Maxwell Chern-Simons theory in a geometric representation

Lorenzo Leal* and Oswaldo Zapata
Departamento de Física, Facultad de Ciencias, Universidad Central de Venezuela, Apartado Postal 47270, Caracas 1041-A, Venezuela
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We quantize the Maxwell Chern-Simons theory in a geometric representation that generalizes the Abelian loop representation of Maxwell theory. We find that in the physical sector the model can be seen as the theory of a massles scalar field with a topological interaction that enforces the wave functional to be multivalued. This feature allows us to relate the Maxwell Chern-Simons theory with the quantum mechanics of particles interacting through a Chern-Simons field.

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I. INTRODUCTION

The Maxwell Chern-Simons theory (MCST) [1] presents the interesting property of being a massive theory, while also being gauge invariant. The mass is provided by the topological Chern-Simons (CS) term, which, in turn, has been widely considered both in the Abelian and non-Abelian cases as a pure gauge theory [1–5].

The purpose of this paper is to study the geometric representation appropriate for the Abelian MCST, within the spirit of the loop representation of Maxwell theory [6–8]. Our motivation is mainly to obtain further insight into the loop and path representations that could be useful for later developments in more realistic theories, such as quantum gravity in the Ashthekar formulation [19–21]. Within this program, the path representation of the Proca-Stueckelberg theory was studied recently [9]. Also, the Maxwell field coupled to point particles has been quantized in a geometric representation [10]. A common feature of these models, which is also shared by the free Maxwell theory, is that the introduction of loops or open paths (depending on the case) automatically solves the Gauss constraint. As we shall see, this is not the case for the MCST. Instead, the Gauss constraint further restricts the path space, leaving the boundary of the paths as the relevant geometric structures, except by the fact that the theory is sensible to the number of times hat the paths wind around their own boundaries. This feature lead us to deal with multivalued wave functionals. A similar result was obtained for the Chern-Simons field coupled to a scalar field several years ago [11].

The multivaluedness of wave functions due to topological interactions is the hallmark of anyonic behavior within the context of quantum mechanics [12–18]. Hence one could interpret the MCST as one of point particles, lying at the boundaries of the paths, and obeying fractional statistics. The statistical parameter results to be related to the mass of the model. Indeed, the mass term can be gauged away by the singular gauge transformation that maps the ordinary wave function into the multivalued one. At last, the geometric approach allows to display the following equivalence: the MCST may be mapped into a massles scalar field theory with fractional statistics.

*Email address: lleal@tierra.ciens.ucv.ve

The organization of the paper is as follows. In Sec. II we recall some basic results about the MCST and its canonical quantization. In Sec. III we review the Abelian path space, and study the path representation of the quantum MCST, paying special interest to the geometric resolution of the Gauss constraint. Section IV is devoted to explore the relation between the MCST, the massles scalar field theory, and the quantum mechanics of nonrelativistic particles with CS interaction. A short discussion is presented in Sec. V.

II. MODEL

The MCS Lagrangian density is given by [1]

$$\mathcal{L}=-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}+\frac{k}{4\pi}\epsilon^{\alpha\beta\gamma}\partial_{\alpha}A_{\beta}A_{\gamma}, \tag{1}$$

where $F_{\alpha\beta}=\partial_{\alpha}A_{\beta}-\partial_{\beta}A_{\alpha}$. We take $g_{\mu\nu}=\mathrm{diag}(1,-1,-1)$. The equation of motion

$$\partial_{\alpha}F^{\alpha\gamma}+\frac{k}{2\pi}\epsilon^{\alpha\beta\gamma}\partial_{\alpha}A_{\beta}=0 \tag{2}$$

leads to

$$\left[\Box+\left(\frac{k}{2\pi}\right)^{2}\right]F_{\alpha\beta}=0 \tag{3}$$

which shows that the MCS gauge field is massive [k has units of mass, as can be readily seen from Eq. (2)]. Moreover, it can be shown that the theory posseses a single excitation with mass $|k|/2\pi$ and spin 1 ($k>0$) or -1 ($k<0$) [1].

The canonical quantization in the manner of Dirac yields the following results. In the Weyl gauge ($A_0=0$) there is a first class constraint generating the time independent gauge transformations

$$\partial_{i}\Pi^{i}(\mathbf{x})+\frac{k}{4\pi}B(\mathbf{x})=0 \tag{4}$$

with $B=\epsilon^{ij}\partial_{i}A_{j}$, and Π^{i} being the canonical momentum satisfying

$$[A_{i}(\mathbf{x}),\Pi^{j}(\mathbf{y})]=i\,\delta_{i}^{j}\delta^{2}(\mathbf{x}-\mathbf{y}). \tag{5}$$

The remaining canonical commutators vanish.

Unlike the pure Maxwell case, the momentum Π^i does not coincide with the electric field. Indeed,

$$E^i = F^{i0} = \Pi^i - \frac{k}{4\pi}\epsilon^{ij}A_j . \tag{6}$$

It is easily verified that both E^i and B are gauge invariant quantities, in contrast with Π^i and A_i. The algebra of the fundamental observables results to be

$$[E^i(\mathbf{x}),B(\mathbf{y})] = -i\epsilon^{ij}\partial_j\delta^2(\mathbf{x}-\mathbf{y}), \tag{7}$$

$$[E^i(\mathbf{x}),E^j(\mathbf{y})] = -i\frac{k}{2\pi}\epsilon^{ij}\delta^2(\mathbf{x}-\mathbf{y}), \tag{8}$$

$$[B(\mathbf{x}),B(\mathbf{y})] = 0. \tag{9}$$

The Hamiltonian of the theory, on the physical sector, is given by

$$H = \int d^2x\frac{1}{2}[(E^i)^2 + B^2] \tag{10}$$

which, together with the conserved momentum

$$P^i = -\int d^2x\, \epsilon^{ij}E^jB, \tag{11}$$

the angular momentum

$$J = \int d^2x\, x^iE^iB, \tag{12}$$

and the generator of Lorentz boosts

$$M^{i0} = \frac{1}{2}\int d^2x[(E^i)^2 + B^2] - tP^i, \tag{13}$$

provide a representation of the Poincaré algebra in 2+1 dimensions [1].

III. PATH SPACE REPRESENTATION

Now we focus on the geometric representation appropriate to the MCST. To this end, we recall some basic facts about the path representation [9]. Given a curve γ in R^n, we define its form factor

$$T^i(x,\gamma) = \int_\gamma \delta^n(x-y)dy^i \tag{14}$$

which is independent of the parametrization chosen. It should be said that γ could consist of several disjoint pieces, some of which could be closed. Expression (14) allows us to group the curves in equivalence classes: two curves γ and γ' are said to be equivalent if $T^i(\gamma) = T^i(\gamma')$. It is a simple matter to show that this is indeed an equivalence relation. The equivalence classes of curves $[\gamma]$ are denominated paths. From now on, we shall not make a distinction between a path and any of its representatives.

The usual composition of curves can be lifted to a group product among paths as follows. Given two curves γ_1,γ_2, the form factor of their composition $T^i(\gamma_1\cdot\gamma_2)$ does not depend on the representatives, i.e., $T^i(\gamma_1'\cdot\gamma_2') = T^i(\gamma_1\cdot\gamma_2)$, whenever $\gamma_1'\sim\gamma_1$ and $\gamma_2'\sim\gamma_2$. Hence we define the product of two paths as the class to which the composition of their representative curves belong. Furthermore, the equivalence class of the opposite curve $-\gamma$ plays the role of the inverse path, while the equivalence class of the null curve amounts to the identity. The group so defined is Abelian, as may be readily seen.

As in the study of the Proca field [9], we shall use the path derivative $\Delta_i(x)$, which measures the change of a path-dependent functional $\Psi(\gamma)$ when a small open path $\delta\gamma_x^{x+h}$ starting at x and ending at $x+h$ $(h\to 0)$ is attached to γ:

$$\Psi(\delta\gamma.\gamma) \equiv [1 + h^i\Delta_i(x)]\Psi(\gamma). \tag{15}$$

Equation (15), defining $\Delta_i(x)$, must be thought to hold up to first order in h^i. The path derivative is related to the Abelian loop derivative $\Delta_{ij}(x)$ of Gambini-Trias [7] by

$$\Delta_{ij}(x) = \frac{\partial}{\partial x^i}\Delta_j(x) - \frac{\partial}{\partial x^j}\Delta_i(x). \tag{16}$$

This last object, also known as the area derivative, serves to compute how a path (or loop) dependent functional changes when a small plaquette is attached to it at the point x.

Using definition (15) it is a trivial matter to show that (we return to 2+1 dimensions)

$$\Delta_i(\mathbf{x})T^j(\mathbf{x}',\gamma) = \delta_i^j\delta^2(\mathbf{x}-\mathbf{x}') \tag{17}$$

hence the canonical algebra (5) is satisfied if we set

$$A_i(\mathbf{x}) \to \frac{i}{e}\Delta_i(\mathbf{x}), \tag{18}$$

$$\Pi^i(\mathbf{x}) \to eT^i(\mathbf{x},\gamma), \tag{19}$$

which constitutes a realization of the canonical operators onto path dependent wave functionals $\Psi(\gamma)$. In Eqs. (18) and (19) the constant e with units of $[mass]^{1/2}$ was introduced to properly adjust the dimensions.

To write down the constraint Eq. (4) in the path representation, we need to calculate

$$\frac{\partial}{\partial x^i}T^i(\mathbf{x},\gamma) = -\sum_s [\delta^2(\mathbf{x}-\boldsymbol{\beta}_\mathbf{s}) - \delta^2(\mathbf{x}-\alpha_\mathbf{s})]$$
$$\equiv -\rho(x,\gamma), \tag{20}$$

where β_s (α_s) is the ending (starting) point of the piece $s\text{-}th$ which contributes to the whole path γ (remember that γ may consist of several disjoint pieces). Thus $\rho(\mathbf{x},\gamma)$ can be thought as the ''form factor'' of the boundary of the path. The first class constraint of the theory Eq. (4) demands that the physical (i.e., gauge invariant) wave functionals obey

$$\left(-\rho(\mathbf{x},\gamma)+\frac{ik}{8\pi e^2}\epsilon^{ij}\Delta_{ij}(\mathbf{x})\right)\Psi(\gamma)=0. \qquad (21)$$

It is worth mentioning a major difference between the geometric representation of the MCS and the pure Maxwell or Proca theories. In the Maxwell case [6], the introduction of loop-dependent functionals automatically solves the gauge constraint. Similarly, the use of path-dependent wave functionals fulfills gauge invariance in the Proca-Stueckelberg theory [9]. However, this is not the case with the MCST. Further restrictions on the path dependence of $\Psi(\gamma)$ which are to be dictated by the constraint Eq. (21) remain to be considered. To this end, we set, without loss of generality,

$$\Psi(\gamma)=\exp[i\chi(\gamma)]\Phi(\gamma) \qquad (22)$$

and ask $\chi(\gamma)$ to obey

$$\epsilon^{ij}\Delta_{ij}(\mathbf{x})\chi(\gamma)=-\frac{8\pi e^2}{k}\rho(\mathbf{x},\gamma) \qquad (23)$$

then Eq. (21) reduces to

$$\epsilon^{ij}\Delta_{ij}(\mathbf{x})\Phi(\gamma)=0. \qquad (24)$$

Equation (23) is solved by

$$\chi(\gamma)=\frac{2e^2}{k}\int d^2x\int dx'^2\partial_i\partial_l\ln|\mathbf{x}-\mathbf{x}'|\epsilon^{lk}T^i(\mathbf{x}',\gamma)T^k(\mathbf{x},\gamma)$$
$$=-\frac{2e^2}{k}\sum_s\int_\gamma dx^k\epsilon^{lk}\left[\frac{(x-\beta_s)^l}{|\mathbf{x}-\beta_\mathbf{s}|^2}-\frac{(x-\alpha_s)^l}{|\mathbf{x}-\alpha_\mathbf{s}|^2}\right] \qquad (25)$$

as a careful application of the area derivative shows.

Since

$$\theta=\int_\gamma dx'^k\epsilon^{lk}\frac{(x'-x)^l}{|\mathbf{x}'-\mathbf{x}|^2} \qquad (26)$$

is the angle subtended by the path γ from the point $\mathbf{x}$, we see that Eq. (25) yields

$$\chi(\gamma)=-\frac{2e^2}{k}\Delta\Theta, \qquad (27)$$

where $\Delta\Theta$ is equal to the sum of the angles subtended by the pieces of the path from their final points β_s, minus the angles subtended by the same pieces measured from their starting points α_s. Hence we see that $\chi(\gamma)$ depends on γ through their boundary, and through the way that the diverse pieces of γ wind around these boundary points α_s's and β_s's.

Equation (24), on the other hand, states that $\Phi(\gamma)$ is insensitive to the addition of closed paths, i.e., $\Phi(C\cdot\gamma)=\Phi(\gamma)$, where C is a loop. Thus $\Phi(\gamma)$ only depends on the boundary of the path:

$$\Phi(\gamma)=\Phi(\alpha_s;\beta_s). \qquad (28)$$

Summarizing, we have that on the physical sector

$$\Psi(\gamma_\alpha^\beta)_{Physical}=\exp\left(-i\frac{2e^2}{k}\Delta\Theta\right)\Phi(\alpha_s;\beta_s) \qquad (29)$$

with $\Phi(\alpha_s;\beta_s)$ an arbitrary functional of the boundary of the path.

Expression (29) is then the solution to the gauge constraint in path space Eq. (21). We see that although the introduction of paths does not solve automatically the constraint, it does allow to characterize the physical sector in a geometrically appealing form. To write down the physical observables of the theory in the path space representation, one needs to know how the gauge invariant operators B and E^i act onto the physical sector of the Hilbert space. One has, after some calculations,

$$E^i(\mathbf{x})\exp[i\chi(\gamma)]\Phi(\alpha_s;\beta_s)$$
$$=\exp[i\chi(\gamma)]\left[-\frac{e}{\pi}\sum_s\left(\frac{(x-\beta_s)^i}{(\mathbf{x}-\beta_\mathbf{s}|^2}-\frac{(x-\alpha_s)^i}{|\mathbf{x}-\alpha_\mathbf{s}|^2}\right)\right.$$
$$\left.-\frac{ik}{4\pi e}\epsilon^{ij}\Delta_j(\mathbf{x})\right]\Phi(\alpha_s;\beta_s) \qquad (30)$$

and

$$B(\mathbf{x})\exp[i\chi(\gamma)]\Phi(\alpha_s;\beta_s)$$
$$=\frac{4\pi e}{k}\exp[i\chi(\gamma)]\rho[\mathbf{x},(\alpha;\beta)]\Phi(\alpha_s;\beta_s), \qquad (31)$$

where we have set $\rho(\mathbf{x},\gamma)=\rho[\mathbf{x},(\alpha;\beta)]$ to stress the fact that ρ depends on γ just through its boundary, the set of starting points α_s, and ending points β_s.

We thus see that the physical sector is invariant under the action of both B and E^i, as expected. It should be remarked that the path derivative $\Delta_i(\mathbf{x})$ acting on $\Phi(\alpha_s;\beta_s)$ is a well defined object, since a boundary dependent function $\Phi(\partial\gamma)$ is a special kind of a path-dependent one. From Eqs. (29)–(31) we also see that there is a simple unitary transformation which allows to eliminate the path dependent phase: namely,

$$\Psi(\gamma)_{Physical}\to\tilde{\Psi}(\gamma)=\exp[-i\chi(\gamma)]\Psi(\gamma)_{Physical}$$
$$=\Phi(\alpha,\beta),$$
$$A_{Physical}\to\tilde{A}=\exp[-i\chi(\gamma)]A_{Physical}\exp[i\chi(\gamma)], \qquad (32)$$

where $A_{Physical}$ is any gauge invariant operator of the theory. After the unitary transformation is performed, what is left is a dependence on the set of ''signed'' points α_s and β_s, corresponding to the boundary of the missed path. It can be shown that these sets of signed points inherit a group structure due to the paths where they come from. In fact, when a starting and ending point meet at the same place, they annihilate each other. Therefore we shall refer to them as ''points'' and ''antipoints,'' respectively. It is worth saying that there is a non-Abelian version of this group of points, which encodes the kinematics of the principal chiral fields, and that will be discussed elsewhere.

From Eqs. (10), (30), and (31), and taking into account the unitary transformation Eq. (32), we can write down the Schrödinger equation in the geometric representation

$$i\frac{\partial}{\partial t}\Phi[(\alpha_s;\beta_s),t] = \int dx^2\left\{\left[-\frac{e}{\pi}\sum_s\left(\frac{(x-\beta_s)^i}{|\mathbf{x}-\beta_\mathbf{s}|^2}-\frac{(x-\alpha_s)^i}{|\mathbf{x}-\alpha_\mathbf{s}|^2}\right) - \frac{ik}{4\pi e}\epsilon^{ij}\Delta_j(x)\right]^2 + \left(\frac{4\pi e}{k}\right)^2\rho^2[x,(\alpha;\beta)]\right\}\Phi[(\alpha_s;\beta_s),t]. \tag{33}$$

In a similar way, the conserved momentum P^i, angular momentum J, and the boosts generators M^{0i} can be realized in the geometric representation. It may be seen that the operators P^i and J act by translating and rotating the argument of the wave functional $\Phi(\alpha;\beta)$; for instance,

$$(1+u^iP_i)\Phi(\alpha;\beta)=\Phi(\alpha+\mathbf{u};\beta+\mathbf{u}) \tag{34}$$

with $\mathbf{u}$ being an infinitesimal constant spatial vector. It must be said that both P^i and J, inasmuch H should be properly regularized, since they involve ill defined products of distributions (needless to say that this feature is not a consequence of the geometric representation).

IV. RELATION WITH THE MASSLES SCALAR FIELD AND NONRELATIVISTIC ANYONS

The Schrödinger Eq. (33) resembles the wave equation of a collection of point particles interacting through a Chern-Simons term [15,17,18] in the sense that there appears a ''covariant derivative''

$$-iD_l(\mathbf{x})\equiv -i\Delta_l(\mathbf{x})-\frac{4e^2}{k}\sum_s \epsilon_{il}\left(\frac{(x-\beta_s)^i}{|\mathbf{x}-\beta_s|^2}-\frac{(x-\alpha_s)^i}{|\mathbf{x}-\alpha_s|^2}\right) \tag{35}$$

which comprises, besides the path derivative $\Delta_j(\mathbf{x})$, a term of statistical interaction among the points α and antipoints β, which should play the role of the particles. This observation can be made more precise as follows. Let us consider the *singular* gauge transformation

$$\Phi(\alpha;\beta)\rightarrow\overline{\Phi}(\alpha;\beta)\equiv\exp[i\Lambda(\alpha;\beta)]\Phi(\alpha;\beta) \tag{36}$$

with

$$\Lambda(\alpha;\beta)=-\frac{2e^2}{k}\int d^2x\int d^2y\,\rho[\mathbf{x},(\alpha;\beta)]\theta(\mathbf{x}-\mathbf{y})\times\rho[\mathbf{y},(\alpha;\beta)] = -\frac{2e^2}{k}\sum_s\sum_{s'}[\theta(\beta_\mathbf{s}-\beta_{s'})-\theta(\beta_s-\alpha_{s'}) + \theta(\alpha_s-\alpha_{s'})-\theta(\alpha_s-\beta_{s'})], \tag{37}$$

$\theta(\mathbf{x})$ being the angle that the vector $\mathbf{x}$ makes with the positive x axis. With the aid of the expressions

$$\Delta_i(\mathbf{z})\rho[\mathbf{x},(\alpha;\beta)]=\frac{\partial}{\partial z^i}\delta^2(\mathbf{z}-\mathbf{x}) \tag{38}$$

and

$$\frac{\partial}{\partial x^l}\theta(\mathbf{x})=-\epsilon_{lk}\frac{x^k}{|\mathbf{x}|^2} \tag{39}$$

it can be seen that

$$-iD_j(\mathbf{x})\Phi(\alpha;\beta)=-i\exp[i\Lambda(\alpha;\beta)]\Delta_j(\mathbf{x})\overline{\Phi}(\alpha;\beta). \tag{40}$$

Thus, in the covariant derivative and in the Schrödinger equation, the interaction may be removed at the expense of dealing with redefined wave functionals $\overline{\Phi}$, which result to be multivalued. In fact, the Schrödinger equation for that multivalued wave functional may be written as

$$i\frac{\partial}{\partial t}\overline{\Phi}[(\alpha_s;\beta_s),t]=\int dx^2\left[-\left(\frac{k}{4\pi e}\right)^2[\Delta_j(\mathbf{x})]^2 + \left(\frac{4\pi e}{k}\right)^2\rho^2[\mathbf{x},(\alpha;\beta)]\right]\overline{\Phi}[(\alpha_s;\beta_s),t] \tag{41}$$

which corresponds to the Schrödinger equation for the massles scalar field theory, with Lagrangian density

$$\mathcal{L}_\phi=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \tag{42}$$

in a geometric representation, as we briefly discuss. The associated Hamiltonian is

$$H_\phi=\int d^2x\frac{1}{2}(\Pi^2+\partial_i\phi\partial_i\phi) \tag{43}$$

with Π being the canonical momentum

$$[\phi(\mathbf{x}),\Pi(\mathbf{y})]=i\delta^2(\mathbf{x}-\mathbf{y}). \tag{44}$$

If we prescribe the realization

$$\partial_i\phi(\mathbf{x})\rightarrow -i\Delta_i(\mathbf{x}), \tag{45}$$

$$\Pi(\mathbf{x})\rightarrow\rho[\mathbf{x},(\alpha;\beta)] \tag{46}$$

onto wave functionals $\Phi_\phi(\alpha;\beta)$, the commutator (44) is verified, as well as

$$[\partial_i\phi(\mathbf{x}),\partial_j\phi(\mathbf{y})]=[\Pi(\mathbf{x}),\Pi(\mathbf{y})]=0 \tag{47}$$

while the corresponding Schrödinger equation reads

$$i\frac{\partial}{\partial t}\Phi_\phi[(\alpha_s;\beta_s),t]=\int d^2x\left[-\left(\frac{k}{4\pi e}\right)^2\Delta_j(\mathbf{x})^2+\left(\frac{4\pi e}{k}\right)^2\rho^2[\mathbf{x},(\alpha;\beta)]\right]\Phi_\phi[(\alpha_s;\beta_s),t] \tag{48}$$

which coincides with Eq. (41), as claimed, except by the fact that here the wave functional Φ_ϕ is single valued. It should be observed that there is no need to realize ϕ, since only its derivative $\partial_i\phi$ appears in the expressions for the observables of the theory. This reflects the invariance of the model under the shift $\phi\to\phi+\text{const}$.

The fact that the path dependence is only manifested through the boundary $(\alpha;\beta)$ of the path γ evidences that, indeed, there is a simpler geometry underlying both the MCS and the massles scalar field theories: the appropriate geometric representation is one of sets of ''points'' and ''antipoints'' [see comment after Eq. (36)]. This signed point group is the first member in a list of geometric structures related to gauge theories of p forms, to which paths and loops (for the case $p=1$) belong.

We are ready to compare the MCS Schrödinger equation in path space, Eq. (41), and its multivalued wave function $\bar{\Phi}$, Eq. (36), with what results from the quantization of a collection of N nonrelativistic particles interacting through a CS term [15,17,18]. The corresponding Schrödinger equation may be written as

$$i\partial_t\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_\mathbf{N},t)=\sum_{p=1}^{N}-\frac{1}{2m_p}(\nabla_\mathbf{p}-ie_p\mathbf{a}_\mathbf{p})^2\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_\mathbf{N},t) \tag{49}$$

with

$$\mathbf{a}_\mathbf{p}=\frac{1}{k}\nabla_\mathbf{p}\sum_{p\neq q}^{N}e_q\theta_{pq} \tag{50}$$

while θ_{pq} is the angle that the vector $\mathbf{x}_\mathbf{p}-\mathbf{x}_\mathbf{q}$ makes with the x axis.

Equation (49) may be written in the form

$$i\partial_t\Psi_0(\mathbf{r}_1,\ldots,\mathbf{r}_\mathbf{N},t)=\sum_{p=1}^{N}-\frac{1}{2m_p}(\nabla_\mathbf{p})^2\Psi_0(\mathbf{r}_1,\ldots,\mathbf{r}_N,t) \tag{51}$$

with

$$\Psi_0=\exp\left(-i\sum_{p<q}\frac{e_pe_q}{k}\Theta_{pq}\right)\Psi. \tag{52}$$

The multivalued function Ψ_0, which converts the the multiparticle Schrödinger equation into a ''free'' equation, presents remarkable coincidences with the functional $\bar{\Phi}(\alpha;\beta)$ [Eqs. (36) and (37)]. In fact, since $\theta(\beta_s-\beta_{s'})=\theta(\beta_{s'}-\beta_s)\pm2\pi$, Eq. (37) can be written as

$$\Lambda(\alpha;\beta)=-\frac{e^2}{k}\sum_{s'<s}[\theta(\beta_\mathbf{s}-\beta_{s'})-\theta(\beta_s-\alpha_{s'})+\theta(\alpha_s-\alpha_{s'})-\theta(\alpha_s-\beta_{s'})]+\text{const}. \tag{53}$$

In writing Eq. (53) we have omited the undetermined ''self-interaction'' terms of the type $\theta(\beta_s-\beta_s)=\theta(0)$. If the charges of the particles in the CS point-particles theory are restricted by $|e_p|=e$, we see that the phase in Eq. (52) coincides with $\Lambda(\alpha;\beta)$. In both cases, exchange of two ''particles'' makes the wave function to pick up a phase factor which is a multiple of $\exp[\pm i(e^2\pi/k)]$, depending on the route followed to exchange the ''points'' and ''antipoints'' (or the particles), and on their relative sign.

V. DISCUSSION

We have studied the canonical quantization of the MCST in a path representation. The physical sector of the theory, the basic gauge invariant operators, and the Hamiltonian were explicitly calculated in this geometric representation. The resolution [Eq. (29)] of the Gauss constraint (21), provides a nontrivial example of path-space calculation. Also, it shows the advantages of employing this formulation to deal with the geometrical content of the theory, which allows us to relate it with the quantum mechanics of point particles with anyonic behavior.

More precisely, it was shown that the MCST is equivalent to the theory of a massles scalar field whose wave functional obeys anyonic boundary conditions. This anyonic behavior is manifested in a simple form within the path-representation framework, since the ends of the paths (''points'' and ''antipoints'') just play the role of the particles whose exchanges give rise to the nonconventional statistical phase factor that reveals the anyonic content of the theory. In other words, it is due to the fact that we are working in a path representation, instead of a ''shape representation'' $|A_i\rangle$, that we can make an easy contact with the model of anyonic particles.

It would be interesting to explore the non-Abelian counterpart of the present theory in the corresponding geometric representation. One can suspect that a non-Abelian ''signed-points'' representation, which arises when dealing with the principal chiral field, could be the key to carrying out this program [see comment after Eq. (32)]. It would also be interesting to study the self-dual (i.e., massive Chern-Simons) theory [22] in the path representation. This model is dual (and henceforth equivalent) to the MCST [23], and probably there exists an underlying geometry supporting this duality that could be made explicit with the aid of an appropriate geometric representation.

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