New family of inhomogeneous γ-law cosmologies: Example of gravitational waves in a homogeneous $p=\varrho/3$ background

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(Received 17 October 2000; published 16 March 2001)

We present an explicit three-parameter class of $p=\gamma\varrho$ $(-\frac{1}{3}\leq\gamma<1)$ cosmological models admitting a two-dimensional group G_2 of isometries acting on spacelike surfaces. The family is self-similar in the sense that it has a further homothetic vector field and it contains subfamilies of both (previously unknown) tilted and nontilted Bianchi models with that equation of state. This is the first algebraically general class of solutions of this kind including dust inhomogeneous solutions. The whole class presents a universal spacelike big-bang singularity in the finite past. More interestingly, the case $p=\varrho/3$ constitutes a new two-parameter inhomogeneous subfamily which can be viewed as a Bianchi type V background with a gravitational wave traveling orthogonally to the surfaces of transitivity of the G_2 group. This wave generates the inhomogeneity of the spacetime and is related to the sound waves tilting the perfect fluid. It seems to be the first explicit exact example of a gravitational wave traveling along a homogeneous background that has a realistic equation of state $p=\varrho/3$.

DOI: 10.1103/PhysRevD.63.084008 PACS number(s): 04.20.Jb, 04.30.−w, 04.40.Nr

I. INTRODUCTION

There is no need to mention that among the various methods used for the study of spatially inhomogeneous cosmological models, the research on exact solutions of Einstein field equations plays a crucial role. Because of the high nonlinearity of the equations, the exact solutions are necessary for the understanding of particular qualitative features that may constitute a guide in the study of general situations. Indeed, this has been the way in which many new kind of unexpected behaviors have been found. Of course, the exact solutions properties must be related and compared with the results obtained from other methods, such a combination usually leads to very powerful and general conclusions. For instance, when using the dynamical systems techniques in cosmology [1], some special exact solutions are shown to be asymptotic states of general classes of models. Exact solutions can also be compared with approximations or perturbations to check the validity of the involved expansions [2]. Yet another example could be the study of the structure and appearance of singularities, which complements and sheds some light onto the singularity theorems and their conclusions, see Ref. [3], and references therein.

The research on exact solutions is based on some physically reasonable restrictions used to simplify the Einstein equations. As an outstanding example, and with regard to geometrical properties, the existence of symmetries described by n-dimensional groups of motions G_n (see Ref. [2], and references therein) was the first assumption treated in a systematic way, see, e.g., [4,5,2,6], leading to classifications of solutions as well as to fruitful techniques for their finding. In this sense, and with the study of spatially inhomogeneous cosmologies in mind, an important and particularly fruitful line of research during the last two decades has been the consideration of the class of spacetimes admitting a maximal two-dimensional group of isometries G_2 acting on spacelike surfaces. This line was somehow launched in Ref. [7] with a classification scheme for the particular Abelian case of these so-called "G_2 spacetimes" based solely on the properties and relations of the Killing vector fields. The classification was generalized for the non-Abelian case in Ref. [8], see Ref. [9] for a complete review. Among the classes defined in the Abelian case, the most simple subcase arises when there exists a family of surfaces orthogonal to the orbits of the group (it is then said that the group acts orthogonally transitively) and the two Killing vectors are mutually orthogonal, which implies that they are in fact hypersurface orthogonal, so that the metric can be cast in diagonal form in coordinates adapted to the Killing vectors. Focusing the attention on these diagonal G_2 spacetimes, some additional assumptions have been made in order to simplify the field equations for a perfect fluid source, as, for example, the existence of additional homothetic or proper conformal symmetries, see Refs. [10,11], and references therein. Let us recall here that there have also been general studies on orthogonally transitive G_2 cosmologies from a qualitative point of view, analyzing the autonomous system of first-order partial differential equations derivable from the Einstein field equations by using methods from the theory of dynamical systems [12,13,1]. The relations between some of the known explicit solutions and these theoretical studies were widely analyzed in Ref. [1], and many references therein.

Another important simplifying assumption for the perfect-

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fluid diagonal G_2 spacetimes, which has received systematic attention, corresponds to the separability of the metric functions in coordinates that keep the diagonal form of the metric, so called canonical coordinates. The case when these canonical coordinates that bring the metric functions to a separate form are also adapted to the velocity vector of the fluid, that is to say, they are comoving coordinates too, was exhausted in Refs. [14,13,15], except for a very particular case identified in Ref. [9] that did not appear in Ref. [13] accidentally. The general treatment of the separability in noncomoving canonical coordinates can be found in Refs. [16,9], where a classification for separable diagonal G_2 on S_2 perfect-fluid solutions was obtained depending on the number of linearly independent functions appearing in the metric, leading to a systematic procedure for the obtaining of solutions. The classification was exhausted, but not wholly solved, because once the machinery for the systematic derivation of solutions was established, the main effort was focused on finding solutions with special interest or physical relevance in order to study them in detail.

Thus, for instance, an interesting solution was singled out in Ref. [9] (named 22BIIc) because of its γ-law equation of state which includes the relevant cases $\gamma=0$ (dust models) and $\gamma=\frac{1}{3}$ (models for relativistic radiation). It is also interesting because it provides inhomogeneous generalizations of some Bianchi III, V, and VI_h models found in Refs. [17,18], see also Ref. [1]. The particular dust solutions belonging to this family are actually included in one of the two classes of dust spacetimes studied in Ref. [19]. The solutions with nonzero γ, including those with $\gamma=\frac{1}{3}$, are new, though. In fact, the number of exact solutions for inhomogeneous spacetimes with a $p=\varrho/3$ equation of state is rather scarce: as far as we know the first one appeared in the Wainwright-Goode family [14] to which followed the Feinstein-Senovilla solution [20], Davidson's [21], the singularity-free metric of Ref. [22], their common generalization in the Ruiz-Senovilla class [13], and the nondiagonal $p=\gamma\varrho$ family found by Mars and Senovilla [23,24].

The aim of this paper is to present the explicit family of solutions mentioned in the previous paragraph, as well as to perform an extensive detailed geometrical and physical study of its main features. The solutions constitute a three-parameter class of $p=\gamma\varrho$, $-\frac{1}{3}\leqslant\gamma<1$, cosmological models admitting a maximal G_2 acting on spacelike surfaces. The whole family is self-similar in the sense that it has a further homothetic vector field and it contains subfamilies of both tilted and nontilted Bianchi models. This is the first inhomogeneous family with a γ-law equation of state having a free γ which includes the $\gamma=0$ case, something which may be very useful in order to study perturbations of the dust case.

The structure of the paper is as follows. In Sec. II we introduce the line element for the whole family in noncomoving canonical coordinates and show the ranges of the free parameters and the perfect-fluid variables, which are in turn expressed in terms of its velocity potential (Secs. II A and II B). The kinematical quantities of the fluid flow and their properties, as well as the deceleration parameter and the Weyl tensor are given in Secs. II C and II D. Then, in Sec. II E, we study the general symmetries of the spacetimes including the analysis of the special cases that arise, which include previously known exact ''nontilted'' Bianchi spacetimes together with some other new ''tilted'' ones. Next, comoving canonical coordinates are introduced in Sec. II F, on the one hand to show that the metric is nonseparable in comoving coordinates in general, and on the other hand, to construct a half-null coordinate system that will be used to make manifest the singularity structure and its type in Sec. II G. The result is that the whole class presents a universal spacelike big-bang singularity in the finite past, which turns out to be of Kasner type [25,26]. Similarly, the future asymptotic behavior of the solutions is shown in Sec. II H.

In Sec. III we present the most interesting particular subfamilies and limits of the general spacetime. In particular, two vacuum limits in the half-null coordinates are found in Sec. III A, providing two two-parameter families of pure gravitational *pp*-wave solutions. The $p=\varrho/3$ subfamily is studied then in Sec. III B, and it is given the interpretation of a Bianchi V background inhomogenized by means of a pure gravitational wave traveling along the direction orthogonal to the surfaces of transitivity of the G_2 group. This gravitational wave is closely related to some acoustic waves which travel along and tilt the perfect fluid. This result is new in the sense that all previous works concerning propagation of waves in curved backgrounds were developed in the case of vacuum or massless minimally coupled scalar fields without potential (the latter includes the stiff fluid $p=\varrho$ case, see Ref. [27] for a pioneering treatment of the subject), see, e.g., Refs. [28–31], and references therein, for a selection of main results, or in an anisotropic generalization of the stiff fluid in which the energy density equals the pressure on the direction of propagation of the waves [32], and finally, in the case of electromagnetic fields, see Ref. [33]. There have also been other works on solutions describing exact solitonic perturbations of γ-law perfect fluid backgrounds [34,35,30], but the formalism consists in translating the solutions of the Einstein field equations into equivalent five-dimensional massless scalar field spacetimes, and thus the backgrounds are severely restricted by some conditions on the matter content so that eventually only Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes were used. Moreover, these perturbations give rise to anisotropies in the energy-momentum tensor. In our case, the gravitational wave is exact and travels on a spatially homogeneous but anisotropic background, and both the background and the resulting inhomogeneous spacetime satisfy the same realistic equation of state $p=\varrho/3$. This is the first known example of such a situation.

Finally, the dust subfamily is identified within the general classes found in Ref. [19] in sub-Sec. III C. Throughout the paper we follow the following conventions and notations. The metric has signature $+2$. $\mathcal{L}_{\vec{v}}$ denotes the Lie derivative with respect to the vector field $\vec{v}$. Primes and dots will stand for derivatives with respect to x and t, respectively. Greek indices run from 0 to 3. We take units with $8\pi G=c=1$.

II. THE MODELS

This section is devoted to presenting, in as compact a way as possible, the new family of spacetimes and their main geometrical and physical properties.

A. The line element

The line element was derived using separability of the metric functions in noncomoving coordinates [16] [more precisely, it arises as a particular case of the $m=n=2(22\mathrm{BII}c)$ class as defined in Ref. [9]] and it can be written as follows:

$$ds^2=F^2(t,x)(-dt^2+dx^2)+e^{\mu[a/(a+b)](t-x)}(1-e^{-\mu t})\times\{[e^{\mu[b/(a+b)](t-x)}(1-e^{-\mu t})]^{2l}dy^2+[e^{\mu[b/(a+b)](t-x)}\times(1-e^{-\mu t})]^{-2l}dz^2\}, \qquad (1)$$

with

$$F(t,x)\equiv\exp\left[\frac{\mu}{a+b}(td-cx)\right](1-e^{-\mu t})^{\lambda},$$

and where we have defined the following constants:

$$a\equiv(4\lambda+1)(\nu^2-1)(\nu-1),$$

$$b\equiv(6\lambda\nu-2\lambda+\nu+1)(\nu^2-1),$$

$$c\equiv\nu(2\lambda+1)[\nu^2+1+2(2\nu-1)(\nu+1)\lambda],$$

$$d\equiv\nu^2+1+2(\nu+3)\lambda\nu^2+2(5\nu^2-4\nu+1)(\nu+1)\lambda^2,$$

with λ being

$$\lambda\equiv l^2-1/4$$

an auxiliary constant that will be used for the sake of simplicity. Actually, the constants satisfy the relation $c=\nu[d-\lambda(a+b)]\equiv\nu\hat{c}$ that allows to cast the function $F(t,x)$ in the alternative and possibly more convenient form given by

$$F(t,x)=\exp\left[\frac{\mu\hat{c}}{a+b}(t-\nu x)\right](e^{\mu t}-1)^{\lambda}.$$

Nevertheless, and for the sake of simplicity in some expressions, we prefer to keep the four constants a, b, c, d, and F as given previously.

The family of solutions has then three free parameters l (or λ), ν, and μ, although the latter simply provides the coordinate scaling. We obviously have $\lambda\geqslant-1/4$, and we can choose $\mu>0$ without loss of generality (see below). Furthermore, we must have $a+b\neq0$, which eventually will be equivalent to

$$(5\nu-3)\lambda+\nu\neq0.$$

B. The perfect fluid

The line element (1) is a solution of Einstein's field equations for a perfect-fluid energy-momentum tensor $T_{\alpha\beta}=(\varrho+p)u_\alpha u_\beta+pg_{\alpha\beta}(\varrho+p\neq0)$ whenever ν is restricted by

$$1-\nu^2>0, \qquad (2)$$

which in turn implies the last condition of the previous subsection. The unit velocity vector field $\vec{u}$ then reads

$$\vec{u}=\frac{1}{F\sqrt{1-\nu^2}}\left[\frac{\partial}{\partial t}+\nu\frac{\partial}{\partial x}\right].$$

The ranges for ν and λ immediately imply $a>0$. One can also deduce that $d>0$ as follows: we have $5\nu^2-4\nu+1>0,\forall\nu$, and thus $d>d|_{\lambda=-1/4}=(3-\nu)(3-\nu^2)/8>0$.

The energy density is given by

$$\varrho=F^{-2}\left(\frac{a-b}{a+b}\right)\frac{\mu^2(\lambda+1)}{(e^{\mu t}-1)},$$

and the equation of state is barotropic and obeys the gamma law

$$p=\gamma\varrho,$$

where γ is given explicitly in terms of λ by

$$\gamma=\frac{\lambda}{\lambda+1}=\frac{l^2-1/4}{l^2+3/4},$$

so that we have $-1/3\leqslant\gamma<1$. Notice that in the $\varrho+3p=0$ case $\lambda=-1/4\Leftrightarrow l=0$ and the solutions admit a plane G_3 on S_2.

From the above expressions is clear that the solutions have an initial big-bang singularity at $t=0$, and this is why we have taken $\mu>0$ without loss of generality. Section II G is devoted to studying the singularity structure of the family, and in particular it will be shown that the big-bang singularity at $t=0$ is the only one for the whole family.

The perfect-fluid region covers the entire manifold, and we have $\varrho>0$ everywhere whenever $a^2-b^2>0$, which is equivalent to

$$(5\nu-3)\lambda+\nu<0, \qquad (3)$$

where we have taken into account that $(a-b)/[2(1-\nu^2)]=(\nu+1)\lambda+1>0$ (so that $\varrho\neq0$) which follows from the ranges for ν and λ. In fact, the previous condition (3) implies $a+b=-2(1-\nu^2)[(5\nu-3)\lambda+\nu]>0$, and this ensures the fulfillment of both the dominant and the strong energy conditions on the whole spacetime.

Since the fluid flow is irrotational it can be expressed as the normalized gradient of a scalar field, the so-called velocity potential σ [26], that is,

$$u_\alpha=\sigma_{,\alpha}/\sqrt{-\sigma_{,\beta}\sigma^{,\beta}},$$

where the commas stand for the partial derivative. Because of the gamma law equation of state, the corresponding energy density reads $\varrho=(-\sigma_{,\alpha}\sigma^{,\alpha})^{(\gamma+1)/2\gamma}$, and the velocity potential satisfies a homogeneous wave equation (nonlinear whenever $p\neq\varrho$) which gives the sound wave equation once it is linearized [26]. The velocity potential for the whole family is given by

$$\sigma=-\left(\mu^2(\lambda+1)\frac{a-b}{a+b}\right)^{\lambda/(2\lambda+1)}$$
$$\times\frac{(2\lambda+1)(a+b)}{\mu\hat{c}\sqrt{1-\nu^2}}\exp\left[\frac{\mu\hat{c}}{(2\lambda+1)(a+b)}(t-\nu x)\right]$$

apart from an additive constant, so that the parameter ν is nothing but the peculiar spatial fluid velocity $-\sigma'/\dot{\sigma}$. The range given in Eq. (2) for the perfect fluid is consistent with this interpretation, so that given any γ the corresponding subfamily contains all the possible values the ratio $\sigma'/\dot{\sigma}$ can achieve. Indeed, since this ratio is constant, this family of solutions does not follow an asymptotically velocity-dominated regime near the initial singularity except for the cases $\nu\rightarrow 0$, which could be seen as perturbations of homogeneous models (see Sec. II E).

C. The kinematical quantities

In order to compute the kinematical (and other) quantities for the fluid congruence defined by $\vec{u}$, let us take the orthonormal tetrad $\{\boldsymbol{\theta}^\alpha\}$ with $\boldsymbol{\theta}^\alpha\propto dx^\alpha$ in the above coordinate system $\{x^\alpha\}=\{t,x,y,z\}$. Of course, the vorticity of the fluid congruence vanishes identically. Regarding its acceleration, its nonvanishing components are

$$a_0=-\nu a_1,\quad a_1=F^{-1}\mu\lambda\frac{\nu}{1-\nu^2}\frac{1}{(1-e^{-\mu t})},\qquad(4)$$

so that the fluid flow does not follow geodesic trajectories at any point of the spacetime, except for the special cases $\nu=0$ or $\lambda=0$, in which the acceleration vanishes everywhere. These special cases will be discussed later in Sec. II E.

With respect to the expansion and the nonzero components of the shear tensor we have

$$\theta=F^{-1}\frac{\mu}{\sqrt{1-\nu^2}(a+b)(e^{\mu t}-1)}\times[\alpha^2(e^{\mu t}-1)+(a+b)(\lambda+1)],$$

$$\sigma_{00}=\nu^2\sigma_{11},\quad \sigma_{01}=-\nu\sigma_{11},$$

$$\sigma_{11}=-F^{-1}\frac{\mu[a\nu+b-e^{\mu t}a(\nu-1)]}{(1-\nu^2)^{3/2}(a+b)(e^{\mu t}-1)}+\frac{2}{3(1-\nu^2)}\theta,$$

$$\sigma_{22}+\sigma_{33}=F^{-1}\frac{\mu}{3\sqrt{1-\nu^2}(a+b)(e^{\mu t}-1)}\times\{[-3(d-\nu c)-\alpha^2]\times(e^{\mu t}-1)-(a+b)(2\lambda-1)\},$$

$$\sigma_{22}-\sigma_{33}=F^{-1}\frac{2\mu l[a+\nu b-e^{\mu t}b(\nu-1)]}{\sqrt{1-\nu^2}(a+b)(e^{\mu t}-1)},\qquad(5)$$

where we have defined $\alpha^2\equiv d-\nu c+a(1-\nu)$, which is indeed a positive constant for the given ranges of ν and λ. This can be easily deduced from its explicit expression

$$\alpha^2=2(1-\nu^2)(1+\lambda)[(4\nu^2-3\nu+1)\lambda+(\nu^2)+(1-\nu)],$$

as every term between round brackets is strictly positive because of Eq. (2) (in fact $4\nu^2-3\nu+1>0,\forall\nu$), and so the less favorable case would correspond to $\lambda=-1/4$, which gives a positive value for the term in square brackets for the valid range of ν.

Therefore, from expression (5) we see that the fluid congruence is expanding everywhere, $\theta>0$, starting with an unbounded value at the initial singularity $t=0$ and decreasing continuously from then on arriving eventually to zero as t tends to infinity. A straightforward calculation also shows that

$$\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{2\theta^2}(t\rightarrow\infty)\rightarrow\frac{2(1-\nu^2)^2}{3\alpha^2}\left[\frac{1}{4}[a(1-\nu)-2(d-\nu c)]^2+3(1-\nu)^2b^2l^2\right],$$

so that the only case in which the solutions isotropize in the future is given by $\lambda=\frac{1}{2}$ and $\nu=0$ ($\sigma_{11}=0,\sigma_{22}=-\sigma_{33}\neq 0$), which is a ''comoving'' family with an additional symmetry, as we will see later in Sec. II E.

Finally, we present the expression of the deceleration parameter q, whose general definition is

$$\vec{u}(\theta^{-1})\equiv\frac{1}{3}(1+q),$$

so that it reads

$$\frac{1}{3}(1+q)=\frac{\mu}{F\sqrt{1-\nu^2}\theta}\left[\frac{d-\nu c}{a+b}+\frac{e^{\mu t}+\lambda}{(e^{\mu t}-1)}-\frac{\alpha^2e^{\mu t}}{\alpha^2(e^{\mu t}-1)+(a+b)(\lambda+1)}\right],$$

from where it can be checked that at the singularity $q(t\rightarrow 0)=2$. It is interesting to remark that, as follows from the previous expression and Eq. (5), q is independent of x, despite the inhomogeneity of the solutions. It should be stressed that this result holds in the above coordinate system, which is not comoving, and it is a simple consequence of the separability of the metric components in these coordinates. In the comoving coordinates adapted to the fluid flow, which will be given in Sec. II F, the deceleration parameter certainly depends on the corresponding spatial variable.

D. The Weyl tensor

Concerning the Weyl tensor, the nonvanishing scalars computed in the null tetrad (see Ref. [2]) $\mathbf{k}=2^{-1/2}(\boldsymbol{\theta}^0-\boldsymbol{\theta}^1)$, $\boldsymbol{l}=2^{-1/2}(\boldsymbol{\theta}^0+\boldsymbol{\theta}^1)$, $\boldsymbol{m}=2^{-1/2}(\boldsymbol{\theta}^2+i\boldsymbol{\theta}^3)$, are given by

$$\Psi_0-\Psi_4=-\frac{\mu^2 l}{F^2(a+b)^2(e^{\mu t}-1)}\{[2(b\lambda+c)-a]a+[(2\lambda-1)b-2d]b+2e^{\mu t}(c+d-a)b\},$$

$$\Psi_2=\frac{\mu^2}{12(a+b)F^2(e^{\mu t}-1)^2}\{e^{\mu t}[(10\lambda+1)b+(2\lambda-1)a]+(a-b)(4\lambda+1)\},$$

$$\Psi_0=\frac{\mu^2 l}{2(a+b)F^2(e^{\mu t}-1)^2}\{(a+b)(2\lambda-1)+2(c-d)-e^{\mu t}[2(c-d)-b-a]\},$$

and therefore, the Weyl tensor does not vanish and there is no possible particularization to a conformally flat solution within the whole family. In particular, flat spacetime is not included in the family.

The Petrov type is I at generic points for the general case with a maximal G_2, and type D for the G_3 on S_2 case ($l=0$) as well as for the case $\nu=0$, $\lambda=2$, which admits two additional isometries, belonging then to the class of LRS models (see Sec. II E). The nonzero component of the magnetic part of the Weyl tensor with respect to $\vec{u}$ is given by

$$H_{23}(\vec{u})=\frac{1+\nu}{1-\nu}\Psi_0-\frac{1-\nu}{1+\nu}\Psi_4,$$

so that $H(\vec{u})$ only vanishes in the type D cases: $\nu=0$, $\lambda=2$ or $l=0$.

E. The symmetries: Special cases

The line element (1) admits in general a two-parameter group of isometries, generated by the two Killing vectors

$$\vec{\xi}=\frac{\partial}{\partial y},\quad \vec{\eta}=\frac{\partial}{\partial z},$$

which obviously commute, so that the G_2 is Abelian. Moreover, the metric admits a homothetic vector field given by

$$\vec{\zeta}=2\frac{\partial}{\partial x}+\frac{\mu}{a+b}(a+2bl-2c)y\frac{\partial}{\partial y}+\frac{\mu}{a+b}(a-2bl-2c)z\frac{\partial}{\partial z}, \tag{6}$$

which satisfies

$$\mathcal{L}_{\vec{\zeta}}g_{\alpha\beta}=-4c\mu g_{\alpha\beta}.$$

It follows that the general family of solutions with $c\neq0$ belongs to the class of so-called ''tilted'' inhomogeneous self-similar perfect-fluid models [10]: the velocity vector $\vec{u}$ is neither tangential nor orthogonal to the orbits of the three-dimensional homothetic group H_3 generated by $\{\vec{\xi},\vec{\eta},\vec{\zeta}\}$, which is acting on spacelike hypersurfaces (S_3). The algebraic structure of H_3 is described by the following Bianchi types: Bianchi VI$_h$ with $h=-[(a-2c)/2lb]^2$ (Bianchi III is indeed included when $h=-1$) whenever $lb\neq0$, and Bianchi V when $lb=0$. In this last possibility, by taking into account that the case $l=0$ already admits a G_3 on S_2 group of isometries, the resulting group is an H_4 acting on S_3 if $l=0$.

Coming to the possible particular cases with further symmetry we first find the already mentioned case with $l=0$ ($\lambda=-1/4$), so that the equation of state has the form $\varrho+3p=0$. This solution admits a G_3 group of isometries acting multiply-transitively on spacelike plane S_2-orbits, and the Petrov type is D.

When $l\neq0$, the only possible cases with additional Killing vectors are all given by $c=0$, in which case the line-element (1) admits the vector field $\vec{\zeta}$ given by Eq. (6) restricted to $c=0$, that is,

$$\vec{\zeta}=2\frac{\partial}{\partial x}+\frac{\mu}{a+b}(a+2bl)y\frac{\partial}{\partial y}+\frac{\mu}{a+b}(a-2bl)z\frac{\partial}{\partial z},$$

so that the H_3 becomes a G_3 on S_3 with the same Bianchi types as indicated above (see Refs. [17,5,1], and references therein). The explicit expression of c leads to two possibilities for $c=0$.

(1) Case with $\nu=0$. Now the velocity vector $\vec{u}$ is orthogonal to the orbits of the simply transitive G_3 group, so that the resulting solutions belong to the following nontilted Bianchi classes of spacetimes.

Bianchi V when $\lambda=\frac{1}{2}$. This homogeneous spacetime is a special case of the general Bianchi V family with $p=\varrho/3$ found by Ruban in Ref. [18] [line element (9.20) in Ref. [1] with $\alpha=m^2$]. This is the only solution of the whole family such that the flow generated by $\vec{u}$ isotropizes in the future (see Sec. II C).

Bianchi III when $\lambda=2$. As already mentioned, this case corresponds to another algebraically special solution (Petrov type D), which actually admits a fourth Killing vector given by

$$\vec{\chi}=y\left(\frac{\partial}{\partial x}+\frac{3}{8}\mu\frac{\partial}{\partial y}\right),$$

so that this case belongs to the LRS (G_4 on S_3) models of class II in Ref. [36]. As we said in Sec. II C, this is the only solution in Eq. (1), apart from the plane G_3 on S_2 case ($l=0$), having a purely electric Weyl tensor with respect to the fluid vector $\vec{u}$, $H(\vec{u})=0$ (see Ref. [37]).

A one-parameter family of Bianchi type VI$_h$ spacetimes when $\lambda\neq\frac{1}{2},2$. This family is included in the class of evolving nontilted Bianchi VI$_h$ spacetimes for $p=\gamma\varrho$ (Table 9.4 in Ref. [1]). Following the notation in Ref. [1] (using a tilde for the quantities in Ref. [1]), the present families are included in the general cases with $\tilde{k}=|4-3\tilde{\gamma}|/2$ (Table 9.4 in Ref. [1]), discovered by Uggla and Rosquist [38] up to quadratures.

(2) Case with

$$\lambda=\frac{1+\nu^2}{2(1-2\nu)(\nu+1)},$$

which satisfies $\varrho>0$ and $1/\sqrt{10}<\gamma<1$ [for $\nu\in(-1,1/2)$]. In this case, and for $\nu\neq 0$, the perfect-fluid flow has a nonvanishing projection onto the G_3 orbits, so that it constitutes a one-parameter family of exact ''tilted'' Bianchi solutions. The free parameter can be chosen to be γ with the restriction above and taking into account that the case $\gamma=\frac{1}{3}$ (and $\nu=0$) falls onto the previous ''nontilted'' Bianchi V case. The Bianchi type for this ''tilted'' homogeneous solutions is VI_h with $h=-[\nu^2(4\lambda+1)]^{-1}$ (including Bianchi III).

F. The comoving coordinates

As is known [7], every perfect-fluid diagonal G_2 solution can be written in comoving coordinates $\{T,X,y,z\}$ keeping the diagonal form of the metric. By comoving coordinates we mean those such that $\vec{u}\propto\partial_T$. The comoving coordinates will be useful for the study of the singularities that will be performed in Sec. II G. Also, this will prove that the use of comoving coordinates may sometimes be not well adapted to writing some solutions in explicit form, or even to look for them.

The explicit change to comoving coordinates for the metric (1) is easily found to be

$$t=\frac{1}{\sqrt{1-\nu^2}}(T+\nu X),\quad x=\frac{1}{\sqrt{1-\nu^2}}(X+\nu T),\qquad (7)$$

where the Jacobian of the change is 1. By writing $\hat{\mu}\equiv\mu/\sqrt{1-\nu^2}$, the line element becomes

$$\begin{aligned}ds^2=&F^2(T,X)(-dT^2+dX^2)+e^{\hat{\mu}[a/(a+b)](1-\nu)(T-X)}J(T,X)\\&\times\{[e^{\hat{\mu}[b/(a+b)](1-\nu)(T-X)}J(T,X)]^{2l}dy^2\\&+[e^{\hat{\mu}[b/(a+b)](1-\nu)(T-X)}J(T,X)]^{-2l}dz^2\},\end{aligned}\qquad (8)$$

where now

$$F(T,X)=\exp\left[\frac{\hat{\mu}}{a+b}[(d-c\nu)T-(c-\nu d)X]\right]J^{\lambda}(T,X),$$

and we have that

$$J(T,X)=1-\exp[-\hat{\mu}(T+\nu X)].\qquad (9)$$

Then the fluid velocity vector field simply reads $\vec{u}=F^{-1}\partial_T$. As is obvious, this family of solutions is not separable in comoving coordinates for $\nu\neq 0$. See Ref. [9] for a study of the loss of separability when performing arbitrary coordinate changes (within the two-spaces orthogonal to the G_2-group orbits) which keep the diagonal form of the metric.

As we can see, the structure of the line-element is perhaps not too cumbersome in comoving coordinates, but it is complicated enough so that the solutions had not been found until the *Ansätz* of separability in noncomoving coordinates was used. The structure shown in Eq. (8) may indicate some new *Ansätze* providing, perhaps, generalizations of these G_2 spacetimes with $p=\gamma\varrho$.

G. The half-null coordinates: Singularity structure

As mentioned previously, the solutions present an initial big-bang singularity at the spacelike hypersurface $t=0$ coming from the vanishing of the function J given in Eq. (9) for the metric (8). Nevertheless, the form of the function F/J^{λ} may suggest that other singularities could be present, as, for instance, at $x\to\pm\infty$. We are going to show that this is not the case, and consequently the only singularity of the solutions is the reachable universal spacelike singularity at $t=0$, which in the comoving coordinates is given by $T+\nu X=0$.

To that end, let us start by noticing that the coordinate ranges of Eq. (8) are in principle only restricted by

$$T+\nu X>0.\qquad (10)$$

Simple inspection on the expressions for the energy density and the Weyl scalars indicates that in this coordinate range the only other possible singular points would be those where the function F/J^{λ} vanishes, and that the singularities of both the Ricci and Weyl tensors coincide. At this point, it is very useful to perform the change to null coordinates $\{U,V\}$ in the surfaces orthogonal to the orbits of the G_2 group, given by

$$U=\frac{1}{\sqrt{2}}(T-X),\quad V=\frac{1}{\sqrt{2}}(T+X),\qquad (11)$$

so that Eq. (10) now becomes

$$V(1+\nu)+U(1-\nu)>0.\qquad (12)$$

Let us define now the following two constants:

$$\kappa_u\equiv\sqrt{2}\frac{\hat{\mu}}{a+b}(1-\nu)(d+c),\quad \kappa_v\equiv\sqrt{2}\frac{\hat{\mu}}{a+b}(1+\nu)(d-c),$$

whose fundamental property will turn out to be their positivity. Indeed, we have first of all that

$$(1-\nu)(d+c)=(1-\nu^2)[2(3\nu-1)^2\lambda^2+(8\lambda+1)\nu^2+1].$$

Since $\lambda\geqslant -1/4$ and $\nu^2<1$ we have that $\nu^2(8\lambda+1)>-1$, which, together with $a+b>0$, easily leads to $\kappa_u>0$. On the other hand, the following explicit expression

$$(1+\nu)(d-c)=(1-\nu^2)[2\lambda^2(1-\nu^2)+(4\lambda+1)\nu^2+1],$$

directly shows that $\kappa_v>0$ too. The function F is now such that

$$F(U,V)J^{-\lambda}(U,V)=e^{(1/2)(\kappa_u U+\kappa_v V)},$$

and the usefulness of the change is now clear since the region with $F/J^{\lambda}\to 0$, that is $\kappa_u U+\kappa_v V\to-\infty$, can be reached within the range (12) only if $U\to-\infty$ or $V\to-\infty$. In order to ascertain whether or not they are reachable, let us then bring them to finite values by making the typical coordinate change

$$U=\frac{1}{\kappa_u}\log(\kappa_u u), \quad V=\frac{1}{\kappa_v}\log(\kappa_v v),$$

so that the range of the new null coordinates (u,v) is given, in principle, by $u>0$, $v>0$ and the restriction coming from Eq. (12), which reads

$$(\kappa_u u)^{(1-\nu)/\kappa_u}(\kappa_v v)^{(1+\nu)/\kappa_v}>1. \tag{13}$$

The line element then becomes

$$\begin{aligned} ds^2 = & -2J^{2\lambda}(u,v)du dv+(\kappa_u u)^{a/(d+c)}J(u,v) \\ & \times\{[(\kappa_u u)^{b/(d+c)}J(u,v)]^{2l}dy^2 \\ & +[(\kappa_u u)^{b/(d+c)}J(u,v)]^{-2l}dz^2\}, \end{aligned} \tag{14}$$

where now

$$J(u,v)=1-(\kappa_u u)^{-(a+b)/[2(d+c)]}(\kappa_v v)^{-(a+b)/[2(d-c)]},$$

and $F/J^{\lambda}=(\kappa_u u)(\kappa_v v)$, so that the other possible singularities have been transported to $uv=0$ in the new coordinates.

The point now is that the singularity at $t=0$, which lies in the limit of the restriction given in Eq. (13) and has the form (for some constant A)

$$u=A^2 v^{-(d+c)/(d-c)}, \tag{15}$$

does hide the other possible singularities at $uv=0$ in the sense that any endless past-directed causal curve from any point in our manifold terminates necessarily at $t=0$ (this is why it is called a universal big-bang singularity, see Ref. [3]). In other words, $uv=0$ is not accessible within the physical spacetime, see Fig. 1.

Singularity type. The three (nonvanishing) eigenvalues of the distortion tensor of the fluid congruence defined by $\vec{u}$,

$$\theta_{\alpha\beta}\equiv\sigma_{\alpha\beta}+\frac{1}{3}\theta P_{\alpha\beta},$$

where $P_{\alpha\beta}\equiv g_{\alpha\beta}+u_\alpha u_\beta$ is the projector orthogonal to $\vec{u}$, read as follows at the limit $t\to 0$:

$$\theta_1=f\lambda, \quad \theta_2=f\left(\frac{1}{2}+l\right), \quad \theta_3=f\left(\frac{1}{2}-l\right),$$

where we have defined $f\equiv\lim_{t\to 0}\mu/(\sqrt{1-\nu^2}FJ)$. The beahviour of the fluid congruence is described then by the scale factors $l_i(i:1,2,3)$ defined by $\vec{u}(\log l_i)=\theta_i$, giving three different possibilities depending on the values of λ at the limit $t\to 0$:

$\lambda<0$: $l_1\to\infty$, $l_2,l_3\to 0$; $\lambda=0$: $l_2\to 0$, l_1 and l_3 tend to a finite value; $\lambda>0$: $l_3\to\infty, l_1,l_2\to 0$.

Therefore, the initial singularity is of cigar type whenever $\lambda\neq 0$ and of pancake type for $\lambda=0$. This can be also inferred by means of the limit $t\to 0$ on the line-element in Eq. (1), which after changing $(\mu t)^{\lambda}dt=d\tau$, redefining μ and absorbing some constants in y and z reads

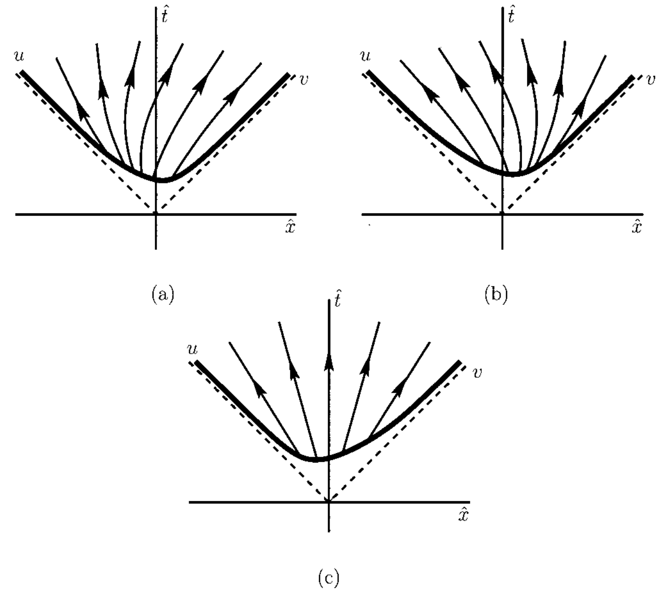

FIG. 1. Diagrams showing the singularity in the (u,v) surfaces for the three possibilities (a) $\kappa_v>\kappa_u$, (b) $\kappa_v<\kappa_u$, (c) $\kappa_v=\kappa_u$. The whole spacetime is the product of these surfaces with the group orbits. As usual, null lines are at 45°. The fluid flow is indicated by arrowed lines in the region given by expression (13). Notice that the shown coordinates $\{\hat{t},\hat{x}\}$ would correspond to the Lorentzian coordinates related to $\{u,v\}$ in the same manner as $\{T,X\}$ are related to $\{U,V\}$ (11). The big-bang singularity at $t=0$, or equivalently Eq. (15), which is a spacelike hypersurface, is denoted by a thick curve which tends asymptotically to $uv=0$ (denoted by dashed lines). Of course, it is evident that $uv=0$ is hidden "in the past" of the big-bang singularity in the physically meaningful spacetime ($\varrho>0$), and therefore the values $u=0$ or $v=0$ are unreachable.

$$\begin{aligned} ds^2|_{t\to 0} = & e^{-2c[\mu/(a+b)]x}[-d\tau^2+(\mu\tau)^{2\lambda/(1+\lambda)}dx^2] \\ & +e^{-[\mu/(a+b)](a+2bl)x}(\mu\tau)^{(1+2l)/(1+\lambda)}dy^2 \\ & +e^{-[\mu/(a+b)](a-2bl)x}(\mu\tau)^{(1-2l)/(1+\lambda)}dz^2. \end{aligned}$$

In other words, the exponents p_i such that $l_i\propto\tau^{p_i}$ (Ref. [1], p. 121) are $\{2p_1=2\lambda/(1+\lambda),\ 2p_2=(1+2l)/(1+\lambda),\ 2p_3=(1-2l)/(1+\lambda)\}$ so that

$$p_1+p_2+p_3=p_1^2+p_2^2+p_3^2=1.$$

Therefore, the singularity is of Kasner type [25,26].

H. Future asymptotic behavior of the solutions

The future asymptotic behavior is to be computed here as the limit $T\to\infty$ in comoving coordinates. (The same result is obtained, of course, by performing the limit $t\to\infty$, but the final expressions will be eventually better related to well-known homogeneous spacetimes by using the comoving coordinates.) The line-element (8) in that limit and for finite values of the spacelike coordinate X reads

$$ds^2|_{T\to\infty}=e^{2[\mu/(a+b)](d-c\nu)T}(-dT^2+dX^2)+e^{[\mu/(a+b)](1-\nu)(a+2bl)T}dy^2+e^{[\mu/(a+b)](1-\nu)(a-2bl)T}dz^2,$$

and thus the whole family tends to the perfect-fluid self-similar Bianchi I solutions [1], the three p_i exponents reading then $p_1=1$, $p_2=(1-\nu)(a+2bl)/[2(d-c\nu)]$, $p_3=(1-\nu)(a-2bl)/[2(d-c\nu)]$. The asymptotic behavior for the only case in which the solutions isotropize in the future ($\lambda=1/2$, $\nu=0$) is easily identified here as the flat FLRW solution with $p=\varrho/3$ ($p_1=p_2=p_3=1$).

III. INTERESTING PARTICULAR SUBFAMILIES

In this section we present some of the solutions included in the general family which are of some physical interest. We have selected three types of subfamilies: the dust subfamily, which was actually considered at length in Ref. [19] as a particular family of a broader class of dust solutions; the $p=\varrho/3$ family, which has a physically realistic equation of state for radiation-dominated epochs and, as we will see, it provides a simple example of how a gravitational wave can give rise to the inhomogeneization of the underlying perfect fluid; and the vacuum limits, which include some plane-wave spacetimes and, therefore, will also be relevant for the discussion about the inhomogeneization of the spacetime by the gravitational waves just mentioned. These three cases are treated in separate subsections.

A. The vacuum limits

The half-null coordinates of the previous section are very useful to find vacuum limits of the general solution. For the line-element as written in Eq. (14), the expression for the fluid velocity vector $\vec{u}$ transforms to

$$\vec{u}=J^{-\lambda}(u,v)\frac{1}{\sqrt{2(\kappa_u u)(\kappa_v v)}}\left[(\kappa_u u)\frac{\partial}{\partial u}+(\kappa_v v)\frac{\partial}{\partial v}\right], \quad (16)$$

and the energy density now reads

$$\varrho=J^{-2\lambda-1}\left(\frac{a-b}{a+b}\right)\frac{(1-\nu^2)\hat{\mu}^2(\lambda+1)}{(\kappa_u u)^{1+(a+b)/[2(d+c)]}(\kappa_v v)^{1+(a+b)/[2(d-c)]}}.$$

From this it follows that $\nu^2=1$ provides vacuum limits in this coordinate system. Performing them in the form (14) we arrive at the following.

(1) Case with $\nu=1$. The line element becomes

$$ds^2=-2J^{2\lambda}du\,dv+J(J^{2l}dy^2+J^{-2l}dz^2),$$

where $J=1+\sqrt{2}\hat{\mu}v$. This is a particular *pp*-wave [2] with ∂_u as a null Killing vector. The Petrov type is N and the Weyl tensor takes the simple form

$$\Psi_0=4\lambda l\hat{\mu}^2J^{-2\lambda-2}.$$

(2) Case with $\nu=-1$. Now the line element reads

$$ds^2=-2J^{2\lambda}du\,dv+(\kappa_u u)^{\tilde{a}}J([(\kappa_u u)^{\tilde{b}}J]^{2l}dy^2+[(\kappa_u u)^{\tilde{b}}J]^{-2l}dz^2),$$

where we have put $\tilde{a}\equiv 2(1+4\lambda)/(1+4\lambda+16\lambda^2)$ and $\tilde{b}\equiv 8\lambda/(1+4\lambda+16\lambda^2)$ and the function J becomes

$$J=1-(\kappa_u u)^{-(\tilde{a}+\tilde{b})/2}.$$

This is again a pure gravitational *pp*-wave, and the type-N Weyl tensor has the only nonvanishing scalar

$$\Psi_4=-\frac{4\lambda l\hat{\mu}^2}{J^{2\lambda+2}(\kappa_u u)^{\tilde{a}+\tilde{b}+2}(8\lambda+1)}\times[1-4(16\lambda^2-4\lambda-1)(\kappa_u u)^{\tilde{a}+\tilde{b}}-2(16\lambda+3)\times(\kappa_u u)^{\tilde{a}+\tilde{b}}-2(16\lambda+3)(\kappa_u u)^{(\tilde{a}+\tilde{b})/2}].$$

The previous vacuum solutions correspond to limits of the family of perfect fluid spacetimes of Sec. II B, where ν was restricted by Eq. (2). Nevertheless, the line element given by Eq. (1) contains further vacuum spacetimes arising whenever $a-b=0$ apart from the cases $\nu=\pm 1$, that is, if

$$\nu\lambda+\lambda+1=0,$$

which together with the restriction $\lambda\geqslant -\frac{1}{4}$ requires $\nu>3$ or $\nu<-1$. The line element for these cases reads

$$ds^2=F^2(t,x)(-dt^2+dx^2)+(e^{(\mu/2)(t-x)}-e^{-(\mu/2)(t+x)})\times[(e^{(\mu/2)(t-x)}-e^{-(\mu/2)(t+x)})^{2l}dy^2+(e^{(\mu/2)(t-x)}-e^{-(\mu/2)(t+x)})^{-2l}dz^2], \quad (17)$$

with

$$F(t,x)\equiv e^{-(\mu/2)x}(e^{(\mu/2)(t-x)}-e^{-(\mu/2)(t+x)})^{\lambda}.$$

This vacuum subfamily has in general a maximal Abelian G_3 acting simply transitively on spacelike hypersurfaces, the additional spacelike Killing vector field given by

$$\vec{\eta}=e^{(\mu/2)x}\left[\sinh\left(\frac{\mu}{2}t\right)\frac{\partial}{\partial t}+\cosh\left(\frac{\mu}{2}t\right)\frac{\partial}{\partial x}\right],$$

and is of Petrov type I in general, so that it is the well-known Kasner metric [2].

B. The $p=\varrho/3$ subfamily: Propagation of gravitational waves in a homogeneous background

This section is devoted to the radiation case $\gamma=\frac{1}{3}$ ($\lambda=\frac{1}{2}$). The condition (3) then reads $\nu<\frac{3}{7}$. As has been shown, this family constitutes a generalization of the Ruban ''nontilted'' Bianchi V solution, and we are now going to prove that, in fact, the general family can be interpreted as the inhomogeneization, via the propagation of plane gravitational waves, of the mentioned Bianchi V Ruban solution.

To that end, first of all it is necessary to choose the appropriate coordinate system in which the inhomogeneization will be more transparent. In this case, and despite what one might try at first, the natural choice is *not* the comoving coordinates (nor their half-null counterpart), but rather the original noncomoving coordinates $\{t,x,y,z\}$. The reason for this is that the change to comoving coordinates (7) depends explicitly on the parameter ν, but this very parameter is the one defining the inhomogeneization. Thus, for the several different inhomogeneous metrics (which are selected by the particular values of ν), a different change to comoving coordinates is needed. In other words, in order to use a common coordinate system which is valid for the general subfamily, as well as for the particular Bianchi V metrics, one has to resort to the system $\{t,x,y,z\}$. This is an explicit example of the adequacy of using noncomoving coordinates in some occasions based on physical grounds.

Now, we show how to write the line element in a form that makes it explicit the homogeneous Bianchi V background and the traveling waves leading to its inhomogeneization. By setting $\lambda=\frac{1}{2}$, the metric (1) can be rewritten as

$$ds^2=e^{f_V+f_{\rm inh}}(-dt^2+dx^2)+e^{g_V+p_{\rm inh}}(e^{p_V-\sqrt{3}p_{\rm inh}}dy^2 +e^{-p_V+\sqrt{3}p_{\rm inh}}dz^2), \tag{18}$$

where

$$e^{f_V}\equiv e^{\mu t}-1,\quad e^{g_V}\equiv e^{f_V-\mu x},$$
$$e^{p_V}\equiv(1-e^{-\mu t})^{\sqrt{3}}, \tag{19}$$

correspond to the functions of the Bianchi V homogeneous solution, defined by $\nu=0$, whereas

$$f_{\rm inh}\equiv\mu\left(\frac{4\nu(3\nu+1)}{(1-\nu^2)(3-7\nu)}\right)(t-\nu x),$$

$$p_{\rm inh}\equiv\mu\left(\frac{4\nu}{3-7\nu}\right)(t-x),$$

are the functions linked to the inhomogeneities of this $p=\varrho/3$ family of solutions. The energy density for this family takes the following expression:

$$\varrho=\frac{3}{2}\mu^2e^{-(2f_V+f_{\rm inh})}\left(\frac{3+\nu}{3-7\nu}\right).$$

As we see, the inhomogeneity is driven by the same parameter ν that ''tilts'' the perfect fluids, that is, the peculiar spatial velocity. This inhomogeneity appears primarily in the transversal part of the metric as the function $p_{\rm inh}$, which depends only on the null coordinate $t-x$, so that $p_{\rm inh}$ is obviously a solution of the flat-space wave equation

$$\Box_\eta p_{\rm inh}=0, \tag{20}$$

where $\Box_\eta$ denotes the d'Alembertian for the Minkowski (η) metric. Thus, this inhomogeneity is constant at the null hypersurfaces $t-x=$const, or in other words, it propagates at the speed of light orthogonally to the surfaces of transitivity of the G_2 group of motions. We interpret this as signalling the existence of gravitational waves traveling in the Bianchi V background whose spacelike propagation direction is orthogonal to the S_2 orbits spanned by $\{y,z\}$. This interpretation seems to be in accordance with that given for the family of inhomogeneous nondiagonal stiff fluid solutions found by Wainwright and Marshman in Ref. [27] (see also Ref. [30]), where the inhomogeneization is driven by an arbitrary function depending on $t-x$. In our case, the effect of the waves is *a fortiori* revealed also by the imprint which leave on the longitudinal part of the line element, given by $f_{\rm inh}$. Actually, since $f_{\rm inh}$ is functionally dependent on the velocity potential via

$$\sigma\propto e^{f_{\rm inh}/4},$$

the implicit relation between the gravitational waves and the propagation of inhomogeneities through the acoustic waves is manifest. The propagation of the gravitational and acoustic waves has thus two effects: it breaks the spatial homogeneity of the spacetime and at the same time tilts the velocity vector of the matter but keeping the perfect-fluid character of the matter content.

Of course, all this has to be considered in a more careful way. For instance, in the case $\nu=-\frac{1}{3}$ we fall into the special cases analyzed in the final point of Sec. II E, so that the line element (18) is in fact a Bianchi VI_{-3} spatially homogeneous solution, but now with $p_{\rm inh}\neq0$ and $f_{\rm inh}=0$. The form of the longitudinal part is kept, but there appears a nontrivial wavelike inhomogeneity in the transversal part given by $p_{\rm inh}\neq0$. It would seem that, in this case, the propagation of the waves would not give rise to the inhomogeneous trace left in $f_{\rm inh}$. However, this can be seen to arise as a rather exceptional situation because, by imposing $f_{\rm inh}=0$ but setting $p_{\rm inh}=\beta(t-x)$, one obtains a one-parameter family of Bianchi V–VI solutions for a nonperfect fluid matter content. The third Killing vector is given by

$$\vec{\zeta}=2\frac{\partial}{\partial x}+[\mu+(1-\sqrt{3})\beta]y\frac{\partial}{\partial y}+[\mu+(1+\sqrt{3})\beta]z\frac{\partial}{\partial z}.$$

The energy-momentum tensor of this family can be interpreted as a fluid with a nonzero energy flux in the direction of propagation of the wavelike inhomogeneity. Thus, in these cases, the waves seem to keep the spatially homogeneous character of the spacetime but breaking the perfect-fluid character of the matter. The restriction to the particular value $\beta=-\mu/4$ leads to the mentioned case corresponding to $\nu=-\frac{1}{3}$, in which the matter content is a perfect fluid. In this exceptional case, the inhomogeneization effects of the traveling waves, together with that of the flux of energy and of the acoustic waves shown by the tilting of the fluid, seem altogether to balance in a final outcome of $p_{\rm inh}$ which simply changes the Bianchi type of the solution.

Summarizing, one could better say it is the inhomogeneity that generates the gravitational waves and not the other way round. Indeed, clearly $f_{\rm inh}\neq0\Rightarrow p_{\rm inh}\neq0$, but as we have just seen $p_{\rm inh}\neq0\not\Rightarrow f_{\rm inh}\neq0$. This preferable point of view would state that the tilting (or the acoustic waves) generates the inhomogeneity in $f_{\rm inh}$ which, in turn, is responsible for the appearance of the gravitational waves.

The above paragraphs seriously indicate that one cannot talk about gravitational-wave propagation and its inhomogeneization properties in a naive manner. Therefore, we have tried to support the interpretation of $p_{\rm inh}$ representing gravitational waves propagating in the homogeneous background in two other independent ways. These are presented in what follows.

The first way is the comparison with the available systematic studies for the description of exact gravitational waves on homogeneous backgrounds. These studies were initially developed in Refs. [29,39]. The exact formalism presented in that reference, later used in Ref. [40], applies to perfect-fluid solutions of the Einstein field equations, as well as the degeneracies thereof, containing gravitational waves propagating over Bianchi I to VII backgrounds along an ''algebraically'' preferred direction. The case we are interested in is that of Bianchi V cosmologies which, after inhomogeneization takes place, become diagonal G_2 perfect-fluid solutions. This case was termed as the single polarization (+) waves in Bianchi V, see Refs. [29,40]. Using the explicit form of the metric as given in Ref. [18] where p_V, g_V, p_V correspond to functions of a fully general Bianchi V spacetime and taking $p_{\rm inh}(t,x)$ and $f_{\rm inh}(t,x)$ to be free functions giving the inhomogeneous generalization, the Einstein equations for a perfect fluid can be split as follows:

$$\Box_\eta (Re^{-\mu x}) = -(\varrho - p) Re^{-\mu x} e^{f_V + f_{\rm inh}}, \tag{21}$$

$$\Box \psi = 0, \tag{22}$$

plus two other first order differential equations for $f_{\rm inh}$ (longitudinal scale equations [29]). Here $\Box$ denotes the d'Alambertian, $R \equiv e^{f_v + p_{\rm inh}}$ describes the transverse scale expansion, and $\psi \equiv p_V - \sqrt{3} p_{\rm inh}$ corresponds to the so called wave amplitude [29]. It must be noticed that, in this formalism, the function satisfying a source-free massless scalar field equation is $p_V - \sqrt{3} p_{\rm inh}$, that is, the full combination of the function corresponding to the background plus the function carrying the inhomogeneity. This is so for the cases with a single (+) polarization, which restricts the Bianchi types to be I, III, V, or VI, and furthermore is in accordance with previous works in which the diagonal Einstein-Rosen [41] form of the line element is taken and ψ is the function appearing in the transverse part once the transitivity surface area element, which corresponds to $Re^{-\mu x}$ in Eq. (18), is factorized, see Refs. [26,28,30,31]. Nevertheless, as we see in the previous case, the transverse scale expansion R as defined in Ref. [29] *is not* equivalent to the transitivity surface area element in general. Actually, for some other Bianchi types this implies that the definition of ψ in Refs. [29,40] differs from that in the ''Einstein-Rosen view,'' and thereby the wave equation they satisfy are different.

The function $p_{\rm inh}(t,x)$ will actually satisfy the homogeneous wave equation $\Box p_{\rm inh} = 0$ for those cases with $\varrho = p$, that is, for stiff fluids including the vacuum and the minimally coupled massless scalar field solutions. This is so primarily because the so-called ''transverse scale'' Eq. (21) couples R to the matter source through the function $\varrho - p$, so as long as this function vanishes, $Re^{-\mu x}$ is completely independent of the matter content and satisfies a flat-space wave equation. One can argue then that R has to be in fact the ''homogeneous'' function $R = R_V(t)$ in order to have models that reduce to the Bianchi spacetime when the waves reduce to zero, see Ref. [40]. In other occasions, as was the case in the ''Einstein Rosen view,'' the condition $R_{,\mu} R^{,\mu} < 0$ everywhere was often imposed, see Ref. [26]. These simplifications, for the stiff fluid case, imply eventually that the d'Alambertian $\Box$ coincides with $\Box_V$, that is, with the d'Alambertian of the homogeneous background metric,[1] and thus, since $\Box p_V = \Box_V p_V = 0$, Eq. (22) implies $\Box p_{\rm inh} = 0$. That is, both p_V and $p_{\rm inh}$ are solutions of the same wave equation (22).

Fortunately, despite all the above, the simplification on R is not necessary and, moreover, for more realistic equations of state R evolves (as must be) coupled with the matter. The former statement follows trivially in general because, in Eq. (18), the homogeneous spacetime is recovered when $p_{\rm inh} = 0$, while the latter immediately implies that, in fact, $\Box \neq \Box_V$. In Eq. (18) they are related as follows:

$$\Box = e^{-f_{\rm inh}} [\Box_V - \dot{p}_{\rm inh} \partial_t + p'_{\rm inh} \partial_x],$$

and therefore the coupling of R with the matter leads to the appearance of $p_{\rm inh}$, causing the inhomogeneization of the transverse scale expansion, which in turn drives the coupling of the function ψ with the matter through the operator $\Box$. The longitudinal scale equations would account then for $f_{\rm inh}$. Notice that, as remarked before, in the present family (18) with (19), it is $f_{\rm inh}$, or equivalently σ, that actually switches on the inhomogeneization. In Eq. (18) together with Eq. (19) and because $p_{\rm inh}$ is a function of $t - x$, we have that

$$\Box p_{\rm inh} = e^{-f_{\rm inh}} \Box_V p_{\rm inh} = \Box p_V / \sqrt{3}$$
$$= -e^{-(f_{\rm inh} + f_V)} \frac{\mu}{\sqrt{3}} \left(\frac{4\nu}{3 - 7\nu} \right) \dot{p}_V,$$

the second equality coming from Eq. (22). Thus, the propagation of $p_{\rm inh}$ is not that of a source-free massless scalar field, rather it is driven by the interaction with the background geometry transverse part, clearly showing the nonlinearity on the splitting of the background and the wave. The important thing here, however, is that the above equation is manifestly hyperbolic in character, as is obvious in its simpler form (20), and that the characteristic propagation speed of its solutions is the speed of light. Having no electromagnetic field present, the propagation of gravitational waves seems the best possibility.

The second way to support our claim comes from the vacuum limits of the solutions presented in Sec. III A. Because of the nonlinear interaction of the waves with the matter proper to general relativity, and as we have seen in the

[1] We are concentrating on the Bianchi V case, but similar scenarios arise in the rest of the Bianchi I–VII spacetimes [29].

previous paragraph for this particular case, it is not possible nor desirable to separate the metric into two linear terms representing the background and the waves, respectively. However, one can try to annihilate completely one of the two terms, and then only the other must survive. Of course, by construction, if we set the wavelike part defined by $p_{\rm inh}$ to zero we obtain the Bianchi V, $p=\varrho/3$, homogeneous background. This corresponds to putting $\nu=0$. But can we also get rid of the matter, so that only the gravitational wave remains? A striking and beautiful answer would come from finding that the vacuum limits in the coordinates of Eq. (18) correspond to plane gravitational waves, as was the case for the limits found in Sec. III A. Unfortunately, there are no vacuum limits in these coordinates. To achieve the vacuum limits one has to make use of other coordinate systems, but then again the limit depends on the coordinate system chosen [39]. In spite of this, the existence of coordinate systems in which the limits $\nu=\pm 1$ correspond to a gravitational *pp*-wave has already been shown in Sec. III A. At this point, the question giving unequivocal sense to the gravitational wave inhomogeneity interpretation for this family of $p=\varrho/3$ solutions would thus be: do all the vacuum limits of this family with $\nu=-1$ correspond to a plane gravitational wave? We do not have a rigorous answer for this question yet, but we do believe that the answer is positive. In this sense, we claim that getting rid of the matter, whenever this is possible, and within the allowed range of the parameter ν, provides a pure and simple gravitational *pp*-wave, which we interpret as the remanent of the mixed case.

Hitherto, the work on gravitational waves in cosmological backgrounds based on exact solutions has been mainly tackled in the cases of vacuum, scalar and electromagnetic fields, and stiff fluid (and its anisotropic generalization [32]) as sources in G_2 on S_2 spacetimes, see the reviews [28–31], and references therein. As mentioned in the Introduction, there are other works presenting solitonic perturbations of $p=\varrho/3$ FLRW spacetimes [34], later generalized to $p=\gamma\varrho$ in Ref. [35], but the energy-momentum tensor of the inhomogeneous spacetime turns out to be a nonperfect fluid. The general formalism used in these papers assumes that the backgrounds in which the perturbations propagate have a restricted type of energy-momentum tensors, so it has been applied only to FLRW spacetimes. The present family of solutions may constitute then the first exact solution in general relativity for a perfect fluid with a realistic $p=\varrho/3$ equation of state describing gravitational waves traveling on a spatially homogeneous background.

C. The dust subfamily

The particular dust cases, defined by $\lambda=0$ [so that Eq. (3) requires now $\nu<0$], belong to a more general family of algebraically general dust spacetimes already presented in Ref. [19]. We devote this short subsection to identifying the present dust one-parameter subfamily within the more general dust family appearing in Ref. [19].

By taking profit of the form of the line-element in the half-null coordinates $\{u,v,y,z\}$ (14), and by making a coordinate change to Lorentzian coordinates analogous to that in Eq. (11), the line element (14) with $\lambda=0$ coincides exactly with expression (7) of Ref. [19] for the following particular values of the constants (here, we denote with a tilde the quantities which appear in Ref. [19], if necessary).

$\tilde{a}=\kappa_u/\sqrt{2}$. This can be seen as the free parameter of the solution.

$c_2=0$, $c_1<0$. This implies the relation $\kappa_u=\kappa_v$, which follows from $\lambda=0$.

$\tilde{b}=\nu(1+\nu)/(1+\nu^2)$. By remembering that $\nu\in(-1,0)$, this leads to the restriction $\tilde{b}\in[\tilde{b}_-,0)$ for the parameter $\tilde{b}$ of Ref. [19].

The diagram for this subfamily corresponds to the Fig. 2(f) of Ref. [19] with the point $\tilde{p}$ at $\tilde{p}=\tilde{x}=0$, in agreement with the particular case (c) of the present Fig. 1.

ACKNOWLEDGMENTS

We are grateful to Alex Feinstein, Filipe C. Mena, and Reza Tavakol for some fruitful comments and helpful indications concerning references. The authors also thank financial support from the Basque Country University, from the Generalitat de Catalunya, and from the Ministerio de Educación y Cultura, under Project Nos. UPV 172.310-G02/99, 98SGR 00015, and PB96-0384, respectively. R.V. thanks the Spanish Secretaría de Estado de Universidades, Investigación y Desarrollo, Ministerio de Educación y Cultura, Grant No. EX99 52155527.

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