

Remarks on Dirac-like monopoles, Maxwell and Maxwell-Chern-Simons electrodynamics in $D=2+1$ dimensions

Winder A. Moura-Melo^{1,*} and J. A. Helayël-Neto^{1,2,†}

¹*Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150-Urca, 22290-180, Rio de Janeiro, RJ, Brazil*

²*Grupo de Física Teórica, Universidade Católica de Petrópolis (GFT-UCP), Petrópolis, RJ, Brazil*

(Received 11 July 2000; published 21 February 2001)

Classical Maxwell and Maxwell-Chern-Simons electrodynamics in (2+1) dimensions are studied in some detail. General expressions for the potential and fields are obtained for both models, and some particular cases are explicitly solved. Conceptual and technical difficulties arise, however, for accelerated charges. The propagation of electromagnetic signals is also studied and their reverberation is worked out and discussed. Furthermore, we show that a Dirac-like monopole yields a (static) tangential electric field. We also discuss some classical and quantum consequences of the field created by such a monopole when acting upon a usual electric charge. In particular, we show that at large distances the dynamics of one single charged particle, under the action of such a potential and a constant (external) magnetic field as well, reduces to that of one central harmonic oscillator, presenting, however, an interesting angular sector which admits energy eigenvalues. Among other peculiarities, both sectors, the radial and the angular one, present nonvanishing energy eigenvalues for their lowest levels. Moreover, those associated to the angle are shown to respond to discrete shifts of such a variable. We also raise the question on the possibility of the formation of bound states in this system.

DOI: 10.1103/PhysRevD.63.065013

PACS number(s): 11.10.Kk

I. INTRODUCTION

Field-theoretic models defined in a (2+1)-dimensional space time have been studied for nearly two decades [1,2]. Actually, lower-dimensional models have provided many interesting results which do not take place in the (3+1) dimensional world, e.g., Schwinger's mechanism in two-dimensional QED (QED₂) [3] and fractional statistics in three dimensions [4]. Consequently, lower-dimensional theories cannot be considered as mere lower limits of four-dimensional ones; they have rather revealed characteristics that are intrinsic to its dimensionality.

On the other hand, some (2+1)D theories, whenever supplemented by a Chern-Simons term, turn out to exhibit a new interesting physical content, such as, for example, Maxwell and Einstein-Hilbert actions [2,5]. Furthermore, it has been claimed that such models [mainly those in the context of Maxwell-Chern-Simons (MCS) terms] have relevance for a deeper understanding of some condensed matter phenomena, such as the quantum Hall effect (QHE) [6] and high- T_c superconductivity [7] (see also, Refs. [8,9]).

Although Maxwell and Maxwell-Chern-Simons (mainly the latter, in both Abelian and non-Abelian frameworks) have attracted a great deal of effort, it is curious that one has not provided an "electrodynamical body" (Liénard-Wiechert-type potentials, Larmor-like formula, and so forth)

for such (say, Abelian) theories which would be similar to the one we have for (3+1)D Maxwell.¹ Thus we shall try to draw the attention to the fact that the "lack" of a complete "electrodynamical body" is related to some serious difficulties, for instance, in calculating A_μ (and $F_{\mu\nu}$) for a single accelerated pointlike charge. In view of that, a Larmor-like expression relating energy-flux (radiation) and the acceleration of the sources is still missing.

We start the present work by studying the Maxwell (massless) case. Some results are discussed and a number of difficulties are pointed out. Following, we add a Chern-Simons term to the former model and some consequences of such a procedure are worked out. Going on, we analyze the issue concerning the introduction of a Dirac-like monopole within both models and some properties of its field. Some effects of its potential on a usual electric charge are discussed in both classical and quantum (nonrelativistic) frameworks. We close this paper by pointing out some conclusions and prospects.

II. CLASSICAL MAXWELL ELECTRODYNAMICS IN $D=(2+1)$

Let us consider the $D=(2+1)$ Maxwell electrodynamics (MED₃) Lagrangian:²

$$\mathcal{L}_{MED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j_\mu A^\mu. \quad (1)$$

The invariance of the action under local Abelian gauge transformations, $A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \Lambda(x)$, is ensured by the conservation of the three current, say, $\partial_\mu j^\mu = 0$. Moreover with

²Our conventions read $\text{diag}(\eta_{\mu\nu}) = (+, -, -)$, greek letters running 0,1,2; the 2D spatial coordinates are labeled by latin letters running 1,2, and $\epsilon_{012} = \epsilon^{012} = \epsilon_{12} = \epsilon^{12} = +1$.

*Email address: winder@cbpf.br; present address after March 1, 2001: Departamento de Ciências Exatas, Universidade Federal de Lavras, Caixa Postal 37, Cep. 36200-000, Lavras, Minas Gerais, Brazil.

†Email address: helayel@cbpf.br

¹Although in a different approach, a classical analysis of the non-Abelian case [SU(2), more precisely] was performed by D'Hoker and Vinet [10].

the usual definition of the field strength, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we get $F_{\mu\nu} = [F_{0i} = +(\vec{E})_i; F_{12} = B]$. Next, the field strength clearly satisfies $\partial_\mu F^{\mu\nu} = j^\nu$ and $\partial_\mu \tilde{F}^\mu = 0$, whence there follow

$$\nabla B = \partial_t \vec{E}^* + \vec{j}^*, \quad \nabla \cdot \vec{E} = \rho, \quad \text{and} \quad \nabla \cdot \vec{E}^* = \partial_t B,$$

where we have defined $\tilde{F}^\mu = \frac{1}{2} \epsilon^{\mu\nu\kappa} F_{\nu\kappa} = (+B; -\vec{E}^*)$, with the components of a dual vector given by $(\vec{U}^*)_i = \epsilon_{ij} U_j$.

The dynamical equation for the more basic quantity A_μ reads (in the gauge $\partial_\mu A^\mu = 0$): $\partial^2 A_\mu(x) = j_\mu(x)$. The solutions to this wave equation may be readily obtained by means of the well-known Green's function method (or by applying the Hadamard's *descent method*, see Ref. [11] for further details). Such a function, $G^{2+1}(x-y)$, may be explicitly worked out and reads [the advanced function is easily obtained by introducing a $\Theta(-\tau)$; Θ is the usual step function]

$$\begin{aligned} G_{ret}^{2+1}(x-y) &= -\frac{\Theta(\tau)}{2\pi} \int_0^\infty J_0(kr) \sin(k\tau) dk \\ &= -\frac{\Theta(\tau)}{2\pi} \frac{\Theta(\tau^2 - r^2)}{\sqrt{\tau^2 - r^2}}, \end{aligned} \quad (2)$$

where $\tau = x^0 - y^0$ and $r = |\vec{x} - \vec{y}|$. The integral above may be found, for example, in Ref. [12] (on page 731 and Eq. 6.671-7). It is worth noticing that G^{2+1} presents a quite different behavior with respect to its (3+1)D counterpart, G^{3+1} : the support of G^{2+1} lies no longer on the surface of the light cone, where $(x-y)^2 = 0$, as is the case for $G^{3+1}(x-y) = \delta[(x^0 - y^0)^2 - |\vec{x} - \vec{y}|^2] / 2\pi$. Indeed, it rather spreads throughout the whole internal region of the light cone, where $(x-y)^2 > 0$ [blowing up as $(x-y)^2 \rightarrow 0_+$ and vanishing for space-like intervals, $(x-y)^2 < 0$]. Thus the Huyghens principle is satisfied by G^{3+1} and violated by G^{2+1} .

As we shall see, this will lead to profound modifications in planar electrodynamics with respect to the (3+1)D Maxwell theory. For example, by virtue of the failure of Huyghens principle, electromagnetic signals reverberate in (2+1) dimensions, and a Larmor-like formula for the radiated power appears to be a highly nontrivial task.

Next, by taking a single pointlike charge, $j^\mu(y) = q \int_{-\infty}^{+\infty} \dot{z}^\mu(s) \delta^{2+1}[y - z(s)] ds$, we get the general form for its potential (we have omitted the homogeneous part of the potential):

$$A_{ret}^\mu(x) = +\frac{q}{2\pi} \int_{-\infty}^{+\infty} \Theta[x^0 - z^0(s)] \frac{\Theta\{[x - z(s)]^2\}}{\sqrt{[x - z(s)]^2}} \dot{z}^\mu(s) ds, \quad (3)$$

with $(x-z)^2 = [(x^0 - z^0)^2 - |\vec{x} - \vec{z}|^2]$. The expression for the field strength is also obtained in the usual way, and reads [with $P = (x-z)^\alpha \dot{z}_\alpha$ and $Q = (x-z)^\alpha \ddot{z}_\alpha$]

$$\begin{aligned} F_{\mu\nu}(x) &= \frac{q}{2\pi} \int_{-\infty}^{+\infty} \frac{\Theta(x^0 - z^0) \Theta[(x-z)^2]}{P^2 \sqrt{(x-z)^2}} [\ddot{z}_\nu(x-z)_\mu P \\ &\quad + \dot{z}_\nu(x-z)_\mu (1-Q) - \mu \leftrightarrow \nu] ds. \end{aligned} \quad (4)$$

Here, it is worth noticing that, in general, we do not get to solve the expressions above. Actually, we have tried to solve elementary accelerated motions, say parabolic and hyperbolic ones. Unfortunately, we have found serious difficulties in performing some integrals that are highly nontrivial and plagued with serious divergences that have to be suitably handled.³ In (3+1)D, the scenario is quite different, because we have a $\delta^{3+1}[(x-z)^2]$ (instead of $\Theta[(x-z)^2]/\sqrt{(x-z)^2}$) which, in turn, implies in a straightforward factorization of the integral in s variable, by picking up only those points for which $(x-z)^2 = 0$.

Hence we conclude that the ‘‘lack’’ of closed analytic expressions for A_μ (and $F_{\mu\nu}$) in the case of an arbitrary motion (Liénard-Wiechert-type expressions) is deeply related to the failure of the Huyghens' principle, since the solutions to the ∂^2 operator in (2+1)D, G^{2+1} , do not satisfy such a principle (indeed, the same happens for any G^{n+1} , n even; see, for example, Refs. [11] and [14–16]).

On the other hand, even the static case (the constant motion may be easily obtained by a Lorentz boost) reveals some of the new characteristics of the model. Thus, by taking $z^\mu = (s, \vec{0}) \Rightarrow \dot{z}^\mu = (1, \vec{0})$, we get

$$\begin{aligned} A^\mu(x) &= \begin{cases} A^0(\vec{r}, t) = -\frac{q}{2\pi} \ln|\vec{r}| + \frac{q}{2\pi} \lim_{\tau \rightarrow +\infty} (\ln|\tau + \sqrt{\tau^2 - r^2}|), \\ \vec{A}(\vec{r}, t) = 0, \end{cases} \end{aligned} \quad (5)$$

$$F_{\mu\nu}(x) = \begin{cases} F_{0i}(\vec{r}, t) = +\frac{q}{2\pi} \frac{r^i}{r^2} - \frac{q}{2\pi} r^i \lim_{\tau \rightarrow r^+} \left(\frac{\tau}{r^2 \sqrt{\tau^2 - r^2}} \right), \\ F_{ij}(\vec{r}, t) = 0. \end{cases} \quad (6)$$

Here we notice that, besides the well-known $\ln|\vec{x}|$ behavior of the potential in planar electrodynamics, there is an extra term which explicitly diverges. Such a term clearly represents the asymptotic value of the potential as $|\vec{x}| \rightarrow +\infty$ and is directly related to the *infrared divergence* of the theory. Indeed, by

³It was already pointed out in the literature that (2+1)D electrodynamics indeed imposes additional troubles in calculating some quantities; for example, in Ref. [13], the author discusses some difficulties brought about by the logarithmic behavior of the potential.

calculating $A_0(x)$ by means of $\tilde{A}_0(k)$ (its Fourier transform), we may clearly see that such a term arises when the mass term is set to zero, as below:

$$\begin{aligned} A^0(\vec{r}, t) &= + \frac{q}{2\pi} \int_0^\infty \frac{J_0(pr)}{p} dp \\ &= \lim_{\mu \rightarrow 0_+} K_0(\mu r) \\ &\approx -\ln|\vec{r}| - \lim_{\mu \rightarrow 0_+} \ln(\mu/2), \end{aligned}$$

whence, we see that as $m \rightarrow 0_+$, the last term above blows up. On the other hand, the explicitly divergent term appearing in the $F_{\mu\nu}$ above may be removed by a suitable subtraction procedure, which is possible because such a quantity vanishes asymptotically. (Among others, such subtleties shall be more explicitly discussed in Ref. [17].)

It is interesting to pay attention to the appearance of such an infrared divergence at the classical level; indeed, infrared problems in (2+1)D are much more severe than in four dimensions. For example, the non-Abelian case, even in the presence of massive matter, makes sense only for very special gauge choices [18].

Still concerning the general $F_{\mu\nu}$ form, Eq. (4), there remains an interesting issue to be pointed out. By taking into account the terms proportional to the acceleration $\ddot{z}(s)$, which are those that effectively contribute to the energy flux and so, to a Larmor-like formula, we notice that such terms are proportional to $\int ds / \sqrt{(x-z)^2}$, and might surprisingly lead us to the result that radiation in (2+1)D no longer falls off with r^{-1} . Indeed it may increase proportionally to $\ln|\vec{r}|$, as long as $z(s)$ depends on s^2 , which is the case for constant accelerated motions.

Next, let us point out a rather peculiar characteristic of the model as long as the propagation of electromagnetic signals is concerned. Let us start by considering the charge configuration: $\rho(\vec{y}, t') = q \delta^2(\vec{y}) \delta(t')$. Its potential reads

$$\Phi_{pulse}(\vec{x}, t) = \frac{q}{2\pi} \frac{\Theta(t - |\vec{x}|)}{\sqrt{t^2 - |\vec{x}|^2}}, \quad (7)$$

in contrast with its (3+1)D counterpart $\Phi_{pulse}(\vec{x}, t) = -q \delta(t - |\vec{x}|) / 4\pi |\vec{x}|$. Clearly, although such a signal has been sharply sent (at $t=0$ it was just at $|\vec{x}|=0$) it cannot later be recorded as a sharp one: the pulse develops a ‘‘tail’’ (its spreading in time) and so it reverberates. Therefore we now need a *very long time to record* a sharp signal sent at an earlier time. Next, we obtain the superposed case, which is obtained from $\rho(\vec{y}, t') = q \delta^2(\vec{y}) \Theta(t')$, and reads

$$\Phi_{sup}(\vec{x}, t) = + \frac{q}{2\pi} \ln \left(\frac{t + \sqrt{t^2 - |\vec{x}|^2}}{|\vec{x}|} \right) \Theta(t - |\vec{x}|), \quad (8)$$

whence we see that these signals superpose in a logarithmic way, differently from the (3+1) dimensions, where such a superposition takes place linearly, $\Phi_{sup}(\vec{x}, t) = -q \Theta(t - |\vec{x}|) / 4\pi |\vec{x}|$.

The logarithmic superposition leads us to an interesting point if we compare with previous results when t is equal or slightly greater than $|\vec{x}|$: while the single pulse potential, Eq. (7), appears to be very strong, the contrary happens to the superposed case, which is very weak there. However, as time goes by, things straighten up: while single pulses fall off, their superposition appears to broaden the potential. (The expressions for the electric field are also easily obtained and exhibit similar phenomenon concerning reverberation, while the superposition is ‘‘better behaved’’ than the Φ potential.) Moreover, notice that as (and only as) $t \rightarrow \infty$, we recover the static potential, Eq. (5).

Thus the results discussed above bring an additional complication to the (classical, at least) electrodynamics of a system of interacting charges, since even single pulses emitted by an electric charge will demand a very long time to be completely *felt* by another one. In other words, even the static feature of the potentials and fields *will be no longer* determined only by the static configuration of the charges. It rather demands a very long time to actually happen, since at finite times the electromagnetic quantities are time dependent.

Indeed, in (2+1)D, we may regard the classical propagation of a signal as if the wave front travels with velocity c , and decreasing in a such a way that the back point of the signal has null velocity [this is exactly what Eq. (7) says].

Actually, similar conclusions concerning the reverberation of signals were already discussed by other authors [11,15]. For instance, Courant and Hilbert in their classical book [14] analyze such a propagation and, by virtue of the failure of the Huyghens principle, they conclude that D’Alembertian waves (in general), even if sharply produced, cannot be later recorded with the same sharpness.

Furthermore, we would like here to raise a question in view of what we have understood about the spreading that unavoidably affects the classical propagation of sharp signals in (2+1)D. By facing an electromagnetic signal rather as a wave, reverberation affects its propagation and we can no longer speak of sharp pulses; on the other hand, if we are to give the electromagnetic signal the status of a particle, we wonder whether the concept of photon as a localized energy packet should not be reassessed in the framework of planar electromagnetism.⁴

⁴An analogous question is pertinent in the MCS case (next section). There, however, by virtue of the mass gap, reverberation is more expected to happen, since massive (Klein-Gordon or Proca-like) fields exhibit such a phenomenon even in (3+1) dimensions [19,20]. [See also Ref. [21] in which is studied a modification of the standard electromagnetism, by the inclusion of a Lorentz- and Parity-violating Chern-Simons-like term in (3+1) dimensions.]

III. MAXWELL-CHERN-SIMONS MODEL

Let us write the Lagrangian for the Maxwell-Chern-Simons electrodynamics (MCS):

$$\mathcal{L}_{MCS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m}{2}\epsilon^{\mu\nu\kappa}A_{\mu}\partial_{\nu}A_{\kappa} + j_{\mu}A^{\mu}, \quad (9)$$

where $(m/2)\epsilon^{\mu\nu\kappa}A_{\mu}\partial_{\nu}A_{\kappa} = (m/2)A_{\mu}\tilde{F}^{\mu}$ is the (Abelian) Chern-Simons term, which provides a mass for the boson A_{μ} without breaking the original local gauge symmetry of the action [2], $S_{MCS} = \int d^{2+1}x \mathcal{L}_{MCS}(x)$. Moreover, the mass parameter m may be taken to be positive or negative. Depending on the choice of its signal, the ‘‘massive photon’’ will carry polarization equal to $+1$ ($m > 0$) or -1 ($m < 0$).⁵ Notice, however, that in both cases, massless or massive, the ‘‘photon’’ carries *only one physical degree of freedom*, which highlights its ‘‘scalar nature.’’ Actually, since its mass is given by means of a topological mass term, we do not expect to have any additional degree of freedom.

In a similar way to the massless case, A_{μ} potential can be worked out and reads as below:

$$A^{\mu}(x) = \int d^{2+1}y \left(G^{2+1}(x-y) \eta^{\mu\nu} + \frac{m}{m^2} [G^{mass}(x-y) - G^{2+1}(x-y)] \epsilon^{\mu\nu\kappa} \partial_{\kappa} \right) j_{\nu}(y), \quad (10)$$

where the massive Green’s function is given by

$$G_{adv}^{mass}(x-y) = -\frac{1}{2\pi} \frac{\Theta[t^2 - r^2] \cos(m\sqrt{t^2 - r^2})}{\sqrt{t^2 - r^2}} \Theta[\pm t]$$

with $t = x^0 - y^0$ and $r = |\vec{x} - \vec{y}|$. We clearly see that, as $m \rightarrow 0$, then $G^{mass} \rightarrow G^{2+1}$. Similarly to its massless counterpart, G^{mass} does not satisfy the Huyghens’ principle: again, the support spreads throughout the whole region $(x-y)^2 \geq 0$.

Next, the general expression for A_{μ} , as produced by a single pointlike charge, takes the form

⁵Talking about *spin* in (2+1) dimensions, we should be careful, since its meaning is rather different from its (3+1)D counterpart. In fact, for a massive particle, its ‘‘spin’’ in (2+1)D has some similarities with the helicity of its massless correspondent in (3+1)D: only the positive, $+1$, or negative, -1 , polarizations may take place, while no component of zero polarization appears. See Refs. [22] and [23].

$$A^{\mu}(x) = +\frac{q}{2\pi} \int_{-\infty}^{+\infty} ds \Theta[x^0 - z^0(s)] \Theta[(x-z)^2] \times \left\{ \frac{\cos[m\sqrt{(x-z)^2}]}{\sqrt{(x-z)^2}} \dot{z}^{\mu} + \frac{m}{m^2} \epsilon^{\mu\nu\kappa} \left[\dot{z}_{\nu}(x-z)_{\kappa} \times \left(\frac{m \sin[m\sqrt{(x-z)^2}]}{[\sqrt{(x-z)^2}]^2} + \frac{\cos[m\sqrt{(x-z)^2}] - 1}{[\sqrt{(x-z)^2}]^3} \right) + \ddot{z}_{\nu} \dot{z}_{\kappa} \left(\frac{\cos[m\sqrt{(x-z)^2}] - 1}{\sqrt{(x-z)^2}} \right) \right] \right\}, \quad (11)$$

from which we may notice the difficulties which arise in trying to solve it for arbitrary motions of the charge (indeed, the general solution to such an expression deeply depends on the massless one). There is also a new sort of term, not present in the massless case, which is explicitly acceleration dependent (a radiationlike term, the last one in the equation above). Such a term, in turn, will lead to another one that explicitly depends on d^3z/ds^3 in the expression for $F_{\mu\nu}$: a back-reaction-like term. By virtue of its length, we shall not give the explicit form for this field here. We refer the reader to Ref. [17], where a detailed derivation of the results above will be presented. We only anticipate that the possibility that the radiation increases like a $\ln|\vec{r}|$ also takes place here.

Even though a general solution for A_{μ} (and $F_{\mu\nu}$) for arbitrary motions appears to be far off our possibilities, it is instructive to work out static quantities which already exhibit some of the new properties brought about by the Chern-Simons term. They read as follows:

$$A^{\mu}(x) = \begin{cases} \Phi(\vec{x}) = +\frac{q}{2\pi} K_0(m|\vec{x}|) \\ A^i(\vec{x}) = -\frac{q}{2\pi} \frac{m}{m^2} \frac{\epsilon^{ij} x^j}{|\vec{x}|} \left(\frac{1}{|\vec{x}|} - m K_1(m|\vec{x}|) \right), \end{cases} \quad (12)$$

$$F_{\mu\nu}(x) = \begin{cases} E^i(\vec{x}) = -\frac{q}{2\pi} \frac{m x^i}{|\vec{x}|} K_1(m|\vec{x}|) \\ B(\vec{x}) = +\frac{q}{2\pi} m K_0(m|\vec{x}|) = m\Phi(\vec{x}). \end{cases} \quad (13)$$

Now, we see that A_{μ} acquires a better asymptotic behavior: $A_{\mu} \rightarrow 0$ as $|\vec{x}| \rightarrow \infty$ (at large distances, K_0 and K_1 roughly behave as $e^{-|m\vec{x}|/\sqrt{|m\vec{x}|}}$). Indeed, even the long-range sector of \vec{A} now decreases as $|\vec{x}|^{-1}$. Such a sector is related to the well-known nondynamical massless pole and also to the possibility of topological objects such as vortexlike magnetic field. In addition, due to the Chern-Simons term, the charge now produces a nonvanishing static magnetic field. Nevertheless, this does not lead to radiation at all. Indeed, it is easy to show that $\nabla \cdot \vec{S}^* = \nabla \cdot (\vec{E}^* B) = 0$, with \vec{S}^* being the Poynting vector.

We should now comment on the short-distance behavior of these quantities. By recalling that, for $|z| \ll 1$, the modified

Bessel functions behave as $K_0(z) \approx -\ln(z/2)$ and $K_1(z) \approx z^{-1}$, we see that, near the charge, Φ and B diverge as $\ln|\vec{m}\vec{x}|$ while \vec{E} blows up as $|\vec{x}|^{-1}$. The vector potential, on the other hand, exhibits a very peculiar behavior: it vanishes as $|\vec{x}| \rightarrow 0$! Such a result is actually in accordance with Eq. (10): the A^i components should vanish as $\sqrt{t^2 - |\vec{x}|^2} \rightarrow 0$.

Moreover, the fact that, as $|\vec{x}| \rightarrow 0$, $B \approx \ln|\vec{m}\vec{x}|$ implies that a charge within the Chern-Simons framework is a richer object than within a pure Maxwell context: along with its massive electric field, it also produces a flux “tube” of magnetic field, of width m^{-1} and strength q/m (what demands m to be sufficiently large). It is precisely in this nonvanishing character of B that there lies the possibility of the fractional statistics exhibited by such “charges” [24].

Furthermore, it is easy to conclude, using Eq. (10) for example, that upon $m \rightarrow -m$, $A^0 = \Phi$ and \vec{E} remains unchanged while \vec{A} and B changes their signals.

Next, we shall treat the propagation of signals in the Maxwell-Chern-Simons framework. We shall start by obtaining and analyzing the single pulse case, which is produced by $\rho(\vec{y}, t') = q \delta^2(\vec{y}) \delta(t')$. The quantities read [we have omitted $\Theta(t - |\vec{x}|)$ in all expressions below]

$$\begin{aligned} \Phi_{pulse}(\vec{x}, t) &= + \frac{q}{2\pi} \frac{\cos(m\sqrt{t^2 - |\vec{x}|^2})}{\sqrt{t^2 - |\vec{x}|^2}}, \\ A_{pulse}^i(\vec{x}, t) &= - \frac{q}{2\pi} \frac{m}{m^2} \epsilon^{ij} \partial_j \left(\frac{\cos(m\sqrt{t^2 - |\vec{x}|^2}) - 1}{\sqrt{t^2 - |\vec{x}|^2}} \right), \end{aligned} \quad (14)$$

for the potentials, while the fields are

$$\begin{aligned} E_{pulse}^i(\vec{x}, t) &= + \frac{q}{2\pi} \partial^i \left(\frac{\cos(m\sqrt{t^2 - |\vec{x}|^2})}{\sqrt{t^2 - |\vec{x}|^2}} \right) \\ &\quad + \frac{q}{2\pi} \frac{m}{m^2} \epsilon^{ij} \partial_i \partial_j \left(\frac{\cos(m\sqrt{t^2 - |\vec{x}|^2}) - 1}{\sqrt{t^2 - |\vec{x}|^2}} \right) \\ B_{pulse}(\vec{x}, t) &= - \frac{q}{2\pi} \frac{m}{m^2} \nabla_x^2 \left(\frac{\cos(m\sqrt{t^2 - |\vec{x}|^2}) - 1}{\sqrt{t^2 - |\vec{x}|^2}} \right). \end{aligned}$$

The reverberation of the pulse is evident: it is very strong when t is equal or slightly greater than $|\vec{x}|$ and decreases as time goes by, vanishing as $t \rightarrow \infty$. The superposed case is obtained by integrating expressions above from $|\vec{x}|$ to t . For example, the scalar potential superposes as

$$\Phi_{sup}(\vec{x}, t) = \int_{|\vec{x}|}^t \Phi(\vec{x}, \tau) d\tau = + \frac{q}{2\pi} \int_{|\vec{x}|}^t \frac{\cos(m\sqrt{\tau^2 - |\vec{x}|^2})}{\sqrt{\tau^2 - |\vec{x}|^2}} d\tau. \quad (15)$$

Here, a new result takes place in the MCS framework: we cannot exactly evaluate how electromagnetic signals superpose for arbitrary cases (say, finite times), since the integral above is not available, in closed form, unless $t \rightarrow \infty$ (the other electromagnetic quantities also depend on the same integral). At this limit, we get (see, for example, Ref. [12], page 419, Eq. 3.754-2):

$$\lim_{t \rightarrow \infty} \int_{|\vec{x}|}^t \frac{\cos(m\sqrt{\tau^2 - |\vec{x}|^2})}{\sqrt{\tau^2 - |\vec{x}|^2}} d\tau = K_0(m|\vec{x}|),$$

which, in turn, leads us to the static potential, Eq. (12), as $t \rightarrow \infty$. A similar scenario holds for the other quantities. Thus we see that, in the case of the \vec{E} field, only its longitudinal component survives asymptotically.

IV. DIRAC-LIKE MONOPOLE AND ITS TANGENTIAL ELECTRIC FIELD

Now, let us draw attention to the introduction of a Dirac-like object into the previously studied models and to discuss some characteristics and consequences of the fields produced by this sort of monopole.

As it is well known, such an (pointlike) object shows up by breaking the Bianchi identity [25],⁶ $\partial_\mu \tilde{F}^\mu = g$, which in terms of the potentials gets the form

$$\int_t dt \int_{xy} d^2x \{ \epsilon_{ij} [\partial_i, \partial_t] A_j(\vec{x}, t) - [\partial_x, \partial_y] \Phi(\vec{x}, t) \} = g; \quad (16)$$

in the static limit, it reduces to

$$[\partial_x, \partial_y] \Phi(\vec{x}) = -g \delta^2(\vec{x}). \quad (17)$$

Now, the above equation may be satisfied only if Φ carries a “singular structure.” Indeed, by recalling that

$$[\partial_x, \partial_y] \arctan\left(\frac{y}{x}\right) = \partial_x \left(\frac{x}{x^2 + y^2} \right) + \partial_y \left(\frac{y}{x^2 + y^2} \right)$$

exactly coincides with

$$\nabla^2 \ln \sqrt{x^2 + y^2} = +2\pi \delta(x) \delta(y),$$

we identically solve Eq. (17) by taking [as usual $r = \sqrt{x^2 + y^2}$ and $\varphi = \arctan(y/x)$]

$$\Phi(\vec{x}) = - \frac{g}{2\pi} \arctan\left(\frac{y}{x}\right) \Rightarrow \Phi(r, \varphi) = - \frac{g}{2\pi} \varphi. \quad (18)$$

⁶In the Maxwell-Chern-Simons case, the naïve breaking of such an identity yields the breaking of gauge invariance. Thus one should take into account that the monopole induces an extra electric current in order to balance $\partial_\mu j^\mu = 0$, and so restores gauge invariance [see Refs. [26] and [27] for details. See also Ref. [28] for an alternative approach to a similar problem in (3+1) dimensions].

The appearance of the angle function above suggests us the need for a single-valuedness requirement: $\Phi(\varphi) = \Phi(\varphi + 2\pi n)$. Its remarkable angular (instead of being radial) dependence leads to a very interesting (static) electric field [$\vec{\mathcal{E}} = -(\nabla\Phi + \partial_t\vec{A})$, as usual]:

$$\vec{\mathcal{E}}(x,y) = + \frac{g}{2\pi} \frac{x\hat{y} - y\hat{x}}{x^2 + y^2} \Rightarrow \vec{\mathcal{E}}(r,\varphi) = + \frac{g}{2\pi} \frac{\hat{e}_\varphi}{r}, \quad (19)$$

whence we clearly see the announced property of the g monopole: it yields a (static) tangential electric field.⁷ [As far as we have seen, such a peculiarity takes place only in (2+1)D electrodynamics. Furthermore, we do expect that such a property survives at time-dependent regimes.] Moreover, it is worth noticing that a pointlike magnetic vortex is characterized by a vector potential identical in structure to the tangential electric field above [29]. Thus we may identify a ‘‘duality’’ between both objects: the vortex is obtained from the monopole (more precisely, from its ‘‘string’’ —see below) by taking the electric field and the charge of the first to be respectively the vector potential and the magnetic flux associated to the latter.

On the other hand, it is a well-known fact that in (2+1)D the ‘‘worldline’’ of a monopole is reduced to a point in (2+1)-dimensional space time (see, for example, Refs. [26] and [27]; see also Ref. [30], where is presented a study of monopoles in non-Abelian Chern-Simons theory supplemented by Higgs mechanism). Therefore the singular point above cannot be identified with the monopole itself. Actually, the modified Bianchi equation, $\partial_\mu \tilde{F}^{\mu\nu} = g \delta^2(\vec{x})$, has to be rather viewed as an equation for the ‘‘string’’ of the monopole. What happens is that at static limit the ‘‘string’’ [indeed, reduced to a spatial point in the (2+1)D case] appears to be localized at the origin.

Although such a localization seems to state us that g should be rather faced as a peculiar electric charge, we stress that this is not so. Indeed, what occurs is that, at static limit, the vanishing of radiation, $\int \nabla \cdot \vec{S}^* d^2x = \int \nabla \cdot (\vec{\mathcal{E}}^* \mathcal{B}) d^2x = 0$, demands that the monopole magnetic field must also vanish. (Notice that such a requirement, $\mathcal{B} = 0$, is intimately related to the tangential feature of $\vec{\mathcal{E}}$, once that $\vec{\mathcal{E}}^*$ becomes radial, and so $\nabla \cdot \vec{\mathcal{E}}^* \neq 0$.) Hence what we may state is that such an object yields only nonvanishing (tangential) electric field at the static limit.

Next, we analyze the (classical) dynamics of a usual electric charge q with mass m , moving under the action of such a tangential field. Its equations of motion are easily obtained and read as follows:

$$\frac{2\pi m}{gq} \ddot{x} = - \frac{y}{x^2 + y^2} \quad \text{and} \quad \frac{2\pi m}{gq} \ddot{y} = + \frac{x}{x^2 + y^2}, \quad (20)$$

⁷Strictly speaking, such a field does not produce a genuine Newton’s force on another charge (usual or peculiar one), since the force between them does not lie on the line that links both particles, as may be readily seen.

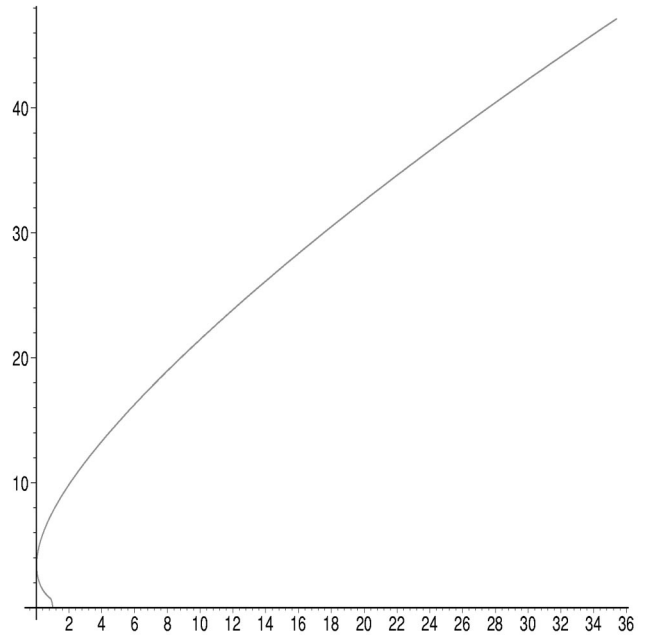


FIG. 1. Typical plot (x - y coordinates) of the trajectory of a charged particle (initially at rest in $x=1, y=0$) under the action of the tangential electric field alone (with $2\pi m/qg=1$).

or in (r, φ) coordinates:

$$\frac{2\pi m}{gq} (\ddot{r} - r\dot{\varphi}^2) = 0 \quad \text{and} \quad \frac{2\pi m}{gq} \frac{d}{dt}(r^2\dot{\varphi}) = 1. \quad (21)$$

Now, due to the angle-dependent feature of the potential, we notice that the particle ‘‘angular momentum’’ is clearly not conserved. As far as we have seen, such a nonconservation imposes an intricate coupling between the coordinates, what implies in serious difficulties towards analytical resolution of the differential equations. A typical plot of the motion (x - y coordinates) of the charged particle is shown in Fig. 1. By virtue of the tangentially repulsive nature of the electric field, the particle is quickly drifted away, despite the signals of the charges.

A further system which deserves more attention is that in which we also have the presence of an external (constant, for concreteness) magnetic field. A realistic planar system may be obtained at very low temperatures (around or less than 1 K) and sufficiently strong magnetic field (at least 10 T) perpendicular to a very thin plate.⁸ Such a perpendicular field is got by taking a vector potential entirely confined to the 2D-spatial plane, for example, $\vec{A} = \vec{A}_1 = B_0 x \hat{j}$, $\vec{A} = \vec{A}_2 = -B_0 y \hat{i}$ (Landau gauges) or still $\vec{A} = \frac{1}{2} \vec{A}_1 + \vec{A}_2$ (symmetric gauge).

⁸Such systems may be realized, for instance, in the interface between two semiconductors. Furthermore, since the motion of the charges (electrons, for concreteness) takes place as if the third dimension (perpendicular to the plane of motion) were frozen, the generally employed 2D (spatial) treatment is justified, and has been shown to give us a very good explanation of the physical phenomena which occur within such systems, e.g., the quantum Hall effect.

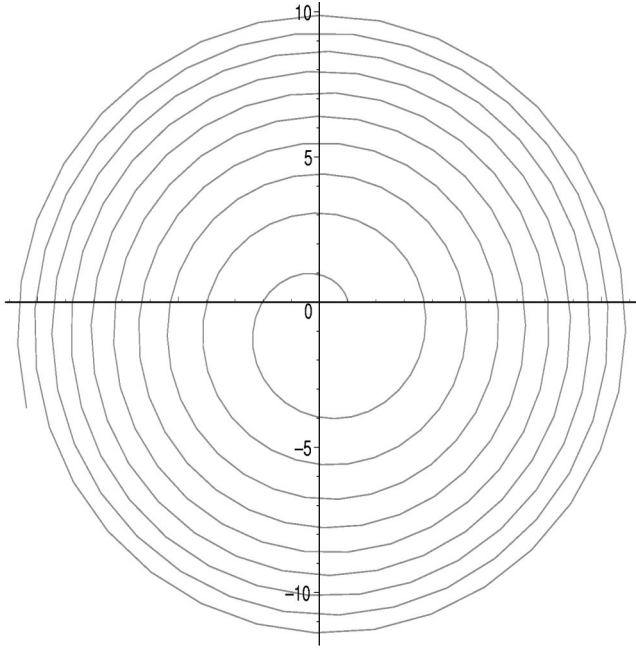


FIG. 2. Now, the system is supplemented by a homogeneous external magnetic field, $qB_0=1$ (we take the same initial conditions and values of the parameters of the preceding case).

Now, our present system is composed by the electric charge subject to the external magnetic and to the tangential electric field as well. Again, the classical equations of motion are easy to be obtained and read (equations of motion in r, φ immediately follow)

$$\frac{m\ddot{x}}{q} = -\frac{q}{2\pi} \frac{y}{x^2+y^2} + B_0\dot{y} \quad \text{and} \quad \frac{m\ddot{y}}{q} = +\frac{g}{2\pi} \frac{x}{x^2+y^2} - B_0\dot{x}. \quad (22)$$

Or, by defining complex dynamical variables as $\eta = x + iy$ and $\eta^* = x - iy$, we get

$$2m(\ddot{\eta}\eta^* + \dot{\eta}\dot{\eta}^*) + iqB_0(\dot{\eta}\eta^* - \eta\dot{\eta}^*) = 0 \quad \text{and}$$

$$4\pi m(\ddot{\eta}\dot{\eta}^* + \dot{\eta}\ddot{\eta}^*) + iqg \frac{(\dot{\eta}\eta^* - \eta\dot{\eta}^*)}{\eta\eta^*} = 0.$$

Despite their symmetric appearance, the resolution of the equations above is not too easy. Indeed, we expect that they may be even more difficult to be solved than those in the absence of magnetic field (previous case).

On the other hand, numerical resolution shows us that the magnetic field tends to compensate the repulsive effect of the electric one so that the (classical) motion of the particle appears to drift in a slower way, describing an almost regular spiral-like pattern (see Fig. 2). Notice also that the distance between two neighbor arms of such a pattern decreases as the radial distance increases: the particle asymptotically ‘‘approaches’’ to perform a closed trajectory (in the next section,

we shall see that, the quantum dynamics of the charged particle asymptotically, $r \rightarrow \infty$, reduces to that of one central harmonic oscillator).

There is, however, at least one important information which may be analytically obtained: in both cases, $B_0=0$ and $B_0 \neq 0$, the velocity of the charged particle is bounded by the angle, as below:

$$(\vec{v})^2(t) = \frac{qg}{m\pi} \varphi(t) + (\vec{v}_0)^2. \quad (23)$$

It is worth noticing that the number of windings of the charge around the origin must be taken into account, i.e., the kinetic energy is determined by the total angle described by the charge. [As a sort of quantum counterpart, we shall see that as $r \rightarrow \infty$ the (angular) energy eigenvalues have to be shifted as $\varphi \rightarrow \varphi + 2\pi$ (see the next section for details).]

V. PRELIMINARY ANALYSIS OF THE QUANTUM CHARGE-MONOPOLE SYSTEM

Next, we shall present a preliminary (nonrelativistic) quantum analysis of the system above: one electric charge q moving under the action of the monopole scalar potential and of an external constant magnetic field B_0 . The Hamiltonian (the pure gq system is readily obtained by setting $\vec{A}=0$),

$$H = \frac{1}{2m} (\vec{p} - q\vec{A})^2 + qV$$

for this system is obtained by taking \vec{A} in a particular gauge (Landau or symmetric), as well as $V(x,y) = -(g/2\pi)\arctan(y/x) = -(g/2\pi)\arg(\vec{r})$. [Notice that the potential remains invariant under general scale transformation, say: $x \rightarrow f(x,y)x$ and $y \rightarrow f(x,y)y$, but the same symmetry is not present in the full Hamiltonian, even for $f(x,y) = a = \text{const.}$]

For the analysis to be presented here, concerning the non-conservation of the angular momentum and some of its consequences, as well as asymptotic behaviors of the present system, it will be more convenient to write the Hamiltonian above in polar coordinates, r, φ , and \vec{A} in the symmetric gauge, as below:

$$H = \frac{1}{2m} \left[p_r^2 + \frac{p_r}{r} + (qB_0)^2 r^2 \right] + \frac{1}{2m} \frac{p_\varphi^2}{r^2} + \frac{qB_0}{2m} p_\varphi - \frac{gq}{2\pi} \varphi, \quad (24)$$

with r and φ defined as before and $\vec{p} = p_r \hat{e}_r + (p_\varphi/r) \hat{e}_\varphi$, whence there follows that $p_r \leftrightarrow -i\hbar(\partial/\partial r)$ and $p_\varphi \leftrightarrow -i\hbar(\partial/\partial \varphi)$.

Now, we notice the first remarkable feature of this Hamiltonian: H is explicitly angle dependent and so noninvariant under rotations; conversely, the angular momentum operator, $J = p_\varphi = -i\hbar(\partial/\partial \varphi)$, is not conserved, $[J, H] = +i\hbar gq/2\pi \neq 0$.

Although other angle-dependent Hamiltonians have been studied and shown to be relevant in physics (see, for example, Ref. [31]), a remarkable difference between them and the one presented here is that the latter is not separable.

Indeed, as far as we have seen, the system appears to present an intricate coupling between its degrees of freedom, despite of the coordinates chosen. (Perhaps, some nonstandard transformation could lead us to such a separation, but could also lead us, on the other hand, to results which were of hard physical interpretation. Such an issue remains to be investigated.)

It is clear, from the Hamiltonian (24) and also from the fundamental commutation relations, $[r, \varphi] = [p_r, p_\varphi] = 0$ and $[r, p_r] = [\varphi, p_\varphi] = +i\hbar$, that the nonseparability arises from the nonconservation of the angular momentum, $[J, H] \neq 0$. Indeed, as it may be easily checked, such an angular sector would be separable if it had the general form $(1/r^2)(J^2 + aJ + b\varphi)$. So, it is the lack of a $1/r^2$ factor in J and in φ terms that prevents us from having a split of variables.

On the other hand, by facing H as being nonseparable, the analytical resolution of the eigenvalue problem, $H|\psi\rangle = E|\psi\rangle$, appears to be of very hard achievement. [Actually, the presence of the terms proportional to φ and r —or powers of r —in H prevents us from solving this eigenvalue problem by means of, for example, hypergeometric functions (see, for example Ref. [32]).]

Therefore a numerical resolution appears to be a more suitable (and direct) attempt towards solving the problem (results will be communicated as soon as they were obtained). Here, however, we shall deal with some analytical results at asymptotic limits, even though some of them appear to be quite qualitative. We shall mainly discuss the limits $r \rightarrow 0$ and $r \rightarrow \infty$:

(i) $r \rightarrow 0$: we have seen that near the origin (where the string is localized), the charged particle experiences a very strong tangentially repulsive electric field (see previous section for details). Since as $r \rightarrow 0$ this field blows up, it is expected that q can never reach the origin, say, its wave function must vanish there: $|\psi(r=0, \varphi)\rangle \equiv 0$. Such a requirement may be viewed as the counterpart of the Dirac veto in (3+1)D: a single charge moving under the action of the magnetic monopole field could not cross the string of its associated vector potential [25]. [In addition, such a requirement will impose (see $r \rightarrow \infty$ limit below) severe restrictions on the asymptotic wave solutions.]

Thus what remains to be determined is how quickly $|\psi\rangle$ vanishes as $r \rightarrow 0$. Nevertheless, contrary to the $r \rightarrow \infty$ limit, in which the Hamiltonian gets separable (see below), here the variables are not naïvely separated. This arises because p_φ^2/r^2 is one of the leading terms, similarly to the original problem, described by the Hamiltonian (24). (In this sense numerical techniques could help us in order to get some information about the present system as $r \rightarrow 0$, say, the form of the wave functions and eigenvalues.)

(ii) $r \rightarrow \infty$: supposing that the canonical momenta remain finite in this limit, we get

$$H(r, \varphi)_{r \rightarrow \infty} \approx \frac{1}{2m}(p_r^2 + q^2 B_0^2 r^2) + \frac{qB_0}{2m} p_\varphi - \frac{gq}{2\pi} \varphi, \quad (25)$$

in which the variables appear explicitly split, say, $H_{r \rightarrow \infty} = H_{r \rightarrow \infty}^r + H_{r \rightarrow \infty}^\varphi$. Thus, at this limit, we have that (the limit $r \rightarrow \infty$ is implicit hereafter)

$$\begin{aligned} [H|\psi(r, \varphi)\rangle] &= [E_n|\psi(r, \varphi)\rangle] \Rightarrow (H^r + H^\varphi)|R(r)\Phi(\varphi)\rangle \\ &= [(E^r + E^\varphi)|R\Phi\rangle], \end{aligned} \quad (26)$$

which leads us to

$$H^r R_k(r) = E_k^r R_k(r) \quad \text{and} \quad H^\varphi \Phi_l(\varphi) = E_l^\varphi \Phi_l(\varphi). \quad (27)$$

Therefore, as $r \rightarrow \infty$, we get the following set of differential equations:

$$\hbar^2 \frac{d^2}{dr^2} R + (2mE^r - q^2 B_0^2 r^2) R = 0, \quad (28)$$

$$i\hbar \frac{d}{d\varphi} \Phi + (\epsilon^\varphi + \beta\varphi)\Phi = 0, \quad (29)$$

with $\beta = +mg/\pi B_0$ and $\epsilon^\varphi = 2mE^\varphi/qB_0$.

We notice that, at this limit, the radial part of the Hamiltonian reduces to that of one central harmonic oscillator, whose solutions may be written in terms of Hermite polynomials, H_n :

$$R_k(u) = R_0 e^{-u^2/2} H_k(u),$$

with $u = qB_0 r$. This implies the well-known eigenvalues $E_k^r = \hbar\omega_c(k+1/2)$, where $\omega_c = qB_0/m$ is the cyclotron frequency. Now, by virtue of the requirement $|\psi(r=0, \varphi)\rangle \equiv 0$, only the k =odd solutions will survive. This implies in a nonvanishing value for the lowest (renormalized) energy level, $E_k^1 = \hbar\omega_c$.

On the other hand, the angular sector appears to be quite unusual. Indeed, by solving the differential equation in φ , we readily obtain

$$\Phi_l(\varphi) = \Phi_0 \exp\left[\frac{i}{\hbar} \left(\frac{\beta\varphi}{2} + \epsilon_l^\varphi\right) \varphi\right]. \quad (30)$$

It is worth noticing the new φ^2 -like phase factor, along with the usual linear one. As a first remark, we should stress that it cannot be removed by any suitable gauge transformation; indeed, it must rather be faced as a consequence of the φ -like scalar potential. Although quite unusual, it leads us to new and interesting results. First, notice that $\Phi(\varphi)$ has periodicity $2\pi(\beta\pi + \epsilon_l^\varphi)$. Thus the requirement that Φ be single valued, i.e., continuous, is equivalent to set

$$2\pi(\beta\pi + \epsilon_l^\varphi) = 2\pi l\hbar \Rightarrow E_l^\varphi = E_0^\varphi + \frac{\omega_c}{2} l\hbar. \quad (31)$$

Now, if we identify the parameter l as the number of windings q gives around g (for example, in the counter clockwise sense), then l shall be taken as a non-negative integer (indeed, the negative values would be associated to the clockwise sense). Therefore the eigenvalues associated to the angular variable feel whether it is running between 0 and 2π , 2π and 4π , and so forth. In other words, whenever φ is shifted, say, by 2π , its associated eigenvalues respond to this change by shifting up their values.

However, before completing a winding around g , q particle would have vanishing energy, since $l=0$. It is precisely

here that $E_0^\varphi = -gq/2$ enters: since E_0^φ is a classical value, the lowest (angular) energy level of q charge is nonvanishing. In other words, since the angular potential is acting on q , it is expected that, by conservation of energy, this charge has nonvanishing (also angular) kinetic energy, as long as it has started its motion. This is the reason why there appears an E_0^φ with an intrinsically classical nature.

The results above may be viewed as a quantum analog of Eq. (23). For example, the fact that \vec{v}^2 be given by the full angle (including many windings) is now represented by parameter l ; and the initial kinetic energy, \vec{v}_0^2 (which may be classically set to zero) ‘survives’ at quantum level, but acquiring a intrinsic nonvanishing value.

Moreover, we could be tempted to naively apply J operator on $|\psi_{kl}\rangle$ above, to get

$$J|\psi_{kl}\rangle = (\beta\varphi + \epsilon_l^\varphi)|\psi_{kl}\rangle = [\beta(\varphi + \pi) + l]|\psi_{kl}\rangle,$$

and hence to guess that $|\psi_{kl}\rangle$ carry continuous angular momentum. However, this is not a legitimate procedure, because $|\psi_{kl}\rangle$ are not eigenvectors of J (recall that $[J, H] \neq 0$). Actually, as far as we have seen, the only two quantities which may be simultaneously diagonalized in $|\psi_{kl}\rangle$ -basis are H_r and H_φ [the components of the asymptotic Hamiltonian, Eq. (25)].

Clearly, the results and remarks above are strictly valid only at the asymptotic limits specified previously. Whether similar scenario does happen at arbitrary distances [as the classical result (23) does], remains to be studied and will be strongly dependent on the separation of variables in the original Hamiltonian, Eq. (24).

A naive analysis of the limits discussed above would lead us to conclude that, since the charged particle is repelled from the origin by the φ potential and since as $r \rightarrow \infty$ its dynamics reduces to that of one central harmonic oscillator (whose wave functions fall off exponentially), it is expected that the system yields physical bound states. Therefore even though the pure gq system does not admit bound states [once that the confining r^2 -type potential is absent, for this case, in Eq. (25), we get indeed a radially free particle], when it is supplemented by a suitable external magnetic field, the possibility for such states may be raised.

Nevertheless, when electrons are moving on the plane subject only to a perpendicular magnetic field, then the choice of Landau gauge immediately reduces the quantum problem to that of one harmonic oscillator in one dimension, and a free particle motion in the other direction. In this case, we cannot have bound states.⁹ However, when the system is supplemented by an extra, say, scalar potential (as in the present case), it is also well known that bound states show

⁹Nevertheless, whenever two species of fermions are combined into a unique four-component spinor, the presence of a constant magnetic field induces flavor symmetry breakdown and fermion condensates appear [33]. Such condensates are, however, quite sensible to thermal effects and disappear at finite temperature [34].

up, even in the case of repulsive potential [35]. Here, we have just raised such a question, and a precise answer demands further investigation.

VI. CONCLUSION AND PROSPECTS

We have shown that classical (2+1)D Maxwell and Maxwell-Chern-Simons electrodynamics present some interesting novelties as compared to Maxwell theory in (3+1)D, namely, the reverberation of signals and the far-from-trivial question of a Larmor-like formula. As we have seen, such phenomena are intimately related to the failure of the Huyghens principle. Namely, the latter is very difficult to be obtained even for constant accelerated motions (parabolic and hyperbolic ones). The integrals involved are highly non-trivial and appear to diverge, so demanding some suitable regularization scheme. On the other hand, we hope that some hints about such a Larmor formula could be obtained with the help of numerical calculations. Next, as a natural extension of our present results, we shall pursue an investigation of the canonical quantization of the electromagnetic radiation for the models contemplated here [36].

Concerning the Dirac-like monopole, it also presents some new properties whenever compared to its (3+1)D counterpart; for instance, its static tangential electric field. Furthermore, acting on a single charged particle, it leads us to interesting classical and quantum results. For example, the gq system (with B_0) has been shown to give rise, at least asymptotically and at nonrelativistic regimes, to a central harmonic oscillator, with an interesting angular sector which contributes to the energy eigenvalues.

As future prospects, solutions to the Hamiltonian of Eq. (24) in its general form shall be the object of a further investigation [37]. It would be also of relevance to compute possible effects of this peculiar potential on spin particles, for instance, planar Dirac fermions. Moreover, by virtue of its peculiar scalar potential (and unusual consequences), such a monopole could be relevant to condensed matter problems. For instance, by looking at this object as a sort of impurity (scatter) within a sample, could its presence modify the Hall conductivity? And eventually, what would such a modification actually look like?

ACKNOWLEDGMENTS

The authors are grateful to Professor S.A. Dias, Professor B. Schroer, F. Araruna, H. Belich, J. L. Boldo, R. Casana, G. Cuba Castillo, O. Del Cima, R. Klippert, L. Moraes, A. Nogueira, R. Paunov, and R. Rodrigues for useful discussions. They also express their gratitude to Professor M. Henneaux for a careful reading of an earlier version of this manuscript and for having drawn their attention to the work of Ref. [10]. Professor Ashok Das and Professor M. Plyushchay are deeply acknowledged for a number of very pertinent comments and for having drawn our attention to some relevant references. Finally, the authors would also like to thank Professor R. Jackiw for having pointed out the work of Ref. [21]. CNPq-Brasil is also acknowledged for financial support.

- [1] W. Siegel, Nucl. Phys. **B156**, 135 (1979); J. Schonfeld, *ibid.* **B185**, 157 (1981). For more recent references, see, for example, R. Jackiw, *Lectures Notes*, in Proc. V Jorge A. Swieca School '89, Campos do Jordão, Brasil (Sociedade Brasileira de Física, 1989); R. Jackiw, *Topics in Planar Physics* (1989); G. Dunne, *Aspects of Chern-Simons Theory*, Lectures given at Les Houches Summer School '98, also available via hep-th/9902115; A. Lerda, *Anyons: Quantum Mechanics of Particles with Fractional Statistics*, Lectures Notes in Physics (Springer International, Berlin, 1992).
- [2] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982); Phys. Rev. Lett. **48**, 975 (1982).
- [3] J. Schwinger, Phys. Rev. **128**, 2425 (1962).
- [4] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); **49**, 957 (1982); see also F. Nobre and C. Almeida, Phys. Lett. B **455**, 213 (1999); C. Hagen, *ibid.* **470**, 119 (1999).
- [5] J.L. Boldo, L.M. de Moraes, and J.A. Helayël-Neto, Class. Quantum Grav. **17**, 813 (2000); J.L. Boldo, N. Panza, and J.A. Helayël-Neto, "Propagating Torsion in 3D-Gravity and Dynamical Mass Generation," hep-th/9905110.
- [6] G. Morandi, *Quantum Hall Effect* (Bibliopolis, Naples, 1988); H. Aoki, Rep. Prog. Phys. **50**, 655 (1987); R. Prange and S. Girvin, *The Quantum Hall Effect* (Springer, New York, 1987).
- [7] J. Bednorz and A. Müller, Z. Phys. B: Condens. Matter **64**, 89 (1986); R. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988); A. Fetter, C. Hanna, and R. Laughlin, Phys. Rev. B **39**, 9679 (1989); Y.-H. Chen, F. Wilczek, E. Witten, and B. Halperin, Int. J. Mod. Phys. B **3**, 1001 (1989).
- [8] S. Randjbar, A. Salam, and J. Strathdee, Nucl. Phys. **B340**, 403 (1990); Int. J. Mod. Phys. B **5**, 845 (1991); N. Dorey and N. Mavromatos, Phys. Lett. B **266**, 163 (1991); Nucl. Phys. **B386**, 614 (1992).
- [9] M. De Andrade, O. Del Cima, and J.A. Helayël-Neto, Nuovo Cimento A **111**, 1145 (1998); O. Del Cima, Ph.D. thesis, CBPF, 1997.
- [10] E. D'Hoker and L. Vinet, Ann. Phys. (N.Y.) **162**, 413 (1985).
- [11] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, New York, 1962), Vol. 2, mainly pp. 202–206 and 686–698.
- [12] I. Gradshteyn and R. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, Orlando, 1980).
- [13] S. Ghosh, Phys. Rev. D **57**, 6342 (1998).
- [14] R. Courant and D. Hilbert, in Ref. [11], see the topics related to Huyghens' principle and propagation, namely, see pp. 202–206, 695–698, and 760–766.
- [15] B. Baker and E. Copson, *The Mathematical Theory of Huygens' Principle* (Oxford University Press, London, 1953).
- [16] J.J. Giambiagi, Nuovo Cimento A **104**, 1841 (1991); C. Bollini and J.J. Giambiagi, J. Math. Phys. **34**, 610 (1993); D. Barci, C. Bollini, L. Oxman, and M. Rocca, Int. J. Theor. Phys. **37**, 3015 (1998); R. Amaral and E. Marino, J. Phys. A **25**, 5183 (1992).
- [17] W.A. Moura-Melo and J.A. Helayël-Neto (in preparation).
- [18] Ashok Das (private communication).
- [19] B. Schroer (private communication).
- [20] A.O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Dover Edition, New York, 1980), for the Green functions for massive scalar fields, see pp. 158–160.
- [21] S. Carroll, G. Field, and R. Jackiw, Phys. Rev. D **41**, 1231 (1990).
- [22] B. Binengar, J. Math. Phys. **23**, 1511 (1982); S. Deser and R. Jackiw, Phys. Lett. B **263**, 431 (1991).
- [23] M. Plyushchay, Nucl. Phys. **B589**, 413 (2000); **B362**, 54 (1991).
- [24] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); **49**, 957 (1982); R. Pisarski and S. Rao, Phys. Rev. D **32**, 2081 (1985), Appendix A, where some classical aspects of MCS model and their relevance to a quantum analysis, including fractional statistics, are pointed out.
- [25] P.A.M. Dirac, Proc. R. Soc. London **A133**, 60 (1931).
- [26] M. Henneaux and C. Teitelboim, Phys. Rev. Lett. **56**, 689 (1986).
- [27] R. Pisarski, Phys. Rev. D **34**, 3851 (1986).
- [28] W.A. Moura-Melo, N. Panza, and J.A. Helayël-Neto, Int. J. Mod. Phys. A **14**, 3949 (1999); in Proceedings of the XVIII Brazilian National Meeting of Particles and Fields (Sociedade Brasileira Física, 1997), p. 146; Winder A. Moura-Melo, M.S. thesis, CBPF, 1997.
- [29] R. Jackiw, Ann. Phys. (N.Y.) **201**, 83 (1990). Electrically charged vortices may also appear in Chern-Simons models provided that Higgs mechanism be implemented: S. Paul and A. Khare, Phys. Lett. B **174**, 420 (1986); H. de Vega and F. Schaposnik, Phys. Rev. Lett. **56**, 2564 (1986); multivortex solutions for an Abelian Chern-Simons-Higgs theory were first discussed in J. Hong, Y. Kim, and P. Pac, *ibid.* **64**, 2230 (1990); R. Jackiw and E. Weinberg, *ibid.* **64**, 2234 (1990); while the non-Abelian case appeared in L. Cugliandolo, G. Lozano, L. Manias, and F. Schaposnik, Mod. Phys. Lett. A **6**, 479 (1991).
- [30] J. Edelstein and F. Schaposnik, Nucl. Phys. **B425**, 137 (1994).
- [31] K. Fujikawa, Nucl. Phys. **B484**, 495 (1997); B. Mandal, Int. J. Mod. Phys. A **15**, 1225 (2000), and related references cited therein.
- [32] A. Erdélyi *et al.*, *Higher Transcendental Functions* (Bateman Project) (McGraw-Hill, New York, 1953), Vol. I, namely, see pp. 232–237.
- [33] K. Klimenko, Z. Phys. C **54**, 323 (1992); V. Gusynin, V. Miransky, and I. Shovkovy, Phys. Rev. Lett. **73**, 3499 (1994).
- [34] A. Das and M. Hott, Phys. Rev. D **53**, 2252 (1996).
- [35] See, for example, G. Morandi, in Ref. [6], Chap. 5; R. Prange, Phys. Rev. B **23**, 4802 (1981); see also, P. Giacconi and R. Soldati, J. Phys. A **33**, 5193 (2000).
- [36] W.A. Moura-Melo and J.A. Helayël-Neto (work in progress).
- [37] P. Haberler, J.A. Helayël-Neto, and W.A. Moura-Melo (work in progress).