Nonsymmetric unified field theory. II. Phenomenological aspects

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The nonsymmetric unified field theory of gravitation and electromagnetism developed in a previous paper in vacuum is here supplemented by introducing the sources. The sources of the field, the matter energy-momentum tensor and the electromagnetic current, are introduced explicitly into the Lagrangian providing a close contact with elementary particle physics concepts in the linear approximation of the theory and an explicit form for the conservation laws. The theory is shown to be free of ghost-negative energy particles and tachyons as well. The equations of motion of test charged particles are established through the invariance of the interaction Lagrangian.

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I. INTRODUCTION

In a previous paper [1], herefrom referred to as I, a nonsymmetric unified field theory of gravitation and electromagnetism has been developed in vacuum. The antisymmetric part of the metric $g_{[\alpha\beta]}$ has been made to describe a massless spin-1 field obeying Maxwell's equations in the flat space linear approximation of the theory, supporting then its identification to the electromagnetic strength tensor $F_{\alpha\beta}$, as in Eq. (2.6) below. By having this flat space Maxwellian behavior of $g_{[\alpha\beta]}$ guaranteed the theory was shown to be free of negative-energy radiative modes even when expanded about a Riemannian background space. The Einstein-Maxwell theory appears to lowest order about a general relativity (GR) curved space. The theory provides a new version of Einstein's unified theory [2], by modifying the Bonnor [3] and Moffat-Boal [4] (MB) unified theories. Bonnor introduced an extra term in the Einstein Lagrangian in such a way that the Coulomb force could be obtained in the equation of motion to lowest order and MB offered later a different interpretation of the Bonnor theory by suggesting the identification of the antisymmetric part of the metric $g_{[\alpha\beta]}$ to the electromagnetic field tensor $F_{\alpha\beta}$, within a universal constant p as in Eq. (2.6) below, instead of to its dual as Bonnor had it, after Einstein. Yet, although the divergence Maxwell vacuum equation is present in the MB theory, the curl equation appears only in the limit of a vanishing p this being only a formal limit however. We could have it for fixed p, as it actually is. The Maxwellian behavior of $g_{[\alpha\beta]}$ was made possible by modifying the Einstein part of the Bonnor Lagrangian, by keeping only that piece of the Einstein tensor [2] which contains the symmetric part of the connection only. By doing so the Bonnor term end up to play the decisive role in the curl-type field equation, determining then the appearance of Maxwell's curl equation in the linear approximation for fixed p.

Here we introduce the sources of the field, the phenomenological matter energy-momentum-stress tensor and electromagnetic current explicitly into the Lagrangian. As pointed out in I, our approach to the unified theory follows the procedure that we have adopted previously [5] to develop a nonsymmetric theory of gravitation (pure gravitation with no association of the antisymmetric part of the metric to the electromagnetic field tensor), where the sources of the metric field are the matter energy-momentum tensor and the matter fermionic number current. The role of this current will be played here by the electromagnetic current.

The introduction of the sources is actually not in the spirit of Einstein's thoughts on his unified theory because these are phenomenological quantities of non-gravitational character, as emphasized by him, which are being put into a theory from which, in principle, everything should follow. However, we have done so, at least in the actual stage of the theory. With the sources at hand we shall then be able to study the particle content of the theory when going to the flat space linear approximation, where field theoretical concepts of particle physics are to be discussed, showing that it is free of ghost-negative energy particles and tachyons. Also, we shall be able to obtain the explicit form of the conservation laws, from which we shall be in position to obtain the equation of motion of test charged point particles in the theory which generalizes the Einstein-Maxwell equation of motion. Therefore, in the present theory the proposed modification of the Einstein Lagrangian is crucial: it permits us to obtain a massless spin-1 Maxwellian character for $g_{[\alpha\beta]}$ in the flat space linear approximation, to avoid the appearance of negative-energy radiative modes when expanded about a Riemannian background space and to prevent the appearance of unphysical particles in the flat space limit, all at the same time.

Lastly, the equation of motion of charged point test particles is obtained, this being accomplished directly from the coordinate invariance of the matter interaction Lagrangian.

The paper is organized as follows. In Sec. II we write the Lagrangian, reviewing the origin of the field part of it as studied in I, and display the field equations in Sec. III. In Sec. IV we analyze the particle content of the theory. The equations of motion of test charged particles are discussed in Sec. V and in Sec. VI we draw our conclusions.

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II. THE LAGRANGIAN

We write the Lagrangian density as

$$\mathbf{L}=\frac{1}{16\pi}\left[-\mathbf{g}^{\alpha\beta}(U_{\alpha\beta}+\Gamma_{[\alpha,\beta]})+\frac{1}{p^{2}}\mathbf{g}^{[\alpha\beta]}g_{[\alpha\beta]}\right]+\mathbf{L}_{M}. \tag{2.1}$$

The first term is the field part of the Lagrangian written in I, here with the factor $(16\pi)^{-1}$. We use the notation $\mathbf{X}=\sqrt{-g}X$ where g is the determinant of $g_{\alpha\beta}$ whose inverse $g^{\alpha\beta}$ is defined by

$$g^{\alpha\beta}g_{\alpha\gamma}=g^{\beta\alpha}g_{\gamma\alpha}=\delta^{\beta}_{\gamma}. \tag{2.2}$$

Next,

$$U_{\alpha\beta}=\Gamma^{\sigma}_{(\alpha\beta),\sigma}-\Gamma^{\sigma}_{(\sigma\alpha),\beta}+\Gamma^{\sigma}_{(\alpha\beta)}\Gamma^{\lambda}_{(\sigma\lambda)}-\Gamma^{\sigma}_{(\alpha\lambda)}\Gamma^{\lambda}_{(\sigma\beta)}, \tag{2.3}$$

symmetric and containing only the symmetric part of the connection, is the analogue of the usual Ricci tensor and $\Gamma_{\alpha}=\Gamma^{\gamma}_{[\alpha\gamma]}=\frac{1}{2}(\Gamma^{\gamma}_{\alpha\gamma}-\Gamma^{\gamma}_{\gamma\alpha})$ is the torsion vector. $U_{\alpha\beta}$ is actually that piece of the Einstein tensor [2] which contains only the symmetric part of the connection. The second term in the square brackets is the term introduced by Bonnor (p being here the inverse of his p) in his modification of the Einstein unified theory so as to have the Coulomb force present in the theory. We have taken the multiplicative parameter to $\Gamma_{[\alpha,\beta]}$ of I with value $d=1$ without any loss of generality because Γ_{α}, working as a Lagrange multiplier, will not appear in the field equations. Next, $\mathbf{L}_{M}$ is the matter part of the Lagrangian [6] modeled after the one of GR, containing here the generalized nonsymmetric (Hermitian) matter energy-momentum-stress tensor $T_{\alpha\beta}$ and electromagnetic current J^{α}, as given by

$$\delta\mathbf{L}_{M}=\frac{1}{2}\sqrt{-g}T_{\alpha\beta}\delta g^{\alpha\beta}+\frac{1}{4}p\sqrt{-g}J^{\alpha}\delta\Gamma_{\alpha}, \tag{2.4}$$

that is,

$$T_{\alpha\beta}=\frac{2}{\sqrt{-g}}\frac{\delta\mathbf{L}_{M}}{\delta g^{\alpha\beta}} \tag{2.5}$$

and $J^{\alpha}=(4/p\sqrt{-g})\delta\mathbf{L}_{M}/\delta\Gamma_{\alpha}$. The second term in Eq. (2.4) has the same form of the usual electromagnetic coupling term of GR, $-\mathbf{J}^{\alpha}\delta A_{\alpha}$, if we recall the relation $A_{\alpha}=-4^{-1}p\Gamma_{\alpha}$ derived in I for the vector potential, when $d=1$.

We recall that the vacuum field Lagrangian was built out so as to make the identification of $g_{[\alpha\beta]}$ to $F_{\alpha\beta}$, defined by

$$g_{[\alpha\beta]}=pF_{\alpha\beta}, \tag{2.6}$$

a possibly consistent procedure. It was built out from considering in the first place the most general form of the second-order tensor, containing at most first-order derivatives and quadratic products of the affine connection, design to play the role of the Einstein tensor, satisfying Einstein's condition of Hermiticity [2]. This means invariance under transposition, which is defined as the transformation that exchanges the indices of the metric tensor and the lower ones of the connection, followed by an exchange of the two indices of any second-order tensor that depends on the metric and connection. This symmetry property has the physical meaning [2] that the same field equations are satisfied for positive and negative charges, the transformation taking one into the other. One is then left with four out of an initial seven parameters. Next, by requiring that the vacuum Maxwell's equations should hold in the linear flat space limit the remaining parameters present in the field equations were all forced to have specific values, leading to the final vacuum field equations of the theory. These equations were then shown to be derivable directly from the free part of the Lagrangian written above, in Eq. (2.1), by varying the corresponding action with respect to $g^{\alpha\beta}$, $\Gamma^{\sigma}_{(\alpha\beta)}$, and Γ_{α}. This is what we shall do now in the present context, with the sources present. Let us note that as Γ_{α} changes sign under transposition, invariance of $\mathbf{L}$ under this operation will demand that J^{α} in the second term on the right of Eq. (2.4) changes sign. This charge conjugation transformation materializes then Einstein's assertion that Hermiticity reflects the symmetry of the theory in describing positive and negative charges.

We close this section with a few comments concerning the nonsymmetric stress-energy. Together with the down-indices stress $T_{\alpha\beta}$ we shall be working with the upper-indices $T^{\mu\nu}$ defined by the variation with respect to $g_{\mu\nu}$,

$$T^{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta\mathbf{L}_{M}}{\delta g_{\mu\nu}}, \tag{2.7}$$

as in GR. This second stress is related to the first one by

$$T_{\alpha\beta}=g_{\alpha\nu}g_{\mu\beta}T^{\mu\nu}, \tag{2.8}$$

which follows from the relation $\delta g_{\mu\nu}/\delta g^{\alpha\beta}=-g_{\alpha\nu}g_{\mu\beta}$ resulting from the variation of Eq. (2.2). It should be kept in mind that Eq. (2.8) does not imply a rule for lowering indices because this operation is not defined for a nonsymmetric metric. A better name for the upper-indices stress tensor would probably be $S^{\mu\nu}$ but we shall use the same T for both tensors. Notice that the inverse relation is $T^{\mu\nu}=g^{\mu\beta}g^{\alpha\nu}T_{\alpha\beta}$ and that both have the same trace $g^{\alpha\beta}T_{\alpha\beta}=g_{\mu\nu}T^{\mu\nu}$. Notice also that Eq. (2.8) preserves the Hermiticity of both tensors.

Now consider the situation in which we are dealing with a perfect pressureless fluid, as in fact we shall when studying the equation of motion in Sec. V. Then it is natural to take for $T^{\mu\nu}$ the symmetric dust-like energy tensor $T^{\mu\nu}_{d}=\rho u^{\alpha}u^{\beta}$, ρ being the matter rest densiy and u^{α} the velocity. Actually, as we shall see, once the symmetry of this tensor is assumed its form will be determined by the conservation laws. Thence, Eq. (2.8) tell us that even for this symmetric tensor, $T_{\alpha\beta}$ will have a symmetric and an antisymmetric part as well, both involving the symmetric and the antisymmetric part of the metric. Another situation in which a nonsymmetric $T_{\alpha\beta}$ appears is when one consider Lagrangians of matter fields in the new scheme, oriented by those of GR as a guide but with a nonsymmetric metric. We shall do so in Sec. IV considering the Lagrangian of the massive charged boson field as a working example.

III. FIELD EQUATIONS

Variations of the action $\int \mathbf{L}\, d^4x$ with respect to $g^{\alpha\beta}$, $\Gamma^{\gamma}_{(\alpha\beta)}$, and Γ_α yields the field equations. The former leads to

$$U_{\alpha\beta}+\Gamma_{[\alpha,\beta]}-K_{\alpha\beta}=8\pi\overline{T}_{\alpha\beta}, \tag{3.1}$$

where

$$\overline{T}_{\alpha\beta}=T_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}T, \tag{3.2}$$

with $T=g^{\alpha\beta}T_{\alpha\beta}$, and [3]

$$K_{\alpha\beta}=\frac{1}{p^2}\left(g_{[\alpha\beta]}+g_{\alpha\mu}g^{[\mu\nu]}g_{\nu\beta}+\frac{1}{2}g_{\alpha\beta}g^{[\mu\nu]}g_{[\mu\nu]}\right). \tag{3.3}$$

The symmetric and antisymmetric parts of Eq. (3.1) are

$$U_{\alpha\beta}-K_{(\alpha\beta)}=8\pi\overline{T}_{(\alpha\beta)} \tag{3.4}$$

and

$$\Gamma_{[\alpha,\beta]}-K_{[\alpha\beta]}=8\pi\overline{T}_{[\alpha\beta]}, \tag{3.5}$$

which, upon taking its curl, gives

$$K_{[\alpha\beta,\gamma]}=-8\pi\overline{T}_{[\alpha\beta,\gamma]}. \tag{3.6}$$

Here we have used the indication $X_{[\alpha\beta,\mu]}=X_{[\alpha\beta],\mu}+X_{[\mu\alpha],\beta}+X_{[\beta\mu],\alpha}$ for the curl of $X_{[\alpha\beta]}$. Of course, the curl of $\Gamma_{[\alpha,\beta]}$ is zero. The variation with respect to $\Gamma^{\gamma}_{(\alpha\beta)}$ will give the same result as in I,

$$\mathbf{g}^{(\alpha\beta)}{}_{,\gamma}+\mathbf{g}^{(\alpha\sigma)}\Gamma^{\beta}_{(\gamma\sigma)}+\mathbf{g}^{(\beta\sigma)}\Gamma^{\alpha}_{(\gamma\sigma)}-\mathbf{g}^{(\alpha\beta)}\Gamma^{\sigma}_{(\sigma\gamma)}=0. \tag{3.7}$$

Next, as the Γ_α term of Eq. (2.1) can be written $(16\pi)^{-1}\mathbf{g}^{[\alpha\beta]}{}_{,\beta}\Gamma_\alpha$ up to a total derivative, the variation with respect to Γ_α gives

$$\frac{1}{p}\mathbf{g}^{[\alpha\beta]}{}_{,\beta}=-4\pi\mathbf{J}^{\alpha}, \tag{3.8}$$

which is Maxwell's inhomogeneous generalized equation. Equations (3.4), (3.6), (3.7), and (3.8) are the field equations of the theory. Equation (3.6) gives in vacuum the generalized homogeneous Maxwell equation. Inside matter this equation gets a coupling to the antisymmetric part of the energy-momentum tensor. As discussed in I for the vacuum case, in a first-order expansion about a Riemannian space the field equations reduce to the corresponding equations of the Einstein-Maxwell theory which, we briefly discuss at the end of this section. *A fortiori* they reduce to the usual Maxwell's equations when expanded about a Minkowski flat space. Equation (3.7) can be solved for $\Gamma^{\sigma}_{(\alpha\beta)}$. We get [1]

$$\Gamma^{\sigma}_{(\alpha\beta)}=\tfrac{1}{2}g^{(\sigma\lambda)}(s_{\alpha\lambda,\beta}+s_{\lambda\beta,\alpha}-s_{\alpha\beta,\lambda})+\Omega^{\sigma}_{\alpha\beta}, \tag{3.9}$$

where

$$\Omega^{\sigma}_{\alpha\beta}=\frac{1}{4}(g^{(\sigma\lambda)}s_{\alpha\beta}-\delta^{\sigma}_{\alpha}\delta^{\lambda}_{\beta}-\delta^{\lambda}_{\alpha}\delta^{\sigma}_{\beta})\left(\ln\frac{s}{g}\right)_{,\lambda} \tag{3.10}$$

and $s_{\alpha\beta}$, symmetric, and with determinant s is the inverse of $g^{(\alpha\beta)}$, as defined by

$$s_{\alpha\beta}g^{(\alpha\gamma)}=\delta_{\beta}{}^{\gamma}. \tag{3.11}$$

When the antisymmetric part of $g_{\alpha\beta}$ vanishes, $s_{\alpha\beta}$ will be equal to $g_{\alpha\beta}$ and the right-hand side of Eq. (3.9) becomes the usual Christoffel symbol, as it should. The symmetric and antisymmetric parts of $K_{\alpha\beta}$ are, from Eq. (3.3),

$$K_{(\alpha\beta)}=\frac{1}{p^2}\left(g_{(\alpha\mu)}g^{[\mu\nu]}g_{[\nu\beta]}+g_{(\beta\mu)}g^{[\mu\nu]}g_{[\nu\alpha]}+\frac{1}{2}g_{(\alpha\beta)}g^{[\mu\nu]}g_{[\mu\nu]}\right) \tag{3.12}$$

and

$$K_{[\alpha\beta]}=\frac{1}{p^2}\left(g_{[\alpha\beta]}+g_{(\alpha\mu)}g^{[\mu\nu]}g_{(\nu\beta)}+g_{[\alpha\mu]}g^{[\mu\nu]}g_{[\nu\beta]}+\frac{1}{2}g_{[\alpha\beta]}g^{[\mu\nu]}g_{[\mu\nu]}\right). \tag{3.13}$$

This completes the discussion of the unified theory with sources. The field equations could also be obtained from the general considerations developed in I, by requiring that the usual Maxwell inhomogeneous divergence equation and the vacuum curl equation should be present in the flat space linear approximation of the theory.

We briefly mention now the linearization of the field equations about a Riemannian background with metric $g^{(0)}_{\alpha\beta}$, as discussed in I in vacuum. This can be achieved by the expansion

$$g_{\alpha\beta}=g^{(0)}_{\alpha\beta}+g^{(1)}_{\alpha\beta}, \tag{3.14}$$

where $g^{(1)}_{\alpha\beta}$ is the perturbation. The inverse of $g_{\alpha\beta}$, as defined in Eq. (2.2), is then

$$g^{\alpha\beta}=g^{(0)\alpha\beta}-g^{(1)\beta\alpha}, \tag{3.15}$$

where the sub- and superscripts are moved by the initial metric tensor $g^{(0)}_{\alpha\beta}$, that is, $g^{(1)\alpha\beta}=g^{(0)\alpha\mu}g^{(0)\beta\nu}g^{(1)}_{\mu\nu}$. We then have $g^{(\alpha\beta)}=g^{(0)\alpha\beta}-g^{(1)(\alpha\beta)}$ and $g^{[\alpha\beta]}=g^{(1)[\alpha\beta]}$. Thence, to first order Eq. (3.8) reads

$$\frac{1}{p}(\sqrt{-g^{(0)}}g^{(0)\alpha\mu}g^{(0)\sigma\nu}g^{(1)}_{[\mu\nu]})_{,\sigma}=-4\pi\sqrt{-g^{(0)}}J^{\alpha}. \tag{3.16}$$

On the other hand, as Eq. (3.13) gives

$$K^{(1)}_{[\alpha\beta]}=\frac{2}{p^2}g^{(1)}_{[\alpha\beta]}, \tag{3.17}$$

we find that Eq. (3.5) yields

$$\Gamma^{(1)}_{[\alpha,\beta]}-\frac{2}{p^2}g^{(1)}_{[\alpha\beta]}=8\pi T_{[\alpha\beta]}, \tag{3.18a}$$

or

$$g^{(1)}_{[\alpha\beta,\gamma]}=8\pi T_{[\alpha\beta,\gamma]}. \tag{3.18b}$$

The conclusion from Eq. (3.16) is then that $g^{(1)}_{[\alpha\beta]}$ satisfies the inhomogeneous Maxwell equation of the Einstein-Maxwell theory and Eq. (3.18b) gives *in vacuum* the Maxwell curl equation. They are then recuperated with the identification of $g^{(1)}_{[\alpha\beta]}$ to the electromagnetic field strength $\tilde{F}_{\alpha\beta}$ of that theory,

$$g^{(1)}_{[\alpha\beta]}=p\tilde{F}_{\alpha\beta}, \tag{3.19}$$

corresponding to the first-order part of Eq. (2.6).

We end this section by writing the relation between $g^{[\alpha\beta]}$ and $g_{[\alpha\beta]}=pF_{\alpha\beta}$, showing then the explicit form of the inhomogeneous equation that generalizes the corresponding Maxwell first equation. We have [7]

$$g^{[\alpha\beta]}=\frac{1}{g}\left(g_S a^{\alpha\mu}a^{\beta\nu}g_{[\mu\nu]}+\frac{1}{2}\sqrt{g_A}\varepsilon^{\alpha\beta\mu\nu}g_{[\mu\nu]}\right), \tag{3.20}$$

where $a^{\alpha\beta}$ is the inverse of $g_{(\alpha\beta)}$ as defined by

$$a^{\alpha\beta}g_{(\alpha\gamma)}=\delta^{\beta}_{\gamma}, \tag{3.21}$$

$g_S=\det(g_{(\alpha\beta)})$ and $g_A=\det(g_{[\alpha\beta]})$, which is given by $\sqrt{g_A}=8^{-1}\varepsilon^{\alpha\beta\mu\nu}g_{[\alpha\beta]}g_{[\mu\nu]}$. The determinants are related by $g=g_S(1+\frac{1}{2}a^{\mu\nu}a^{\alpha\beta}g_{[\mu\alpha]}g_{[\nu\beta]})+g_A$. If $g_A\neq 0$ the second term inside the parentheses of Eq. (3.20) is equal to $g_A m^{\alpha\beta}$, where $m^{\alpha\beta}_A$ is the inverse of $g_{[\alpha\beta]}$ as defined by $m^{\alpha\beta}g_{[\alpha\gamma]}=\delta^{\beta}_{\gamma}$. The explicit form of the generalized inhomogeneus equation is then

$$\left(-\frac{1}{\sqrt{-g}}\left(g_S a^{\alpha\mu}a^{\beta\nu}F_{\mu\nu}+\frac{1}{16}p^2\varepsilon^{\gamma\delta\rho\sigma}F_{\gamma\delta}F_{\rho\sigma}\varepsilon^{\alpha\beta\mu\nu}F_{\mu\nu}\right)\right)_{,\beta}=-4\pi\sqrt{-g}J^{\alpha}, \tag{3.22}$$

where

$$g=g_S(1+\tfrac{1}{2}p^2a^{\mu\nu}a^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta})+64^{-1}p^4(\varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu})^2.$$

On the other hand, the explicit form of $K_{[\alpha\beta]}$ will result from the substitution of Eq. (3.20) into Eq. (3.13), to be then taken into the curl equation Eq. (3.6). We are then facing highly nonlinear equations involving $F_{\mu\nu}$. It is to be noted that in itself this $F_{\mu\nu}$ is expected to be related nonlinearly to the Reissner-Nordström $\tilde{F}_{\alpha\beta}$.

In the next section we shall study the particle content of the theory by analyzing the form of the propagator. This will be obtained from the expansion of the Lagrangian to second order about a Minkowski flat space, where particle physics concepts will be discussed.

IV. PARTICLE CONTENT: A GHOST FREE THEORY

In this section we shall study the particle content of the theory through the study of the propagator. For that purpose we shall expand the Lagrangian in Eq. (2.1) to second order about a Minkowski flat space with metric $\eta_{\alpha\beta}=(1,-1,-1,-1)$, by writing

$$g_{\alpha\beta}=\eta_{\alpha\beta}+h_{\alpha\beta}, \tag{4.1}$$

where $|h_{\alpha\beta}|\ll 1$. The inverse of this equation is

$$g^{\alpha\beta}=\eta^{\alpha\beta}-h^{\beta\alpha}, \tag{4.2}$$

where the sub- and superscripts are moved by the metric $\eta_{\alpha\beta}$, that is, $h^{\beta\alpha}=\eta^{\beta\mu}\eta^{\alpha\nu}h_{\mu\nu}$. Notice that $g^{(\alpha\beta)}=\eta^{\alpha\beta}-h^{(\beta\alpha)}$ and $g^{[\alpha\beta]}=h^{[\alpha\beta]}$. To second order the U term will be identical to the result of GR. This can be checked by direct calculation by using [1] the first-order result coming from Eq. (3.9),

$$\Gamma^{\sigma(1)}_{(\alpha\beta)}=\tfrac{1}{2}\eta^{\sigma\rho}(h_{(\alpha\rho),\beta}+h_{(\beta\rho),\alpha}-h_{(\alpha\beta),\rho}), \tag{4.3}$$

as in GR. Adopting the convention $a_\alpha b_\alpha=\eta^{\alpha\beta}a_\alpha b_\beta$ and writing $h=h^{\alpha}{}_{\alpha}$, one finds, for the second-order part of $\mathbf{L}$,

$$\mathbf{L}=\mathbf{L}_{GR}+\mathbf{L}', \tag{4.4}$$

where

$$\mathbf{L}_{GR}=\frac{1}{16\pi}\left(\frac{1}{4}h_{(\mu\nu),\lambda}h_{(\mu\nu),\lambda}+\frac{1}{2}h_{,\mu}h_{(\mu\alpha),\alpha}-\frac{1}{4}h_{,\mu}h_{,\mu}-\frac{1}{2}h_{(\mu\alpha),\alpha}h_{(\mu\beta),\beta}\right)-\frac{1}{2}h_{(\alpha\beta)}T_{(\alpha\beta)}, \tag{4.5}$$

is precisely the Lagrangian of GR and, up to a total derivative,

$$\mathbf{L}'=\frac{1}{16\pi}\left(h_\alpha\Gamma_\alpha+\frac{1}{p^2}h_{[\alpha\beta]}h_{[\alpha\beta]}\right)+\frac{1}{4}p\Gamma_\alpha J_\alpha+\frac{1}{2}h_{[\alpha\beta]}T_{[\alpha\beta]}, \tag{4.6}$$

where $h_\alpha=h_{[\alpha\beta],}{}^{\beta}$. The graviton spectrum, which is contained in $\mathbf{L}_{GR}$, is known to be free of ghosts so we consider only $\mathbf{L}'$. Let us note that if it were not for the last, $T_{[\alpha\beta]}$, term of Eq. (4.6) the theory would be obviously ghost-free because $\mathbf{L}'$ would be just that of electromagnetism in first order form, as discussed in vacuum in I, but now with the current present. In fact, without it $h_{[\alpha\beta]}$ could be eliminated in favor of $\Gamma_{[\alpha,\beta]}$ by using Eq. (3.18a) to yield Maxwell's Lagrangian in second order form with the current present. Going back to our $\mathbf{L}'$, we follow the study of the ghost properties of generalized theories of gravitation of Mann and Moffat [8] who followed the analysis of Sezgin and Nieuwenhuizen [9] on higher-derivative gravity, extending the results of Neville [10]. To keep in close contact with the notation of Ref. [8] we put

$$\tfrac{1}{4}\, p\Gamma_{\alpha}=\Lambda_{\alpha}; \quad \tau_{[\alpha\beta]}=\tfrac{1}{2}\, T_{[\alpha\beta]}, \tag{4.7a}$$

and

$$C=\frac{1}{4\pi p}; \quad D=-\frac{1}{8\pi p^2}. \tag{4.7b}$$

Then $\mathbf{L}'=\mathbf{L}'_0+\mathbf{L}'_M$ where

$$\mathbf{L}'_0=Ch_{\alpha}\Lambda_{\alpha}-\tfrac{1}{2}\, Dh_{[\alpha\beta]}h_{[\alpha\beta]} \tag{4.8}$$

is the free part of the Lagrangian and

$$\mathbf{L}'_{\mathbf{M}}=\Lambda_{\alpha}J_{\alpha}+h_{[\alpha\beta]}\tau_{[\alpha\beta]} \tag{4.9}$$

is the matter interaction part. Its form depends on what matter field the fields $h_{[\alpha\beta]}$ and Λ_{α} are taken to interact with. After discusssing the ghost properties of the propagator we shall consider the massive charged pion field. The free part of the Lagrangian in Fourier, momentum (k), space is

$$\mathbf{L}'_0=\tfrac{1}{2}\, iCk_{\beta}(h^*_{[\alpha\beta]}\Lambda_{\alpha}-\Lambda^*_{\alpha}h_{[\alpha\beta]})-\tfrac{1}{2}\, Dh^*_{[\alpha\beta]}h_{[\alpha\beta]}, \tag{4.10}$$

which is written under the form [10],

$$\mathbf{L}'_0=\frac{1}{2}\sum_{A,B} F^*_A O_{AB}F_B, \tag{4.11}$$

where in our case, $F_A=(h_{[\mu\nu]},\Lambda_{\mu})$ and $F_B=(h_{[\alpha\beta]},\Lambda_{\alpha})$, O_{AB} being the wave operator. The ghost properties are contained in its inverse. Next one uses the spin-projection operator formalism [11] and invert the wave operator O_{AB} to get the saturated propagator,

$$\Pi=-\sum_{A,B} S_A O^{-1}_{AB}S_B, \tag{4.12}$$

where $S_A=(\tau_{[\mu\nu]},J_{\mu})$. In Appendix A we show how to get the saturated propagator in such a simple situation by using a more down to earth approach of semi-classical field theory, offering in this way a little more insight into the problem. To go on, as $h_{[\alpha\beta]}$ decomposes into a spin-1^- part $(h_{[0i]})$ and a spin-1^+ part $(h_{[ij]})$, and as Λ_{α} decomposes into a spin -1^- part (Λ_i) and a spin-0^+ part (Λ_0), the relevant spin-projection operators with which O_{AB} is to be constructed are [8]

$$P(1^+)_{\mu\nu\alpha\beta}=\tfrac{1}{2}\,(\theta_{\mu\alpha}\theta_{\nu\beta}-\theta_{\nu\alpha}\theta_{\mu\beta}), \tag{4.13}$$

$$P(0^+)_{\alpha\beta}=\omega_{\alpha\beta}, \tag{4.14}$$

and $P(1^-)$ is the 2×2 matrix corresponding to the two spin -1^- fields $h_{[0i]}$ and Λ_i, with elements

$$P_{11}(1^-)_{\mu\nu\alpha\beta}=\tfrac{1}{2}(\theta_{\mu\alpha}\omega_{\nu\beta}-\theta_{\mu\beta}\omega_{\nu\alpha}-\theta_{\nu\alpha}\omega_{\mu\beta}+\theta_{\nu\beta}\omega_{\mu\alpha}), \tag{4.15a}$$

$$P_{12}(1^-)_{\mu\nu\alpha}=\frac{1}{\sqrt{2k^2}}(k_{\nu}\theta_{\mu\alpha}-k_{\mu}\theta_{\nu\alpha}), \tag{4.15b}$$

$$P_{21}(1^-)_{\mu\alpha\beta}=P_{12}(1^-)_{\alpha\beta\mu}, \tag{4.15c}$$

and

$$P_{22}(1^-)_{\mu\alpha}=\theta_{\mu\alpha}. \tag{4.15d}$$

Here

$$\theta_{\alpha\beta}=\eta_{\alpha\beta}-\frac{k_{\alpha}k_{\beta}}{k^2}, \tag{4.16a}$$

and

$$\omega_{\alpha\beta}=\frac{k_{\alpha}k_{\beta}}{k^2}. \tag{4.16b}$$

In terms of these quantities, Eq. (4.10) can be written as

$$\mathbf{L}'_0=\frac{1}{2}\Big[a(1^+)h^*_{[\mu\nu]}P(1^+)_{\mu\nu\alpha\beta}h_{[\alpha\beta]} +\sum_{m,n} a_{mn}(1^-)F^*_m P_{mn}(1^-)F_n\Big], \tag{4.17}$$

where the coefficients are

$$a(1^+)=-D, \tag{4.18a}$$

next, with the indication (1^-) being implied,

$$a_{11}=-D; \quad a_{22}=0,$$
$$a_{12}=-a_{21}=iC\sqrt{\frac{1}{2}k^2}, \tag{4.18b}$$

and $a(0^+)=0$. The propagator for each component is obtained by inverting the nonzero coefficient matrices in momentum space. The saturated propagator is then, from Eq. (4.12),

$$\Pi=-a^{-1}(1^+)\tau_{[\mu\nu]}P(1^+)_{\mu\nu\alpha\beta}\tau_{[\alpha\beta]} -\sum_{m,n} a^{-1}_{mn}(1^-)S_m P_{mn}(1^-)S_n, \tag{4.19}$$

and the coefficients are

$$a^{-1}(1^+)=-\frac{1}{D}, \tag{4.20a}$$

and

$$a_{11}^{-1}=0;\quad a_{22}^{-1}=\frac{2D}{C^2k^2},$$

$$a_{12}^{-1}=-a_{21}^{-1}=\frac{i}{C}\sqrt{2/k^2}. \tag{4.20b}$$

From Eq. (4.20a) we see that the spin-1^+ sector does not propagate: only a contact term appears here. Next, the criterium for freedom of ghosts in the massless spin-1^- sector is that [9] the residue of the trace of the matrix $a^{-1}(1^-)$ at $k^2=0$, which is equal to $2DC^{-2}$, be negative: and that is exactly what we have, because, from Eq. (4.7b),

$$\frac{2D}{C^2}=-4\pi. \tag{4.21}$$

Therefore the theory is free of tachyons and ghosts. Written in full, the propagator is

$$\Pi=-2\pi p^2T_{[\mu\nu]}P(1^+)_{\mu\nu\alpha\beta}T_{[\alpha\beta]}-\frac{2\pi ip}{k^2}T_{[\mu\nu]}\times(k_\nu\theta_{\mu\alpha}-k_\mu\theta_{\nu\alpha})J_\alpha+\frac{2\pi ip}{k^2}J_\alpha(k_\nu\theta_{\mu\alpha}-k_\mu\theta_{\nu\alpha})T_{[\mu\nu]}+\frac{4\pi}{k^2}J_\alpha\theta_{\alpha\beta}J_\beta. \tag{4.22}$$

We call attention for some facts. First, the two middle terms on the right of this last equation do not cancel since the source terms attached to the left (right) belong to the left (right)-hand side of the propagator. For instance, if we have the scattering of particles 1 and 2, and if we attach the left source terms to 1, the right ones will be attached to 2, that is, we shall have the two combinations $T_{1[\mu\nu]}J_{2\alpha}$ and $-J_{1\alpha}T_{2[\mu\nu]}$. Consider now the last term. As $k_\beta J_\beta=0$, which is the equation of continuity in k-space, that term can be written as $4\pi k^{-2}J_\alpha J_\alpha$ on account of Eq. (4.16a). Well, this is just what we obtain from field theory in lowest order, as it should. The discussion in Appendix A will make all these things very clear. Finally, by again making use of Eq. (4.16a), the propagator can be simplified to

$$\Pi=-2\pi p^2T_{[\mu\nu]}\left(\eta_{\nu\beta}-\frac{2}{k^2}k_\nu k_\beta\right)T_{[\mu\beta]}-\frac{4\pi ip}{k^2}(T_{[\alpha\nu]}k_\nu J_\alpha-J_\alpha k_\nu T_{[\alpha\nu]})+\frac{4\pi}{k^2}J_\alpha J_\alpha. \tag{4.23}$$

One remark before we continue. The free part of the Lagrangian studied in [8] contains in the antisymmetric sector two additional terms with coefficients A and B (here uppercases) which with C and D make a set of four constants which assume values depending on the type of theory considered. Then, with the coefficient of the first term satisfying the condition $A>0$ it is concluded that the Bonnor term cannot be present because it leads to a ghost (if $C>0$) or a tachyon (if $C<0$). This the case of the Bonnor and Moffat-Boal theories as it can easily be checked. Then, without the Bonnor term one is back to the Einstein Lagrangian, leading then to a spin-0, scalar, character for $h_{[\alpha\beta]}$, as it has been shown by Moffat and Mann [12]. This type of theory would then doubly damage a unified theory of gravitation and electromagnetism: no Coulomb force and no Maxwellian spin-1 character for $h_{[\alpha\beta]}$. In the present theory those two coefficients A and B are absent and that is why we could keep the Bonnor term, which actually turn out to be responsible for the desired Maxwellian behavior of $h_{[\alpha\beta]}$ in vacuum.

We end this section by considering a possible structure of the sources terms, discussing the massive charged π-meson field as a working example. We shall assume that the Lagrangian has a form similar to the one of GR but now with a nonsymmetric metric, with a possible extra nonsymmetric term and with the vector potencial $A_\alpha=-\Lambda_\alpha$, from Eq. (4.7a) and the remark after Eq. (2.5), that is,

$$\mathbf{L}_M=\sqrt{-g}[g^{(\mu\nu)}(D_\mu\phi)^*D_\nu\phi+ig^{[\mu\nu]}(D_\mu\phi)^*D_\nu\phi-m^2\phi^*\phi], \tag{4.24}$$

where

$$D_\mu=\partial_\mu+ieA_\mu=\partial_\mu-ie\Lambda_\mu, \tag{4.25}$$

and with the extra nonsymmetric term having an i factor for the Lagrangian to be real. Before going on let us calculate the stress tensor of Eq. (2.5) and the electric current. We get

$$T_{\alpha\beta}=[(D_\alpha\phi)^*D_\beta\phi+\alpha\leftrightarrow\beta]+i[(D_\alpha\phi)^*D_\beta\phi-\alpha\leftrightarrow\beta]-g_{\alpha\beta}[g^{(\mu\nu)}(D_\mu\phi)^*D_\nu\phi+ig^{[\mu\nu]}(D_\mu\phi)^*D_\nu\phi-m^2\phi^*\phi], \tag{4.26}$$

with a symmetric and an antisymmetric part. On the other hand, the current $J^\alpha=+(1/\sqrt{-g})\delta\mathbf{L}_M/\delta\Lambda_\alpha$ is given by

$$J^\alpha=-ieg^{(\mu\alpha)}[(D_\mu\phi)^*\phi-\phi^*D_\mu\phi]+eg^{[\mu\alpha]}[(D_\mu\phi)^*\phi+\phi^*D_\mu\phi]. \tag{4.27}$$

These are the quantities to be used on the right-hand side of Eqs. (3.1) and (3.8).

To lowest order the antisymmetric part of Eq. (4.26) is

$$T_{[\alpha\beta]}=i(\partial_\alpha\phi^*\,\partial_\beta\phi-\alpha\leftrightarrow\beta), \tag{4.28}$$

and Eq. (4.27) gives, to lowest order,

$$J_\alpha=-ie(\partial_\alpha\phi^*\,\phi-\phi^*\,\partial_\alpha\phi). \tag{4.29}$$

These are the quantities to be placed on the right-hand side of Eq. (4.9). Notice that from Eq. (4.24), the part of the interaction Lagrangian containing the antisymmetric sector is, to lowest order,

$$\mathbf{L}'_M = -ie(\partial_\mu \phi^* \phi - \phi^* \partial_\mu \phi)\Lambda_\mu + ih_{[\mu\nu]}(\partial_\mu \phi^* \partial_\nu \phi), \tag{4.30}$$

which, on account of Eqs. (4.28) and (4.29), reproduces Eq. (4.9), as it should. In Appendix A we write the full saturated propagator for this π-scattering process.

V. THE EQUATION OF MOTION OF TEST CHARGED PARTICLES

From the coordinate invariance of the field Lagrangian density $\mathbf{L}_0 = \mathbf{L} - \mathbf{L}_M$ in Eq. (2.1), we obtain the four generalized Bianchi identities

$$(\mathbf{g}^{\alpha\beta} G_{\lambda\beta} + \mathbf{g}^{\beta\alpha} G_{\beta\lambda})_{,\alpha} + g^{\alpha\beta}{}_{,\lambda} \mathbf{G}_{\alpha\beta} = 0, \tag{5.1}$$

where $G_{\lambda\beta} = U_{\lambda\beta} - K_{\lambda\beta} - \frac{1}{2} g_{\lambda\beta}(U-K)$. On the other hand, from the coordinate invariance of the matter part of the Lagrangian density $\mathbf{L}_M$, in Eq. (2.4), and using the fact that the variation of a vector under the infinitesimal coordinate transformation $x^\alpha \to x'^\alpha = x^\alpha + \varepsilon^\alpha(x)$ is $\delta\Gamma_\alpha = -\varepsilon^\lambda{}_{,\alpha}\Gamma_\lambda - \Gamma_{\alpha,\lambda}\varepsilon^\lambda$, we get by direct calculation [13] the four conservation laws

$$(g^{\alpha\beta}\mathbf{T}_{\lambda\beta} + g^{\beta\alpha}\mathbf{T}_{\beta\lambda})_{,\alpha} + g^{\alpha\beta}{}_{,\lambda}\mathbf{T}_{\alpha\beta} - p\Gamma_{[\lambda,\beta]}\mathbf{J}^\beta = 0, \tag{5.2}$$

which can also be obtained through the use of the field equations in the Bianchi identities. In terms of the upper-indices stress tensor $T^{\mu\nu}$ we get, from Eqs. (2.8) and (2.2),

$$g_{\alpha\lambda}\mathbf{T}^{\alpha\beta}{}_{,\beta} + g_{\lambda\alpha}\mathbf{T}^{\beta\alpha}{}_{,\beta} + 2[\alpha\beta,\lambda]\mathbf{T}^{\alpha\beta} - p\Gamma_{[\lambda,\beta]}\mathbf{J}^\beta = 0, \tag{5.3}$$

where

$$[\alpha\beta,\lambda] = \tfrac{1}{2}(g_{\alpha\lambda,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda}). \tag{5.4}$$

Following the method of Papapetrou [14], one can now establish the equation of motion of test particles. We will quote here only the final result and give the details in Appendix B. There we show that by using Papapetrou's method there is no need to assume the full, dust-type, form of $T^{\alpha\beta}$ but only that it is symmetric, its form actually being determined by the method itself. On the other hand, if one does assume the dust-type form, the calculation is straightforward without any need of the method, as it has been done by Fock [15] for a point mass in the case of GR. The equation of motion of the particle with mass m and charge e is

$$\frac{du^\alpha}{d\tau} + C^\alpha_{\beta\gamma} u^\beta u^\gamma = \frac{ep}{2m} a^{\alpha\beta} K_{[\beta\gamma]} u^\gamma, \tag{5.5}$$

where $u^\alpha = dX^\alpha/d\tau$ is the velocity of the particle, $a^{\alpha\beta}$ is the inverse of $g_{(\alpha\sigma)}$ as defined by

$$a^{\alpha\beta} g_{(\sigma\beta)} = \delta^\alpha_\beta, \tag{5.6}$$

and

$$C^\alpha_{\beta\gamma} = \tfrac{1}{2} a^{\alpha\sigma}(g_{(\beta\sigma),\gamma} + g_{(\gamma\sigma),\beta} - g_{(\beta\gamma),\sigma}) \tag{5.7}$$

is the Christoffel symbol formed with the symmetric part of the metric, with $g_{(\alpha\beta)}$ referring to the background non-Riemannian field where the test particle moves. To first-order in an expansion about a Riemannian background with metric $g^{(0)}_{\alpha\beta}$, Eq. (5.5) becomes the Einstein- Lorentz equation of motion. This is so because in that limit we have, from Eqs. (3.17) and (3.19), $pK^{(1)}_{[\alpha\beta]} = 2\tilde{F}_{\alpha\beta}$ and $a^{\alpha\sigma} \to g^{(0)\alpha\sigma}$.

VI. CONCLUSIONS

We have developed a unified field theory of gravitation and electromagnetism with sources by introducing the matter energy-momentum tensor and the electromagnetic current explicitly into the Lagrangian. This is actually not in the spirit of Einstein's thoughts, because these are phenomenological quantities of non-gravitational character as by him emphasized, which are being put into a theory from which, in principle, everything should follow. However, we have done so at least in the actual stage of the theory, in order to get in contact with the field theoretical concepts of particle physics, when going to the linearization of the theory, and in order to get an explicit form for the conservation laws. We could then study the particle content of the theory, showing that no unphysical particles appear: the theory is shown to be free of ghost-negative energy particles and tachyons.

The equation of motion of a test charged particle has been established through the invariance of the interaction Lagrangian. It is found that the deviation of the geodesic path of the non-Riemannian space is due solely to the gravito-electromagnetic contribution coming from the Bonnor term. In the first order of approximation about a Riemannian space the equation goes into the Einstein-Lorentz equation of motion.

In a forthcoming paper we shall study the solution of the field equations for a pointlike charged source with a spherically symmetric field. Once we have this solution we shall be able to get the explicit form of the equation of motion of a test charged particle and thus determine the deviation of Coulomb's law in the new theory. This deviation should depend on the universal parameter p and could probably be used to determine p to some degree.

APPENDIX A: THE PROPAGATOR FROM SEMI-CLASSICAL FIELD THEORY

To illustrate the method let us consider electrodynamics. The Lagrangian density is

$$L = -\frac{1}{16\pi} F_{\mu\nu} F_{\mu\nu} + L_{int}, \tag{A1}$$

with the same indices convention of Sec. IV, where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ and

$$L_{int} = -J_\mu A_\mu \tag{A2}$$

is the interaction Lagrangian density. The Euler-Lagrange equation gives $F_{\mu\nu,\nu}=-4\pi J_\mu$, which leads to

$$\Box A_\mu = 4\pi J_\mu \tag{A3}$$

for the vector potential satisfying $A_{\nu,\nu}=0$. Equation (A3) can be solved by the usual Green's function method. One writes

$$A_\mu(x)=\int D(x,x')J_\mu(x')d^4x', \tag{A4}$$

where the Green's function D, the propagator, satisfies

$$\Box D(x,x')=4\pi\delta(x-x') \tag{A5}$$

or, in its integral form,

$$D(x,x')=-\frac{4\pi}{(2\pi)^4}\int \frac{1}{q^2}e^{-iq(x-x')}d^4q, \tag{A6}$$

as it can easily be checked by applying $\Box$ to it. (It is to be understood the Feynman definition of the poles, $q^2\to q^2+i\varepsilon$, when going to the semi-classical arguments.) Substitution of Eq. (A4) into Eq. (A2) gives, for the interaction Lagrangian itself,

$$\int L_{int}d^3x=\int\int J_\mu(x)D(x,x')J_\mu(x')d^4x'\,d^3x, \tag{A7}$$

showing the current-current interaction structure mediated by the propagator. Following Möller [16] let us see how one can get the scattering amplitude of two charged particles, 1 and 2, in lowest order of perturbation theory by using field theoretical semi-classical arguments. If p_1 and p_2 (p_1' and p_2') are the momenta of the initial (final) particles we associate each current to each particle and write

$$J_\mu(x)=J_{1\mu}(0)e^{i(p_1'-p_1)x} \tag{A8}$$

as the $p_1\rightharpoonup p_1'$ transition current for particle 1 and

$$J_\mu(x')=J_{2\mu}(0)e^{i(p_2'-p_2)x'} \tag{A9}$$

the $p_2\rightharpoonup p_2'$ one for particle 2. The signs in the exponentials are chosen to give the right energy-momentum conservation for the process, with the exponentials proportional to $\exp(-iEt)$ for destruction of a particle and $\exp(iEt)$ for creation, as in usual quantum mechanics for the transitions between two energy levels (we are using units with $\hbar=1$). With these prescriptions the probability for the transition $(p_1,p_2)\to(p_1',p_2')$, or scattering amplitude, is [16]

$$S=i\int L_{int}(x)d^4x. \tag{A10}$$

Using Eqs. (A6), (A8), and (A9) in Eq. (A7) we get

$$S=i(2\pi)^4\delta(p_1'+p_2'-p_1-p_2)\Pi \tag{A11}$$

where, with the indication

$$k=p_1'-p_1=p_2-p_2', \tag{A12}$$

$$\Pi=J_{1\mu}(0)\frac{4\pi}{k^2}J_{2\mu}(0) \tag{A13}$$

is the saturated propagator of quantum electrodynamics to lowest order. The propagator itself is $4\pi k^{-2}$, in momentum space. To come back to the whole discussion, what we are saying in a short way is that we are taking Eq. (A2) at point x with $J_\mu(x)$ as the transition current for particle 1 and with $A_\mu(x)$ representing the potential due to particle 2, that is,

$$L_{int}=-J_{1\mu}(0)e^{i(p_1'-p_1)x}A_{2\mu}(x), \tag{A14}$$

where, from Eqs. (A4), (A6), and (A9)

$$A_{2\mu}(x)=-\frac{4\pi}{k^2}J_{2\mu}(0)e^{-ikx}, \tag{A15}$$

with $k=p_2-p_2'$. Notice that, with Eq. (A9), this result also follows directly from Eq. (A3) calculated at point x' [that is $\Box' A_\mu(x')=4\pi J_\mu(x')$], after taking the final answer at point x. Substituting Eq. (A15) into the previous one leads to Eq. (A11) with Π given by Eq. (A13).

Now, all this can dramatically be abbreviated if we work directly in momentum (k) space. Just notice that in momentum space the interaction is, from Eq. (A2), $-J_{1\mu}(0)A_{2\mu}(0)$ while Eq. (A3) gives $-k^2A_{2\mu}(0)=4\pi J_{2\mu}(0)$. By eliminating $A_{2\mu}(0)$, one is immediately led to Eq. (A13). If, for instance, we have Dirac particles one writes the currents in terms of spinors depending on the momenta and the calculation to obtain the cross-section goes on.

Consider now our Lagrangian in Eq. (4.6). The Euler-Lagrange equations for $h_{[\alpha\beta]}$ and Γ_α are, respectively,

$$\frac{1}{2}(\Gamma_{\alpha,\beta}-\Gamma_{\beta,\alpha})=\frac{2}{p^2}h_{[\alpha\beta]}+8\pi T_{[\alpha\beta]} \tag{A16}$$

and

$$h_{[\alpha\beta],\beta}=-4\pi pJ_\alpha, \tag{A17}$$

which are, respectively, Eqs. (3.5) and (3.8) to lowest order. Contracting the first one of these equations with ∂_β and using the second one we obtain the equation for Γ_α, using $\Gamma_{\beta,\beta}=0$,

$$\Box\Gamma_\alpha=-16\pi\left(\frac{1}{p}J_\alpha-T_{[\alpha\beta],\beta}\right). \tag{A18}$$

Solving this we can obtain $h_{[\alpha\beta]}$ from Eq. (A16). Lets go now to the semi-classical arguments as illustrated in the electromagnetic case to get the propagator, working directly in momentum space. The interaction Lagrangian for the scattering of particles 1 and 2 is, from Eq. (4.6) with sources due to particle 1,

$$L_{int}=\tfrac{1}{4}\,pJ_{1\alpha}(0)\Gamma_{2\alpha}(0)+\tfrac{1}{2}\,T_{1[\alpha\beta]}(0)h_{2[\alpha\beta]}(0). \tag{A19}$$

The solution of Eq. (A18) is, for the torsion due to particle 2,

$$\Gamma_{2\alpha}=\frac{16\pi}{k^2}\left(\frac{1}{p}J_{2\alpha}+ik_\beta T_{2[\alpha\beta]}\right)(0)e^{-ikx}. \tag{A20}$$

Using this result in Eq. (A16), the field due to particle 2 is then

$$h_{2[\alpha\beta]}(0)=-4\pi p^2\left\{T_{2[\alpha\beta]}+\frac{i}{k^2}\times\left[k_\beta\left(\frac{1}{p}J_{2\alpha}+ik_\gamma T_{2[\alpha\gamma]}\right)-\alpha\leftrightarrow\beta\right)\right]. \tag{A21}$$

Substituting Eqs. (A20) and (A21) into Eq. (A19) and using Eq. (A10) yields the saturated propagator

$$\Pi=\frac{4\pi}{k^2}J_{1\alpha}J_{2\alpha}-\frac{4\pi ip}{k^2}(k_\lambda T_{1[\alpha\lambda]}J_{2\alpha}-J_{1\alpha}k_\lambda T_{2[\alpha\lambda]}) -2\pi p^2\left(T_{1[\alpha\beta]}-2\frac{k_\beta k_\gamma}{k^2}T_{1[\alpha\gamma]}\right)T_{2[\alpha\beta]}, \tag{A22}$$

with the zeros being implied, which reproduces Eq. (4.23). In the case of the charged π-meson scattering process discussed in Sec. IV, the $p_a\rightharpoonup p'_a$ transition current for particle $a=1,2$ is, from Eq. (4.29) in momentum space,

$$J_{a\alpha}(0)=e(p'_{a\alpha}+p_{a\alpha})\phi'_a(0)\phi_a(0), \tag{A23}$$

and the corresponding transition stress is

$$T_{a[\alpha\beta]}=i(p'_{a\alpha}p_{a\beta}-\alpha\leftrightarrow\beta)\phi'_a(0)\phi_a(0), \tag{A24}$$

where $\phi_a(0)=(2\pi)^{-3/2}(2E_a)^{-1/2}$ with $E_a=(p_a^2+m^2)^{1/2}$ [16]. These are the quantities to be substituted on the right of Eq. (4.23) on the left-hand side source terms for $a=1$ and right-hand side source terms for $a=2$, for the π-scattering process at hand.

APPENDIX B: EQUATION OF MOTION OF TEST CHARGED PARTICLES

We shall describe the motion of a charged test particle moving outside massive bodies. Then the energy-momentum tensor and the current of the massive bodies vanish at the position of the test particle and near it. Therefore, $T^{\alpha\beta}$ and J^α in Eq. (5.3) reduce to those of the particle. Also, this being a test particle we shall neglect its contribution to the metric and torsion vector. Therefore $g_{\alpha\beta}$ and Γ_α in Eqs. (3.8) and (5.3) refer only to the background field produced by the massive bodies.

Following the moment method of Papapetrou [14], we shall derive the equation of the charged test particle from Eqs. (3.8) and (5.3). We then consider an extended, small, system with reference point X^α with velocity $u^\alpha=dX^\alpha/d\tau$ and we shall take moments of $T^{\alpha\beta}$ and J^α around X^α. By demanding that the dimensions of the system tend to zero at the very end of the calculation, this point will give us the world line of our pointlike charged particle. For such a simple system, we shall assume that $T^{\alpha\beta}$ is symmetric without fixing its form however. The method will give it to us. Then the first two terms of Eq. (5.3) can be written $2g_{(\alpha\lambda)}\mathbf{T}^{\alpha\beta}{}_{,\beta}$. After contraction of the equation with the inverse $a^{\sigma\lambda}$ to $g_{(\sigma\lambda)}$, as defined in Eq. (5.6), we obtain

$$\mathbf{T}^{\sigma\beta}{}_{,\beta}+C^\sigma_{\alpha\beta}\mathbf{T}^{\alpha\beta}-\tfrac{1}{2}\,pa^{\sigma\lambda}\Gamma_{[\lambda,\beta]}\mathbf{J}^\beta=0, \tag{B1}$$

where, because of the symmetry of $\mathbf{T}^{\alpha\beta}$,

$$C^\sigma_{\alpha\beta}=\tfrac{1}{2}\,a^{\sigma\lambda}(g_{(\alpha\lambda),\beta}+g_{(\beta\lambda),\alpha}-g_{(\alpha\beta),\lambda}) \tag{B2}$$

is the Christoffel symbol formed with the symmetric part of the background metric only. We shall need also the relation

$$(x^\alpha\mathbf{T}^{\sigma\beta})_{,\beta}=\mathbf{T}^{\sigma\alpha}+x^\alpha\mathbf{T}^{\sigma\beta}{}_{,\beta}. \tag{B3}$$

Next we write $x^\alpha=X^\alpha+\delta x^\alpha$ and neglect first-order moments of $\mathbf{T}^{\alpha\beta}$. We then integrate both Eqs. (B1) and (B3) over the three dimensional space for constant t. Space divergences integrate to zero so that we get, from Eq. (B1),

$$\frac{d}{dt}\int\mathbf{T}^{\sigma 0}\,d^3x+C^\sigma_{\alpha\beta}(X)\int\mathbf{T}^{\alpha\beta}\,d^3x-\frac{1}{2}p(a^{\sigma\lambda}\Gamma_{[\lambda,\beta]})(X)\times\int\mathbf{J}^\beta\,d^3x=0, \tag{B4}$$

where we have already taken the Christoffel symbol and the Γ term at the reference point $X^\alpha(\tau)$. From Eq. (B3) we obtain

$$\frac{d}{dt}\left(X^\alpha\int\mathbf{T}^{\sigma 0}\,d^3x\right)=\int\mathbf{T}^{\sigma\alpha}\,d^3x+X^\alpha\frac{d}{dt}\int\mathbf{T}^{\sigma 0}\,d^3x, \tag{B5}$$

where, again, x^α inside the integrands have been put equal to X^α. From here we get

$$v^\alpha\int\mathbf{T}^{\sigma 0}\,d^3x=\int\mathbf{T}^{\sigma\alpha}\,d^3x, \tag{B6}$$

where, $v^\alpha=dX^\alpha/dt$. Putting $\sigma=0$ in this equation we get

$$v^\alpha\int\mathbf{T}^{00}\,d^3x=\int\mathbf{T}^{0\alpha}\,d^3x. \tag{B7}$$

Therefore, due to the symmetry of $\mathbf{T}^{\sigma\alpha}$, Eq. (B6) can be written

$$\int\mathbf{T}^{\sigma\alpha}\,d^3x=v^\alpha v^\sigma\int\mathbf{T}^{00}\,d^3x. \tag{B8}$$

Both these last two relations are to be substituted in Eq. (B4). Before we do so, we shall prove the relation

$$\int \mathbf{J}^{\alpha} d^3x = e v^{\alpha}, \tag{B9}$$

where e is the charge of the particle. For that purpose we first integrate the equation of continuity $\mathbf{J}^{\alpha}{}_{,\alpha}=0$, which follows from Eq. (3.8), and then the relation $(x^{\alpha}\mathbf{J}^{\beta})_{,\beta}=\mathbf{J}^{\alpha}$. From the first integration, we obtain $de/dt=0$, where $e=\int \mathbf{J}^0\, d^3x$ is the charge of the particle, which is then a constant. Then by integrating the second one and putting $x^{\alpha}=X^{\alpha}$ inside the integrand we are led to Eq. (B9). Now we substitute Eqs. (B8) and (B9) into Eq. (B4), multiply the resulting equation by $u^0=dt/d\tau$ and notice that $u^0 v^{\sigma}=dX^{\sigma}/d\tau=u^{\sigma}$. After using the relation $\Gamma_{[\alpha,\beta]}=K_{[\alpha\beta]}$, which follows from Eq. (3.5) outside the massive bodies where the particle moves, we get

$$\frac{d}{d\tau}(mu^{\sigma})+mC^{\sigma}_{\alpha\beta}u^{\alpha}u^{\beta}-\frac{1}{2}epa^{\sigma\lambda}K_{[\lambda\beta]}u^{\beta}=0, \tag{B10}$$

where we have used the indication

$$m=\frac{1}{u^0}\int \mathbf{T}^{00}\, d^3x. \tag{B11}$$

Contracting Eq. (B10) with $g_{(\alpha\sigma)}u^{\alpha}$ and using $g_{(\alpha\sigma)}u^{\alpha}u^{\sigma}=g_{\alpha\sigma}u^{\alpha}u^{\sigma}=1$ together with Eqs. (B2) and (5.6), we get $dm/d\tau=0$. We then conclude that m is a constant, which we identify with the rest mass of the particle. Then going back to Eq. (B10) we finally get

$$\frac{du^{\sigma}}{d\tau}+C^{\sigma}_{\alpha\beta}u^{\alpha}u^{\beta}=\frac{ep}{2m}a^{\sigma\lambda}K_{[\lambda\beta]}u^{\beta}, \tag{B12}$$

which is the equation of motion, in Eq. (5.5). Notice now that Eq. (B8) can be written

$$u^o\int \mathbf{T}^{\alpha\sigma}\, d^3x = mu^{\alpha}u^{\sigma}. \tag{B13}$$

Thence, calling ρ the rest state mass density of the system, defined by

$$m=u^o\int \rho\sqrt{-g}\, d^3x, \tag{B14}$$

we can write

$$T^{\alpha\sigma}=\rho u^{\alpha}u^{\sigma}. \tag{B15}$$

This gives, in fact, the form (dust-like) of the energy-momentum tensor of the system.

We show now that if we assume Eq. (B15) together with the corresponding relation $J^{\alpha}=\rho_e u^{\alpha}$ for the current, where ρ_e is the rest state charge density, the calculation is straightforward, as it has been shown by Fock [15] for a point mass in the case of GR. Taking these results into Eq. (B1) and using again the relation $\Gamma_{[\alpha,\beta]}=K_{[\alpha\beta]}$, outside the massive bodies, we get

$$(\rho u^{\sigma}u^{\beta})_{,\beta}+C^{\sigma}_{\alpha\beta}\rho u^{\alpha}u^{\beta}-\tfrac{1}{2}pa^{\sigma\lambda}K_{[\lambda\beta]}\rho_e u^{\beta}=0. \tag{B16}$$

By making use of the equation of continuity for the matter part, $(\rho u^{\beta})_{,\beta}=0$, the first term becomes $\rho u^{\beta}u^{\sigma}{}_{,\beta}=\rho\, du^{\alpha}/d\tau$. For a point particle at $\mathbf{X}(t)$ one has $\rho(\mathbf{x})=m\delta[\mathbf{x}-\mathbf{X}(t)]$ and $\rho_e(\mathbf{x})=e\delta[\mathbf{x}-\mathbf{X}(t)]$, and upon integration the equation of motion, Eq. (B12), follows.

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