Gauge field copies

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The problem of Wu-Yang ambiguities in three dimensions is related to the problem of the existence of torsion-free driebeins for an arbitrary potential. The ambiguity is only at the level of boundary conditions. We also find that in three dimensions any smooth Yang-Mills field tensor can be uniquely written as the non-Abelian magnetic field of a smooth Yang-Mills potential.

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Wu and Yang [1] gave an explicit example of two (gauge inequivalent) Yang-Mills potentials $\vec{A}_i(x)=\{A_i^a(x),a=1,2,3\}$ generating the same non-Abelian magnetic field

$$\vec{B}_i[A](x)=\epsilon_{ijk}(\partial_j\vec{A}_k+\tfrac{1}{2}\vec{A}_j\times\vec{A}_k). \tag{1}$$

Since then there has been a wide discussion of the phenomenon in the literature [2–15]. We may refer to gauge potentials giving the same non-Abelian magnetic field, as gauge field copies in contrast with gauge equivalent potentials that generate magnetic fields related by a homogeneous gauge transformation. If we require all higher covariant derivatives of B_i^a also match then there are effectively no gauge copies [11].

Deser and Wilczek [4] first pointed out the consistency condition for $\vec{A}_\mu$ and $\vec{A}'_\mu=\vec{A}_\mu+\vec{\Delta}_\mu$ to generate the same field strength. Using the Bianchi identity, they obtained that $\vec{\Delta}_\mu$ had to satisfy the equation

$$[\tilde{\mathbf{F}}_{\mu\nu},\Delta_\nu]=0, \tag{2}$$

where in two dimensions,

$$\tilde{F}^{\mu\nu ab}=\frac{1}{2}\epsilon^{\mu\nu}\epsilon^{abc}F^c{}_{\mu\nu}=M^{ab}, \tag{3}$$

and in four dimensions

$$\tilde{F}^{\mu\nu ab}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\epsilon^{abc}F^c{}_{\rho\sigma}=M^{a\mu,b\nu}. \tag{4}$$

Treating this as an eigenvalue equation for Δ, we have the condition for existence of nontrivial solutions of Δ is that the determinant of **M** is zero. In two dimensions the determinant corresponding to **M** vanishes identically and there Δ necessarily has nontrivial solutions. However in four dimensions this determinant is generically nonzero and there are hardly any gauge copies.

This sort of analysis exists only in even dimensions. In three Euclidean dimensions, we only get the constraint $\vec{B}_i[A]\times\vec{\Delta}_i=0$. This equation has many solutions, but this is only a consisitency condition. It does not mean that any $\vec{\Delta}_i$ satisfying this equation gives a gauge copy. Recently Freedman and Khuri [15] have exhibited several examples of continuous families of gauge field copies in $d=3$. Their technique was to use a local map of the gauge field system into a spatial geometry with a second rank symmetric tensor $G_{ij}=B_i^aB_j^a\det B$ and a connection with torsion constructed from it.

We adopt a different method and directly ask the question as to how many different solutions (if any), does the system of equations defined by Eq. (1) have for any specified $\vec{B}_i(x)$. For that we proceed with the analysis using the Cauchy-Kowalevsky existence theorems on systems of partial differential equations. The equations for the gauge field copies are not *a priori* in the form where this theorem can be applied. However, by reorganizing the equations a bit they can be brought to the form so that these theorems can be applied to that system.

I. EXISTENCE OF *A* FOR ARBITRARY *B*

Let us first state the Cauchy-Kowalevsky existence theorem that we use [16].

Let a set of partial equations be given in the form

$$\frac{\partial z_i}{\partial x_1}=\sum_{j=1}^{m}\sum_{r=2}^{n}G_{ijr}\frac{\partial z_j}{\partial x_r}+G_i \tag{5}$$

for values $i=1,\ldots,m$, being m equations in m dependent variables; the coefficients G_{ijr} and the quantities G_i are functions of all the variables, dependent and independent. Let $c_1,\ldots,c_m,a_1,\ldots,a_n$ be a set of values of $z_1,\ldots,z_m,x_1,\ldots,x_n$, respectively, in the vicinity of which all the functions G_{ijr} and G_i are regular; and let $\phi_1,\ldots,\phi_m$ be a be a set of functions of $x_2,\ldots,x_n$, which acquire values $c_1,\ldots,c_m$ respectively when $x_2=a_2,\ldots,x_n=a_n$, which are regular in the vicinity of these values of $x_2,\ldots,x_n$, and which are otherwise arbitrary. Then a system of integrals of the equations can be determined, which are regular functions of $x_1,\ldots,x_n$ in the vicinity of the values $x_1=a_1,x_2=a_2,\ldots,x_n=a_n$, and which acquire the values $\phi_1,\ldots,\phi_m$ when $x_1=a_1$; moreover, the system of integrals determined in accordance with these conditions, is the only system of integrals that can be determined as regular functions.

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Our system of equations is

$$\vec{B}_1=\partial_2\vec{A}_3-\partial_3\vec{A}_2+\vec{A}_2\times\vec{A}_3, \tag{6}$$

$$\vec{B}_2=\partial_3\vec{A}_1-\partial_1\vec{A}_3+\vec{A}_3\times\vec{A}_1, \tag{7}$$

$$\vec{B}_3=\partial_1\vec{A}_2-\partial_2\vec{A}_1+\vec{A}_1\times\vec{A}_2, \tag{8}$$

where $\vec{B}_1$, $\vec{B}_2$, and $\vec{B}_3$ are treated as given variables and we want to solve for $\vec{A}_1$, $\vec{A}_2$, and $\vec{A}_3$. With this definition of the B's, the Bianchi identity $D_iB_i=0$ follows automatically. However the existence theorem cannot be applied directly to this set of equations. For that we rewrite the equations in a different way. Consider

$$\partial_3\vec{A}_2=\partial_2\vec{A}_3+\vec{A}_2\times\vec{A}_3-\vec{B}_1, \tag{9}$$

$$\partial_3\vec{A}_1=\partial_1\vec{A}_3-\vec{A}_3\times\vec{A}_1+\vec{B}_2. \tag{10}$$

The existence theorem implies that we have solution for $\vec{A}_1$ and $\vec{A}_2$ for any specified $\vec{B}_1$, $\vec{B}_2$, and $\vec{A}_3$. But $\vec{A}_1$ and $\vec{A}_2$ so obtained have to satisfy Eq. (8). Is this always possible with some choice of $\vec{A}_3$, and if yes, is the choice of $\vec{A}_3$ unique? To address this question, we presume that the initial data on $x_3=0$ satisfies Eqs. (6)–(8). This is always possible for any given $\vec{B}_i(x)$ as follows from the analysis of the 1+1-dimensional case. Then Eq. (8) may be equivalently replaced by another equation obtained by applying ∂_3 on it and using Eqs. (6) and (7). This is just the Bianchi identity. We write it in the form

$$\vec{A}_3\times\vec{B}_3=-\partial_3\vec{B}_3-\partial_2\vec{B}_2-\vec{A}_2\times\vec{B}_2-\partial_1\vec{B}_1-\vec{A}_1\times\vec{B}_1. \tag{11}$$

Now let us decompose $\vec{A}_3$ in directions parallel and perpendicular to $\vec{B}_3$,

$$\vec{A}_3=\alpha\vec{B}_3+\vec{A}_{3\perp}. \tag{12}$$

In the generic case, where $|\vec{B}|\neq 0$, Eq. (11) determines $\vec{A}_{3\perp}$ entirely. Taking the cross product of Eq. (11) with $\vec{B}_3$, we get

$$\vec{A}_3=\alpha\vec{B}_3-\frac{1}{|\vec{B}_3|^2}\vec{B}_3\times[(\vec{A}_2\times\vec{B}_2)+(\vec{A}_1\times\vec{B}_1)+(\partial_i\vec{B}_i)], \tag{13}$$

where α can be arbitrarily chosen.

We now address the question whether α can also be determined. Taking the dot product of (11) with $\vec{B}_3$, we get

$$\vec{B}_3\cdot\partial_i\vec{B}_i+(\vec{B}_3\times\vec{B}_1)\cdot\vec{A}_1+(\vec{B}_3\times\vec{B}_2)\cdot\vec{A}_2=0. \tag{14}$$

This is a constraint that $\vec{A}_1$ and $\vec{A}_2$ have to satisfy. It is satisfied on $x_3=0$. In order that it is satisfied at all x_3, we apply ∂_3 on Eq. (14) and use Eqs. (9) and (10). We obtain

$$\begin{aligned}&-(\partial_1\vec{A}_3-\vec{A}_3\times\vec{A}_1+\vec{B}_2)\cdot(\vec{B}_1\times\vec{B}_3)-\vec{A}_1\cdot\partial_3(\vec{B}_1\times\vec{B}_3)\\&-(\partial_2\vec{A}_3+\vec{A}_2\times\vec{A}_3-\vec{B}_1)\cdot(\vec{B}_2\times\vec{B}_3)-\vec{A}_2\cdot\partial_3(\vec{B}_2\times\vec{B}_3)\\&+\partial_3(\partial_i\vec{B}_i)\cdot\vec{B}_3+(\partial_i\vec{B}_i)\cdot(\partial_3\vec{B}_3)=0.\end{aligned} \tag{15}$$

Now we can substitute the expression for $\vec{A}_3$ from Eq. (13). Note that in this substitution, the derivatives do not act on α since in that case we get terms $\vec{B}_3\cdot\vec{B}_1\times\vec{B}_3$ and $\vec{B}_3\cdot\vec{B}_2\times\vec{B}_3$ that vanish. We get the coefficient of α as $(D_1[A]\vec{B}_3)\cdot(\vec{B}_1\times\vec{B}_3)+(D_2[A]\vec{B}_3)\cdot(\vec{B}_2\times\vec{B}_3)$. Whenever this coefficient is nonzero, the linear equation for α is invertible and this explicitly gives us α as a function of $\vec{A}_1$, $\vec{A}_2$, and $\vec{B}_i$. Generically we do not expect any problem in solving for α.

We now have $\vec{A}_3$ as a local function of $\vec{A}_1$, $\vec{A}_2$, and $\vec{B}_i$'s and we can substitute for it in Eqs. (9) and (10). We further expect that the field configurations are mostly nonvanishing so that the coefficients G_{ijr} and G_i are regular and we can apply the theorem to get $\vec{A}_1$, $\vec{A}_2$ and hence $\vec{A}_3$ as unique functionals of $\vec{B}_i(x)$.

Alternatively, we could consider the system of equations

$$\partial_3\vec{A}_2=\partial_2\vec{A}_3+\vec{A}_2\times\vec{A}_3-\vec{B}_1, \tag{16}$$

$$\partial_3\vec{A}_1=\partial_1\vec{A}_3-\vec{A}_3\times\vec{A}_1+\vec{B}_2, \tag{17}$$

$$\begin{aligned}\partial_3(\vec{A}_3\times\vec{B}_3)=&-(\partial_1\vec{A}_3-\vec{A}_3\times\vec{A}_1+\vec{B}_2)\times\vec{B}_1\\&-(\partial_2\vec{A}_3+\vec{A}_2\times\vec{A}_3-\vec{B}_1)\times\vec{B}_2\\&-\vec{A}_1\times\partial_3\vec{B}_1-\partial_3(\partial_i\vec{B}_i)-\vec{A}_2\times\partial_3\vec{B}_2,\end{aligned} \tag{18}$$

$$\partial_3(\vec{A}_3\cdot\vec{B}_3)=\partial_3(|\vec{B}_3|^2\alpha(\vec{A}_1,\vec{A}_2,\vec{B}_i)). \tag{19}$$

Here in the last equation $\alpha(\vec{A}_1,\vec{A}_2,\vec{B}_i)$ is to be replaced by the expression obtained for α from Eq. (15) and $\partial_3\vec{A}_1$ and $\partial_3\vec{A}_2$ are to be replaced using Eqs. (16) and (17). This system of equations is in the form where the Cauchy-Kowalevsky theorem can be applied and this system uniquely determines all the unknown variables once the initial data is specified. The first two equations contain the six unknowns $\vec{A}_1$ and $\vec{A}_2$. The third one contains the two components of $\vec{A}_3$ transverse to $\vec{B}_3$ and the fourth one has the component of $\vec{A}_3$ parallel to $\vec{B}_3$. Thus all the nine degrees of freedom are uniquely determined. Therefore generically there are no gauge field copies. The only ambiguity in the choice of the potential is limited to a subspace that specifies the initial conditions as required in the theorem.

II. EXISTENCE OF TORSION FREE DRIEBEINS FOR ARBITRARY *A*

In this section we address the question whether there exists any continuous family of potentials that generate the same magnetic field. Let $\vec{A}_i$ and $\vec{A}_i+\epsilon\vec{e}_i$ generate the same magnetic field, where ϵ is a small parameter. Then $\vec{e}_i$ satisfies the equation

$$\epsilon_{ijk}(\partial_j \vec{e}_k + \vec{A}_j \times \vec{e}_k) = 0. \tag{20}$$

This is precisely the equation for a driebein $\vec{e}_i$ to have zero torsion with respect to the connection one form $\vec{A}_i$. Thus we are asking if there exists a driebein with zero torsion for a given arbitrary connection one form. This is an important question in the context of general relativity. We also have a consistency condition by taking the covariant derivative of this equation, that is given by

$$\vec{B}_k \times \vec{e}_k = 0. \tag{21}$$

Let us rewrite the equations in a more convenient way. We take our system of equations as

$$\partial_3 \vec{e}_2 = \partial_2 \vec{e}_3 + \vec{A}_2 \times \vec{e}_3 - \vec{A}_3 \times \vec{e}_2, \tag{22}$$

$$\partial_3 \vec{e}_1 = \partial_1 \vec{e}_3 + \vec{A}_1 \times \vec{e}_3 - \vec{A}_3 \times \vec{e}_1, \tag{23}$$

and the consistency condition (21). This set is equivalent to the set of Eq. (20). As in the previous case, we first look at the consistency condition. Let us decompose $\vec{e}_3$ as

$$\vec{e}_3 = \beta \vec{B}_3 + \vec{e}_{3\perp}. \tag{24}$$

Again Eq. (21) fixes for us $\vec{e}_{3\perp}$ in terms of the magnetic fields (in the generic cases $\vec{B}_3 \neq 0$). We get

$$\vec{e}_3 = \beta \vec{B}_3 - \frac{1}{|\vec{B}_3|} \vec{B}_I \times \vec{e}_I, \tag{25}$$

where I goes over 1,2. Now we can substitute this form of $\vec{e}_3$ in Eqs. (22) and (23). We obtain $\vec{e}_1$ and $\vec{e}_2$ as unique functions of β and the magnetic fields. However this $\vec{e}_1$ and $\vec{e}_2$ has to satisfy the consistency conditions

$$\vec{B}_3 \cdot \vec{B}_I \times \vec{e}_I = 0, \tag{26}$$

where again I goes over 1,2. Taking ∂_3 of Eq. (26), we get, using Eqs. (22) and (23)

$$D_3(\vec{B}_3 \times \vec{B}_I) \cdot \vec{e}_I + \vec{B}_I \cdot \vec{B}_3 \times D_I \vec{e}_3 = 0. \tag{27}$$

Putting in the expression of $\vec{e}_3$, we get a linear equation for β,

$$D_3(\vec{B}_3 \times \vec{B}_I) \cdot \vec{e}_I + (\vec{B}_I \times \vec{B}_3) \cdot (D_I \vec{B}_3) \beta - (\vec{B}_I \times \vec{B}_3) \cdot D_I \left[\frac{1}{|\vec{B}_3|} (\vec{B}_J \times \vec{e}_J) \right] = 0. \tag{28}$$

This equation can be inverted to solve for β as a function of $\vec{e}_1$, $\vec{e}_2$, $\vec{A}_1$, $\vec{A}_2$, and $\vec{B}_i$ whenever $(\vec{B}_I \times \vec{B}_3) \cdot (D_I \vec{B}_3)$ is nonzero.

Formally we could have also looked at the set of equations

$$\partial_3 \vec{e}_2 = \partial_2 \vec{e}_3 + \vec{A}_2 \times \vec{e}_3 - \vec{A}_3 \times \vec{e}_2, \tag{29}$$

$$\partial_3 \vec{e}_1 = \partial_1 \vec{e}_3 + \vec{A}_1 \times \vec{e}_3 - \vec{A}_3 \times \vec{e}_1, \tag{30}$$

$$\begin{aligned} \partial_3(\vec{B}_3 \times \vec{e}_3) = & -(\partial_3 \vec{B}_2) \times \vec{e}_2 - (\partial_3 \vec{B}_1) \times \vec{e}_1 \\ & - \vec{B}_2 \times (\partial_2 \vec{e}_3 + \vec{A}_2 \times \vec{e}_3 - \vec{A}_3 \times \vec{e}_2) \\ & - \vec{B}_1 \times (\partial_1 \vec{e}_3 + \vec{A}_1 \times \vec{e}_3 - \vec{A}_3 \times \vec{e}_1), \end{aligned} \tag{31}$$

$$\partial_3(\vec{B}_3 \cdot \vec{e}_3) = \partial_3(|\vec{B}_3|^2) \beta(\vec{e}_1, \vec{e}_2, \vec{A}_1, \vec{A}_2, \vec{B}_i). \tag{32}$$

In the last equation, β has to be replaced by its solution from Eq. (28) and $\partial_3 e_I$ is to be substituted from Eqs. (29) and (30).

We expect the non-Abelian potentials and magnetic fields are smooth and nonvanishing so that the coeffiecient functions for the set of differential equations are regular. Applying the Cauchy-Kowalevsky theorem to this set, we get a unique smooth solution for $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$. Thus, for any potential there is a torsion-free driebein, and the only ambiguity is in the choice of the driebein to fix the initial conditions required by the theorem.

III. AN EXPLICIT CALCULATION

We now illustrate these results by an explicit calculation for the special case $A_i^a = \delta_i^a$. In momentum space, the equation looks like

$$\epsilon_{ijk}(-i p_j \delta^{ac} + \epsilon_{abc} \delta_j^b) e_k^c(p) = 0 \tag{33}$$

or

$$(-i \epsilon_{ijk} p_j \delta^{ac} + \delta_i^a \delta_k^c - \delta_i^c \delta_k^a) e_k^c(p) = 0. \tag{34}$$

In three dimensions we can choose three orthogonal vectors. We choose three such vectors as $(\vec{p}, \vec{n}, \vec{m})$ where $\vec{p}$ coincides with the $\vec{p}$ that appears in the equation and $\vec{n}$ and $\vec{m}$ are unit vectors. We also orient $(\vec{p}, \vec{n}, \vec{m})$ such that $\vec{p} \times \vec{m} = |\vec{p}| \vec{n}$ and $\vec{p} \times \vec{n} = -|\vec{p}| \vec{m}$. Next we write a general solution for e_k^c in terms of the dyad basis as

$$\begin{aligned} e_{kc} = & a_1 n_c m_k + a_2 n_k m_c + a_3 n_k n_c + a_4 m_k m_c + a_5 p_c m_k \\ & + a_6 p_k m_c + a_7 p_c n_k + a_8 p_k n_c + a_9 p_k p_c, \end{aligned} \tag{35}$$

where a_i's are unknown coefficients to be determined.

Substituting the solution in the equation, we get various relations among the coefficients. a_5, a_6, a_7, a_8, and a_9 turn out to be zero identically. In addition, we get

$$-i|\vec{p}| a_1 = -i|\vec{p}|^3 a_2 = a_3 = |\vec{p}|^2 a_4. \tag{36}$$

Therefore, we get a nonzero solution only if

$$|\vec{p}|=1, \tag{37}$$

in which case,

$$-ia_1=-ia_2=a_3=a_4=a. \tag{38}$$

Thus the general solution is,

$$e_{ib}(x)=\int d\Omega\, a(\Omega)e^{i\hat{p}\cdot x}(\hat{m}+i\hat{n})_i(\hat{m}-i\hat{n})_b. \tag{39}$$

Here the integration is over all directions of the vector $\hat{p}$. The solutions have an arbitrary function $a(\Omega)$. We may fix $a(\Omega)$ by using initial data on $x_3=0$ surface. This may be interpreted as the arbitrary choice of $\vec{e}_i(x)$ at the boundary. However if we require $\vec{e}_i(x)$ vanishes rapidly at infinity, there may not be any solutions. Thus gauge copies would be absent in this case.

A similar exercise can be carried out for any constant vector potential and gives an identical result.

IV. CONCLUSIONS

In this paper we have looked at two problems regarding the existence of non-Abelian vector potentials. First we asked the question if there exists a vector potential for any arbitrary magnetic field. We found that there are many choices of $\vec{A}_i(x)$ on the $x_3=0$ surface that reproduces $\vec{B}_i(x)$ on the surface. (This is the gauge field ambiguity in 1+1 dimensions.) For each such boundary condition on $\vec{A}_i(x)$ we have seen (in the generic case) that there is a unique potential $\vec{A}_i(x)$ that reproduces the given magnetic field everywhere. The non-Abelian Bianchi identity does not constrain the non-Abelian magnetic fields in contrast to the Abelian case. The ambiguity in the choice of the potentials is (in the generic case) only due to the ambiguity in $\vec{A}_i(x)$ on the $x_3=0$ surface. Thus it is related to the gauge copy problem in 1+1 dimensions.

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