Marginal stability and the metamorphosis of Bogomol'nyi-Prasad-Sommerfield states

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We discuss the restructuring of the BPS spectrum which occurs on certain submanifolds of the moduli or parameter space—the curves of the marginal stability (CMS)—using quasiclassical methods. We argue that in general a ''composite'' BPS soliton swells in coordinate space as one approaches the CMS and that, as a bound state of two ''primary'' solitons, its dynamics in this region is determined by nonrelativistic supersymmetric quantum mechanics. Near the CMS the bound state has a wave function which is highly spread out. Precisely on the CMS the bound state level reaches the continuum, the composite state delocalizes in coordinate space, and restructuring of the spectrum can occur. We present a detailed analysis of this behavior in a two-dimensional $\mathcal{N}=2$ Wess-Zumino model with two chiral fields, and then discuss how it arises in the context of ''composite'' dyons near weak coupling CMS curves in $\mathcal{N}=2$ supersymmetric gauge theories. We also consider cases where some states become massless on the CMS.

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I. INTRODUCTION

Centrally extended supersymmetry algebras admit a special class of massive representations which preserve some fraction of the supersymmetry of the vacuum, and consequently form multiplets which are smaller than a generic massive representation. The states lying in these shortened [or Bogomol'nyi-Prasad-Sommerfield (BPS)] multiplets are extremely useful probes of the theory because on one hand their spectrum is determined almost entirely by kinematical constraints (i.e., by the central charges) while on the other the multiplet structure ensures their generic stability. More precisely, the fact that BPS states lie in short multiplets implies that they must remain BPS states, unless a degeneracy of several BPS multiplets is achieved which can then combine to form a generic massive multiplet. In the absence of such an exotic scenario, the dynamics of the BPS sector forms a closed subsystem.

The stability of BPS (particle) states follows from the fact that their masses are determined by the superalgebra to be the expectation values of the central charge $\mathcal{Z}_i$, $M_i=|\mathcal{Z}_i|$. Since the central charges $\mathcal{Z}_i$ are additive, this implies via the triangle inequality that a BPS state whose mass is

$$M=\left|\sum_i \mathcal{Z}_i\right| \tag{1}$$

is stable with respect to decay into BPS ''constituents'' with masses $M_i=|\mathcal{Z}_i|$,

$$M\leqslant\sum_i M_i\,. \tag{2}$$

Even at points where this equality is saturated there is no phase space for a physical decay. Thus one concludes that BPS particles are indeed stable.

However, restructuring of the spectrum is nonetheless possible because of the existence of special submanifolds of the moduli or parameter space where the inequality (2) is saturated. Specifically, this allows for discontinuities of the spectrum with respect to changes in these moduli. Such changes are ''unphysical'' in the sense that one shifts between different superselection sectors. Nonetheless, one is often interested in considering such an evolution, as it may correspond to the extrapolation from a weakly coupled to a strongly coupled regime. In this case, the stability of BPS states can often be used to infer information about the strongly coupled region. The caveat of course is that one should not cross a submanifold where the bound (2) is saturated, and where restructuring of the spectrum may occur and BPS states may for example disappear. Such submanifolds are consequently known as curves of marginal stability (CMS), although their actual co-dimension in the moduli or parameter space will vary.

Marginal stability curves, and the corresponding discontinuities of the BPS spectrum, are quite ubiquitous in theories with centrally extended supersymmetry algebras. Examples include: the existence of a CMS for the BPS soliton spectrum in general classes of two dimensional models discussed by Cecotti and Vafa [1]; and the CMS for the BPS particle spectrum in $\mathcal{N}=2$ supersymmetric gauge theories [2–5]. In the latter case an explicit demonstration of the discontinuity of the spectrum across these curves in the vacuum moduli space was provided in generic SU(2) theories by Bilal and Ferrari [4,5]. The realization of these dyonic states in terms of type IIB string junctions has also led to the appearance of marginal stability conditions in this context [6,7]. Furthermore, a discontinuity in the BPS spectrum of wall solutions in a Wess-Zumino model with the Taylor-Veneziano-Yankielowicz superpotential [8], which leads to a glued potential, was also observed recently by Smilga and Veselov [9,10]. The discontinuity arises in this case as a function of the mass parameter—a feature also observed in some of the

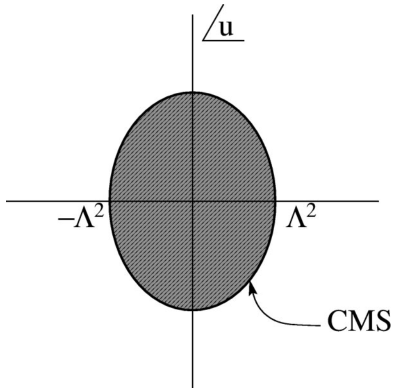

FIG. 1. A schematic representation of the moduli space for $\mathcal{N}=2$ SYM with gauge group SU(2) in terms of the VEV $u=\langle \mathrm{tr}\phi^2\rangle$ of the adjoint scalar ϕ. The W bosons only exist outside the shaded region, which consequently determines their stability domain.

models to be considered in this paper. Finally, we also mention that marginal stability conditions have more recently been studied in the context of string compactification on manifolds with nontrivial cycles [11].

In order to illustrate the discussion with a particular example, we recall that the notion of marginal stability arises, in particular, in the Seiberg-Witten solution [2] of $\mathcal{N}=2$ supersymmetric (SUSY) gauge theories (see e.g. [4]). In the simplest example of pure super Yang-Mills (SYM) theory with gauge group SU(2), there is a one-dimensional elliptical curve of marginal stability in the moduli space (see Fig. 1).

On crossing this curve by varying the moduli a restructuring of the spectrum of BPS states takes place. For instance, the electrically charged vector bosons $W^{\pm}$ only exist outside the CMS, and disappear from the spectrum in the interior region. To make these notions a little more general, we can define a ''stability domain'' as a submanifold of the moduli space in which a particular BPS state exists. This domain will always be bounded by a CMS. In this example, the W boson has a stability domain in the exterior region illustrated in Fig. 1. On crossing the CMS from the stability domain, it is usually stated that the W bosons ''decay'' into a two particle state consisting of a monopole and a dyon with unit electric charge. This interpretation is a little awkward because for a particle to properly decay it must exist in the spectrum, at least as a quasi-stationary state, and this is not true after crossing the CMS. The question then arises as to exactly what happens on the CMS resulting in the apparent discontinuity of the BPS spectrum.

In this paper we will suggest a physical interpretation of this phenomenon, which we summarize below. For this purpose, its convenient to continue with the W boson example to make the ideas more concrete. However, one should bear in mind that this system is not directly accessible to the semiclassical techniques that are used in this paper, because the CMS curve lies at strong coupling. Nonetheless, we will argue that there are several constraints ensuring, at least qualitatively, the generality of this behavior.

Specifically, the emerging picture is that when the moduli approach the CMS, the $W^{\pm}$ states swell in coordinate space as they become more weakly bound. Near the CMS, but still within the stability domain, one can interpret the $W^{\pm}$ as a composite particle built from two primary constituents (a monopole and a dyon of electric charge one), whose interaction can be described by a nonrelativistic (super)potential depending on the relative separation, within the framework of supersymmetric quantum mechanics (SQM) [12]. As one approaches the CMS, the separation of the two primary constituents diverges, while the bound state level reaches the continuum (i.e. the binding energy vanishes). Further motion after crossing the CMS leads to an SQM superpotential which fails to exhibit a supersymmetric ground state separated by a gap from the continuum. Consequently, the ground state in the sector with unit electric charge is no longer the one-particle W boson state but rather a set of non-BPS two-particle states forming a ''long'' multiplet.

We will argue that this picture is the general situation for CMS curves associated with BPS particle states. Namely, whenever a discontinuity occurs in the BPS spectrum at a point in the parameter space, then certain BPS states delocalize in coordinate space. Indeed, the phenomenon of marginal stability of BPS states involves, by definition, the alignment of central charges of primary states in such a way that the binding energy vanishes. In this context it is quite natural that crossing the CMS involves infrared effects, and the 'size' of the marginally stable state should diverge as the CMS is approached. However, within this general picture of *delocalization* one can identify several different mechanisms underlying this behavior.

The features are somewhat dimension dependent, so it is convenient to focus first on 1+1D which will be our primary concern in this paper. We will then remark on certain aspects which distinguish the behavior in 3+1D in particular. Moreover, for solitons in 1+1D its convenient to distinguish two delocalization mechanisms.

(1) The first is when there are no massless fields relevant to the problem, and consequently one can describe the interactions of the primary constituents using non-relativistic collective coordinate dynamics with linearly realized supersymmetry and short range potentials. For a large class of systems (including the ones to be considered here), its possible to limit attention to just one collective coordinate – the relative separation r of the primary solitons. We then observe two characteristic dynamical scenarios for the near CMS dynamics:

In the first case, the short range potential is of deuteron-type which remains finite on the CMS but possesses a single bound state, whose wave function spreads out as the CMS is approached, while the level reaches the continuum at this point. On crossing the CMS, there is no longer a supersymmetric ground state reflecting the fact that the BPS state has disappeared from the spectrum to be replaced as a ground state by a non-BPS two-particle state with the same quantum numbers.

In the second scenario, the relative separation becomes an exact modulus on the CMS, and the potential therefore vanishes at this point. In this case, the composite state still delocalizes as the CMS is approached as the wave function becomes highly spread out. The state is however highly quantum mechanical and has no classical analogue. In this case, we also observe that the potential may support (in general different) composite states on each side of the CMS.

We will study a two-field model exhibiting both these dynamical regimes in subsequent sections.

(2) The second delocalization mechanism arises when there are massless fields involved, these being either the primary states themselves, or the fields via which their interactions are mediated.

First, in situations where massive primary states interact via massless exchange, we shall argue in Sec. II that attractive Coulomb-like interactions between the primary constituents must vanish on the CMS as a consequence of the structure of the BPS mass spectrum.

Second, a new mechanism arises when one or more of the primary states is massless. This scenario may be taken as a special case of (1) in that massless points arise generically as co-dimension one submanifolds on the CMS curve. In this case it is not possible to reduce the effective dynamics to non-relativistic quantum mechanics, and one must consider the effective theory of the massless state. We note that more exotic examples of this behavior may arise (in higher dimensions) at Argyres-Douglas points [13] in $\mathcal{N}=2$ SYM theory, or more generally at second order critical points in supersymmetric theories.

Although we have framed this discussion mostly in the context of 1+1D field theories many of the features apply also in higher dimensions. In particular, restructuring of the BPS spectrum via delocalization is apparently a generic phenomenon. However, an important distinction between 1+1D and, say, 3+1D is that in 1+1D an arbitrarily small attraction is sufficient to form a bound state while in 3+1D this is not the case. For this reason long range forces play a special role in 3+1D, where Coulomb-like attractive potentials are required to form bound states at an arbitrarily small effective coupling. We will discuss this scenario in the form of composite dyons in $\mathcal{N}=2$ SYM theory. The general arguments outlined above will be presented in Sec. II, while particular examples of different scenarios will be discussed in subsequent sections.

In this paper we focus first on class (1) and present an exhaustive study of a particular two-dimensional Wess-Zumino model [14] with $\mathcal{N}=2$ supersymmetry of the type considered previously [15] in a related context. This is a simple model which exhibits composite solitons (kinks) and a corresponding CMS curve accessible to quasiclassical techniques. Thus it serves as an ideal arena to study in detail the effective SQM which determines the presence or otherwise of the composite soliton. The model involves two weakly interacting chiral fields. In the decoupling limit there are ''primary'' BPS kinks for each field, which when quantized lead to short BPS multiplets containing one bosonic and one fermionic state (plus antiparticles). There are also ''composite'' solitons which are combinations of the primary configurations.

Switching on an interaction between the fields we see that the primary BPS solitons exist throughout the parameter space, while the composite solitons exhibit a finite stability domain bounded by the CMS. We analyze in detail the effective SQM which exhibits the composite soliton as a supersymmetric bound state, and verify the behavior described earlier with regard to the approach to the CMS. Its worth noting that in this model the structure of the stability domain is quite complex. In particular, there are different dynamical regimes depending on which part of the CMS curve is crossed. In most cases, a composite state only exists on one side of the CMS. However, there are regions in the parameter space where stability domains for particular composite states meet on a CMS, and consequently (in general different) composite states can exist on either side. In this latter regime we observe that the relative separation of the primary states becomes a modulus on the CMS, as the potential vanishes. Therefore this model exhibits both scenarios outlined in (1) above.

The advantage of dealing with the Wess-Zumino model is that all the features of the non-relativistic quantum dynamics can be calculated analytically. In the vicinity of the CMS we obtain the explicit form for the (super)potential describing the interaction of the primary solitons and are able to track the form of the bound state wave function right onto the CMS. Moreover, certain qualitative aspects are apparently rather model independent due to the constraints imposed by the BPS spectrum.

To investigate the situation in 3+1D we consider explicitly $\mathcal{N}=2$ SYM theory with gauge group SU(3) which contains a spectrum of primary and composite monopole (dyon) solutions. The two ''primary'' monopole solutions are embedded along each of the simple roots of the algebra. An embedding along the additional positive root leads to a ''composite'' dyon which becomes marginally stable on a CMS accessible in the semiclassical region. The major difference between the monopole case in four dimensions in comparison to the two-field model in two dimensions is the presence of massless exchanges resulting in a long range Coulomb-like interaction, which can lead to bound states as noted above. Recently there has been considerable interest in this system [16–22], in part because the composite dyon configuration is an example of a 1/4-BPS state in $\mathcal{N}=4$ SYM theory. This work, which has centered on the moduli space formulation of the low energy dynamics, has resulted in the detailed form of the long range interaction. We observe that, in accord with the general expectations of Sec. II, the attractive component of the long range force (the term $\propto 1/r$ in the effective potential) vanishes on the CMS, while a repulsive component ($\propto 1/r^2$) remains. There is no attraction on the other side of the CMS, the term $\propto 1/r$ changes its sign. Thus, a BPS bound state which exists in the stability domain on one side of the CMS becomes more and more delocalized when approaching the CMS, and ceases to exist on the other side. Accounting for the fact that long range forces are now crucial, we observe that the qualitative picture is nonetheless quite similar to the two-field Wess-Zumino model, in that the composite state delocalizes on approach to the CMS.

The layout of the paper is as follows: In Sec. II we present some general arguments constraining the dynamics of primary solitons near a CMS. Using these results we discuss, in a simplified setting, the underlying mechanism involved in restructuring the spectrum, introducing the necessary notation and definitions in passing. In this section, we also consider the embedding of the effective SQM superalgebra within the superalgebra of the field theory. As a specific example, we consider the realization of the $\mathcal{N}=2$ superalgebra with central charges in two dimensions in the two soliton sector. In this regard its worth remarking that this embedding shows explicitly how the presence of field theoretic central charges is crucial in allowing a linear realization of supersymmetry in the effective non-relativistic dynamics.

Section III presents a detailed analysis of the (quasiclassical) solitons in the two-field Wess Zumino model with regard to their BPS properties. We calculate the form of the CMS, and prove that outside the stability domain the BPS solution corresponding to the composite state ceases to exist. In Sec. IV we derive and discuss the SQM which describes the interaction between the primary solitons in the vicinity of CMS and determines whether or not a supersymmetric bound state exists. We obtain analytic solutions for the superpotential and the bound state wave function.

In Sec. V we consider the more complex situation of monopoles and dyons in $\mathcal{N}=2$ SYM theory with gauge group SU(3), and review the form of the long range potential near the CMS [16–22]. The attractive component vanishes on the CMS, in agreement with the general arguments of Sec. II, while a repulsive component remains leading to delocalization on the CMS even at the classical level.

In Sec. VI we turn to the class (2) delocalization mechanism which involves delocalization due to a field becoming massless on the CMS. We consider a restriction of the two-field model, discussed in Sec. III which, when perturbed by a term which breaks $\mathcal{N}=2$ to $\mathcal{N}=1$supersymmetry, provides a simple example of this phenomenon.

We collect some concluding remarks in Sec. VII, and discuss in particular the applicability of our results to marginal stability of the W boson, and also subtleties associated with extended BPS objects.

II. SOLITON DYNAMICS NEAR THE CMS

Before considering a specific model in detail, we first discuss some simple but quite general constraints which are useful in providing a qualitative guide to the dynamics appropriate to the near-CMS regime.

A. Dominant interactions

Consider the dynamics of two primary BPS solitons with masses M_1 and M_2 near a CMS curve for the composite BPS soliton with mass M_{1+2}. From the CMS condition that the binding energy vanishes, $M_{1+2}=M_1+M_2$, it is clear that by going sufficiently close to the CMS, the relevant energy scales—kinetic and binding energy—can be made much smaller than the soliton masses. The system is then nonrelativistic, and the effective dynamics is described by supersymmetric quantum mechanics on the space of collective coordinates of the configuration. With spherically symmetric interactions, the relevant part of this system can be reduced to one-dimensional SQM associated with the relative separation r between the primary solitons.

We can also deduce some generic features of the potential, in part from knowledge of the BPS mass spectrum. First, it is inconsistent for the potential to be of attractive Coulomb-like form on the CMS itself. This result follows straightforwardly from the incompatibility of the BPS mass spectrum with the structure of the bound state energy levels associated with a Coulomb-like potential. Indeed, the quantum mechanical spectrum associated with the attractive $1/r$ potential will exhibit towers of closely spaced bound states, only the lowest of which can be BPS saturated. The Coulomb wave functions $\psi\sim r^n e^{-r/n}$ lead to bound state levels of the form $\epsilon_n\propto -1/n^2$. In contrast, we know from the form of the BPS mass spectrum that on the CMS the lowest level in the tower must reach the continuum. Clearly the only way this can happen is if the $1/r$ attractive interaction vanishes on the CMS.

In other words, if attractive Coulomb-like forces are generically present, there must be a coefficient which we may identify as the distance to the CMS,

$$V(r)\underset{r\to\infty}{\to}\mathrm{const}-(q^2-f)\frac{1}{4\pi r}+\cdots, \tag{3}$$

where q is used to denote the appropriate charge and f is a certain function of the moduli equal to q^2 on the CMS. The ellipsis denotes higher order terms in $1/r$.

A second constraint is the requirement that the potential admits a normalizable bound state arbitrarily close to the CMS (inside the stability domain). This constraint is dimension dependent. While in 1+1D and 2+1D an arbitrarily small attraction can result in such bound state, this is not the case in higher dimensions. In 3+1D, in order to form a bound state the attraction must be strong enough, $\int dr r(-V(r))>\hbar^2/M$. In particular, for the 3+1D dynamics of dyons in SU(3) SYM, as we will see in Sec. V, the bound state is due to a Coulomb-like attraction at large distances [16–22]. Although according to Eq. (3) the effective Coulomb coupling diminishes on approach to the CMS, the bound state does exist even for an arbitrarily small coupling.

In conclusion, from the simple arguments above we deduce that close to the CMS the dynamics is nonrelativistic and the long range component of the potential controlling the restructuring of the spectrum satisfies the following constraints. First, on the CMS it either vanishes, or is repulsive. Secondly, the simplest way to form bound states in dimensions higher than 2+1D is for the potential to have an attractive long range form off the CMS.

B. The restructuring mechanism

To understand what happens to the spectrum in the near-CMS regime it will be useful to present a simple model which exhibits the relevant features. Specifically, we consider below the mechanism via which a restructuring of the spectrum can occur.

Assume that the model under consideration contains a set of parameters (to be denoted generically as $\{\mu\}$), and admits BPS solitons at a certain value $\{\mu_0\}$. The parameters $\{\mu\}$ can be moduli, or some parameters in the action. The question is how can BPS solitons disappear from the spectrum under continuous variations of $\{\mu\}$? Generally speaking, we would expect that if the BPS state exists at $\{\mu_0\}$, it remains in the spectrum at least in some finite domain in the vicinity of $\{\mu_0\}$.

The argument is based on the multiplicity of the corresponding supermultiplet. Indeed, in the models to be considered below, the number of states in the BPS multiplet is twice smaller than the number of states in the non-BPS multiplet (this type of ''shortening'' is typical). This means that if a BPS state is to become non-BPS, a factor of two jump in the number of states must occur. Generally speaking, this will not happen under continuous deformations of $\{\mu\}$, unless from the very beginning we had *two* BPS multiplets which become degenerate at a certain point in the parameter space and combine together to leave the BPS spectrum as a joint non-BPS multiplet.

We are more interested in another scenario—when a BPS state becomes non-BPS at a certain critical point $\{\mu_*\}$, without the pre-arranged doubling of the type mentioned above. Are we aware of any simple analogs of this phenomenon?

The answer is yes, a simple example has been known for a long time. We will discuss it here for two reasons: first, it nicely illustrates the generalities of the dynamical phenomenon discussed in the preceding subsection; and secondly, we will need to introduce the corresponding notation later anyway. The example can be found in supersymmetric quantum mechanics (SQM) with two supercharges introduced by Witten [12]. Consider a system (as motivated above) described by the Hamiltonian

$$H=\frac{1}{2}[p^2+(W')^2+\sigma_3 W''], \tag{4}$$

where $p=-i\mathrm{d}/\mathrm{d}x$, and W is a function of x with the prime denoting differentiation by x. Moreover, σ_3 is the third Pauli matrix corresponding to the fact that $[\sigma_1,\sigma_2]$ forms an appropriate representation of the Grassmann bilinear. The function W will be referred to as the SQM superpotential. Two conserved (real) supercharges are

$$Q_1=\frac{1}{\sqrt{2}}(p\sigma_1+W'\sigma_2), \quad Q_2=\frac{1}{\sqrt{2}}(p\sigma_2-W'\sigma_1). \tag{5}$$

They form the following superalgebra:

$$(Q_1)^2=(Q_2)^2=H, \quad \{Q_1,Q_2\}=0. \tag{6}$$

If W' has an odd number of zeros then the ground state of the system (4) is supersymmetric (i.e., the supercharges annihilate it) and unique. This is the analog of the BPS soliton. If W' has no zeros or even number of zeros, the ground state is doubly degenerate and is not annihilated by the supercharges. The ground states in this case are analogs of non-BPS solitons. The unique versus doubly degenerate ground state in the problem (4) imitates ''multiplet shortening.'' The transition from the first case to the second under continuous deformations of parameters is easy to visualize. Indeed, let us assume, for definiteness, that

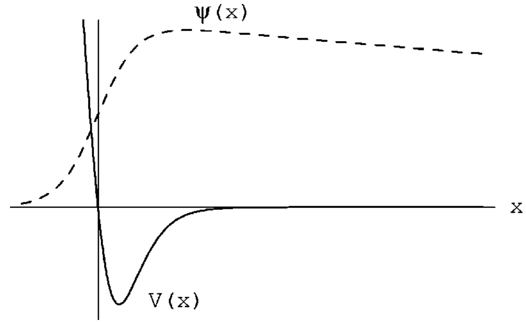

FIG. 2. The potential $V(x)$ in the problem (4), (7) (solid line) and the corresponding ground state wave function (dashed line). The parameter $\mu=0.98$. The units on the vertical and horizontal axes are arbitrary.

$$W(x)=\ln\cosh x-\mu x, \quad W'=\tanh x-\mu. \tag{7}$$

At $\mu=0$ the derivative of the superpotential vanishes at the origin. As μ grows (remaining positive), the point where W' vanishes shifts to the right, towards large positive values of x. The ground state wave function is supersymmetric and unique,

$$\Psi_0=\mathrm{e}^{-W(x)}|\downarrow\rangle=\frac{e^{\mu x}}{\cosh x}|\downarrow\rangle. \tag{8}$$

As one approaches $\mu_*=1$ from below this wave function becomes flatter on the right semi-axis; representing a swelling of the bound state in coordinate space. The corresponding scalar potential

$$V(x)=\frac{1}{2}[(W')^2-W'']$$

at $\mu=0.98$ and the ground state wave function are depicted in Fig. 2.

The point $\mu=1$ is critical. At $\mu>1$ the wave function (8) at $E=0$ becomes non-normalizable, and the true ground state, coinciding with the continuum threshold, is doubly degenerate. The transition from one regime to another occurs through delocalization in that the zero of W', the equilibrium point x_0, escapes to infinity. Note that dynamically the SQM problem under consideration is similar to that of deuterium. The potential well in Fig. 2 is at $x<1$, but the tail of the wave function stretches very far to the right due to the fact that the $E=0$ level is very close to the continuum spectrum.

We can make this somewhat more precise by introducing a ''classicality parameter'' ξ defined as

$$\xi\equiv\left.\frac{W''''(x)}{(W''(x))^2}\right|_{x=x_0}, \tag{9}$$

where x_0 is the classical minimum of the potential: $W'(x_0)=0$. The parameter ξ may be interpreted as measuring the quantum correction to the curvature of the potential at the classical equilibrium point. i.e., the system is essentially classical if $\xi\ll 1$, while it is highly quantum if $\xi\gg 1$.

In the current example, we find that as we approach the critical point,

$$\xi \sim \frac{1}{2(1-\mu)} + \cdots, \tag{10}$$

and so the system indeed becomes highly quantum in this regime. In fact this feature is quite generic for short range potentials and may be viewed as an artifact of the remnant supersymmetry. Specifically, since the mass term $V_F \sim W''(x)$ is linear in the superpotential, while the bosonic potential is quadratic, $V_B \sim (W'(x))^2$, for short range interactions the fermionic W'' term in the superpotential (II B) will dominate for large separations. This is despite the fact that the fermionic term is a quantum effect (in field theory it corresponds to the 1-loop correction to the effective potential through integrating out the fermions). Thus, although the system becomes more and more weakly bound, in the CMS region the system enters a highly quantum regime where the classical minimum of the bosonic potential need not be relevant. Below we will see that exactly the same phenomenon occurs for BPS solitons near the CMS in a 1+1D Wess-Zumino model.

C. Embedding of SQM within the field theory superalgebra

To establish a link between the field-theoretical description of solitons on the one hand and the supersymmetric quantum mechanics of two nonrelativistic primary states on the other, we now consider the manner in which the quantum mechanical supercharges emerge from the full field-theoretical superalgebra. The fact that supersymmetry is realized linearly in the two soliton sector may be reinterpreted as the existence of a straightforward embedding of the SQM supercharges. Moreover, near the CMS the system becomes essentially nonrelativistic and we need keep only the leading term in an expansion in velocities.

Although the arguments apply more generally, we consider for definiteness the realization of $\mathcal{N}=2$ supersymmetry in two dimensions in the two soliton sector. Recall that the algebra contains four supercharges Q_α, $Q_\alpha^\dagger$ ($\alpha=1,2$) and has the form [23,1]

$$\{Q_\alpha, Q_\beta^\dagger\} = 2(\gamma^\mu\gamma^0)_{\alpha\beta}P_\mu, \quad \{Q_\alpha, Q_\beta\} = 2i(\gamma^5\gamma^0)_{\alpha\beta}\bar{\mathcal{Z}},$$
$$\{Q_\alpha^\dagger, Q_\beta^\dagger\} = 2i(\gamma^5\gamma^0)_{\alpha\beta}\mathcal{Z}, \tag{11}$$

where $P_\mu = (P_0, P_1)$ are the energy-momentum operators and $\mathcal{Z}$ is a complex central charge. We use the Majorana basis for 2×2 γ-matrices,

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_3, \quad \gamma^5 = \gamma^0\gamma^1 = -\sigma_1. \tag{12}$$

Modulo addition of the central charge, the algebra (11) can be viewed as a dimensional reduction of the $\mathcal{N}=1$ algebra in four dimensions. The SO(3,1) Lorentz symmetry in 3+1 dimensions reduces in 1+1 to the product SO(1,1)$\times$U$(1)_R$ where SO(1,1) is the Lorentz boost in 1+1 and U$(1)_R$ is a global symmetry associated with the fermion charge. More precisely, the Lorentz boost with parameter β acts on the supercharges Q_α as follows:

$$Q_\alpha \to \left[\exp\left(\frac{1}{2}\beta\gamma^5\right)\right]_{\alpha\beta} Q_\beta, \tag{13}$$

while the U$(1)_R$ transformation with parameter η is

$$Q_\alpha \to \left[\exp\left(\frac{i}{2}\eta\gamma^5\right)\right]_{\alpha\beta} Q_\beta. \tag{14}$$

Notice that the U$(1)_R$ transformations can be viewed as a complexification of the Lorentz boost (13).

Its now convenient to introduce the Majorana supercharges Q_α^i [$i=1,2$; $(Q_\alpha^i)^\dagger = Q_\alpha^i$] via the relation

$$Q_\alpha = \frac{e^{-i\alpha/2}}{\sqrt{2}}(Q_\alpha^1 + iQ_\alpha^2), \tag{15}$$

where the phase factor $e^{-i\alpha/2}$ contains an arbitrary parameter α, which we will fix momentarily. In terms of Q_α^i the algebra (11) has the form

$$\{Q_\alpha^i, Q_\beta^j\} = 2\delta^{ij}(\gamma^\mu\gamma^0)_{\alpha\beta}P_\mu + 2i(\gamma^5\gamma^0)_{\alpha\beta}\mathcal{Z}^{ij}, \tag{16}$$

where the 2×2 real matrix of central charges $\mathcal{Z}^{ij}$ is symmetric and traceless. It is related to the original complex $\mathcal{Z}$ as follows:

$$\mathcal{Z}e^{-i\alpha} = \mathcal{Z}^{11} - i\mathcal{Z}^{12}. \tag{17}$$

To consider representations of the algebra we use a Lorentz boost in 1+1 to put the system in the rest frame where $P_1 \to 0$ and $P_0 \to M = \sqrt{P_\mu P^\mu}$. Moreover, we can always choose the basis in U$(1)_R$ to put $\mathcal{Z}^{ij}$ in the form $\mathcal{Z}^{ij} = |\mathcal{Z}|\tau_3^{ij}$. This amounts to fixing the phase α to be equal to the phase of the central charge, $\mathcal{Z} = |\mathcal{Z}|e^{i\alpha}$. Then the algebra (16) takes the following component form:

$$(Q_1^1)^2 = (Q_2^2)^2 = M + |\mathcal{Z}|, \quad (Q_2^1)^2 = (Q_1^2)^2 = M - |\mathcal{Z}|, \tag{18}$$

with all other anticommutators vanishing, so that the algebra splits into two independent subalgebras.

From Eq. (18) we see that $|\mathcal{Z}|$ is a lower bound for the mass, $M \geqslant |\mathcal{Z}|$. When $M > |\mathcal{Z}|$ the irreducible representation has dimension four—two bosonic and two fermionic states. The BPS states saturate the lower bound, $M_{\mathrm{BPS}} = |\mathcal{Z}|$, and in this case the second subalgebra becomes trivial and the representation is two-dimensional—one bosonic and one fermionic state [23].

How do all of these generalities help us with the problem of constructing the SQM near the CMS? In the vicinity of the CMS the difference $M - |\mathcal{Z}|$ is small as compared to $|\mathcal{Z}|$ and can be identified with the nonrelativistic Hamiltonian,[1]

[1]Note that we view M as a Hilbert space operator.

$$H_{\rm SQM}=M-|\mathcal{Z}|. \tag{19}$$

The second subalgebra in Eq. (18) with supercharges Q_2^1 and Q_1^2 then coincides with that of the standard SQM, see Eq. (6). In the first subalgebra the operator $M+|\mathcal{Z}|$ can be substituted by $2|\mathcal{Z}|$ up to relativistic corrections. Consequently, the first subalgebra just leads to a generic multiplet structure (in this case just duplication) for every state found in the SQM.

In Sec. IV we will find all the supercharges in the 1+1 example as explicit functions of the moduli from field-theoretic solutions for two solitons, u and v. Near the CMS, where their relative motion is nonrelativistic, the result can be compared with the quantum mechanical realization of the superalgebra (18). For $H_{\rm SQM}=M-|\mathcal{Z}|$ we take the expression which generalizes Eq. (4) to include a mass parameter,

$$H_{\rm SQM}=\frac{1}{2M_r}[p^2+(W')^2+\sigma_3 W''], \tag{20}$$

where the superpotential W depends on the separation $s=z_u-z_v$, the conjugate momentum $p=-i\mathrm{d}/\mathrm{d}s$, and M_r is the reduced mass,

$$M_r=\frac{M_u M_v}{M_u+M_v}. \tag{21}$$

Then a realization of the superalgebra can be chosen in the form (σ_i and τ_i are two sets of Pauli matrices):

$$Q_1^1=\sqrt{2|\mathcal{Z}|}\,\tau_1\otimes\sigma_3, \quad Q_2^2=\sqrt{2|\mathcal{Z}|}\,\tau_2\otimes\sigma_3,$$

$$Q_2^1=I\otimes\frac{1}{\sqrt{2M_r}}[p\sigma_1+W'(s)\sigma_2],$$

$$Q_1^2=I\otimes\frac{1}{\sqrt{2M_r}}[p\sigma_2-W'(s)\sigma_1]. \tag{22}$$

The realization (22) explicitly indicates a factorization of both the bosonic and fermionic degrees of freedom associated with the center of mass of the system. We can also include dependence on the total spatial momentum P_1 through a Lorentz boost (13) with $\tanh\beta=P_1/\sqrt{M^2+P_1^2}$.

A couple of comments are now in order. First, it is clear from this construction that the SQM can only be realized linearly in BPS sectors with a non-vanishing central charge. Otherwise, one has $Q=\sqrt{M}\psi$ (with ψ a fermionic operator) in the nonrelativistic limit, implying a nonlinear realization. Secondly, we note that the expressions for Q_1^1 and Q_2^2 in the first line of Eq. (22) represent the leading terms in the nonrelativistic v/c expansion. It is not difficult to include higher order terms in this expansion as follows:

$$Q_1^1=\tau_1\otimes\sigma_3\left[2|\mathcal{Z}|+\frac{p^2+(W')^2+\sigma_3 W''}{2M_r}\right]^{1/2},$$

$$Q_2^2=\tau_2\otimes\sigma_3\left[2|\mathcal{Z}|+\frac{p^2+(W')^2+\sigma_3 W''}{2M_r}\right]^{1/2}, \tag{23}$$

where the square root is to be understood as an expansion in $1/|\mathcal{Z}|$.

In concluding this section, we note that within the context of the present $\mathcal{N}=2$ system one can formulate a general statement: given the subalgebra (6) with two supercharges it is always possible to elevate it to a superalgebra with four supercharges and a central charge $\mathcal{Z}$ by adding the two additional supercharges (23).

III. AN $\mathcal{N}=2$ WZ MODEL IN TWO DIMENSIONS

A. Introducing the model

With the aim of concretely illustrating the general arguments of the previous section, we now consider a specific model. A suitable example exists in two dimensions, obtained by dimensional reduction of a four-dimensional Wess-Zumino model with two chiral superfields. The latter is the deformation of a model considered previously in Ref. [15]. The superpotential is

$$\mathcal{W}(\Phi,X)=\frac{m^2}{\lambda}\Phi-\frac{\lambda}{3}\Phi^3-\lambda\Phi X^2+\mu m X^2+\frac{m^2}{\lambda}\nu X, \tag{24}$$

where Φ and X are two chiral superfields, m is a mass parameter, λ is the coupling constant, while μ and ν are deformation parameters. By an appropriate phase rotation of the fields and the superpotential one can always make m and λ real and positive. The parameters μ and ν are in general complex,

$$\mu\equiv\mu_1+i\mu_2, \quad \nu\equiv\nu_1+i\nu_2. \tag{25}$$

The four real dimensionless parameters μ_1, μ_2, ν_1 and ν_2 will form our parameter space $\{\mu\}$. For technical reasons the parameter $\mu\equiv\mu_1+i\mu_2$ will be assumed to be small in what follows, $\mu_{1,2}\ll 1$. Furthermore, we will consistently work in the approximation in which the SQM superpotential is linear in μ; this corresponds to terms of $O(\mu^2)$ in the scalar potential. This limitation is not a matter of principle but, rather, for technical convenience. In the limit of small μ we can obtain all formulas in closed form. We will also take the coupling constant λ to be small, $\lambda/m\ll 1$, so that a quasiclassical treatment is applicable (except on some exceptional submanifolds in the parameter space).

As a two-dimensional model, this theory has extended $\mathcal{N}=2$ supersymmetry, and exhibits solitonic kinks interpolating between the distinct vacua. In two dimensions the solitons are particles (in four dimensions they would be domain walls). The dimensionality of the BPS supermultiplet is two, while that of the non-BPS supermultiplet is four.

Substituting

$$\Phi=\frac{m}{2\lambda}(U+V), \quad X=\frac{m}{2\lambda}(U-V), \tag{26}$$

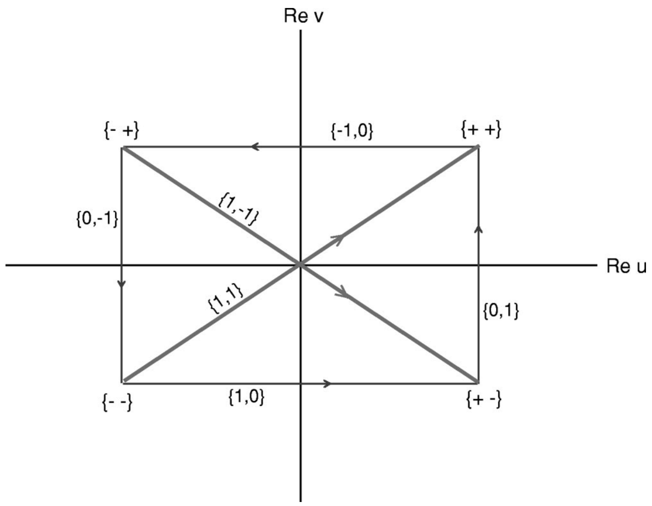

FIG. 3. Structure of vacua and solitons in the Re u, Re v plane for real ν and μ .

we arrive at the following action:

$$S=\frac{m^2}{2\lambda^2}\left\{\frac{1}{4}\int d^2x d^4\theta(\overline{U}U+\overline{V}V)+\left(\frac{m}{2}\int d^2x d^2\theta \mathcal{W}(U,V)+\text{H.c.}\right)\right\}, \tag{27}$$

where the dimensionless superpotential $\mathcal{W}$ is

$$\mathcal{W}(U,V)=U-\frac{1}{3}U^3+V-\frac{1}{3}V^3+\frac{\mu}{2}(U-V)^2+\nu(U-V). \tag{28}$$

The vacua of the model are defined by $\partial\mathcal{W}/\partial u=0$, $\partial\mathcal{W}/\partial v=0$,

$$1+\nu-u^2+\mu(u-v)=0,$$
$$1-\nu-v^2-\mu(u-v)=0. \tag{29}$$

For real μ and ν the solutions to these equations define four different vacua with real values of the fields and real values of the superpotential $\mathcal{W}(u,v)$. The vacuum structure is illustrated in Fig. 3 for small μ. One of these vacua (denoted as $\{++\}$ in Fig. 3) corresponds to a maximum of the real function $\mathcal{W}(u,v)$ on the real section of the variables u and v, the other vacuum (denoted as $\{--\}$) corresponds to a minimum of $\mathcal{W}(u,v)$, and the remaining two vacua ($\{+-\}$ and $\{-+\}$) are saddle points.

In this situation there exists [15] a continuous family of real BPS solitons, i.e., of solutions to the real BPS equations,

$$\frac{1}{m}\frac{d}{dz}u=\frac{\partial\mathcal{W}}{\partial u},\quad \frac{1}{m}\frac{d}{dz}v=\frac{\partial\mathcal{W}}{\partial v}, \tag{30}$$

interpolating between the $\{--\}$ vacuum with $\mathcal{W}_{\min}$ at $z=-\infty$ and the $\{++\}$ vacuum with $\mathcal{W}_{\max}$ at $z=\infty$. All these solitons are degenerate in mass: $M=\mathcal{W}_{\max}-\mathcal{W}_{\min}$, and can be viewed as a superposition of non-interacting primary solitons: one going from the vacuum with $\mathcal{W}_{\min}$ to one of the saddle points, and the other soliton going from the saddle point to the vacuum with $\mathcal{W}_{\max}$. The parameter labeling the solutions in this family can be interpreted in terms of the distance between the basic solitons, and thus the degeneracy in energy implies that there is no interaction between the basic solitons at real μ and ν, at least for some finite range of these parameters.

The decoupling of the dynamics of the primary solitons at $\mu=0$ is trivial, as the superfields U and V are also decoupled within the underlying field theory. However, at $\mu_1\neq 0$, there is no such decoupling within the field theory but, nevertheless, the primary solitons do not interact at rest (provided μ and ν are real). This is a manifestation of the nontrivial ''no-force'' condition for BPS states.

B. Decoupled solitons, $\mu=0$ case

At $\mu=0$ the model is extremely simple: the fields U and V are not coupled. Their VEVs are

$$u=\pm\nu_+,\quad v=\pm\nu_-, \tag{31}$$

where we introduce the notation

$$\nu_\pm=\sqrt{1\pm\nu}. \tag{32}$$

The masses of the BPS solitons are given by

$$M_{n_u,n_v}=\frac{4}{3}\frac{m^3}{\lambda^2}|n_u\nu_+^3+n_v\nu_-^3|, \tag{33}$$

where the topological charges are $n_{u,v}=0,\pm1$ (see Fig. 3).

However, as noted in [15], not all combinations of charges are realized. For a generic value of the complex parameter ν only the $\{1,0\}$ and $\{0,1\}$ solitons and their antiparticles exist as BPS states. To have a BPS state with both n_u and n_v nonvanishing, one needs to align in the complex plane the two terms, ν_+^3 and ν_-^3, contributing to the mass in Eq. (33). The relevant conditions are

$$\text{Im}\left(\frac{\nu_-}{\nu_+}\right)^3=0,\quad \frac{n_v}{n_u}\text{Re}\left(\frac{\nu_-}{\nu_+}\right)^3>0. \tag{34}$$

These conditions define a curve in the complex ν plane presented in Fig. 4.

This curve is the curve of marginal stability for the model. In the case under consideration, with no interaction, the CMS coincides with stability domains for composite solitons, they only exist on this curve.

The curve in Fig. 4 consists of three parts which can be parametrized as

$$\nu=\tanh\sigma,\quad \text{Im}\,\sigma=0,\pm\frac{\pi}{3}. \tag{35}$$

The part sitting on the real axis between $\nu=\pm1$ (corresponding to $\text{Im}\,\sigma=0$) is the stability domain for the $\{1,1\}$

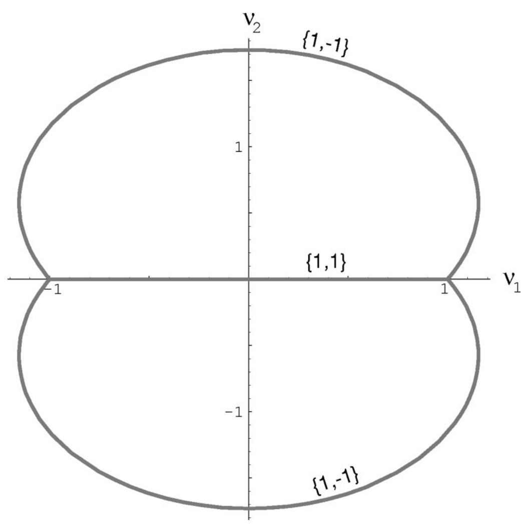

FIG. 4. The curve of marginal stability in the complex plane of ν.

composite solitons (and their antiparticles). The other two parts, $\mathrm{Im}\,\sigma=\pm\pi/3$, give the stability domain for the $\{1,-1\}$ and $\{-1,1\}$ solitons.

The bifurcations at $\nu=\pm 1$ are due to the vanishing of the mass of one of the primary solitons at these points. It is explained by the degeneracy of vacua at these values—instead of four vacua only two remain at $\nu=\pm 1$ (strictly speaking there are still four, but they coalesce in pairs). These are simple analogs of the Argyres-Douglas points [13] in gauge theories.

C. Stabilization by μ

The model at $\mu=0$ is a very degenerate case. Indeed, the extra $\{\pm 1,\pm 1\}$ states exist *only* on the CMS and are nothing but systems of two noninteracting $\{\pm 1,0\}$ and $\{0,\pm 1\}$ solitons. The relative separation between the primary solitons is an extra classical modulus, on quantum level the $\{\pm 1,\pm 1\}$ solitons are not localized states. As we will show, the introduction of a nonvanishing $\mathrm{Im}\,\mu=\mu_2$ expands the domain of stability for the extra BPS state which then occupies a finite area near the original curve. Thus, setting μ_2 nonzero leads to an attraction of the primary solitons.

Using μ as a perturbation parameter we find the VEVS and values of the superpotential $\mathcal{W}$ for the four vacua to first order in μ:

$$\{++\}:\quad u=\nu_+ +\frac{\mu}{2}\left(1-\frac{\nu_-}{\nu_+}\right),\quad v=\nu_- +\frac{\mu}{2}\left(1-\frac{\nu_+}{\nu_-}\right),$$

$$\mathcal{W}_{++}=\frac{2}{3}\nu_+^3+\frac{2}{3}\nu_-^3+\mu(1-\nu_-\nu_+);$$

$$\{+-\}:\quad u=\nu_+ +\frac{\mu}{2}\left(1+\frac{\nu_-}{\nu_+}\right),\quad v=-\nu_- +\frac{\mu}{2}\left(1+\frac{\nu_+}{\nu_-}\right),$$

$$\mathcal{W}_{+-}=\frac{2}{3}\nu_+^3-\frac{2}{3}\nu_-^3+\mu(1+\nu_+\nu_-);$$

$$\{-+\}:\quad u=-\nu_+ +\frac{\mu}{2}\left(1+\frac{\nu_-}{\nu_+}\right),$$

$$v=\nu_- +\frac{\mu}{2}\left(1+\frac{\nu_+}{\nu_-}\right),$$

$$\mathcal{W}_{-+}=-\frac{2}{3}\nu_+^3+\frac{2}{3}\nu_-^3+\mu(1+\nu_+\nu_-);$$

$$\{--\}:\quad u=-\nu_+ +\frac{\mu}{2}\left(1-\frac{\nu_-}{\nu_+}\right),$$

$$v=-\nu_- +\frac{\mu}{2}\left(1-\frac{\nu_+}{\nu_-}\right),$$

$$\mathcal{W}_{--}=-\frac{2}{3}\nu_+^3-\frac{2}{3}\nu_-^3+\mu(1-\nu_+\nu_-). \tag{36}$$

The BPS masses are given by $|\mathcal{W}_{ij}-\mathcal{W}_{i'j'}|$ and the alignment conditions which define the CMS to first order in μ become [cf. Eq. (34)],

$$\mathrm{Im}\left(\frac{\nu_-^2-\mu\nu_+}{\nu_+^2+\mu\nu_-}\right)^{3/2}=0,\quad \mathrm{Im}\left(\frac{\nu_-^2+\mu\nu_+}{\nu_+^2-\mu\nu_-}\right)^{3/2}=0, \tag{37}$$

where the conditions clearly differ only by a choice of the branch of the square root in the terms linear in μ. Analytical expressions for the CMS are simpler in terms of the complex parameter of σ [related to ν by Eq. (35)]. In the complex σ-plane the CMS is given by the curves

$$\sigma_2=\pm\mu_2\cosh\frac{3\sigma_1}{2}\cosh^{1/2}\sigma_1,$$

$$\sigma_2=\frac{\pi}{3}\pm\sinh\frac{3\sigma_1}{2}\mathrm{Re}\left[\mu\cosh^{1/2}\left(\sigma_1+i\frac{\pi}{3}\right)\right],$$

$$\sigma_2=-\frac{\pi}{3}\pm\sinh\frac{3\sigma_1}{2}\mathrm{Re}\left[\mu\cosh^{1/2}\left(\sigma_1-i\frac{\pi}{3}\right)\right], \tag{38}$$

where the indices 1 and 2 refer to the real and imaginary parts, $\sigma=\sigma_1+i\sigma_2$.

The curves of marginal stability in the ν plane are presented in Fig. 5. They form the boundaries of the stability domains for the composite BPS states marked in the figure.

Figure 5 exemplifies different metamorphoses of the composite BPS solitons on the CMS: crossing some boundaries leads to disappearance of the BPS state from the spectrum, on others the original BPS state disappears but a new one appears.

The figure also shows exceptional points on the CMS, where two stability subdomains of the same BPS soliton touch each other. We shall address a dynamical scenario at such points in Sec. IV C. Note also four points of bifurcation (the Argyres-Douglas points) where a pair of the vacuum states collide.

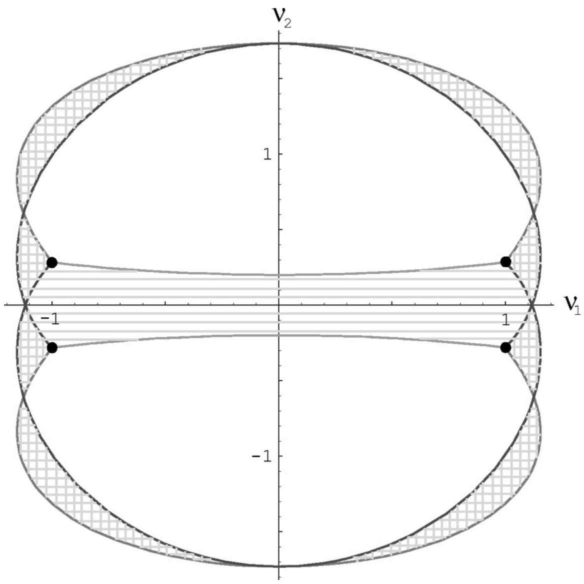

FIG. 5. The domains of stability for the composite BPS states (shown for $\mu_2=0.2$). The hatched region along the real axis is the stability domain for the $\{1,1\}$ solitons and its antiparticles; in the cross hatched one the $\{1,-1\}$ solitons and its antiparticles are stable.

D. A loosely bound composite BPS state

In this subsection we will find a solution to the BPS equations for the composite $\{1,1\}$ soliton. The construction explicitly demonstrates that in the vicinity of the CMS this soliton is a loosely bound state of the primary constituents. For definiteness we choose the region near the real ν-axis and the $\{--\}\to\{++\}$ transition. The BPS equations have the form

$$\frac{1}{m}\frac{du}{dz}=e^{i\alpha}[1+\nu^*-(u^*)^2+\mu^*(u^*-v^*)]$$

$$\frac{1}{m}\frac{dv}{dz}=e^{i\alpha}[1-\nu^*-(v^*)^2-\mu^*(u^*-v^*)] \tag{39}$$

where

$$e^{i\alpha}=\sqrt{\frac{\mathcal{W}_{++}-\mathcal{W}_{--}}{\mathcal{W}^*_{++}-\mathcal{W}^*_{--}}}=\frac{\nu_+^3+\nu_-^3}{|\nu_+^3+\nu_-^3|}. \tag{40}$$

We will use perturbation theory in μ. The part of the CMS chosen for consideration at zeroth order in μ corresponds to real ν: $-1<\nu_1<1$, $\nu_2=0$. Then, at this order, $\alpha=0$ and the solution for u and v reads

$$u^{(0)}=\nu_{1+}\tanh[\nu_{1+}m(z-z_u)],$$

$$v^{(0)}=\nu_{1-}\tanh[\nu_{1-}m(z-z_v)], \tag{41}$$

where $\nu_{1\pm}=\sqrt{\nu_1\pm1}$ [see Eq. (32)] and the parameters z_u and z_v are arbitrary and denote the positions of the centers of the u and v solitons.

At first order in μ, the soliton solutions become complex. With an expansion about the leading order solutions $u^{(0)}$, $v^{(0)}$ of the form

$$u=u^{(0)}+(u_1+iu_2)+\cdots,\quad v=v^{(0)}+(v_1+iv_2)+\cdots, \tag{42}$$

Eqs. (39) lead to

$$\frac{1}{m}\frac{d}{dz}u_1=-2u^{(0)}u_1+\mu_1(u^{(0)}-v^{(0)}),$$

$$\frac{1}{m}\frac{d}{dz}v_1=-2v^{(0)}v_1-\mu_1(u^{(0)}-v^{(0)}),$$

$$\frac{1}{m}\frac{d}{dz}u_2=2u^{(0)}u_2-\nu_2+\alpha[\nu_{1+}^2-(u^{(0)})^2]-\mu_2[u^{(0)}-v^{(0)}],$$

$$\frac{1}{m}\frac{d}{dz}v_2=2v^{(0)}v_2+\nu_2+\alpha[\nu_{1-}^2-(v^{(0)})^2]+\mu_2[u^{(0)}-v^{(0)}], \tag{43}$$

where

$$\alpha=\frac{3}{2}\nu_2\frac{\nu_{1+}-\nu_{1-}}{\nu_{1+}^3+\nu_{1-}^3}. \tag{44}$$

Let us consider the equation for u_2. The function $\cosh^2[\nu_{1+}m(z-z_u)]$ is the solution of the homogeneous part of this equation, and the full solution is

$$u_2(z)=\cosh^2[\nu_{1+}m(z-z_u)]m\int_{-\infty}^{z}dx\frac{-\nu_2+\alpha[\nu_{1+}^2-(u^{(0)}(x))^2]-\mu_2[u^{(0)}(x)-v^{(0)}(x)]}{\cosh^2[\nu_{1+}m(x-z_u)]}. \tag{45}$$

As $z\to-\infty$ the solution satisfies the boundary condition

$$\lim_{z\to-\infty}u_2(z)=-\frac{\nu_2}{2\nu_+}+\frac{\mu_2}{2}\left(1-\frac{\nu_{1-}}{\nu_{1+}}\right) \tag{46}$$

consistent with $\mathrm{Im}\,u_{--}$ in Eq. (36) at the order considered here. As $z\to\infty$ the solution $u_2(z)$ grows exponentially unless the relation

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{-\nu_2 + \alpha[\nu_{1+}^2 - (u^{(0)}(x))^2] - \mu_2[u^{(0)}(x) - v^{(0)}(x)]}{\cosh^2[\nu_{1+} m(x - z_u)]} = 0 \tag{47}$$

is fulfilled. Once this relation is met the $z \to \infty$ boundary condition $u_2 \to \mathrm{Im}\, u_{++}$ is also satisfied.

The relation (47) can be viewed as a constraint ensuring orthogonality of the inhomogeneous part in the u_2-equation and the zero mode $\cosh^{-2}[\nu_{1+} m(z - z_u)]$ in u_1. This approximate zero mode corresponds to a shift of the u-soliton center and at the same time also represents the spatial dependence of the corresponding fermionic zero mode.[2] The relation (47) then fixes the separation $z_u - z_v$ of the two primary solitons and can be presented in the form

$$w_{\nu_1}(z_u - z_v) = \frac{\nu_2}{\mu_2 \kappa}, \tag{48}$$

where

$$\kappa = \frac{1}{2}[\nu_{1+}^3 + \nu_{1-}^3], \tag{49}$$

and the function w_ν of the soliton separation $s = z_u - z_v$ is defined as

$$w_\nu(s) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh^2 x} \tanh\left[\frac{\nu_-}{\nu_+} x + \nu_- m s\right]. \tag{50}$$

It is important that the condition (48) also ensures that the solution for v_2,

$$v_2(z) = \cosh^2[\nu_{1-} m(z - z_v)] m \int_{-\infty}^{z} \mathrm{d}x \frac{\nu_2 + \alpha[\nu_{1-}^2 - (v^{(0)}(x))^2] + \mu_2[u^{(0)}(x) - v^{(0)}(x)]}{\cosh^2[\nu_{1-} m(x - z_v)]}, \tag{51}$$

is finite at both $z \to +\infty$ and $z \to -\infty$, and thus satisfies the proper boundary conditions. As for the solutions for the real parts, u_1 and v_1, described by the first pair of equations in (43), these solutions always exist, due to the existence of real BPS solitons in the model with real parameters [15], as discussed in Sec. III A. Thus no additional constraint arises.

Here we make a few remarks on the properties of the function $w_\nu(s)$, defined by the integral (50). The symmetry properties of this function can easily be seen by writing it as the derivative $w_\nu(s) = \mathrm{d} g_\nu(s)/\mathrm{d}s$ of the function

$$g_\nu(s) = \frac{m}{2} \int_{-\infty}^{\infty} \mathrm{d}z \{1 - \tanh[\nu_+ m z] \tanh[\nu_- m(z + s)]\}, \tag{52}$$

which is symmetric under separate reversal of the sign of ν or s. Thus $w_\nu(s)$ is even in the index ν: $w_{-\nu}(s) = w_\nu(s)$, and is odd in the variable s: $w_\nu(-s) = -w_\nu(s)$, and is monotonically increasing from $w_\nu(s \to -\infty) = -1$ to $w_\nu(s \to +\infty) = +1$. At large positive s its asymptotic behavior is given by

$$w_\nu(s) = 1 - \frac{\nu_+ + \nu_-}{\nu}[\nu_+ \exp(-2\nu_- m s) - \nu_- \exp(-2\nu_+ m s)] + \cdots, \tag{53}$$

where the ellipsis stands for higher powers and mixed products of the two exponents: $\exp(-2\nu_- m s)$ and $\exp(-2\nu_+ m s)$. At $\nu = 0$ the integral in Eq. (50) can be expressed in terms of elementary functions,

$$w_0(s) = \coth ms - \frac{ms}{\sinh^2 ms}, \tag{54}$$

and the asymptotic behavior of $w_0(s)$ as $s \to +\infty$:

$$w_0(s) = 1 - (4ms + 2)e^{-2ms} + \cdots \tag{55}$$

is in agreement with the $\nu \to 0$ limit of the expression (53). Plots of $w_\nu(s)$ for a few values of ν are shown in Fig. 6.

The limited magnitude of $w_\nu(s)$, $|w_\nu(s)| \leqslant 1$, means that the BPS solution we consider only exists in the range

$$|\nu_2| \leqslant |\mu_2| \kappa. \tag{56}$$

As expected the boundaries of this range coincide with the part of CMS found previously by algebraic means near the real ν axis.

It is then simple to find the value for the distance s_0 between the primary solitons in the BPS composite state. Say, for $\nu_1 = 0$, we have when $|\mu_2| - |\nu_2| \ll |\mu_2|$

$$\mathrm{e}^{m|s_0|} = \eta \ln \eta, \quad \text{where} \quad \eta = \sqrt{\frac{|\mu_2|}{|\mu_2| - |\nu_2|}}. \tag{57}$$

[2] In the next section we will show that this orthogonality condition is equivalent to the vanishing of a particular supercharge.

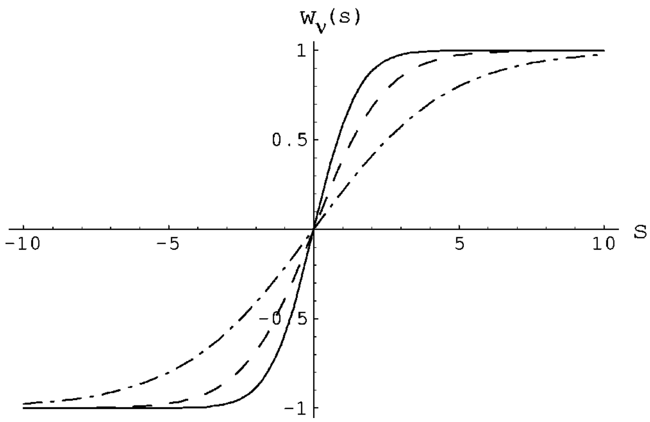

FIG. 6. Plots of $w_\nu(s)$ at $\nu=0$ (solid), $\nu=0.8$ (dashed), and $\nu=0.95$ (dot-dashed); s is measured in units of $1/m$.

IV. QUANTUM MECHANICS OF TWO SOLITONS

The BPS state which connects the vacua $\{++\}$ and $\{--\}$ and exists within the stability domain can be viewed as a bound state of one u-soliton, located at $z=z_u$, and one v-soliton, located at $z=z_v$. The equilibrium separation between the solitons, $s=z_u-z_v$, at which the minimum is achieved at given ν_2 and μ_2, is determined from Eq. (48). In this section we consider the supersymmetric quantum mechanics of the two soliton system. The SQM system describes the BPS bound state (which is the ground state in the problem) within the stability domain, as well as low-lying non-BPS exited states.

The formulation of this problem refers to an effective description of the two solitons as heavy particles with masses M_u and M_v in terms of their coordinates z_u and z_v. This approximation is natural near the CMS where the binding energy is small relative to the soliton masses. For slowly moving solitons $|\dot{z}_u|,|\dot{z}_v|\ll 1$ the nonrelativistic energy can be written as

$$E=M_u\frac{\dot{z}_u^2}{2}+M_v\frac{\dot{z}_v^2}{2}+U(s), \tag{58}$$

where the dot denotes the time derivative, and $U(s)$ is the interaction potential depending on the separation $s=z_u-z_v$ between the solitons. Separating out the center of mass coordinate, we come to a standard quantum mechanical Hamiltonian for the relative motion,

$$H=\frac{p^2}{2M_r}+U(s). \tag{59}$$

The supersymmetric generalization of this Hamiltonian is given in Eq. (20) which depends on the superpotential $W(s)$. Below we will find the expression for this SQM superpotential by comparing the field theoretic supercharges evaluated on the soliton solutions with the SQM realization in Eq. (22). An alternative derivation of W' based solely on conventional bosonic considerations is presented in the Appendix.

A. SQM superpotential from field-theoretic supercharges

The action (27) for the Wess-Zumino model leads to the following expression for the supercharges Q_α:

$$Q=\frac{m^2}{\sqrt{2}\lambda^2}\int dz[\partial_t u\,\psi+\partial_z u\,\gamma^0\gamma^1\psi+im\,\partial_{\bar{u}}\overline{\mathcal{W}}\gamma^0\psi^* +(u\to v,\psi\to\eta)], \tag{60}$$

where u, ψ_α and v, η_α are the bosonic and fermionic components of the U and V superfields, and $\mathcal{W}(u,v)$ is the superpotential of the model. The two remaining supercharges $\overline{Q}_\alpha$ are just the complex conjugates of Q_α.

Let us first evaluate the supercharges for the u-soliton in the leading approximation, i.e., when $\nu_2=\mu_1=\mu_2=0$. The field u is given by Eq. (41), $u=u^{(0)}=\nu_{1+}\tanh[\nu_{1+}m(z-z_u)]$, while v is a constant, $v=-\nu_{1-}$. For the fermionic fields we substitute zero modes, two of which are in the field ψ_α, and there are none in η_α,

$$\psi_{\text{zero modes}}=\begin{pmatrix} ib_u \\ a_u\end{pmatrix}\frac{1}{\sqrt{2M_u}}\partial_z u^{(0)}. \tag{61}$$

In this expression $M_u=(4m^3/3\lambda^2)\nu_{1+}^3$ is the mass of the u-soliton, a_u and b_u are real fermionic operators entering as coefficients of the normalized zero modes, and their algebra is fixed by canonical quantization,

$$a_u^2=b_u^2=1, \quad \{a_u,b_u\}=0. \tag{62}$$

Upon these substitutions, the supercharge Q_α becomes

$$Q=\sqrt{M_u}\begin{pmatrix} a_u \\ ib_u\end{pmatrix}, \tag{63}$$

which can be rewritten in terms of real charges [see Eq. (15) in Sec. II C for definitions],

$$Q_1^1=\sqrt{2M_u}a_u, \quad Q_2^2=\sqrt{2M_u}b_u, \quad Q_2^1=Q_1^2=0. \tag{64}$$

The result for the supercharges matches the general construction of Sec. II C wherein the operators a_u and b_u can be realized as Pauli matrices, e.g., $a_u=\tau_1$ and $b_u=\tau_2$.

Now let us find the supercharges corresponding to the $\{1,1\}$ configuration of the u- and v-solitons at $\nu_2=\mu_1=\mu_2=0$. We choose boosted soliton solutions,

$$u^{(0)}=\nu_{1+}\tanh\left[\nu_{1+}m\left(z-z_u-\frac{p}{M_u}t\right)\right],$$
$$v^{(0)}=\nu_{1-}\tanh\left[\nu_{1-}m\left(z-z_v+\frac{p}{M_v}t\right)\right], \tag{65}$$

where p is their relative momentum, and the total momentum is zero. The fermions are given by Eq. (61) for ψ and by a similar expression for η with the substitution $u\to v$, where u- and v-fermions anticommute. With time-dependent solutions the terms $\partial_t u\,\psi$, $\partial_t v\,\eta$ now contribute to the supercharges (60),

$$Q_1^1=\sqrt{2(M_u+M_v)}(a_u\cos\delta+a_v\sin\delta),$$

$$Q_2^2=\sqrt{2(M_u+M_v)}(b_u\cos\delta+b_v\sin\delta),$$

$$Q_2^1=\frac{p}{\sqrt{2M_r}}(-a_u\sin\delta+a_v\cos\delta),$$

$$Q_1^2=\frac{p}{\sqrt{2M_r}}(-b_u\sin\delta+b_v\cos\delta), \tag{66}$$

where we have defined $\cos\delta=\sqrt{M_u/(M_u+M_v)}$. We observe that the relative motion implies that the ''composite'' state is non-BPS in the absence of any interaction between the solitons.

In order to switch on the interaction we consider nonzero μ and ν_2. To obtain the result to first order in these parameters it is enough to substitute the same leading order expressions for the bosonic and fermionic fields accounting for the terms linear in μ and ν_2 in the expression (60) for the supercharges, as well as for the phase α of the central charge. The linear dependence on μ and ν_2 arises from the terms

$$\frac{m^2}{\sqrt{2}\lambda^2}\int dz[\mu(u^{(0)}-v^{(0)})-i\nu_2]\gamma^0(\psi^*-\eta^*) \tag{67}$$

in Eq. (60). The phase α is also linear in ν_2 [see Eq. (44)], and needs to be taken into account in Eq. (15) when relating Q_α with $Q_\alpha^{1,2}$.

The resulting supercharges are (Q_1^1, Q_2^2 are not changed and are written here for completeness)

$$Q_1^1=\sqrt{2(M_u+M_v)}a,\quad Q_2^2=\sqrt{2(M_u+M_v)}b,$$

$$Q_2^1=\frac{1}{\sqrt{2M_r}}[p\tilde{a}+W'(s)\tilde{b}],\quad Q_1^2=\frac{1}{\sqrt{2M_r}}[p\tilde{b}-W'(s)\tilde{a}], \tag{68}$$

where we denote

$$a=a_u\cos\delta+a_v\sin\delta,\quad b=b_u\cos\delta+b_v\sin\delta,$$

$$\tilde{a}=-a_u\sin\delta+a_v\cos\delta,\quad \tilde{b}=-b_u\sin\delta+b_v\cos\delta. \tag{69}$$

The quantum-mechanical superpotential (more precisely its derivative W') is then read from Q_2^1 (or Q_1^2) to be

$$W'(s)=\frac{3M_r}{1-\nu_1^2}[\mu_2\kappa w_{\nu_1}(s)-\nu_2], \tag{70}$$

where

$$M_r=\frac{M_uM_v}{M_u+M_v}=\frac{2}{3}\frac{m^3}{\lambda^2}\frac{(1-\nu_1^2)^{3/2}}{\kappa},\quad \kappa=\frac{1}{2}(\nu_{1+}^3+\nu_{1-}^3), \tag{71}$$

and the function $w_\nu(s)$ is defined by Eq. (50).

The SQM Hamiltonian then has the form

$$H_{\rm SQM}=(Q_2^1)^2=(Q_1^2)^2=\frac{1}{2M_r}[p^2+(W'(s))^2-iW''(s)\tilde{a}\tilde{b}]. \tag{72}$$

An explicit matrix realization of the four operators $a_{u,v}$ and $b_{u,v}$, satisfying the Clifford algebra, can *a priori* be chosen in the factorized form: $a_u=\tau_1\otimes\sigma_3$, $b_u=\tau_2\otimes\sigma_3$, $a_v=I\otimes\sigma_1$, and $b_v=I\otimes\sigma_2$. This factorized form of the fermionic operators realizes a description in terms of two independent particles. This choice is perfectly acceptable and realizes the $\mathcal{N}=2$ superalgebra (18). However, it differs from the specific realization (22) by an orthogonal rotation by angle δ. In order to match the conventions used in Eq. (22) for a description of the two-soliton system, one has to use an equivalent representation of these operators, obtained by the inverse rotation:

$$a_u=\tau_1\otimes\sigma_3\cos\delta-I\otimes\sigma_1\sin\delta,$$

$$b_u=\tau_2\otimes\sigma_3\cos\delta-I\otimes\sigma_2\sin\delta,$$

$$a_v=\tau_1\otimes\sigma_3\sin\delta+I\otimes\sigma_1\cos\delta,$$

$$b_v=\tau_2\otimes\sigma_3\sin\delta+I\otimes\sigma_2\cos\delta. \tag{73}$$

The final expression for the full quantum Hamiltonian of the two-soliton system can be written as

$$H_{\rm SQM}=\frac{p^2}{2M_r}+\frac{9M_r}{2}\frac{[\mu_2\kappa w_{\nu_1}(s)-\nu_2]^2}{(1-\nu_1^2)^2}+\frac{3}{2}\frac{\mu_2\kappa w'_{\nu_1}(s)}{(1-\nu_1^2)}\sigma_3 \tag{74}$$

with M_r given in Eq. (71) and we use the matrix representation (73) for the fermions (omitting the tensor product with unity in $H_{\rm SQM}$).

It is worth noting a couple of limits in which the potential simplifies, and can be expressed in terms of elementary functions. Recall first of all that when $\nu_1=0$ the function $w_0(s)$ is known analytically [see Eq. (54)]. If ν_1 is not too large, i.e., $\nu_1\leqslant0.5$, there also exists a convenient simplified form in which the superpotential is very closely approximated by the expression

$$W'_{\rm approx}(s)=3M_r[\mu_2\tanh(ms)-\nu_2], \tag{75}$$

where the reduced mass M_r is taken at $\nu_1=0$. In this superpotential we recognize the simplified model discussed in Sec. II B [see Eq. (7)]. Another simple case arises when ν_1^2 is close to 1, i.e., $1-\nu_1^2\ll1$,

$$W'=\frac{3M_r}{\sqrt{1-\nu_1^2}}(\mu_2ms-\nu_2). \tag{76}$$

The potential in this case reduces to that of the harmonic oscillator.

In the limit of large separation, the potential energy in the Hamiltonian (74) tends to a constant which depends on the sign of s,

$$U_{\pm}=U(s\to\pm\infty)=\frac{9M_r}{2}\frac{(\pm\mu_2\kappa-\nu_2)^2}{(1-\nu_1^2)^2}. \tag{77}$$

(Note, however, that the spin dependent σ_3 term does not contribute.) These constants denote the energy levels at which the continuum states appear while the ground state, which is the {1,1} BPS soliton, is a zero energy eigenfunction of $H_{\rm SQM}$.

The origin of the two continuum thresholds is that at nonzero μ the classification for solitons we introduced at $\mu=0$ is no longer sufficient—the u-soliton interpolating between the $\{-+\}$ and $\{++\}$ vacua (see Fig. 3) is different from the $\tilde{u}$-soliton interpolating between the $\{--\}$ and $\{+-\}$ vacua, and a similar distinction arises between the v- and $\tilde{v}$-solitons. Thus, the system under consideration at large s describes two channels: the u plus v solitons at positive s, and the $\tilde{u}$ plus $\tilde{v}$ at negative s. It is straightforward to verify this by calculating the two binding energies,

$$\begin{aligned}\Delta E_+&=M_{1,1}-M_u-M_v\\&=\frac{m^3}{\lambda^2}[|\mathcal{W}_{++}-\mathcal{W}_{--}|-|\mathcal{W}_{++}-\mathcal{W}_{-+}|\\&\quad-|\mathcal{W}_{-+}-\mathcal{W}_{--}|],\\ \Delta E_-&=M_{1,1}-M_{\tilde{u}}+M_{\tilde{v}}\\&=\frac{m^3}{\lambda^2}[|\mathcal{W}_{++}-\mathcal{W}_{--}|-|\mathcal{W}_{+-}-\mathcal{W}_{--}|\\&\quad-|\mathcal{W}_{++}-\mathcal{W}_{+-}|],\end{aligned} \tag{78}$$

from which we observe that $\Delta E_{\pm}=-U_{\pm}$. Note that, although the quantities $\Delta E_{\pm}$ are of second order in μ_2 and ν_2, it is sufficient to use the expressions (36) which are only valid to first order in μ (and are formally exact in ν) for the values of $\mathcal{W}_{ij}$. This is due to the fact that for real ν and μ the values of $\mathcal{W}_{ij}$ are real and $\Delta E_{\pm}$ vanishes. Thus $\Delta E_{\pm}$ arises as an effect quadratic in the imaginary parts of the differences of $\mathcal{W}_{ij}$ which by themselves are linear in ν_2 and μ_2.

Finally, we also write down the asymptotic behavior of the potential as $s\to\pm\infty$,

$$U(s)\to U_{\pm}+K_{\pm}\exp(-2\nu_{1-}m|s|)+\cdots, \tag{79}$$

where the coefficient of the leading exponential term is

$$\begin{aligned}K_{\pm}=&\frac{3(\nu_{1+}+\nu_{1-})\mu_2\kappa}{\nu_1\nu_{1+}(1-\nu_1)(1-\nu_1^2)}\\&\times[-6M_r(\mu_2\kappa\mp\nu_2)+m\sigma_3(1-\nu_1^2)\nu_{1-}].\end{aligned} \tag{80}$$

We have made the assumption here that $\nu_{1-}<\nu_{1+}$. We see that the characteristic distance s is defined by $1/m\nu_{1-}$ which as expected is the wavelength for the lightest particle in the model. We also observe that the spin dependent term contributes to the exponential tail. Moreover, on the CMS where $\mu_2\kappa\mp\nu_2=0$, it is the only contribution. This ''fermionic dominance'' takes place in a very narrow region near the CMS,

$$|\mu_2\kappa\mp\nu_2|\ll\frac{(1-\nu_1^2)\nu_{1-}}{6}\frac{m}{M_r}. \tag{81}$$

The effect of this regime of enhanced quantum corrections near the CMS will be considered in more detail in the next subsection.

B. Properties of the two-soliton system

As expected, the second term in the potential in Eq. (74) is of higher order than the first one in the loop expansion parameter λ. However the second term is of lower order in the small parameter μ_2 and, by tuning μ_2, one can study this potential both in the classical limit corresponding to $\mu_2\gg\lambda^2/m^2$ and in the quantum limit $\mu_2\ll\lambda^2/m^2$, or anywhere in between as long as the condition for validity of the formula (74), $\mu_2\ll 1$, is maintained. Upon a slightly more detailed inspection of classical *vs* quantum effects in the Hamiltonian (74) one can readily see that in fact the quasiclassical parameter in this system is not just the ratio $\lambda^2/(m^2\mu_2)$ but is determined by the parameter

$$\xi=\frac{W''''(s_0)}{(W''(s_0))^2}, \tag{82}$$

introduced in Sec. II, where s_0 is the classical equilibrium separation determined by Eq. (48). Recall that ξ measures the quantum correction to the curvature of the potential near the classical minimum, the system being essentially classical for $\xi\ll 1$, and highly quantum for $\xi\gg 1$.

For the model at hand we find

$$\xi=\frac{w'''_{\nu_1}(s_0)}{2m\mu_2\sqrt{1-\nu_1^2}(w'_{\nu_1}(s_0))^2}\frac{\lambda^2}{m^2}. \tag{83}$$

Near the CMS the equilibrium distance s_0 is large, and ξ takes the form

$$\xi=\frac{\kappa}{\nu_{1+}(\mu_2\kappa\mp\nu_2)}\frac{\lambda^2}{m^2}, \tag{84}$$

where for definiteness we have again assumed that $\nu_{1-}<\nu_{1+}$. Notice that the condition $|\xi|\gg 1$ agrees with Eq. (81) which, as discussed above, defines the essentially quantum regime in the narrow region along the CMS.

When the system admits a supersymmetric ground state, the corresponding wave function $\psi_0(s)$ can always be found as

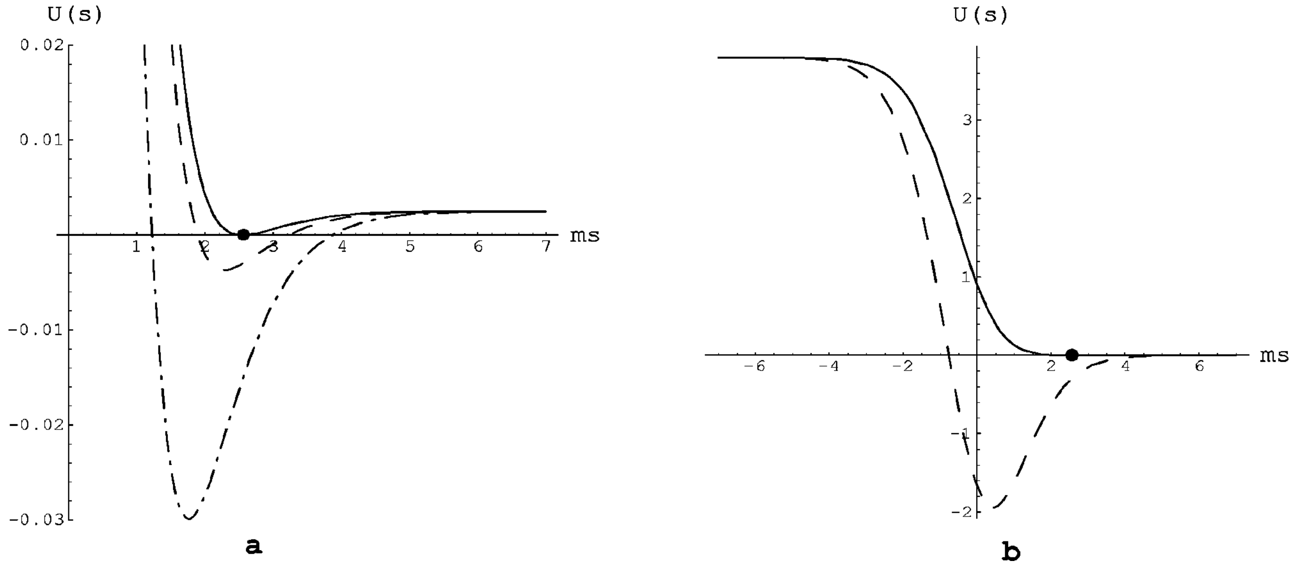

FIG. 7. Plots of the full potential $U(s)$ (arbitrary units) at $\nu_1=0$, $\nu_2/\mu_2=0.95$ for several values of ξ. The classical equilibrium point is at $ms\approx 2.56$ and is shown by heavy dot. (a) Details of the potential near minimum for $\xi=0.009$ (solid), $\xi=0.9$ (dashed), and for $\xi=4.45$ (dot-dashed). (b) The potential shown at a larger scale. The curves for $\xi\leqslant 4.45$ are practically unresolvable and coincide with the solid curve; the dashed curve corresponds to $\xi=125$. It can be noticed that the latter value of ξ still corresponds to moderate values of λ/m: $\lambda^2/m^2\approx 7.7\mu_2$.

$$\psi_0(s)=\text{const}\exp[-W(s)]$$
$$=\text{const}\exp\left[-3\frac{M_r}{m}\,\frac{\mu_2\kappa g_{\nu_1}(s)-\nu_2 ms}{1-\nu_1^2}\right], \quad (85)$$

where for definiteness we again assume $\mu_2>0$, and $g_\nu(s)$, defined by Eq. (52), is the integral of $w_\nu(s)$. Independently of the quasiclassical parameter ξ the maximum of $\psi_0(s)$ is always located at s_0. However the spread of the wave function, i.e., the dispersion of the distance between the solitons in the BPS bound state, essentially depends on the parameter ξ. As $\xi\to 0$ the full potential has a minimum at $s=s_0$, and the system is classically located at the minimum. At larger ξ the minimum of the full potential shifts towards $s=0$, reaching $s=0$ in the limit $\xi\gg 1$, but the maximum of the wave function is still at $s=s_0$. In the latter extreme quantum limit the system resembles the deuteron: the wave function spreads over distances much larger than the size of the interaction region. In the two-soliton system this behavior is even more drastic at large ξ than in the deuteron: the wave function reaches its *maximum* far beyond the interaction region. The classical and the quantum behavior of the system at different values of ξ is illustrated by a series of plots in Figs. 7 and 8.

One may also note that in general the interaction of the two solitons is maximal at distances of order m^{-1} near $s=0$: the potential is asymmetric in s and changes rapidly near $s=0$, i.e., the force is strongest when the solitons substantially overlap. In a narrow region near the CMS, given by Eq. (81), an attraction at short distances creates an essentially quantum state, resembling a deuteron. Deeper into the stability region an exponentially shallow minimum of the potential at large s_0 results in a quasiclassical bound state.

Once one crosses the CMS, the wave function (85) is no longer normalizable, and the physical ground state of the system is non-supersymmetric. The broadening of the wave function for the bound state near the CMS is exhibited in Fig. 8. Thus, as discussed in Sec. II B the bound state level reaches the continuum on the CMS, where it completely delocalizes, and on crossing the CMS the $\{1,1\}$ bound state is no longer present in the physical spectrum.

C. Another dynamical regime: Extra moduli on the CMS

The scenario discussed above, involving a short range superpotential which remains finite on the CMS, is only a generic description for the near CMS dynamics in certain systems. As discussed in Sec. II, a different dynamical scenario is possible if there exist extra moduli on the CMS. However,

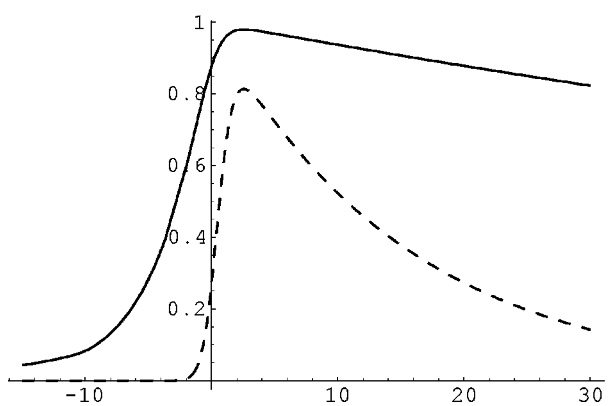

FIG. 8. Plots of the ground state wave function $\psi_0(s)$ for $\xi=17.7$ (dashed) and $\xi=177$ (solid). As above, these parameters correspond to $\nu_2/\mu_2=0.95$ and the classical equilibrium point is at $ms\approx 2.56$.

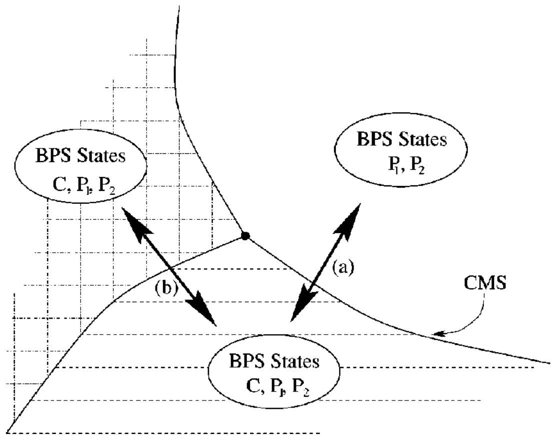

FIG. 9. Possible scenarios for the BPS spectrum, taken from a small region of Fig. 5, where P_1 and P_2 refer to the two primary solitons, while C refers to the composite kink: (a) a composite bound state C exists only on one side of the CMS; (b) a bound state exits on both sides of the CMS.

it turns out that the two-field model also exhibits a dynamical regime of this type, and we observe in this case that the approach to the CMS is still characterized by delocalization of the bound state wave function, albeit in a somewhat different manner to the case considered above.

First, recall that in the example considered above with $\mu_2 \neq 0$, the approach to the CMS was determined by Eq. (48). In the ''interior'' domain, $|\nu_2| < |\mu_2|\kappa$, the equation $W'(s)=0$ has a solution and consequently the composite soliton was BPS saturated. Upon approach to the CMS, the zero of $W'(s)$ runs to infinity and, after crossing the CMS at $|\nu_2| > |\mu_2|\kappa$, there is no longer a solution to $W'(s)=0$ and hence no BPS soliton. This scenario is illustrated by Fig. 9(a).

Now we consider a different dynamical regime, see Fig. 9(b), where, in both the ''interior'' and ''exterior'' regions, the spectrum of BPS states is the same (although possibly rearranged). In this case one still has delocalization on the CMS, although only for the wave function in this case as there is no diverging (classical) separation of the constituents.

To this end let us set $\nu=0$ [i.e., discard the term linear in U,V in the superpotential (24)]. As explained in Sec. III A, at $\nu=0$ the CMS is very simple,

$$\operatorname{Im}\mu \equiv \mu_2 = 0. \tag{86}$$

The SQM system (20) one arrives at in this case is described by the superpotential,

$$W'(s) = 2\frac{m^3}{\lambda^2}\mu_2 w_0(s), \tag{87}$$

where $w_0(s)$ was defined in Eq. (54). For our purposes it is important that $w_0(0)=0$, and that there are no other zeros of $w_0(s)$. Thus the solitons always overlap classically. However, as one approaches the CMS the wave function still spreads out due to the fact that $\mu_2 \to 0$. In particular, at large $|s|$,

$$w_0(s) \to \ \operatorname{sgn}(s), \tag{88}$$

and one observes that the zero energy bound state exists for both positive and negative μ_2 (see Fig. 9b). The wave function at large s is

$$\exp\left(2\frac{m^3}{\lambda^2}\mu_2 s\sigma_3\right) \tag{89}$$

times either $|\downarrow\rangle$ or $|\uparrow\rangle$ depending on the eigenvalue of σ_3. As $|\mu_2| \to 0$ the bound state level approaches the continuum spectrum while the wave function swells. At $\mu_2=0$ the wave function is completely delocalized and there is no binding.

As alluded to above, this dynamical regime is distinct from that considered previously where W' remained finite on the CMS; rather the CMS was characterized by the escape of the root of $W'(s)=0$ to infinity. In contrast, in the example considered here the root of the equation $W'(s)=0$ does not shift at all. Despite this one may note that ξ still diverges near the CMS due to its inverse dependence on μ_2.

In fact, precisely on the CMS the potential vanishes, and thus a new quantum modulus arises corresponding to the relative separation of the constituents.

V. DYONS IN SU(3) $\mathcal{N}=2$ SYM

We turn now to consider similar phenomena in $\mathcal{N}=2$ SYM. To study a model which exhibits a CMS in the weak coupling region, one approach is to extend the gauge group to rank greater than one.[3] Here we consider one of the simplest examples of this kind with gauge group SU(3). In the Coulomb phase this theory exhibits BPS dyon solutions with electric and magnetic charges associated with either of the unbroken U(1)'s of the Cartan torus. After choosing a convenient basis of simple roots for the algebra, one can classify the BPS monopole solutions into one of two types: those whose magnetic charge is aligned along a simple root—''fundamental monopoles''—and those whose magnetic charge is aligned along the non-simple root. These ''composite monopoles'' generically possess CMS curves at weak coupling, and so their dynamics in this regime is amenable to a semi-classical consideration.

Composite dyons in this, and the closely related $\mathcal{N}=4$ system, have recently been studied in some detail [16,18–22,25], with the conclusion that the low energy dynamics of two fundamental dyons at generic points of the Coulomb branch acquires an additional potential term. This term is associated with the misalignment of the adjoint Higgs VEVs

[3]Alternatively, one can add hypermultiplet matter with a large mass. In this case there is a discontinuity in the spectrum of quark monopole bound states on a CMS curve, which has been studied by Henningson [24], and the mechanism involves delocalization in a manner analogous to that discussed in this section.

of the two dyons, and leads to the formation of composite BPS dyons as bound states in this system. We will review some of these results below, and emphasize the implications for the dynamics in the near CMS region. The removal of the composite state on the CMS again arises through delocalization.

A. The BPS mass formula

We first review the features of the classical BPS mass formula for $\mathcal{N}=2$ SYM theories with higher rank gauge groups (see e.g. [26,22]), limiting ourselves to SU(N). For the consideration of solitonic mass bounds, we need consider only the bosonic Hamiltonian which has the form

$$H=2\,\mathrm{Tr}\int d^3x\left\{\frac{1}{2}(E_i)^2+\frac{1}{2}(B_i)^2+D_0\Phi^*D_0\Phi + D_i\Phi^*D_i\Phi+\frac{1}{2}[\Phi^*,\Phi]^2\right\}, \tag{90}$$

where E_i and B_i $(i=1,2,3)$ are the electric and magnetic fields, and Φ is the complex adjoint scalar $(\Phi=(\Phi_1+i\Phi_2)/\sqrt{2}$ in terms of the two real adjoint scalars). We use the normalization $\mathrm{Tr}\,T^aT^b=(1/2)\,\delta^{ab}$ for the generators.

The classical vacua satisfy $[\Phi^*,\Phi]=0$, thus requiring the VEV of Φ to lie in the Cartan subalgebra $\mathbf{H}$,

$$\langle\Phi\rangle=\boldsymbol{\phi}\cdot\mathbf{H}. \tag{91}$$

Note that the remaining Weyl freedom may be fixed by demanding that $\mathrm{Re}\,\boldsymbol{\phi}\cdot\boldsymbol{\beta}^a\geqslant 0$ for a given set of simple roots $\{\boldsymbol{\beta}^a\}$. This defines a region which for $|\boldsymbol{\phi}\cdot\boldsymbol{\beta}^a|\gg\Lambda$ coincides with the semiclassical moduli space of the theory. In this region we can safely neglect field-theoretic perturbative and nonperturbative quantum effects. We will only consider the case where the gauge group is maximally broken to $\mathrm{U}(1)^{N-1}$, for gauge group SU(N).

For a soliton solution we may define the charge vector $\boldsymbol{Q}$

$$\boldsymbol{Q}\cdot\boldsymbol{H}=(\boldsymbol{q}+i\boldsymbol{g})\cdot\boldsymbol{H}=\int_{S^2_\infty}dS^i(E_i+iB_i), \tag{92}$$

where use of the unitary gauge is implied. The real ($\boldsymbol{q}$) and imaginary ($\boldsymbol{g}$) parts of each component of $\boldsymbol{Q}$ have the interpretation of electric and magnetic charges in the corresponding U(1); they are quantized and form a lattice spanned by simple roots,

$$\boldsymbol{q}=q_a\boldsymbol{\beta}^a=en_a^E\boldsymbol{\beta}^a,\quad \boldsymbol{g}=g_a\boldsymbol{\beta}^a=\frac{4\pi}{e}n_a^M\boldsymbol{\beta}^a. \tag{93}$$

Here e is the gauge coupling, and n_a^E and n_a^M are the integral electric and magnetic quantum numbers. We also normalize the simple roots $\boldsymbol{\beta}^a$ with the conventions $(\boldsymbol{\beta}^a)^2=1$, $\boldsymbol{\beta}^{a\pm1}\cdot\boldsymbol{\beta}^a=-1/2$, so that the coroots coincide with the roots.

For general bosonic configurations, there is a Bogomol'nyi mass bound following from Eq. (90) which takes the form

$$M\geqslant\max|\mathcal{Z}_\pm|,\quad \mathcal{Z}_+=\sqrt{2}\,\boldsymbol{\phi}^*\cdot\boldsymbol{Q},\quad \mathcal{Z}_-=\sqrt{2}\,\boldsymbol{\phi}\cdot\boldsymbol{Q}. \tag{94}$$

This bound is saturated by solutions of the Bogomol'nyi equations,

$$B_i=D_ib,\quad E_i=D_ia, \tag{95}$$

along with the equation $D_i^2a\mp e^2[b,[b,a]]=0$ which, making use of Eq. (95), expresses Gauss' law in the gauge $A_0=\mp a$. The fields a,b are real and imaginary parts of $\exp(i\alpha)\Phi$ where the angle of rotation α is defined in terms of the charges $\boldsymbol{Q}$ (see e.g., [27,16,22]).

In the framework of the $\mathcal{N}=4$ supersymmetry algebra the parameters $Z_\pm$ are realized as central charges, and it is advantageous to view the system in this context (implying six instead of two real scalars Φ). Within $\mathcal{N}=4$ SUSY it is generally the case that $|\mathcal{Z}_+|\neq|\mathcal{Z}_-|$, and states which saturate the Bogomol'nyi bound $M\geqslant\max|\mathcal{Z}_\pm|$ will preserve only four of the sixteen supercharges, and will thus be 1/4 supersymmetric. If, however, $|\mathcal{Z}_+|=|\mathcal{Z}_-|$, states which saturate this bound will preserve 1/2 of the $\mathcal{N}=4$ supersymmetry. The possibility of having 1/4 BPS states, which only occurs for gauge groups of rank larger than one, dramatically increases the number of CMS curves accessible to semiclassical analysis, since 1/4 BPS states generically exhibit regions in the parameter space where they become marginally stable with respect to ''decay'' into 1/2 BPS states. In this sense it is useful to think of 1/4 BPS configurations as composite.

The discussion above was framed within $\mathcal{N}=4$ SYM, but this was simply for orientation. In order to preserve any fraction of supersymmetry, four of the six real adjoint scalars must vanish asymptotically, and thus the configurations discussed above all ''descend'' to give classical solutions in $\mathcal{N}=2$ SYM. The difference is that now only $\mathcal{Z}_-$ remains as a central charge [22] and all states saturating the bound $M=|\mathcal{Z}_-|$ are 1/2 BPS states from the point of view of the $\mathcal{N}=2$ SUSY algebra. As noted in [22], those charge sectors for which $|\mathcal{Z}_-|<|\mathcal{Z}_+|$ will have no BPS states from the point of view of the $\mathcal{N}=2$ system. Indeed, in this case states with $M=|\mathcal{Z}_-|$ are not allowed because $M\geqslant|\mathcal{Z}_+|$. Thus $|\mathcal{Z}_-|>|\mathcal{Z}_+|$ is a necessary condition for the existence of $\mathcal{N}=2$ BPS states.

Restricting our attention now to the gauge group SU(3), we notice that on the Coulomb branch the gauge group is broken down to the Cartan subalgebra $\mathrm{U}(1)^2$ and there can be field configurations which are electrically and magnetically charged under either of these U(1)'s [28]. Following Weinberg [28] we use the term ''fundamental dyons'' to refer to those configurations whose charges are aligned along one of these simple roots. Configurations whose charges are aligned along non-simple roots (i.e., $\boldsymbol{\beta}^1+\boldsymbol{\beta}^2$) will be referred to as ''composite.'' We shall focus on a particular composite configuration which has received considerable attention in the recent literature—namely, the composite dyon which has equal magnetic, $n_M^a=(1,1)$, and differing electric, $n_E^a=(q_1/e,q_2/e)$, charges along the simple roots.

B. Marginal stability and Coulomb-like interaction

The BPS mass formula for the (1,1) dyon with $n_M^a = (1,1)$, $n_E^a = (q_1/e, q_2/e)$ takes the form

$$M_{(1,1)} = |\mathcal{Z}_-| = \sqrt{2}|(q_1 + ig)\boldsymbol{\phi}\cdot\boldsymbol{\beta}^1 + (q_2 + ig)\boldsymbol{\phi}\cdot\boldsymbol{\beta}^2|, \tag{96}$$

where $g = 4\pi/e$. This configuration has a CMS curve where the (1,1) dyon is marginally stable with respect to two fundamental dyons: the first is aligned along $\boldsymbol{\beta}^1$, and has $n_M^a = (1,0)$, $n_E^a = (q_1/e, 0)$, while the second is aligned along $\boldsymbol{\beta}^2$, and has $n_M^a = (0,1)$, $n_E^a = (0, q_2/e)$. The masses of the fundamental dyons are

$$M_a = \sqrt{2}|(q_a + ig)\boldsymbol{\phi}\cdot\boldsymbol{\beta}^a| = m_a\left[1 + \frac{q_a^2}{2g^2} + \mathcal{O}\left(\frac{q_a^4}{g^4}\right)\right], \tag{97}$$

where in the second equality we have made use of the fact that $e^2 \ll 1$ in order to write the electric contribution to the dyon mass as a small correction to the mass of the corresponding fundamental monopole, $m_a = \sqrt{2}g|\boldsymbol{\phi}\cdot\boldsymbol{\beta}^a|$. Introducing ω_a as the argument of the VEVs,

$$\boldsymbol{\phi}\cdot\boldsymbol{\beta}^a = |\boldsymbol{\phi}\cdot\boldsymbol{\beta}^a| e^{i\omega_a} \tag{98}$$

we see that the marginal stability condition $M_{(1,1)} = M_1 + M_2$ fixes the argument of the ratio $\boldsymbol{\phi}\cdot\boldsymbol{\beta}^1/\boldsymbol{\phi}\cdot\boldsymbol{\beta}^2$,

$$\omega = \omega_1 - \omega_2, \tag{99}$$

to be equal to the argument of the ratio of complex charges Q_2/Q_1 where $Q_i = q_i + ig$. This implies that the CMS equation is $\omega = \omega_c$ where ω_c is defined as

$$\sin\omega_c = \frac{(q_2 - q_1)g}{\sqrt{g^2 + q_1^2}\sqrt{g^2 + q_2^2}}. \tag{100}$$

Provided n_E^a and n_M^a are of a similar order, the angle ω_c is small in the limit $e^2 \ll 1$, i.e., when the electric corrections to the dyon masses are much smaller than the corresponding monopole mass as in Eq. (97). Thus in this limit the VEVs $\boldsymbol{\phi}\cdot\boldsymbol{\beta}^a$ are only slightly disaligned, and we can make use of an expanded version of Eq. (100),

$$\omega_c = \frac{\Delta q}{g}(1 + \mathcal{O}(q_i^2/g^2)), \quad \Delta q = q_2 - q_1. \tag{101}$$

We are now in a position to verify the general claim of Sec. II A that the Coulombic interaction vanishes on the CMS. At large distances dyons can be viewed as point charges which interact at rest through electrostatic, magnetostatic, and scalar exchange. The electrostatic and magnetostatic interactions are fixed by the corresponding charges, while the scalar exchange can be read off from the asymptotic form of the Higgs field of one of the primary dyons (in a physical gauge where the configuration is a linear superposition of the fundamental dyon solutions [29,27]). The effective Coulombic interaction then takes the form

$$V_{\rm Coul} = -\frac{1}{8\pi r}\left[q_1 q_2 + g^2 - \sqrt{g^2 + q_1^2}\sqrt{g^2 + q_2^2}\cos\omega\right]. \tag{102}$$

Similar expressions have appeared in [27] and [19]. One observes that on the CMS, where the angle ω is given by Eq. (100), the Coulombic potential $V_{\rm Coul}$ vanishes.

When expanded to second order in q_i/g, the potential $V_{\rm Coul}$ takes the form

$$V_{\rm Coul} \approx \frac{(\Delta q)^2 - (g\omega)^2}{16\pi r}, \tag{103}$$

where $\Delta q = q_2 - q_1$ is defined in Eq. (101). If the VEVs were aligned, i.e., $\omega = 0$, we see that to quadratic order, the potential is repulsive [29,30] [as opposed to the SU(2) case [31]] and depends only on the electric charge difference Δq. However, for $(g\omega/\Delta q) > 1$ the potential is attractive and the (1,1) BPS dyon exists with a mass $M_{(1,1)}$ given in the same approximation by

$$M_{(1,1)} = |\mathcal{Z}_-| \approx (M_1 + M_2) - M_r\frac{(\Delta q - g\omega)^2}{2g^2} \quad \left(\frac{g\omega}{\Delta q} > 1\right), \tag{104}$$

where $M_r = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the monopoles, and the corresponding masses $M_{1,2}$ and $m_{1,2}$ are defined in Eq. (97). On the other side of the CMS in the range $|g\omega/\Delta q| < 1$ we have repulsion and the (1,1) BPS dyon does *not* exist.

It is interesting to note that the Coulombic potential reverts to attractive form once more when $(g\omega/\Delta q) < -1$, and the (1,1) bound states reappear. This has a simple interpretation in the framework of $\mathcal{N}=4$ supersymmetry: the lowest mass in this range saturates the $|\mathcal{Z}_+|$ central charge (note that now $|\mathcal{Z}_+| > |\mathcal{Z}_-|$),

$$M_{(1,1)} = |\mathcal{Z}_+| \approx (M_1 + M_2) - M_r\frac{(\Delta q + g\omega)^2}{2g^2} \quad \left(\frac{g\omega}{\Delta q} < -1\right). \tag{105}$$

In terms of $\mathcal{N}=4$ SUSY the (1,1) state at $(g\omega/\Delta q) < -1$ is 1/4 supersymmetric, but preserves a different subalgebra as compared to the $(g\omega/\Delta q) > 1$ case above. Moreover, the generators of this subalgebra are *not* part of the $\mathcal{N}=2$ superalgebra. In terms of $\mathcal{N}=2$ this means that the supermultiplet is not shortened, but nonetheless the Bogomol'nyi bound is saturated at the classical level.[4] Thus, we see an interesting example where the "BPS" nature of the state does not imply multiplet shortening. The presence of these states in $\mathcal{N}=2$ SYM theories was also noted in [22].

We now address the question of what happens to the (1,1) state on the CMS, i.e., when $|g\omega/\Delta q| = 1$ and the $1/r$ terms in the potential vanish? As we will see in Sec. V D the dynamics on the CMS is governed by repulsive $1/r^2$ terms

[4]This saturation will be lifted by field-theoretic quantum corrections.

demonstrating, even at the classical level, that there is no localized bound state on the CMS.

C. Zero modes and moduli spaces

We will shortly consider the low energy dynamics of the fundamental dyons comprising the (1,1) system. However, we first recall a few details regarding the zero mode structure of dyon solutions in $\mathcal{N}=2$ SYM. For BPS dyons in pure $\mathcal{N}=2$ SYM theories the unbroken $\mathcal{N}=1$ supersymmetry is enough in this case to pair the bosonic and fermionic zero modes [32] so we shall focus here just on the bosonic modes. Generic dyon solutions, corresponding to the embedding of the SU(2) monopole along some root of SU(3) have four bosonic zero modes [28,26] parametrizing the moduli space,

$$\mathcal{M}_1=\mathbf{R}^3\times S^1. \tag{106}$$

These modes are naturally identified as the center of mass position in $\mathbf{R}^3$ and the S^1 is an isometry conjugate to the conserved electric charge.

For dyons embedded along a simple root, this is the moduli space for all choices of field-theoretic moduli. However, if we consider composite monopoles, then the monopole moduli space $\mathcal{M}$ enlarges to a space of dimension eight, as is compatible with separating the constituents into two isolated fundamental monopoles [26,27]. This result was obtained in [26] using the index calculations of Weinberg [28] for real Higgs fields.

For the case at hand, the magnetic charge is $\boldsymbol{g}=g(\boldsymbol{\beta}^1+\boldsymbol{\beta}^2)$ and asymptotically the eight dimensional moduli space is simply $\mathcal{M}_1\times\mathcal{M}_1$. However, its exact form has also been deduced in [33–35,29],

$$\mathcal{M}_2=\mathbf{R}^3\times\frac{\mathbf{R}\times\mathcal{M}_{TN}}{Z}. \tag{107}$$

The first $\mathbf{R}^3$ factor corresponds to the center of mass position, while the second $\mathbf{R}$ factor refers to the coordinate conjugate to the total electric charge. The corresponding metric is flat. The relative moduli space $\mathcal{M}_{TN}$ is positive mass Taub-NUT (Newton-Unti-Tamburino) space (which is asymptotically $\mathbf{R}^3\times S^1$). Its four coordinates z^μ describe the relative distance r between the cores, with the corresponding polar and azimuthal angles θ and φ, and also the relative phase χ, conjugate to the relative electric charge Δq. The factor Z denotes a discrete identification for the charge coordinates, ensuring that the asymptotic geometry has a compact factor $S^1\times S^1$, associated with the conserved charges.

The Taub-NUT metric, in our conventions[5] takes the form

$$ds^2_{\rm TN}=g^{\rm TN}_{\mu\nu}dz^\mu dz^\nu=m(r)[dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2]+\frac{g^2}{e^2m(r)}\left[d\chi+\frac{1}{2}\cos\theta d\varphi\right]^2, \tag{108}$$

with a ''running'' mass parameter,

$$m(r)=M_r+\frac{2\pi}{e^2r}, \tag{109}$$

which asymptotes to the reduced mass M_r when the relative separation r diverges.

In terms of the internal U(1) angles ψ_i of the fundamental monopoles, the combination $\xi=\psi_1+\psi_2$ is conjugate to the total electric charge q_t (or more precisely, to q_t/e),

$$q_t=\frac{m_1q_1+m_2q_2}{m_1+m_2}, \tag{110}$$

while $\chi=(m_1\psi_2-m_2\psi_1)/(m_1+m_2)$ is conjugate to $\Delta q/e$, i.e., to the relative electric charge introduced above [34,35].

D. Moduli space dynamics

As first discussed in this context by Manton [36], the low-energy dynamics of fundamental monopoles may be understood as geodesic motion on the underlying moduli space. This picture extends to dyon solutions with aligned charges, but recent work on the dynamics of the fundamental constituents of the (1,1) system [16–22] has shown that for two fundamental dyons with misaligned charges the Lagrangian following from the geodesic approximation needs to be corrected by a new term [19,22]. In this subsection we will partially review these results, while emphasizing the features of the near CMS region.

The construction of Refs. [19,22] can be reformulated in terms of the following Lagrangian for the relative moduli $z^\mu=\{\vec{r},\chi\}$:

$$L_{\rm rel}=\frac{1}{2}g^{\rm TN}_{\mu\nu}\dot{z}^\mu\dot{z}^\nu+g^{\rm TN}_{\mu\nu}\dot{z}^\mu G^\nu, \tag{111}$$

where the metric $g^{TN}_{\mu\nu}$ is given by Eq. (108) and the ''gauge potential'' G^ν, which is a Killing vector generating the χ isometry, is

$$G^\nu=\frac{e}{g}M_r\omega\delta^\nu_\chi. \tag{112}$$

Here ω is the angle of disalignment between the condensates $\boldsymbol{\phi}\cdot\boldsymbol{\beta}^a$; see Eq. (98). The term containing G^ν is dynamically significant due to nontrivial fibering of the S^1 associated with χ in the Taub-NUT metric.

The classical Hamiltonian then has the form

$$H_{\rm rel}=\frac{1}{2}g^{\mu\nu}_{\rm TN}(\pi_\mu-G_\mu)(\pi_\nu-G_\nu), \tag{113}$$

[5]We follow [22] with the exception that χ is rescaled to have a period of 2π rather than 4π, and consequently the conjugate momentum is integer $[(q_2-q_1)/e]$ rather than half-integer $[(q_2-q_1)/2e]$ valued.

where $\pi_\mu = g^{\rm TN}_{\mu\nu}(\dot z^\nu + G^\nu)$ are the canonical momenta. In terms of the original field theory this Hamiltonian is interpreted as $M-|\mathcal{Z}_-|$, and thus BPS states are ''vacuum'' states of this Hamiltonian.

Two of the canonical momenta, namely π_φ and π_χ, are conserved quantities conjugate to isometries along the azimuthal angle φ and the phase χ. The value of π_φ is the z-projection of the angular momentum l_z, and π_χ is equal to $\Delta q/e$. Substituting these and the inverse metric $g^{\mu\nu}_{\rm TN}$ into Eq. (113) we obtain

$$H_{\rm rel} = \frac{\pi_r^2}{2m(r)} + \frac{\pi_\theta^2}{2m(r)r^2} + \frac{1}{2m(r)r^2\sin^2\theta} \times \left(l_z - \frac{\Delta q}{2e}\cos\theta \right)^2 + V(r), \tag{114}$$

where

$$V(r) = \frac{M_r}{2g^2}\left(1+\frac{2\pi}{e^2 M_r r}\right)^{-1}\left(\Delta q - g\omega + \Delta q\frac{2\pi}{e^2 M_r r}\right)^2 \tag{115}$$

is the only term in $H_{\rm rel}$ which depends on the field-theoretic moduli $\boldsymbol{\phi}$ (via ω).

This Hamiltonian vanishes when $\pi_r=0$, $\pi_\theta=0$, and the equilibrium values r_0 and θ_0 of the corresponding coordinates are given by

$$r_0 = \frac{2\pi}{e^2 M_r}\frac{\Delta q}{g\omega - \Delta q}, \quad \cos\theta_0 = 2l_z\frac{e}{\Delta q}, \tag{116}$$

it $(g\omega/\Delta q)>1$ and $|2l_z e/\Delta q|<1$. There is no solution for $(g\omega/\Delta q)<1$, i.e., the BPS state ceases to exist upon crossing the CMS where $g\omega=\Delta q$. We see that the system describes the composite state as a bound state of the dynamics whose spatial size, corresponding to the separation of the primary constituents, diverges on approach to the CMS.

It is instructive to expand the potential $V(r)$ at large r

$$V(r) = M_r\frac{(\Delta q - g\omega)^2}{2g^2} + \frac{(\Delta q)^2 - (g\omega)^2}{16\pi r} + \frac{(g\omega)^2}{8e^2 M_r r^2} + \cdots, \tag{117}$$

where we omitted $1/r^3$ terms and higher powers of $1/r$. The constant term $V(r\to\infty)$ marks the start of the continuum. Indeed, adding $|\mathcal{Z}_-|$ from Eq. (104) we obtain M_1+M_2 in the limit $r\to\infty$. The $1/r$ term coincides with the Coulombic potential (103) discussed earlier. It provides attraction for $|g\omega/\Delta q|>1$, the range where the bound states exist. The range $(g\omega/\Delta q)<-1$ corresponds, as discussed above, to $M_{(1,1)}=|\mathcal{Z}_+|$. What we see in addition is the repulsive $1/r^2$ term which leads to the existence of an equilibrium at large r_0 near the CMS. However, on the CMS it becomes the dominant term, and so there is no localized state on the CMS.

We conclude this section with some brief comments about quantization. The quantization of the dyon system in the context of $\mathcal{N}=2$ SYM theories was first discussed in detail by Gauntlett [32] in the case where the Higgs VEVs are aligned, and this discussion has since been generalized to the case considered here by Bak *et al.* [20] for $\mathcal{N}=4$, and by Gauntlett *et al.* [22] for $\mathcal{N}=2$. The crucial feature of this system is that the relative moduli space inherits a triplet of complex structures $J^{(a)}$, $a=1..3$, and is a hyper-Kähler manifold. Consequently, the system exhibits $\mathcal{N}=4$ supersymmetric quantum mechanics with four real supercharges Q_A where the index A can be associated loosely with a quaternionic structure $J^{(A)}=(\boldsymbol{1},J^a)$. These supercharges satisfy the superalgebra,

$$\{Q_A, Q_B\} = 2\delta_{AB}H_{\rm rel}, \tag{118}$$

where $H_{\rm rel}$ is the supersymmetric completion of the Hamiltonian defined in Eq. (113), to be interpreted as $M-|\mathcal{Z}_-|$. One can compare this with the SQM constructed in Sec. II.

An interesting feature of this system is that the symmetries of the superalgebra and the moduli space combine to ensure that the wave functions have a nontrivial dependence on the angular moduli, as well as the relative separation r. Specifically, the ground state wave function has the functional form [20] $\Psi_0=\Psi_0(r,\sigma_a)$ where σ_a are the basis 1-forms on the S^3 parametrized by (θ,φ,χ). This dependence is hinted at through the θ-dependent terms in the Hamiltonian (114) above. One may speculate that because of the high degree of symmetry in this system—the bosonic system possesses an additional conserved quantity of Runge-Lenz type [25]—a more precise separation of variables may be possible, but we will not pursue this issue here. We note only that, as demonstrated above, the delocalization on the CMS is associated with the cancellation of the terms of $\mathcal{O}(1/r)$ and depends purely on the relative separation. Moreover, this conclusion in the classical bosonic system apparently extends to SQM [20].

VI. DELOCALIZATION VIA MASSLESS FIELDS

On particular submanifolds of the CMS, the discussion we have presented above may be incomplete because certain fields may become massless. Indeed, generically there will be particular points on the CMS where states which are stable on both sides are massless. The presence of these singularities in moduli space can then be thought of as the ''origin'' of the CMS, since marginally stable states may not be single valued on traversing a contour around the singularity, and so a discontinuity in the spectrum becomes necessary for consistency. For example, this point of view provided one of the first arguments for the fact that the W boson must be removed from the spectrum inside the strong coupling CMS in $\mathcal{N}=2$ SYM [3].

In this section we will discuss the behavior of BPS states near these singular points. Within a simple 2D Wess-Zumino model we will find that the discontinuity of the BPS spectrum is explained by the delocalization of fermionic zero modes of the soliton on the CMS. The CMS in this case corresponds to the ''collision'' of two vacua in the parameter space, and thus one might anticipate similar phenomena in

$\mathcal{N}=2$ SYM theory near Argyres-Douglas points [13] when the singularities associated with monopole and quark vacua collide. Unfortunately, this occurs at strong coupling and is out of the range of our semi-classical analysis.

A. Breaking $\mathcal{N}=2$ to $\mathcal{N}=1$ and the restructuring of WZ solitons

The model is a simplified version of the $\mathcal{N}=2$ Wess-Zumino model [14] in two dimensions considered in Sec. III. We shall set the second field X to zero, but consider a new perturbation which breaks $\mathcal{N}=2$ down to $\mathcal{N}=1$. This setup was introduced in Ref. [37] (see Sec. 8).

The superpotential prior to perturbation is then of the standard Landau-Ginzburg form, and its worthwhile recalling a few pertinent details of these theories. A general classification of the $\mathcal{N}=2$ Landau-Ginzburg–type theories in two dimensions was given in [1], while construction of the representations of the $\mathcal{N}=2$ superalgebra with central charges was presented in Refs. [23]. It was shown that the supermultiplet of BPS soliton states is shortened, and this shortened multiplet consists of two states $\{u,d\}$ as we discussed earlier in Sec. III. In particular, in $\mathcal{N}=2$ theories there exists a conserved fermion charge f. The fermion charge of the u and d states is fractional but the difference is unity, $f_u-f_d=1$.

What changes on passing to $\mathcal{N}=1$ in two dimensions? The irreducible representation of the $\mathcal{N}=1$ algebra for BPS states is now one-dimensional (to the best of our knowledge this was first noted in Ref. [37]). The only remnant of the fermion charge is a discrete subgroup Z_2 which is spontaneously broken.

It is natural then to expect a restructuring of the BPS spectrum when $\mathcal{N}=2$ is broken down to $\mathcal{N}=1$. We will study the manner in which restructuring occurs by considering the spectrum of fermionic zero modes of a soliton solution as we vary the soft breaking parameter μ. We observe that for small μ the BPS spectrum remains the same as in the unbroken $\mathcal{N}=2$ theory. However, starting at a critical value μ_*—-corresponding to a ‘‘point of marginal stability’’—half the BPS states disappear from the spectrum. This occurs because quasiclassically the counting of states in the supermultiplet is related to the counting of zero modes of the soliton and when μ reaches μ_* some of the fermionic zero modes become non-normalizable. To follow their fate one can introduce a large box. Then the number of states does not change, but at $\mu=\mu_*$ the identification of states with zero modes implies that half the BPS states spread out all over the box while for $\mu>\mu_*$ they lie on the boundary of the box and are removed from the physical Hilbert space. This picture is quite analogous to the quantum mechanical discussion in Sec. II. However, as we shall see, the quantum description is complicated here by the presence of a massless field.

We take the Kähler metric to be canonical and the cubic superpotential of the model is conveniently represented in terms of real bosonic variables φ_i, $i=1,2$,

$$\mathcal{W}(\varphi_1,\varphi_2)=\frac{m^2}{4\lambda}\varphi_1-\frac{\lambda}{3}\varphi_1^3+\lambda\varphi_1\varphi_2^2. \tag{119}$$

In fact, $\mathcal{W}(\varphi_1,\varphi_2)$ is harmonic

$$\frac{\partial^2\mathcal{W}}{\partial\varphi_i\partial\varphi_i}=0 \quad \text{for } \mathcal{N}=2, \tag{120}$$

as it is the imaginary part of the four dimensional superpotential which is analytic in $\varphi_1+i\varphi_2$. This is therefore a reflection of $\mathcal{N}=2$ supersymmetry in two dimensions.

Now, to break $\mathcal{N}=2$ down to $\mathcal{N}=1$ we consider a more general, nonharmonic, superpotential $\mathcal{W}(\varphi_1,\varphi_2)$,

$$\mathcal{W}(\varphi_1,\varphi_2)=\frac{m^2}{4\lambda}\varphi_1-\frac{\lambda}{3}\varphi_1^3+\lambda\varphi_1\varphi_2^2+\frac{\mu}{2}\varphi_2^2, \tag{121}$$

where μ is the soft breaking parameter. There are two vacuum branches,

$$\left\{\varphi_1=\pm\frac{m}{2\lambda};\quad \varphi_2=0\right\},$$

$$\left\{\varphi_1=-\frac{\mu}{2\lambda};\quad \varphi_2=\pm\frac{\sqrt{\mu^2-m^2}}{2\lambda}\right\}, \tag{122}$$

but the second exists only for $\mu>m$, and vacua collide when $\mu=m$.

This model exhibits a classical kink solution which interpolates between the first set of vacua, and is given by

$$\varphi_1=\frac{m}{2\lambda}\tanh\frac{mz}{2},\quad \varphi_2=0. \tag{123}$$

It satisfies the classical BPS equations,

$$\frac{\partial\varphi_i}{\partial z}=\frac{\partial\mathcal{W}}{\partial\varphi_i}. \tag{124}$$

The zero modes corresponding to this kink are as follows: (a) One bosonic mode:

$$\chi_0=C\frac{1}{\cosh^2(mz/2)} \tag{125}$$

of the field φ_1 corresponds to (the spontaneous breaking of) translational invariance,

$$\chi_0\propto d\varphi_1/dz.$$

The constant C in Eq. (125) is a normalization constant; its explicit numerical value is not important. (Below the normalization constants in the zero modes will be omitted.) (b) Two fermionic modes: The first zero mode of the field $\psi_{1,2}$ (the indices number the superfields and fermionic components in the basis where $\gamma^0=\sigma_2$, $\gamma^1=i\sigma_3$) has the same form χ_0 as the translational mode. It is not accidental, the corresponding differential operators are the same due to $\mathcal{N}=1$ supersymmetry. The second fermionic zero mode

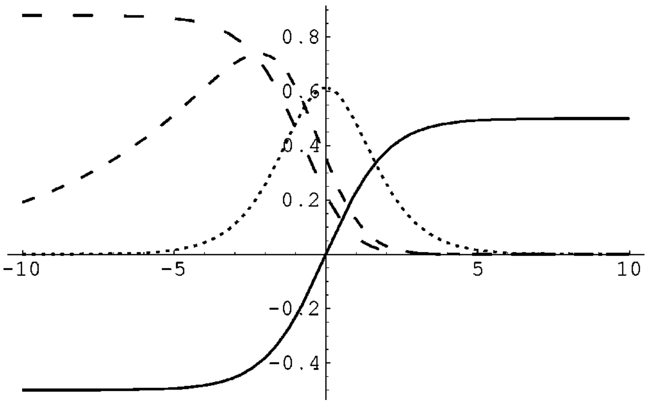

FIG. 10. The kink profile (solid line), with the zero mode ξ_0 for $\mu/m=0$ (dotted line), $\mu/m=0.8$ (short-dashed line), and $\mu/m=1$ (long-dashed line). Note that the vertical scale has been altered for ease of presentation.

$$\xi_0=\frac{\exp(-\mu z)}{\cosh^2(mz/2)} \tag{126}$$

appears in the field $\psi_{2,1}$. At $\mu=0$ the existence of this mode is a consequence of the $\mathcal{N}=2$ SUSY, and ξ_0 coincides with χ_0. At nonvanishing μ, when the extended SUSY is broken, this zero mode is maintained by virtue of the Jackiw-Rebbi index theorem [38].

An interesting feature of the zero mode ξ_0 is that it is asymmetric in z for $\mu\neq 0$. Moreover, this mode is normalizable only for

$$\mu<m. \tag{127}$$

This is readily seen from its asymptotics,

$$\xi_0(z\to\infty)\sim e^{-(\mu+m)z}, \quad \xi_0(z\to-\infty)\sim e^{-(\mu-m)z}. \tag{128}$$

The explicit form of the zero mode for few values of μ is exemplified in Fig. 10. The loss of normalizability occurs at $\mu=m$, when

$$\det\left\{\frac{\partial^2\mathcal{W}}{\partial\varphi_i\partial\varphi_j}\right\}=0$$

in one of the vacua between which the soliton solution interpolates. In other words, one of the vacuum states has gapless excitations at this point. Indeed, in the $z\to-\infty$ vacuum, the eigenvalues of the fermion mass matrix are: m and $\mu-m$, and thus indeed at the point $\mu=m$ where the vacuum branches meet, there is a massless field. This system represents a simplified analog of an Argyres-Douglas point in that the massless field arises through the collision of vacua. Furthermore, we see that this infrared effect destabilizes one of the fermionic zero modes of the soliton.

To study the infrared behavior in detail let us put the system in a large box, i.e., impose boundary conditions at $z=\pm L/2$ where L is large but finite. We choose these conditions in a form which preserves the remnant supersymmetry in the soliton background (i.e., the BPS nature of the soliton),

$$\left(\partial_z\varphi_i-\frac{\partial\mathcal{W}}{\partial\varphi_i}\right)\Big|_{z=\pm L/2}=0,$$

$$\left(\delta_{ij}\partial_z-\frac{\partial^2\mathcal{W}}{\partial\varphi_i\partial\varphi_j}\right)\psi_{j2}\Big|_{z=\pm L/2}=0, \quad \psi_{i1}\big|_{z=\pm L/2}=0 \tag{129}$$

(see Ref. [37] for details). It is easy to check that the soliton solution (123) as well as the zero modes (125), (126) are not deformed by these boundary conditions, i.e., Eq. (123) remains a solution of the classical BPS equations with the appropriate boundary conditions at finite L.

In the finite box there is no problem with normalization; the zero mode (126) remains a solution of the Dirac equation in the soliton background for all μ. However, at $\mu>m$ the mode is localized on the left wall of the box instead of sitting on the soliton as is the case at $\mu<m$. Thus, at $\mu=m$ the critical phenomenon of delocalization starts. As we will show below, upon quantization this means that some BPS soliton states have disappeared from the physical Hilbert space.

B. Quantization

We shall not present a detailed analysis of the quantization of the system here as it requires a somewhat different treatment to the supersymmetric quantum mechanics we have considered thus far. In this case, one needs to consider the dynamics of the light field in addition to the collective coordinates of the soliton.

However, provided we only consider a region somewhat away from the CMS, the spectrum is easily determined. As usual, the remnant $\mathcal{N}=1$ supersymmetry pairs the nonzero modes (one bosonic to two fermionic) around the soliton (see e.g. Sec. 3G of [37]), and the relevant contributions cancel. Thus the soliton spectrum is determined by the zero modes, corresponding to which we have one bosonic collective coordinate z_0 corresponding to the center of the kink, and two real Grassmann collective coordinates α_1 and α_2 determined by the zero modes, $\psi_{1,2}=\alpha_1\chi_0(z)+\cdots$ and $\psi_{2,1}=\alpha_2\xi_0(z)+\cdots$. Combining them into one complex parameter $\eta=(\alpha_1+i\alpha_2)/\sqrt{2}$, the collective coordinate dynamics at $\mu=0$ is determined by the quantum mechanical system

$$\mathcal{L}_{eff}=-M+\frac{1}{2}M\dot{z}_0^2+iM\bar{\eta}\dot{\eta}, \tag{130}$$

where M is the physical kink mass, which we can set to one.

The quantization is carried out in the standard manner. If z_0 and η are the canonical coordinates, we introduce the canonical momenta

$$\pi_{z_0}=\dot{z}_0, \quad \pi_\eta=-i\bar{\eta} \tag{131}$$

and impose the (anti)commutation relations

$$[\pi_{z_0},z_0]=-i, \quad \{\pi_\eta,\eta\}=-i. \tag{132}$$

One then proceeds to construct the raising and lowering operators in the standard manner. From Eqs. (131) and (132) it is clear that η can be viewed as the lowering operator, while $\bar{\eta}$ is the raising operator. One then defines the ''vacuum state'' in the kink sector by the condition that it is annihilated by η,

$$\eta|\text{‘vac}\,'\rangle=0.$$

The application of $\bar{\eta}$ produces a state which is degenerate with the vacuum state. $|\text{‘vac}\,'\rangle$ and $\bar{\eta}|\text{‘vac}\,'\rangle$ are two quantum states which form a (shortened) representation of $\mathcal{N}=2$ supersymmetry (at $\mu=0$). It is clear that these two states, which are degenerate in mass, have fermion numbers differing by unity.

What happens at $\mu\neq 0$? At $\mu<m$ the situation is exactly the same as at $\mu=0$ (apart from the fact that the fermion number is not conserved now and we must classify the states with respect to their Z_2 properties). We have two degenerate quantum states, both are spatially localized and belong to the physical sector of the Hilbert space. At $\mu=m$ only the vacuum state is localized. The spatial structure of the second state $\bar{\eta}|\text{‘vac}\,'\rangle$ has a flat component, which extends to the boundaries of the box. At $\mu>m$ this component is peaked at the boundary. The easiest way to see this is to introduce an external source coupled to $\varepsilon_{ij}\bar{\psi}_i\psi_j$. Thus, the state $\bar{\eta}|\text{‘vac}\,'\rangle$ disappears from the physical sector of the Hilbert space. The supercharge Q_1 acting on the state $|\text{‘vac}\,'\rangle$ produces this state itself, rather than another state. (We recall that Q_2 annihilates $|\text{‘vac}'\rangle$.) Formally this looks like a spontaneous breaking of the remnant supersymmetry.

VII. CONCLUDING REMARKS

Using quasiclassical methods we have argued that the underlying dynamics of marginally stable solitons is determined (generically) by non-relativistic supersymmetric quantum mechanics. Composite BPS states which disappear on the CMS were found to do so through a process of delocalization in coordinate space. Within the quantum mechanical description this process was associated with the bound state level reaching the continuum, while further progress beyond the CMS leads to a potential with a non-supersymmetric ground state. This is a generic picture. In certain cases the CMS can be a boundary between sectors with different composite solitons, the quantum mechanical potential then vanishes on the CMS.

One of the crucial features allowing a detailed investigation of the effective quantum mechanical dynamics in the two field model considered in Secs. III and IV was the linear realization of supersymmetry in the two-soliton sector of the non-relativistic SQM system. This embedding in 1+1D is similar to the 3+1 D effective dynamics of two BPS dyons in $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetric gauge theories [19–22]. In 3+1D the effective Coulombic interaction governs dynamics near the CMS. We demonstrated this in Sec. V for the composite dyon in SU(3), showing how the BPS states swell upon the approach to the CMS.

One may wonder whether some of the conclusions noted above regarding the behavior of composite bound states near the CMS might not be artifacts of the quasiclassical approximation. In particular, returning to pure $\mathcal{N}=2$ SYM with gauge group SU(2), the classic example of marginal stability with which we started this discussion is that of the *W* bosons on a CMS curve at strong coupling [4,5], for which our methods are not directly applicable. We are going to dwell on this issue in a separate publication [39]. Here we will briefly present two suggestive arguments pointing to the conclusion that this phenomenon involves delocalization in the same manner as the examples we have discussed.

The first observation involves duality. If we consider a region very close to the CMS for the *W* boson and not too far from the monopole singularity, we can consider a point particle approximation for the monopoles and dyons within the dual magnetic description. Provided we are close enough to the CMS, a nonrelativistic approximation is reliable. From this viewpoint the dissociation of the *W* is superficially quite similar to that of the bound states of dyons discussed in Sec. V, with the roles of electric and magnetic charge reversed.

The second observation involves the realization of the BPS states considered here in terms of string junctions [6,16,40,7] in type IIB string theory and its extension to F-theory [41]. Although there are still subtleties with this realization, specifically concerning a mismatch between field- and string-theoretic counting of bosonic moduli [17], it is interesting that the disappearance of marginally stable states in this framework appears to universally imply delocalization. The crucial point is that this process involves shrinking one or more of the spokes of the junction to zero length, while it has recently been pointed out [42] that the equilibrium separation of the two constituent states in the field theory is inversely proportional to the length of the shrinking prongs.

As a final remark, it is worth commenting on additional subtleties which arise when considering extended BPS objects. In particular, although we concentrated here on BPS particles, the notion of marginal stability is more general as supersymmetry algebras may also admit central charges supported by extended BPS objects such as strings and domain walls. Indeed, our classical analysis in Sec. III may be lifted to four dimensions where the kink solutions describe BPS domain walls. However, we concentrated on particle states specifically for the reason that quantization in this case leads to quantum mechanics, which is of course well-understood. The main technical difficulty in extending these arguments to solitons such as domain walls is that in addition to the dynamics of relative collective coordinates, one also needs to consider the massless sector of the field theory on the world volume of the soliton.

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APPENDIX: CLASSICAL SOLITON POTENTIAL

In this appendix we present an alternative derivation of the classical potential $(W'(s))^2$ entering the SQM Hamiltonian. This approach is purely bosonic, but requires knowledge of the composite soliton solutions obtained in Sec. III.

We start from the expression (78) for the binding energy ΔE_+,

$$\Delta E_+ = -\frac{m^3}{\lambda^2}\frac{3(\nu_2-\mu_2\kappa)^2}{\kappa\nu_{1+}\nu_{1-}}. \tag{A1}$$

(For $\nu_2/\mu_2<0$ the relative sign between ν_2 and μ_2 in this expression must be reversed.) The formula (A1) gives the minimum of $U(s)-U_+$, where U_+ is the value of potential at $s=\infty$, while the position of the minimum in s is given by Eq. (48). We can combine these two results in order to find the classical expression for $U(s)$ by using the standard Legendre transform approach. We introduce a source term in the original superpotential (24), thus replacing $\mathcal{W}(\Phi,X)$ by

$$\tilde{\mathcal{W}}(\Phi,X;j) = \mathcal{W}(\Phi,X) - \frac{m^2}{\lambda} jX, \tag{A2}$$

where j is a dimensionless (and in general complex) parameter corresponding to the strength of the source. For a static classical configuration described by this superpotential the calculation of the energy in fact gives the minimum of the functional $\mathcal{E}(j)$:

$$\mathcal{E}(j) = E(j) + \left(j\frac{m^2}{\lambda}\frac{1}{2}\int dz d^2\theta X(x,\theta) + \text{H.c.}\right), \tag{A3}$$

where $E(j)$ is the value of the original energy on the configuration which extremizes the action at a given source strength j, and $X(x,\theta)$ is the X superfield evaluated on that configuration. (In fact, being static, X does not depend on time.)

Clearly the effect of the source term is equivalent to a shift in ν: $\nu\to\nu+j$, and for our purposes it is sufficient to consider a purely imaginary source, $j=ij_2$. Then the s dependent part of the functional $\mathcal{E}(j)$ for the two-soliton static configuration is read directly from Eq. (A1) after replacing ν_2 by ν_2+j:

$$\Delta\mathcal{E}(j_2) = -\frac{m^3}{\lambda^2}\frac{3(\nu_2+j_2-\mu_2\kappa)^2}{\kappa\nu_{1+}\nu_{1-}}, \tag{A4}$$

and the relation between the j dependent equilibrium position s and the value of j is derived from Eq. (48),

$$\nu_2 + j_2 = \mu_2\kappa w_{\nu_1}(ms). \tag{A5}$$

The quantity of interest for us here, however, is not the functional $\mathcal{E}(j)$ as a function of j, but rather the binding energy ΔE as a function of s. The latter is found in the standard way from the relation

$$\Delta E = \mathcal{E}(j) - j\frac{d\mathcal{E}}{dj} \tag{A6}$$

with the variable j being eliminated in favor of s, using the relation (A5). Performing this simple operation on the expressions (A4) and (A5) one finds

$$\Delta E(s) = U(s) - U_+ = \frac{m^3}{\lambda^2}\frac{3(\nu_2-\mu_2\kappa w_{\nu_1}(ms))^2}{\kappa\nu_{1+}\nu_{1-}} - \frac{m^3}{\lambda^2}\frac{3(\nu_2-\mu_2\kappa)^2}{\kappa\nu_{1+}\nu_{1-}}, \tag{A7}$$

which represents the classical interaction energy of two primary solitons separated by a distance s. Naturally, the minimum of $\Delta E(s)$, as found from this expression, coincides with that given by Eq. (A1) at the separation s determined by Eq. (48).

Comparing the last term on the right hand side of Eq. (A7) with the expression (77) for U_+ we see that they coincide. Thus, the potential is

$$U_{\rm cl}(s) = \frac{m^3}{\lambda^2}\frac{3(\nu_2-\mu_2\kappa w_{\nu_1}(ms))^2}{\kappa\nu_{1+}\nu_{1-}}, \tag{A8}$$

where the subscript reminds us that this is the potential found at the classical level. This result correctly reproduces the energy difference between the asymptotic states at both infinities. With this normalization, one may readily check that the minimum (zero) of the potential corresponds in Eq. (79) to the mass of the BPS $\{1,1\}$ soliton,

$$M_{1,1} = \frac{8}{3}\frac{m^3}{\lambda^2}\kappa + \frac{3}{4}\frac{m^3}{\lambda^2}\frac{(\nu_{1+}-\nu_{1-})^2}{\kappa}. \tag{A9}$$

Comparing the above expression for the potential with the general form of the SQM Hamiltonian,

$$H_{\rm SQM} = \frac{1}{2M_r}[p^2 + (W'(s))^2 + W''(s)\sigma_3], \tag{A10}$$

we readily derive the superpotential (up to a sign)

$$W'(s) = \sqrt{6M_r}\,\frac{m^{3/2}}{\lambda}\frac{\mu_2\kappa w_{\nu_1}(ms)-\nu_2}{\kappa^{1/2}(\nu_{1+}\nu_{1-})^{1/2}}. \tag{A11}$$

This coincides with the result (70) derived in Sec. IV A, by evaluating the field-theoretic supercharges.

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