This proves that D-theory is indeed a valid regularization of these models. It would be interesting and of practical importance to investigate if the massless phases arise also for small representations of $SU(2N)$. This requires detailed future numerical studies.

The rest of this paper is organized as follows. In Sec. II we use the quantum Heisenberg model to illustrate *D*-theory with a simple example. Section III contains the construction of quantum spin and quantum link models with $SU(N)_L\times SU(N)_R\times U(1)_{L=R}$ symmetry embedded in the algebra of $SU(2N)$. In Sec. IV a coherent state formalism is presented in order to describe the large representation limit of $SU(2N)$. The existence of massless Goldstone bosons for the $SU(N)_L\times SU(N)_R\times U(1)_{L=R}$ symmetric quantum spin model is shown in Sec. V and the existence of a massless Coulomb phase for $SU(N)$ quantum link models is derived in Sec. VI. Finally, Sec. VII contains our conclusions.

Throughout the paper we distinguish classical degrees of freedom, denoted by lower case letters, from quantum operators, denoted by upper case letters.

II. FROM THE $(2+1)$D HEISENBERG QUANTUM SPIN MODEL TO THE 2D $O(3)$ MODEL

To motivate the ideas leading to the *D*-theory formulation of field theory, let us review the well-known relation between the $(2+1)$D Heisenberg quantum spin model and the 2D $O(3)$ model. The 2D $O(3)$ model is asymptotically free, it has a nonperturbatively generated mass gap, as well as instantons and θ-vacua and has been used as a simple toy model for QCD in four dimensions.

The continuum action of the 2D $O(3)$ model is given by

$$S[\vec{s}]=\frac{1}{2g^2}\int d^2x\,\partial_\mu\vec{s}\cdot\partial_\mu\vec{s}, \tag{2.1}$$

where $\vec{s}$ is a classical, 3-component unit vector that assumes continuous values. Clearly, this action is invariant under global $O(3)$ transformations $\vec{s}\,'=R\vec{s}$, where $R^TR=\mathbb{1}$. The standard procedure to regularize this theory beyond perturbation theory is to follow Wilson and introduce a lattice as an ultraviolet cutoff. Partial derivatives are then replaced by finite differences and (after dropping an irrelevant constant), the lattice action takes the form

$$S[\vec{s}]=-\frac{1}{g^2}\sum_{x,\mu}\vec{s}_x\cdot\vec{s}_{x+\hat{\mu}}. \tag{2.2}$$

Here, $\hat{\mu}$ is the unit-vector in the μ-direction. The theory is quantized by writing down the partition function, which is a path integral over classical field configurations,

$$Z=\int\mathcal{D}\vec{s}\exp(-S[\vec{s}]). \tag{2.3}$$

Due to asymptotic freedom, the continuum limit of the lattice-regularized theory is attained by taking the bare coupling constant g to zero. In this limit, the correlation length $\xi\sim\exp(2\pi/g^2)$ diverges exponentially, thus eclipsing any short-distance lattice artifacts.

D-theory follows a radically different approach to field quantization. Instead of performing a path integral over continuous classical fields, those fields are replaced by discrete quantum variables. For example, the above 2D $O(3)$ field theory is formulated in terms of quantum spins with a Heisenberg model Hamiltonian

$$H=J\sum_{x,\hat{\mu}}\vec{S}_x\cdot\vec{S}_{x+\hat{\mu}}. \tag{2.4}$$

The components of the spin vectors $\vec{S}$ are the generators of $SO(3)$ and they satisfy the usual commutation relations

$$[S_x^i,S_y^j]=i\delta_{xy}\varepsilon_{ijk}S_x^k. \tag{2.5}$$

Notice that we are free to choose any representation of $SO(3)$ for the generators $\vec{S}_x$, not just spin 1/2. The $SO(3)$ symmetry of the quantum Hamiltonian is expressed as $[H,\vec{S}]=0$, where $\vec{S}=\Sigma_x\vec{S}_x$.

The case $J<0$ corresponds to a ferromagnet and $J>0$ to an antiferromagnet with a Néel-ordered ground state. We restrict our attention to the former case. The partition function for the Heisenberg model is given by

$$Z=\mathrm{Tr}\exp(-\beta H), \tag{2.6}$$

where the trace is taken in a large Hilbert space, the direct product of the Hilbert spaces corresponding to individual lattice sites. The Hamiltonian evolves the system in an extra dimension, giving rise to a $(2+1)$-dimensional field theory. For a condensed matter quantum spin system the additional dimension is Euclidean time. In D-theory, however, Euclidean time is part of the 2D lattice and the additional Euclidean dimension will ultimately disappear via dimensional reduction.

As discussed in the Introduction, dimensional reduction requires an infinite correlation length, which in this case is due to the existence of massless Goldstone bosons. One way of addressing the question of symmetry breaking and hence dimensional reduction, is to investigate the limit of large spin S. To do this, we set up a spin coherent state representation of the path integral as discussed in [23]. The highest weight vector of the representation with spin S is denoted by $|S,S\rangle$. To generate a system of coherent states, we must therefore act with all group elements on this state. A general $SO(3)$ group element can be parametrized using the three Euler angles as

$$R(\chi,\theta,\phi)=\exp(i\phi S^3)\exp(i\theta S^2)\exp(i\chi S^3). \tag{2.7}$$

Hence, we obtain the system of coherent states, with the following parametrization:

$$\begin{aligned}|\vec{s}\rangle&=R(\chi,\theta,\phi)|S,S\rangle\\&=\exp(i\phi S^3)\exp(i\theta S^2)\exp(i\chi S^3)|S,S\rangle.\end{aligned} \tag{2.8}$$

Notice that $e^{i\chi S^3}|S,S\rangle=e^{i\chi S}|S,S\rangle$, generating the isotropy subgroup for the highest weight vector, $|S,S\rangle$. Choosing $\chi\equiv0$ then corresponds to taking a cross section in the fiber bundle with base $X=SO(3)/SO(2)$ and fiber $SO(2)$.

The coherent states are now parametrized only by θ and ϕ, which fall in the ranges $\theta\in[0,\pi]$ and $\phi\in[-\pi,\pi)$, so that we can think of $\vec{s}$ as parametrizing a vector on the unit sphere, $\vec{s}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$. To obtain an expression for the coherent states as a superposition of S^3-eigenstates, we introduce Schwinger bosons—$a,a^\dagger$ and $b,b^\dagger$—which are two sets of boson creation and annihilation operators, satisfying the usual commutation relations. We can then write

$$S^1+iS^2=a^\dagger b,\quad S^1-iS^2=b^\dagger a,\quad S^3=\frac{1}{2}(a^\dagger a-b^\dagger b). \tag{2.9}$$

The additional constraint $n_a+n_b=2S$ fixes the representation of spin S. We can use the raising and lowering operators to generate the other weight vectors,

$$|S,m\rangle=\frac{(a^\dagger)^{S+m}}{\sqrt{(S+m)!}}\frac{(b^\dagger)^{S-m}}{\sqrt{(S-m)!}}|0\rangle. \tag{2.10}$$

To rotate such a state by R as above, we note that

$$\begin{aligned}\begin{pmatrix}a^\dagger\\b^\dagger\end{pmatrix}'&=\begin{pmatrix}R\,a^\dagger R^{-1}\\R\,b^\dagger R^{-1}\end{pmatrix}\\&=\exp\left(i\frac{\chi}{2}\sigma^z\right)\exp\left(i\frac{\theta}{2}\sigma^y\right)\exp\left(i\frac{\phi}{2}\sigma^z\right)\begin{pmatrix}a^\dagger\\b^\dagger\end{pmatrix}\\&=\begin{pmatrix}u\exp\left(i\frac{\chi}{2}\right)&v\exp\left(i\frac{\chi}{2}\right)\\-v^*\exp\left(-i\frac{\chi}{2}\right)&u^*\exp\left(-i\frac{\chi}{2}\right)\end{pmatrix}\begin{pmatrix}a^\dagger\\b^\dagger\end{pmatrix}.\end{aligned} \tag{2.11}$$

Here, $u(\theta,\phi)=\cos(\theta/2)\exp(i\phi/2)$ and $v(\theta,\phi)=\sin(\theta/2)\exp(-i\phi/2)$. This leads to the following expression for the coherent states (recall that $\chi\equiv0$):

$$|\vec{s}\rangle=\frac{(a^{\dagger\prime})^{2S}}{\sqrt{(2S)!}}|0\rangle=\sqrt{(2S)!}\sum_m \frac{u^{S+m}v^{S-m}}{\sqrt{(S+m)!(S-m)!}}|S,m\rangle. \tag{2.12}$$

These coherent states are not all linearly independent—they form an over-complete set of states. In particular, we can express the identity operator as a superposition of coherent states. The measure of integration we use here is $[(2S+1)/4\pi]d\vec{s}=[(2S+1)/4\pi]\sin\theta d\theta d\phi$,

$$\frac{2S+1}{4\pi}\int d\vec{s}|\vec{s}\rangle\langle\vec{s}|=\sum_m |S,m\rangle\langle S,m|=\mathbb{1}. \tag{2.13}$$

Another important property is the following:

$$\vec{s}\cdot\vec{S}|\vec{s}\rangle=S|\vec{s}\rangle. \tag{2.14}$$

A system of coherent states in the large Hilbert space on which the entire Hamiltonian acts is simply given by a direct product of the coherent state systems we have derived for each lattice site. From property (2.14) it is straightforward to obtain an expression for the expectation value of the Heisenberg Hamiltonian in a coherent state,

$$\mathcal{H}[\vec{s}]=\langle\vec{s}|H|\vec{s}\rangle=\frac{S^2J}{2}\sum_{x,\hat{\mu}}\vec{s}_x\cdot\vec{s}_{x+\hat{\mu}}. \tag{2.15}$$

Using these ingredients, we can express the partition function (2.6) as a path integral over coherent states. The standard procedure is to divide up the ''time'' interval β into N_ε small intervals of width $\varepsilon=\beta/N_\varepsilon$, and to insert a resolution of the identity (2.13) in between each time slice. Eventually, we take $N_\varepsilon\to\infty$. We can manipulate the expression for the path integral using Eq. (2.15). Also, we write

$$\frac{\vec{s}_x(t+\varepsilon)-\vec{s}_x(t)}{\varepsilon}\to\dot{\vec{s}}_x+\mathcal{O}(\varepsilon). \tag{2.16}$$

In our parametrization of the coherent states, the overlap between neighboring states is given by

$$\langle\vec{s}(t+\varepsilon)|\vec{s}(t)\rangle=\exp\left(-iS\varepsilon\sum_x \dot{\phi}_x\cos(\theta_x)\right). \tag{2.17}$$

We thus get the path integral

$$Z=\int \mathcal{D}\vec{s}\exp(-S[\vec{s}]), \tag{2.18}$$

where

$$S[\vec{s}]=iS\sum_x \omega[\vec{s}_x]+\int_0^\beta dt\,\mathcal{H}[\vec{s}], \tag{2.19}$$

and $\omega[\vec{s}]=\int_0^\beta dt\,\dot{\phi}\cos\theta=\oint_{\phi_0}^{\phi_0}d\phi\cos(\theta_\phi)$ is a Berry phase term. The geometric nature of the Berry phase term is evident, as it depends only on the path on the unit sphere traced out by the spin, and not on the explicit dependence of this path on t. In fact, this term measures the area enclosed by the path $\vec{s}(t)$ on S^2. It may be written in gauge invariant form as

$$\omega[\vec{s}]=\int_0^\beta dt\int_0^1 dv\,\vec{s}\cdot(\partial_t\vec{s}\times\partial_v\vec{s}). \tag{2.20}$$

We have introduced the interpolating field $\vec{s}(t,v)$, which depends on an additional (fourth) dimension and obeys the boundary conditions $\vec{s}(t,1)=\vec{s}(t)$, $\vec{s}(t,0)=\vec{s}(t',0)$ and $\vec{s}(0,v)=\vec{s}(\beta,v)$. This field is therefore parametrized on a disk with $v=1$ as the boundary. The only requirement on the interpolation $\vec{s}(t,v)$ from the boundary of the disk to the interior is that it be a smooth function of v. In particular, the value of the Berry phase term differs from one chosen interpolation to another only by an integer multiple of 4π. This is a direct consequence of the fact that the second homotopy group of S^2 is $\Pi_2(S^2)=\mathbb{Z}$. Hence, for the value of the path integral to be independent of the chosen interpolation, the spin S needs to be quantized in half-integer units. With the simple parametrization $s^i(t,v)=vs^i(t)$ for $i=1,2$, we can perform the integral over v to obtain an expression for the Berry phase that is equivalent to the one given in Eq. (2.19), and hence arrive at the following continuum action:

$$S[\vec{s}]=\int_0^\beta dt\int d^2x\left[iS(1+s^3)^{-1}(\partial_t s^1 s^2-\partial_t s^2 s^1)+\frac{\rho_s}{2}\partial_\mu\vec{s}\cdot\partial_\mu\vec{s}\right]. \tag{2.21}$$

Here, $\rho_s=S^2J/2$ is the spin stiffness. This result was also obtained by Leutwyler in [24] using chiral perturbation theory. He showed that this theory has a non-relativistic dispersion relation, attributed to the existence of a conserved order parameter.

Notice that the field $\vec{s}(x)$ is a unit vector field. It therefore lives in S^2, which is the coset space corresponding to a symmetry breaking pattern $SO(3)\to SO(2)$. Thus, the low-energy effective theory in Eq. (2.21) is a theory of Goldstone modes associated with this symmetry breaking.

The mechanism of dimensional reduction was explained for the antiferromagnet in [7]. Here, we adapt the discussion for the ferromagnet. Consider a system of dimensions $L\times L\times\beta$. For $L=\beta=\infty$, the system is in the ordered ground state of the Heisenberg ferromagnet. This breaks the $SO(3)$ symmetry and, as discussed above, we obtain the low-energy theory of Goldstone bosons of Eq. (2.21). If we now consider the case in which the extent of the additional dimension is taken to be finite, then the Coleman-Hohenberg-Mermin-Wagner theorem tells us that there cannot be massless excitations in a slab [5]. The Goldstone bosons must therefore pick up a small, non-perturbatively generated mass. One can use a block spin transformation to map the 3D $O(3)$-model in a slab of finite extent β to a 2D lattice $O(3)$ model. One averages the fields over blocks of size β in the third direction and size a' in the two spatial directions, decreasing the original cutoff $1/a$ down to $1/a'$. To determine a suitable value for a', consider the dispersion relation of the ferromagnet,

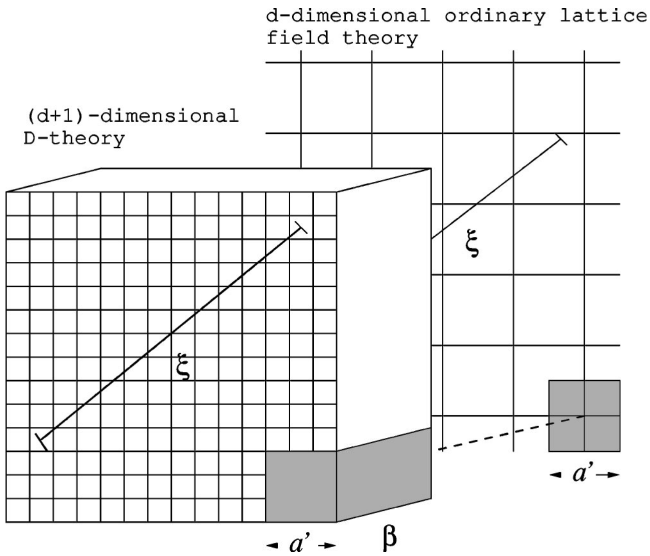

FIG. 1. Dimensional reduction of a D-theory: Averaging the $(d+1)$-dimensional effective field of the D-theory over blocks of size β in the extra dimension and a' in the physical directions results in an effective d-dimensional Wilsonian lattice field theory with lattice spacing $a'=\sqrt{\rho_s\beta/S}$.

$E=(\rho_s/S)p^2$. This implies $1/\beta=(\rho_s/S)(1/a'^2)$, and hence $a'=\sqrt{\rho_s\beta/S}$. One thus obtains a two-dimensional lattice field theory, whose degrees of freedom are the block-averaged fields living at the block centers. The lattice spacing of the new theory is therefore equal to $\sqrt{\rho_s\beta/S}$, which is different from the lattice spacing of the original quantum Heisenberg model (see Fig. 1). The correlation length of the 2D $O(3)$ model in lattice units is given by $\xi/a' \propto g^2\exp(2\pi/g^2)$. The value of a' was found above, and we can identify the coupling constant as $1/g^2=\rho_s\beta$. We thus arrive at the expression for the dependence of the correlation length on the extent β of the additional dimension,

$$\xi\propto(\rho_s\beta)^{-1/2}\exp(2\pi\rho_s\beta). \tag{2.22}$$

This relation was first found by Kopietz and Chakravarty in [8], where they used the same techniques of renormalization-group analysis that had been used for the antiferromagnet in [6] by Chakravarty, Halperin and Nelson. The continuum limit is reached by taking the extent of the additional dimension to infinity, since the correlation length diverges exponentially as $\beta\rightarrow\infty$. However, in this limit the extent of the third dimension is much smaller than the correlation length, i.e., $\xi\gg\sqrt{\rho_s\beta/S}$. Thus, the fields are effectively constant in the t-direction, and the theory undergoes dimensional reduction.

III. $U(N)$ QUANTUM SPINS AND QUANTUM LINKS

In the following sections we will be considering models whose fundamental degrees of freedom in the conventional formulation are elements of unitary $N\times N$ matrices. In D-theory these fields are replaced by quantum operators, so that we have matrices whose entries are operators rather than complex numbers. However, we still want the Hamiltonian constructed from these operators to be invariant under the same symmetries as the conventional action. In particular, we will be considering two models, the principal chiral model with a global $SU(N)_L\times SU(N)_R\times U(1)_{L=R}$ symmetry, and Yang-Mills theory with a local $SU(N)$ symmetry. In order to construct the appropriate D-theory Hamiltonians, let us consider the operators associated with just one lattice site in the case of the principal chiral model, or just one link in the case of Yang-Mills theory. Dropping the site/link indices, we denote these operators by U^{ij}. The appropriate symmetries follow by construction if we also associate with each site/link an $SU(N)_L\times SU(N)_R$ algebra, generated by $\{L^a\}$ and $\{R^a\}$ which satisfy the commutation relations

$$[L^a,L^b]=2if_{abc}L^c,\quad [R^a,R^b]=2if_{abc}R^c,\quad [L^a,R^b]=0. \tag{3.1}$$

Here f_{abc} are the usual structure constants of $SU(N)$. We then require that the site/link operator variables transform as

$$\begin{aligned}U'&=\exp(-i\alpha^aL^a)U\exp(i\beta^bR^b)\\&=\exp(i\alpha^a\lambda^a)U\exp(-i\beta^b\lambda^b),\end{aligned} \tag{3.2}$$

where the λ^a are the Hermitian generators of $SU(N)$ in the fundamental representation. These generators satisfy

$$[\lambda^a,\lambda^b]=2if_{abc}\lambda^c,\quad \mathrm{Tr}\lambda^a\lambda^b=2\delta^{ab}. \tag{3.3}$$

The transformation rule (3.2) is implied by the following commutation relations:

$$[L^a,U^{ij}]=-\lambda^a_{ik}U^{kj},\quad [R^a,U^{ij}]=U^{ik}\lambda^a_{kj}. \tag{3.4}$$

All of these commutation relations can be satisfied by embedding the operators in an $SU(2N)$ algebra. In particular, the aforementioned $SU(N)_L\times SU(N)_R$ algebra is embedded diagonally, while the U^{ij} operators fill in the off-diagonal blocks. To summarize, we get the full set of commutation relations:

$$\begin{aligned}&[L^a,L^b]=2if_{abc}L^c,\quad [R^a,R^b]=2if_{abc}R^c,\\&[R^a,U^{ij}]=U^{ik}\lambda^a_{kj},\quad [L^a,U^{ij}]=-\lambda^a_{ik}U^{kj},\\&[T,U^{ij}]=2U^{ij},\\&[R^a,L^b]=[T,L^a]=[T,R^a]=0,\\&[\mathrm{Re}\,U^{ij},\mathrm{Re}\,U^{kl}]=[\mathrm{Im}\,U^{ij},\mathrm{Im}\,U^{kl}]\\&\qquad=-i(\delta_{ik}\mathrm{Im}\,\lambda^a_{jl}R^a+\delta_{jl}\mathrm{Im}\,\lambda^a_{ik}L^a),\\&[\mathrm{Re}\,U^{ij},\mathrm{Im}\,U^{kl}]=i\Big(\delta_{ik}\mathrm{Re}\,\lambda^a_{jl}R^a-\delta_{jl}\mathrm{Re}\,\lambda^a_{ik}L^a\\&\qquad+\frac{2}{N}\delta_{ik}\delta_{jl}T\Big).\end{aligned} \tag{3.5}$$

Here, T generates an extra $U(1)$ symmetry. [Later, this symmetry needs to be broken explicitly to obtain an $SU(N)$ rather than $U(N)$ Yang-Mills theory.] If we restrict ourselves to representations of $SU(2N)$ which correspond to rectangular Young tableaux with N rows and n columns (as shown in

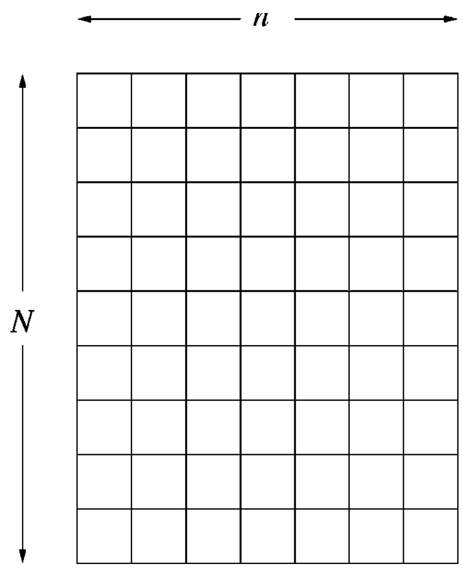

FIG. 2. Young tableau of the $SU(2N)$ representation by which states at each lattice site-link transform.

Fig. 2), we can use a fermionic basis of rishons for our representation [3,14] (we use the summation convention for greek indices),

$$S^{ij}=L^{ij}=\left(l^{i\alpha\dagger}l^{j\alpha}-\frac{n}{2}\delta^{ij}\right),$$

$$S^{N+i,N+j}=R^{ij}=\left(r^{i\alpha\dagger}r^{j\alpha}-\frac{n}{2}\delta^{ij}\right),$$

$$S^{N+j,i}=-U^{ij}=-l^{i\alpha}r^{j\alpha\dagger},$$

$$S^{j,N+i}=-(U^{\dagger})^{ij}=-(U^{ji})^{\dagger}=-r^{i\alpha}l^{j\alpha\dagger},$$

$$T=\sum_i (r^{i\alpha\dagger}r^{i\alpha}-l^{i\alpha\dagger}l^{i\alpha}),$$

$$\sum_i (l^{i\alpha\dagger}l^{i\beta}+r^{i\alpha\dagger}r^{i\beta})=\delta^{\alpha\beta}N, \tag{3.6}$$

where $\alpha=1,\ldots,n$ is an additional rishon flavor index and $i,j=1,\ldots,N$. For convenience, we have chosen these generators not to be traceless. We then have $L^{a}=\lambda^{a}_{ij}L^{ij}$ and $R^{a}=\lambda^{a}_{ij}R^{ij}$. The constraint (3.6) is needed to obtain the correct representation.

For the purpose of deriving systems of coherent states, addressed in the next section, it will be convenient to introduce the following notation:

$$c^{i\alpha}=\begin{cases} l^{i\alpha}, & \text{for } 1\leqslant i\leqslant N;\\ r^{i\alpha}, & \text{for } N+1\leqslant i\leqslant 2N.\end{cases} \tag{3.7}$$

We then have

$$S^{ij}=c^{i\alpha\dagger}c^{j\alpha}-\frac{n}{2}\delta^{ij}, \quad \sum_i c^{i\alpha\dagger}c^{i\beta}=\delta^{\alpha\beta}N. \tag{3.8}$$

Notice that the labels i and j now run from 1 to $2N$.

IV. COHERENT STATES

In this section we first describe how to generate an overcomplete system of states, which we then use to set up a path integral. Moreover, a Berry phase term is generated in the action and we work out its form in terms of the degrees of freedom in the path integral. These degrees of freedom are arranged into $GL(N,\mathbb{C})$ matrices and we decompose such a matrix into its Hermitian and unitary parts. This will allow us to make contact with the respective target theories.

Systems of coherent states for $SU(2N)$

Let us now construct a coherent state system for the types of representations of $SU(2N)$ that we are interested in. The general procedure is described in detail in [25]. We must first pick a vector in the carrier space that the chosen representation acts in. It is convenient to choose a weight vector $|\mu\rangle$ as the initial element of the coherent state system. We will see that for our choice of initial vector, the corresponding isotropy subgroup, i.e. the subgroup of $SU(2N)$ transformations which leave the state defined by this vector unchanged, is $SU(N)_L\times SU(N)_R\times U(1)_{L=R}$. This is a consequence of the chosen type of representation with rectangular Young tableaux of N rows. Coherent states are then characterized by the elements of $SU(2N)/[SU(N)_L\times SU(N)_R\times U(1)_{L=R}]$.

We now proceed to constructing a concrete set of coherent states for representations of $SU(2N)$ with rectangular Young tableaux of the kind previously described. This discussion follows the steps outlined in [14]. Our basis of generators is given by Eq. (3.8). We can choose the Cartan subalgebra to be spanned by the following set of generators,

$$S^{ii}=\sum_{\alpha} c^{i\alpha\dagger}c^{i\alpha}-\frac{n}{2}, \tag{4.1}$$

for $i=1,\ldots,2N$. Notice that Eq. (3.8) imposes one constraint on this set of generators, consistent with the fact that $SU(2N)$ has rank $2N-1$. The remaining operators S^{ij} with $i\neq j$ are the "raising" and "lowering" operators which complete the canonical Cartan basis for the Lie algebra.

For the given representation we obtain the highest-weight vector as follows,

$$|\psi_0\rangle=\mathcal{C}[\varepsilon^{ab\cdots}c^{a\alpha\dagger}c^{b\alpha\dagger}\cdots][\varepsilon^{cd\cdots}c^{c\beta\dagger}c^{d\beta\dagger}\cdots]\cdots|0\rangle, \tag{4.2}$$

where there are N creation operators in each square bracket, and there are n square bracketed terms all together. The indices $a,b,\ldots$ run through all values 1 to N, while $\alpha,\beta,\ldots$ run from 1 to n. We are symmetrizing the column indices of the Young tableau, and antisymmetrizing the row indices. The normalization constant $\mathcal{C}$ is chosen so that $\langle\psi_0|\psi_0\rangle=1$. The weight of this state is given by

$$S^{ii}|\psi_0\rangle=\begin{cases}\frac{n}{2}|\psi_0\rangle, & \text{for } 1\leqslant i\leqslant N,\\ -\frac{n}{2}|\psi_0\rangle, & \text{for } N+1\leqslant i\leqslant 2N.\end{cases} \tag{4.3}$$

We can obtain a coherent state system by applying all possible group transformations modulo the isotropy subgroup of $|\psi_0\rangle$ to our chosen initial vector. In terms of the chosen basis for the Lie algebra we obtain a group element of $SU(2N)$ by exponentiating an anti-Hermitian combination of generators, and so we have

$$|q\rangle=\exp(-q^{ij}S^{ji}+q^{ij*}S^{ij})|\psi_0\rangle, \tag{4.4}$$

where the index j runs through the values 1 to N, and i runs from $N+1$ to $2N$. It is easily checked that if i and j were to fall in the same range of values, i.e., either both are between 1 and N or both are between $N+1$ and $2N$, then $|q\rangle$ would just be equal (or in the case $i=j$ proportional) to $|\psi_0\rangle$. Hence, the isotropy subgroup of the vector $|\psi_0\rangle$ is $SU(N)_L \times SU(N)_R\times U(1)_{L=R}$ as mentioned above. The q^{ij} are N^2 independent complex numbers, which can be arranged into a $GL(N,\mathbb{C})$ matrix. Under $SU(2N)$ rotations, the generators S^{ij} transform in the adjoint representation,

$$\exp(q^{kl}S^{lk}-q^{kl*}S^{kl})S^{ij}\exp(-q^{mn}S^{nm}+q^{mn*}S^{mn})$$
$$=\left[\exp\begin{pmatrix}0 & q^\dagger\\ -q & 0\end{pmatrix}\right]^{ik}S^{kl}\left[\exp\begin{pmatrix}0 & -q^\dagger\\ q & 0\end{pmatrix}\right]^{lj}. \tag{4.5}$$

The states $|q\rangle$ are normalized to one, and from Eq. (4.5) they clearly satisfy the following important identity

$$\langle q|S^{ij}|q\rangle=\frac{n}{2}Q^{ij}, \tag{4.6}$$

where Q is given by

$$Q=\exp\left[\begin{pmatrix}0 & q^\dagger\\ -q & 0\end{pmatrix}\right]\begin{pmatrix}\mathbb{1}_N & 0\\ 0 & -\mathbb{1}_N\end{pmatrix}\exp\left[\begin{pmatrix}0 & -q^\dagger\\ q & 0\end{pmatrix}\right]. \tag{4.7}$$

In particular, if we write the matrix Q as

$$Q=\begin{pmatrix}l & w^\dagger\\ w & r\end{pmatrix}, \tag{4.8}$$

we see that

$$\langle q|U^{ij}|q\rangle=-\frac{n}{2}w^{ij}. \tag{4.9}$$

From the discussion of how to set up a coherent state path integral in the case of the Heisenberg model, we know that a Berry phase term of the form

$$S_B=\int_0^\beta dt\left(\frac{\langle q(t+\varepsilon)|q(t)\rangle-1}{\varepsilon}\right) \tag{4.10}$$

is generated at each site/link as part of the action. To manipulate this term, notice that for any operator M, we have [26]

$$\frac{d}{dx}e^M=\int_0^1 dv\, e^{M(1-v)}\frac{dM}{dx}e^{Mv}. \tag{4.11}$$

Using the expression for $|q\rangle$ found in Eq. (4.4), we can write the Berry phase term as

$$S_B=-\int_0^\beta dt\left\langle q(t)\left|\frac{d}{dt}\right|q(t)\right\rangle=-\int_0^\beta\left\langle \psi_0\left|\exp(q^{ij}S^{ji}-q^{ij*}S^{ij})\frac{d}{dt}\exp(-q^{ij}S^{ji}+q^{ij*}S^{ij})\right|\psi_0\right\rangle$$
$$=-\int_0^\beta dt\int_0^1 dv\left\langle \psi_0\left|\exp[-v(-q^{ij}S^{ji}+q^{ij*}S^{ij})]\left(-\frac{\partial q^{ij}}{\partial t}S^{ji}+\frac{\partial q^{ij*}}{\partial t}S^{ij}\right)\exp[v(-q^{ij}S^{ji}+q^{ij*}S^{ij})]\right|\psi_0\right\rangle. \tag{4.12}$$

If we now define

$$\langle vq(t)|S^{ij}|vq(t)\rangle\equiv\frac{n}{2}Q^{ij}(t,v), \tag{4.13}$$

we can simplify the above to

$$S_B=-\frac{n}{2}\int_0^\beta dt\int_0^1 dv\left(-\frac{\partial q^{ij}}{\partial t}Q^{ji}(t,v)+\frac{\partial q^{ij*}}{\partial t}Q^{ij}(t,v)\right)$$
$$=-\frac{n}{2}\int_0^\beta dt\int_0^1 dv\ \mathrm{Tr}\left[\begin{pmatrix}0 & \frac{\partial q^\dagger}{\partial t}\\ -\frac{\partial q}{\partial t} & 0\end{pmatrix}Q(t,v)\right]. \tag{4.14}$$

As a function of v the matrix $Q(t,v)$ now satisfies

$$Q(t,0)=\begin{pmatrix}\mathbb{1}_N & 0\\ 0 & -\mathbb{1}_N\end{pmatrix}, \tag{4.15}$$

and $Q(t,1)\equiv Q(t)$. Integrating Eq. (4.14) by parts gives

$$S_B=\frac{n}{2}\int_0^\beta dt\int_0^1 dv\ \mathrm{Tr}\left[\begin{pmatrix}0 & q^\dagger(t)\\ -q(t) & 0\end{pmatrix}\frac{\partial}{\partial t}Q(t,v)\right]. \tag{4.16}$$

It is not hard to see that

$$\begin{pmatrix}0 & q^\dagger(t)\\ -q(t) & 0\end{pmatrix}=-\frac{1}{2}Q(t,v)\frac{\partial Q(t,v)}{\partial v}, \tag{4.17}$$

and this leads to the final result for the Berry phase,

$$S_B=-\frac{n}{4}\int_0^\beta dt\int_0^1 dv\left[\mathrm{Tr}\left(Q(t,v)\frac{\partial Q(t,v)}{\partial v}\,\frac{\partial Q(t,v)}{\partial t}\right)\right]. \tag{4.18}$$

In the above derivation we used a specific dependence of $Q(t,v)$ on the variable v, which satisfies the boundary conditions

$$Q(t,0)=Q(t',0),\quad \text{for all}\quad t,t';$$
$$Q(t,1)=Q(t);\quad Q(0,v)=Q(\beta,v). \tag{4.19}$$

Thus, the field $Q(t,v)$ lives in a rectangle $0\leqslant t\leqslant\beta$ and $0\leqslant v\leqslant 1$. From the periodic boundary conditions in the t direction, we can interpret $Q(t)$ as defining a closed curve, parametrized by t, and with the parameter v filling in the space enclosed by the curve to form a disk in the Grassmann manifold $G(N,2N)=SU(2N)/[SU(N)_L\times SU(N)_R\times U(1)_{L=R}]$. In [14] it was shown that S_B is independent of the particular surface that has this boundary, up to multiples of $2\pi nki$ for $k\in\mathbb{Z}$. This result was derived as a direct consequence of the fact that the second homotopy group of this Grassmann manifold is just $\Pi_2(G(N,2N))=\mathbb{Z}$, the group of integers.

Consider now the $GL(N,\mathbb{C})$ matrix q of Eq. (4.4). As shown in the Appendix, we can decompose such a matrix into the product of a left-coset Hermitian matrix b and a unitary matrix u, $q=bu$. Upon substituting this decomposition into Eq. (4.7), we obtain

$$Q=\begin{pmatrix} u^\dagger\cos(2b)u & -u^\dagger\sin(2b)\\ -\sin(2b)u & -\cos(2b)\end{pmatrix}. \tag{4.20}$$

In order to use this result to simplify the Berry phase term in the action, we represent S_B as

$$S_B=\frac{n}{8}\int d^2\xi\,\varepsilon_{pq}\mathrm{Tr}\;[Q\partial_pQ\partial_qQ], \tag{4.21}$$

where p,q take the values 1, 2, and $\xi_1=t$, and $\xi_2=v$, and the integral is over a rectangle in (t,v) space. We parametrize $Q(t,v)$ in the following way,

$$Q(t,v)=\exp\left[\begin{pmatrix}0 & q^\dagger(t,v)\\ -q(t,v) & 0\end{pmatrix}\right]\begin{pmatrix}\mathbb{1}_N & 0\\ 0 & -\mathbb{1}_N\end{pmatrix}$$
$$\times\exp\left[\begin{pmatrix}0 & -q^\dagger(t,v)\\ q(t,v) & 0\end{pmatrix}\right], \tag{4.22}$$

where $q(t,v)$ is a smooth function on the rectangle, such that the boundary conditions (4.19) are satisfied. We can then decompose the matrix $q(t,v)$ as before,

$$q(t,v)=b(t,v)u(t,v), \tag{4.23}$$

and find

$$Q=\begin{pmatrix} u^\dagger(t,v)\cos(2b(t,v))u(t,v) & -u^\dagger(t,v)\sin(2b(t,v))\\ -\sin(2b(t,v))u(t,v) & -\cos(2b(t,v))\end{pmatrix}. \tag{4.24}$$

Substituting this expression into the integrand of Eq. (4.21), we find that after some algebra it reduces to

$$\varepsilon_{pq}\mathrm{Tr}[Q\partial_pQ\partial_qQ]$$
$$=-4\varepsilon_{pq}\mathrm{Tr}[\partial_q[\cos(2b(t,u))u(t,u)\partial_pu^\dagger(t,u)]]. \tag{4.25}$$

Hence, the Berry phase term simplifies to

$$S_B=-\frac{n}{2}\int_0^\beta dt\int_0^1 dv\,(\partial_v\{\mathrm{Tr}[\cos(2b)u\partial_tu^\dagger]\}$$
$$-\partial_t\{\mathrm{Tr}[\cos(2b)u\partial_vu^\dagger]\})$$
$$=-\frac{n}{2}\int_0^\beta dt\;\;\mathrm{Tr}[\cos(2b)u\partial_tu^\dagger], \tag{4.26}$$

where we have used the boundary conditions on $b(t,v)$ and $u(t,v)$ to obtain the last line.

V. PRINCIPAL CHIRAL MODEL

In this section we use symmetry considerations to formulate the principal chiral model as a quantum spin model. We then set up a coherent state path integral using the results of the previous section. The resulting Lagrangian is expanded around its minimum to obtain a 3D principal chiral model as the low-energy effective theory of the quantum spin model. Finally, we explain how the mechanism of dimensional reduction gives rise to the 2D target theory.

A. *D*-theory formulation

The action of the 2D principal chiral model in the continuum is given by

$$S[u]=\frac{1}{2g^2}\int d^2x\;\;\mathrm{Tr}[\partial_\mu u^\dagger(x)\partial_\mu u(x)], \tag{5.1}$$

where the $u(x)$ are unitary $N\times N$ matrices. In Wilson's approach to regularizing the theory, space-time is discretized by introducing a regular lattice. Derivatives are replaced by finite differences to obtain an action of the form

$$S[u]=-\frac{1}{g^2}\sum_{\langle xy\rangle}\mathrm{Tr}[u_x^\dagger u_y]. \tag{5.2}$$

The target theory has a global $SU(N)_L\times SU(N)_R\times U(1)_{L=R}$ symmetry of the form $u_x\rightarrow u_x'=Lu_xR^\dagger$, where L and R are unitary matrices. It is known that this symmetry breaks to an $SU(N)$ vector symmetry $(L=R)$ at $g=0$. Due to the Mermin-Wagner theorem, however, the symmetry cannot break for $g>0$.

Let us now replace the classical fields u_x^{ij} by quantum operators U_x^{ij} and write down a D-theory Hamiltonian, which evolves the two-dimensional system in an additional Euclidean time direction,

$$H=2J\sum_{x,\mu} \mathrm{Re}\,\mathrm{Tr}\,U_x U_{x+\hat{\mu}}^{\dagger}$$
$$=J\sum_{x,\mu}[U_x^{ij}(U_{x+\hat{\mu}}^{ij})^{\dagger}+U_{x+\hat{\mu}}^{ij}(U_x^{ij})^{\dagger}]. \tag{5.3}$$

We would like this Hamiltonian to have an $SU(N)_L \times SU(N)_R\times U(1)_{L=R}$ symmetry, i.e., $[G_L^a,H]=[G_R^a,H]=[T,H]=0$, where G_L^a and G_R^a are mutually commuting sets of $SU(N)$ generators and T generates a $U(1)$ symmetry. As we saw in Sec. III, this can be realized by embedding $SU(N)_L\times SU(N)_R\times U(1)_{L=R}$ diagonally in $SU(2N)$. In particular, we have an $SU(2N)$ algebra of the form (3.6) at each lattice site. Labeling the generators that correspond to the lattice site x by a subscript x, we can write the generators of the algebra of the global $SU(N)_L\times SU(N)_R\times U(1)_{L=R}$ symmetry group as $G_L^a=\Sigma_x L_x^a$, $G_R^a=\Sigma_x R_x^a$ and $T=\Sigma_x T_x$. It then follows that $[G_L^a,H]=[G_R^a,H]=[G_L^a,G_R^b]=[T,H]=0$.

For $J<0$ this model is ferromagnetic. We choose the same representation of $SU(2N)$ for the generators at each site of the lattice, namely the one mentioned above, with a rectangular Young tableau as shown in Fig. 2. Note that the properties of the system defined by the Hamiltonian H are completely determined, once the representation of $SU(2N)$ has been specified.

B. The continuum limit of the theory

Using the results of Sec. IV, we set up a coherent state path integral for the partition function. The Berry phase term of the corresponding action was calculated in Eq. (4.26). The other term in the action is given by

$$\int_0^{\beta} dt\,\mathcal{H}(Q(t))$$
$$=\frac{Jn^2}{4}\int_0^{\beta}dt\sum_{x,\mu}[w_x^{ij}(w_{x+\hat{\mu}}^{ij})^*+(w_{x+\hat{\mu}}^{ij})^* w_x^{ij}]$$
$$=\frac{Jn^2}{4}\int_0^{\beta}dt\sum_{x,\mu}\mathrm{Tr}[w_x w_{x+\hat{\mu}}^{\dagger}+w_{x+\hat{\mu}}w_x^{\dagger}]. \tag{5.4}$$

As discussed above, we have $w=-\sin(2b)u$ and we define $s\equiv\sin(2b)$. Then $w=-su$, where $s=s^{\dagger}$. We can think of s as the radial component and of u as the phase of the matrix w. Due to the sine function and the fact that we can cover the coset space $SU(2N)/[SU(N)_L\times SU(N)_R\times U(1)_{L=R}]$ by limiting the matrix b to have eigenvalues between 0 and $\pi/2$, the eigenvalues of s are constrained to lie between 0 and 1. Substituting the above coset decomposition into Eq. (5.4), we obtain

$$\frac{Jn^2}{4}\int_0^{\beta}dt\sum_{x,\mu}\mathrm{Tr}[s_x u_x u_{x+\hat{\mu}}^{\dagger}s_{x+\hat{\mu}}+s_{x+\hat{\mu}}u_{x+\hat{\mu}}u_x^{\dagger}s_x]$$
$$=-\frac{Jn^2}{4N}\int_0^{\beta}dt\sum_{x,\mu}\mathrm{Tr}\Bigg[\frac{1}{2}(s_x s_{x+\hat{\mu}}-s_{x+\hat{\mu}}s_x)(u_x u_{x+\hat{\mu}}^{\dagger}-u_{x+\hat{\mu}}u_x^{\dagger})$$
$$+\frac{1}{4}(2+u_x u_{x+\hat{\mu}}^{\dagger}+u_{x+\hat{\mu}}u_x^{\dagger})(s_{x+\hat{\mu}}-s_x)(s_{x+\hat{\mu}}-s_x)$$
$$+\frac{1}{4}(s_{x+\hat{\mu}}+s_x)(s_{x+\hat{\mu}}+s_x)(u_{x+\hat{\mu}}-u_x)(u_{x+\hat{\mu}}^{\dagger}-u_x^{\dagger})-2s_x s_x\Bigg]. \tag{5.5}$$

We would like to expand this action around its minimum. Since the eigenvalues of s are bounded by 1, the minimum of the action occurs when $s=\mathbb{1}$ [up to a global $SU(N)_L \times SU(N)_R\times U(1)_{L=R}$ rotation], and u is constant across the lattice. We now introduce fluctuations in the fields, defining the forward lattice derivative

$$\Delta_{\mu}^{f}u_x=\frac{u_{x+\hat{\mu}}-u_x}{a}, \tag{5.6}$$

and writing

$$s_x=\sin(2b_x)=\sin\left(2\left(\frac{\pi}{4}+a\epsilon_x\right)\right)\approx\mathbb{1}-2a^2\epsilon_x^2. \tag{5.7}$$

Substituting these expressions into Eq. (5.5), expanding to quadratic order in a and dropping an irrelevant constant, we obtain

$$S=S_B-\frac{Jn^2}{4}\int_0^{\beta}dt\sum_{x,\mu}a^2\,\mathrm{Tr}[(\Delta_{\mu}^{f}u_x)(\Delta_{\mu}^{f}u_x^{\dagger})+8\epsilon_x^2]. \tag{5.8}$$

Next, we turn to the Berry phase term, and expand the fields in the same way as before,

$$S_B=-\sum_x \frac{n}{2}\int_0^\beta dt\,\mathrm{Tr}(\cos(2b_x)u_x\partial_t u_x^\dagger)$$

$$\approx-\sum_x \frac{n}{2}\int_0^\beta dt\,\mathrm{Tr}\left[\left(\cos\left(\frac{\pi}{2}\right)\right.\right.$$

$$\left.\left.-2a\epsilon_x\sin\left(\frac{\pi}{2}\right)\right)u_x\partial_t u_x^\dagger\right]$$

$$=\sum_x \frac{n}{2}\int_0^\beta dt\,\mathrm{Tr}[2a\epsilon_x u_x\partial_t u_x^\dagger]. \tag{5.9}$$

We thus obtain the final expression for the action in the continuum limit,

$$S=\int_0^\beta dt\int d^2x\,\mathrm{Tr}\left[-\frac{Jn^2}{4}\partial_\mu u(x)\partial_\mu u^\dagger(x)-4Jn^2\epsilon^2(x)\right.$$

$$\left.+\frac{n}{a}\epsilon(x)u(x)\partial_t u^\dagger(x)\right]$$

$$=\int_0^\beta dt\int d^2x\,\frac{\rho_s}{2}\mathrm{Tr}\left[\partial_\mu u\partial_\mu u^\dagger+\frac{1}{c^2}\partial_t u\partial_t u^\dagger\right]. \tag{5.10}$$

In this equation, we have integrated out the shifted field

$$\epsilon'(x)=\epsilon(x)-\frac{1}{8Jna}u(x)\partial_t u^\dagger(x). \tag{5.11}$$

The spin stiffness is given by $\rho_s=|J|n^2/2$, and $c=2na|J|$ is the spin wave velocity.

Notice that we now have a theory with a relativistic dispersion relation. The ferromagnetic $SO(3)$ spin model has a non-relativistic dispersion relation [24], because the order parameter commutes with the Hamiltonian. In the ferromagnetic principal chiral model on the other hand, the order parameter $U^{ij}=\Sigma_x U_x^{ij}$ does not commute with H, so a relativistic dispersion relation comes as no surprise.

The three-dimensional system will dimensionally reduce if the correlation length is much larger than the extent of the third dimension, $\xi\gg\beta c$. If we now assume that this is the case, then the fields u will have no dependence on t, and the integration over t becomes trivial,

$$S=\frac{\beta\rho_s}{2}\int d^2x\,\mathrm{Tr}[\partial_\mu u\partial_\mu u^\dagger]. \tag{5.12}$$

From [27] we know that the correlation for the two-dimensional principal chiral model is given by

$$\xi\propto\exp\left(\frac{2\pi}{g^2N}\right)=\exp\left(\frac{2\pi\beta\rho_s}{N}\right). \tag{5.13}$$

When performing a blockspin transformation in the way described for the Heisenberg ferromagnet, the new lattice spacing for a system with a relativistic dispersion relation, such as the present one, is $a'=\beta c$. Equation (5.13) is consistent with $\xi\gg\beta c$ in the zero temperature ($\beta\to\infty$) limit, so dimensional reduction does indeed occur.

VI. GAUGE THEORY

The quantum link formulation of Yang-Mills theory was worked out in [3] and we shall review it here, before showing how it is related to the classical formulation of the theory. First, recall Wilson's action for lattice gauge theory with gauge group $SU(N)$ [28],

$$S[u]=-\frac{1}{g^2}\sum_{x,\mu\neq\nu}2\,\mathrm{Re}\;\mathrm{Tr}[u_{x,\mu}u_{x+\hat{\mu},\nu}u_{x+\hat{\nu},\mu}^\dagger u_{x,\nu}^\dagger]. \tag{6.1}$$

Here, x labels the sites of a 4D hyper-cubic lattice, and the $u_{x,\mu}$ are $SU(N)$ matrices, associated with each link (x,μ) on the lattice. This action is invariant under local $SU(N)$ transformations of the form $u_{x,\mu}\to u'_{x,\mu}$, where

$$u'_{x,\mu}=\exp(i\alpha_x^a\lambda^a)u_{x,\mu}\exp(-i\alpha_{x+\hat{\mu}}^b\lambda^b). \tag{6.2}$$

The classical partition function for this system is given by

$$Z=\int\mathcal{D}u\,\exp\left(-\frac{1}{g^2}S[u]\right). \tag{6.3}$$

In the D-theory formulation we replace the classical fields that make up the entries $u_{x,\mu}^{ij}$ of the $u_{x,\mu}$ matrices in the action by quantum operators $U_{x,\mu}^{ij}$, to obtain a quantum Hamilton operator that evolves the system in an additional Euclidean direction. The Hamilton operator takes the form

$$H=J\sum_{x,\mu\neq\nu}[U_{x,\mu}^{ij}U_{x+\hat{\mu},\nu}^{jk}(U_{x+\hat{\nu},\mu}^{lk})^\dagger(U_{x,\nu}^{il})^\dagger+\mathrm{H.c.}]$$

$$+J'\sum_{x,\mu}[\det U_{x,\mu}+\det U_{x,\mu}^\dagger]. \tag{6.4}$$

The determinant term is understood to mean

$$\det U_{x,\mu}=\frac{1}{N!}\epsilon_{i_1i_2\cdots i_N}\epsilon_{j_1j_2\cdots j_N}U_{x,\mu}^{i_1j_1}U_{x,\mu}^{i_2j_2}\cdots U_{x,\mu}^{i_Nj_N}. \tag{6.5}$$

It has been introduced into the Hamiltonian to break an extra $U(1)$ symmetry that would otherwise be present and lead to a $U(N)$ rather than an $SU(N)$ gauge invariant model.

This Hamilton operator has to be invariant under gauge transformations, i.e., we require that

$$[H,G_x^a]=0, \tag{6.6}$$

where G_x^a are the generators of an $SU(N)$ algebra at each lattice site x, obeying the commutation relations

$$[G_x^a,G_y^b]=2i\delta_{xy}f_{abc}G_x^c. \tag{6.7}$$

In the Hilbert space, a general gauge transformation is represented by the operator $\Pi_x \exp(i\alpha_x^a G_x^a)$. We can construct gauge covariant transformations of the fields by requiring that

$$U'_{x,\mu} = \prod_y \exp(-i\alpha_y^a G_y^a) U_{x,\mu} \prod_z \exp(i\alpha_z^b G_z^b)$$
$$= \exp(i\alpha_x^a \lambda^a) U_{x,\mu} \exp(-i\alpha_{x+\hat{\mu}}^b \lambda^b). \tag{6.8}$$

This implies commutation relations of the form

$$[G_x^a, U_{y,\mu}] = \delta_{x,y+\hat{\mu}} U_{y,\mu}\lambda^a - \delta_{x,y}\lambda^a U_{y,\mu}. \tag{6.9}$$

In order to satisfy these relations, we write

$$G_x^a = \sum_\mu (R^a_{x-\hat{\mu},\mu} + L^a_{x,\mu}), \tag{6.10}$$

where $R^a_{x,\mu}$ and $L^a_{x,\mu}$ are generators of left and right gauge transformations of the link variable $U_{x,\mu}$. They generate an $SU(N)_R \times SU(N)_L$ algebra on each link, which can be embedded diagonally in the algebra of $SU(2N)$, with the commutation relations as given in Eq. (3.5).

We choose representations for the $SU(2N)$ algebra with rectangular Young tableaux, as we already did in the case of the principal chiral model. In particular, we can use the rishon representation of Eqs. (3.6)–(3.8). In contrast to the principal chiral model, operators now live on the links and not on the lattice sites. The notation is the following:

$$S^{ij}_{x,\mu} = L^{ij}_{x,\mu} = \left(c^{i\alpha\dagger}_{x,\mu} c^{j\alpha}_{x,\mu} - \frac{n}{2}\delta^{ij} \right), \quad S^{N+i,N+j}_{x,\mu} = R^{ij}_{x,\mu} = \left(c^{i\alpha\dagger}_{x+\hat{\mu},-\mu} c^{j\alpha}_{x+\hat{\mu},-\mu} - \frac{n}{2}\delta^{ij} \right),$$
$$S^{N+j,i}_{x,\mu} = -U^{ij}_{x,\mu} = -c^{i\alpha}_{x,\mu} c^{j\alpha\dagger}_{x+\hat{\mu},-\mu}, \quad S^{j,N+i}_{x,\mu} = -(U^{ji}_{x,\mu})^\dagger = -c^{i\alpha}_{x+\hat{\mu},-\mu} c^{j\alpha\dagger}_{x,\mu},$$
$$T_{x,\mu} = \sum_i (c^{i\alpha\dagger}_{x+\hat{\mu},-\mu} c^{i\alpha}_{x+\hat{\mu},-\mu} - c^{i\alpha\dagger}_{x,\mu} c^{i\alpha}_{x,\mu}), \quad \sum_i (c^{i\alpha\dagger}_{x,\mu} c^{i\beta}_{x,\mu} + c^{i\alpha\dagger}_{x+\hat{\mu},-\mu} c^{i\beta}_{x+\hat{\mu},-\mu}) = \delta^{\alpha\beta} N, \tag{6.11}$$

where $\alpha = 1, \ldots, n$ and $i,j = 1, \ldots, N$.

The next step is to set up a coherent state path integral as discussed in Sec. IV. We will consider the analog of Eq. (4.9) and the coset decomposition of Eq. (4.20) to determine some properties of the matrix w that appears in the action of the coherent state path integral. We have

$$U^{ij}_{x,\mu} = -c^{ja\dagger}_{x+\hat{\mu},-\mu} c^{ia}_{x,\mu}. \tag{6.12}$$

From Eq. (4.9) we obtain

$$\langle q | U^{ij}_{x,\mu} | q \rangle = -\frac{n}{2} w^{ij}_{x,\mu}. \tag{6.13}$$

Now consider

$$-\frac{n}{2} w^{ij}_{x+\hat{\mu},-\mu} = \langle q | U^{ij}_{x+\hat{\mu},-\mu} | q \rangle = -\langle q | c^{ja\dagger}_{x,\mu} c^{ia}_{x+\hat{\mu},-\mu} | q \rangle$$
$$= \langle q | (U^{ji}_{x,\mu})^\dagger | q \rangle = \langle q | U^{ji}_{x,\mu} | q \rangle^* = -\frac{n}{2} w^{ji*}_{x,\mu}, \tag{6.14}$$

and hence we see that

$$w_{x+\hat{\mu},-\mu} = w^\dagger_{x,\mu}. \tag{6.15}$$

The coset decomposition is $w_{x,\mu} = -s_{x,\mu} u_{x,\mu}$, where $s = s^\dagger$ and $uu^\dagger = \mathbb{1}$. Taken together with Eq. (6.15), this leads to

$$w_{x+\hat{\mu},-\mu} = -s_{x+\hat{\mu},-\mu} u_{x+\hat{\mu},-\mu} = w^\dagger_{x,\mu}$$
$$= -u^\dagger_{x,\mu} s_{x,\mu} = -u^\dagger_{x,\mu} s_{x,\mu} u_{x,\mu} u^\dagger_{x,\mu}, \tag{6.16}$$

and we deduce that

$$u_{x+\hat{\mu},-\mu} = u^\dagger_{x,\mu}, \quad s_{x+\hat{\mu},-\mu} = u^\dagger_{x,\mu} s_{x,\mu} u_{x,\mu}. \tag{6.17}$$

At this point, the complete action in the coherent state path integral is

$$S = S_B + \frac{Jn^4}{16} \int_0^\beta dt \sum_{x,\mu\neq\nu} 2\,\mathrm{Re}$$
$$\mathrm{Tr}[s_{x,\mu} u_{x,\mu} s_{x+\hat{\mu},\nu} u_{x+\hat{\mu},\nu} u^\dagger_{x+\hat{\nu},\mu} s_{x+\hat{\nu},\mu} u^\dagger_{x,\nu} s_{x,\nu}]$$
$$+ J' \int_0^\beta dt \sum_{x,\mu} \left[\det\left(\frac{n}{2} s_{x,\mu} u_{x,\mu} \right) + \det\left(\frac{n}{2} s_{x,\mu} u^\dagger_{x,\mu} \right) \right]. \tag{6.18}$$

Here

$$S_B = -\frac{n}{2} \sum_{x,\mu} \int_0^\beta dt\, \mathrm{Tr}(\cos(2b_{x,\mu}) u_{x,\mu} \partial_t u^\dagger_{x,\mu}), \tag{6.19}$$

which follows from Eq. (4.26).

We now want to expand around the minimum of the action. From Eq. (6.18), the action is minimized when the eigenvalues of s are largest, i.e., equal to one, and the field u is constant for all links on the lattice. We can use a gauge transformation to rotate these constant fields to the identity matrix. The expansion for s is the same as in Sec. V B,

$$s_{x,\mu} = \sin(2b_{x,\mu}) = \sin(2(b_0 + \epsilon_{x,\mu})) \approx \mathbb{1} - 2\epsilon^2_{x,\mu}. \tag{6.20}$$

We substitute this expression into the action, dropping terms of order $\epsilon_{x,\mu}^3$ and higher. After some rearrangement and relabeling of the summed indices, we find

$$S \approx S_B + \frac{Jn^4}{16}\int_0^\beta dt \sum_{x,\mu\neq\nu} 2\,\mathrm{Re}\,\mathrm{Tr}[(\mathbb{1}-4(\epsilon_{x,\mu}^2+\epsilon_{x,\nu}^2))u_{x,\mu}u_{x+\hat{\mu},\nu}u_{x+\hat{\nu},\mu}^\dagger u_{x,\nu}^\dagger] + 2J'\left(\frac{n}{2}\right)^N \int_0^\beta dt \sum_{x,\mu} (1-2\,\mathrm{Tr}\,\epsilon_{x,\mu}^2)\cos\theta_{x,\mu}. \tag{6.21}$$

Here, we have defined $\exp(i\theta_{x,\mu}) \equiv \det u_{x,\mu}$. If we take $J' < 0$, then the minimum of the action will occur for $\theta_{x,\mu} = 0$. Thus, the matrices $u_{x,\mu}$ will have determinant equal to one, and belong to $SU(N)$ rather than $U(N)$.

In order to be able to take the continuum limit we need to express the unitary matrix field $u_{x,\mu}$, assumed to be close to the identity in our expansion, as the exponential of algebra-valued matrix fields. So let

$$a_{x,\mu} = -a_{x,\mu}^b \lambda^b \tag{6.22}$$

be a Lie algebra valued vector field, λ^a denoting the generators of $SU(N)$, and write

$$u_{x,\mu} \equiv \exp(-ia^2(\theta_{x,\mu}/N)\mathbb{1} - ia\, a_{x,\mu}). \tag{6.23}$$

We also rescale the field $\epsilon_{x,\mu}$, writing $\epsilon_{x,\mu} = a^2 \tilde{\epsilon}_{x,\mu}$. The next step is to expand the action to order a^4, using Eq. (6.23). This leads to

$$S = S_B - \frac{Jn^4}{16}\int_0^\beta dt \sum_{x,\mu\neq\nu} a^4 \mathrm{Tr}[f_{\mu\nu}f_{\mu\nu}] + \int_0^\beta dt \sum_{x,\mu} a^4 \mathrm{Tr}\left[\gamma \tilde{\epsilon}_{x,\mu}^2 - 2J'\left(\frac{n}{2}\right)^N \theta_{x,\mu}^2\right], \tag{6.24}$$

where we have dropped a constant, and

$$\gamma = -3Jn^4 - 4J'\left(\frac{nb}{2}\right)^N. \tag{6.25}$$

Of course, there are not any terms linear in $\tilde{\epsilon}$ in Eq. (6.24), and $\gamma > 0$ since we are expanding about a minimum.

We use the same expansion to manipulate the Berry phase term:

$$\begin{aligned} S_B &= -\frac{n}{2}\int_0^\beta dt \sum_{x,\mu} \mathrm{Tr}[\cos(2b_{x,\mu})u_{x,\mu}\partial_t u_{x,\mu}^\dagger] \\ &\approx -\frac{n}{2}\int_0^\beta dt \sum_{x,\mu} \mathrm{Tr}[(\cos(2b_0) - 2\sin(2b_0)\epsilon_{x,\mu}) \times (\mathbb{1} - ia\,a_{x,\mu})\partial_t(\mathbb{1}+ia\,a_{x,\mu})] \\ &= i\frac{n}{2}\int_0^\beta dt \sum_{x,\mu} a^4\,\mathrm{Tr}\left[\frac{2}{a}\tilde{\epsilon}_{x,\mu}\partial_t a_{x,\mu}\right]. \end{aligned} \tag{6.26}$$

The complete action in the limit $\Sigma_x a^4 \to \int d^4x$ is now

$$\begin{aligned} S &= \frac{1}{2e^2}\int_0^\beta dt \int d^4x\,\mathrm{Tr}\left[f_{\mu\nu}f_{\mu\nu} + 2e^2\gamma\tilde{\epsilon}_\mu\tilde{\epsilon}_\mu + 2ie^2\frac{n}{a}\tilde{\epsilon}_\mu\partial_t a_\mu - 4e^2J'\left(\frac{n}{2}\right)^N \theta_\mu\theta_\mu\right] \\ &= \frac{1}{2e^2}\int_0^\beta dt \int d^4x\,\mathrm{Tr}\left[f_{\mu\nu}f_{\mu\nu} + \frac{1}{c^2}\partial_t a_\mu \partial_t a_\mu + 2e^2\gamma\tilde{\epsilon}'_\mu\tilde{\epsilon}'_\mu - 4e^2J'\left(\frac{n}{2}\right)^N \theta_\mu\theta_\mu\right], \end{aligned} \tag{6.27}$$

where $e^2 = 8/(n^4|J|)$ and $c = (na/2)\sqrt{\gamma|J|}$. We have completed the square in order to integrate out the shifted field

$$\tilde{\epsilon}'_\mu = \tilde{\epsilon}_\mu + i\frac{n}{2\gamma a}\partial_t a_\mu, \tag{6.28}$$

as well as the θ-field, obtaining

$$S = \frac{1}{2e^2}\int_0^\beta dt \int d^4x\,\mathrm{Tr}\left[f_{\mu\nu}f_{\mu\nu} + \frac{1}{c^2}\partial_t a_\mu \partial_t a_\mu\right]. \tag{6.29}$$

If we now again assume that the correlation length is much larger than the extent of the fifth dimension, we can perform the trivial integration over t, to obtain

$$S = \frac{\beta}{2e^2}\int d^4x\,\mathrm{Tr}[f_{\mu\nu}f_{\mu\nu}]. \tag{6.30}$$

It was argued in [1] that a finite correlation length

$$\xi \propto \exp\left(\frac{24\pi^2\beta}{11Ne^2}\right) \tag{6.31}$$

is expected to be generated non-perturbatively. Again, the continuum limit in which the correlation length diverges is achieved by taking the extent of the extra dimension β to infinity. In this limit, we also find that the extent of the extra dimension in physical units is much smaller than the correlation length, $\beta c \ll \xi$. Thus, the theory undergoes dimensional reduction.

VII. CONCLUSIONS

In the D-theory formulation of quantum field theories, a field Lagrangian is replaced by a Hamilton operator and continuous classical fields are replaced by operator fields. The Hamilton operator evolves the system in an additional Euclidean direction. Guided by symmetry considerations, we have formulated the principal chiral model as such a quantum spin system. We then went on to show that with a particular choice of representation for the operators in the Hamiltonian, the theory reduces to a Wilsonian lattice principal chiral model. From numerical simulations we know that the $SU(N)_L \times SU(N)_R \times U(1)_{L=R}$ symmetry of such a model breaks spontaneously to $SU(N)_{L=R}$ at $\beta=\infty$ [22]. We chose representations with rectangular Young tableaux, with N rows and n columns, where n was taken to be large. The Goldstone modes arising from the spontaneous symmetry breaking cause the system to undergo dimensional reduction when we make the extent of the third dimension finite and we thus recover the 2D principal chiral model. We have seen that the continuous degrees of freedom of the low-energy effective theory, which is the same as the standard formulation of the principal chiral model, arise as collective excitations of the discrete degrees of freedom in the D-theory formulation of the model.

We also showed that the quantum link model in $(4+1)$D undergoes dimensional reduction to 4D Yang Mills theory. We chose the quantum link operators to be in the same type of representation that we considered for the operators in the principal chiral model, namely large representations with rectangular Young tableaux. The mechanism for dimensional reduction is different in this case. Instead of Goldstone modes arising from a spontaneously broken global symmetry, the massless modes we need for dimensional reduction result from the fact that a $(4+1)$D gauge theory can exist in a non-Abelian Coulomb phase. We showed that for the aforementioned representations the low-energy effective theory of the D-theory is indeed a 5D Wilson-type lattice gauge theory. It is known from numerical simulations that such a theory is indeed in the non-Abelian Coulomb phase when the extent of the fifth dimension is infinite [2,10]. At finite temperature the gauge bosons form glueballs and acquire mass, due to the confinement hypothesis. The correlation length, however, is exponential in the extent of the fifth dimension, hence leading to dimensional reduction. Again, the continuous fields of the low-energy effective theory after dimensional reduction arise as collective excitations of discrete variables.

In order to be able to get an analytic handle on the behavior of the D-theory formulations of the principal chiral model and non-Abelian gauge theory, we had to consider large representations for the quantum operators in the Hamiltonian. On the other hand, to develop more efficient algorithms for simulating such theories one would like to consider smaller representations, so that each variable can assume only a few discrete values. It is not clear at this point if the mechanism of dimensional reduction also occurs for small representations. Numerical studies are needed to answer this question.

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APPENDIX

For completeness' sake, we show in this section how to decompose a non-singular $GL(N,\mathbb{C})$ matrix q into the product of a Hermitian matrix b and a unitary matrix u. First, let $m=qq^\dagger$, which is Hermitian positive semidefinite. So it can be diagonalized by a unitary transformation,

$$vmv^\dagger = m_D = \mathrm{diag}(m_1, m_2, \ldots, m_N). \tag{A1}$$

We can then define the square root of m_D as

$$\sqrt{m_D} = \mathrm{diag}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_N}). \tag{A2}$$

Now let

$$b = v^\dagger m_D^{1/2} v, \quad u = b^{-1} q = v^\dagger m_D^{-1/2} v q. \tag{A3}$$

It is easily seen that such a u is unitary. We also want to determine the transformation properties of the b and u matrices if q transforms under $U(N)_L \times U(N)_R$ transformations as

$$q \rightarrow q' = LqR^\dagger, \tag{A4}$$

where $LL^\dagger = RR^\dagger = \mathbb{1}$. Then $q' = b'u'$, and

$$m' = q'q'^\dagger = LqR^\dagger Rq^\dagger L^\dagger = LmL^\dagger. \tag{A5}$$

We also have

$$m'_D = v'm'v'^\dagger = v'LmL^\dagger v'^\dagger = m_D. \tag{A6}$$

Thus, $v'L = dv$ or $v' = dvL^\dagger$, where d is a non-degenerate diagonal matrix. So we find the transformation properties

$$b' = v'^\dagger \sqrt{m_D} v' = Lv^\dagger d^\dagger \sqrt{m_D} dvL^\dagger = LbL^\dagger \tag{A7}$$

and

$$u' = b'^{-1} q' = Lb^{-1} L^\dagger LqR^\dagger = LuR^\dagger. \tag{A8}$$

Furthermore, observe that

$$qq^\dagger = m = v^\dagger m_D v = (v^\dagger \sqrt{m_D} v)(v^\dagger \sqrt{m_D} v) = b^2,$$

$$q^\dagger q = u^\dagger b^2 u. \tag{A9}$$

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