6D trace anomalies from quantum mechanical path integrals

Fiorenzo Bastianelli*
Dipartimento di Fisica, Università di Bologna, Italy
and INFN, Sezione di Bologna, via Irnerio 46, I-40126 Bologna, Italy

Olindo Corradini†
C. N. Yang Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11794-3840
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We use the recently developed dimensional regularization (DR) scheme for quantum mechanical path integrals in curved space and with a finite time interval to compute the trace anomalies for a scalar field in six dimensions. This application provides a further test of the DR method applied to quantum mechanics. It shows the efficiency in higher loop computations of having to deal with covariant counterterms only, as required by the DR scheme.

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I. INTRODUCTION

Quantum mechanical (QM) path integrals have been usefully applied to the computation of chiral [1,2] and trace [3,4] anomalies. In these applications the anomalies are identified as certain quantum field theory (QFT) path integral Jacobians [5], first reinterpreted as quantum mechanical traces and then given a path integral representation.

The topological character of chiral anomalies explains the relative easiness of computing the corresponding QM path integrals: the interpretation of chiral anomalies as indices of certain differential operators shows how the leading semiclassical approximation of the corresponding QM path integrals will give directly the desired result. On the contrary, the calculation of trace anomalies requires to control the full perturbative expansion of QM path integrals on curved spaces. The latter has been a favorite topic of study over the years. Most of the early literature dealt with ways of deriving discretized expressions, but the program of taking the continuum limit till the end to identify the correct regularization scheme to be used directly in the continuum was almost never completed.[1]

Thus, starting from Refs. [3,4] a critical re-examination of the correct definitions of QM path integrals was initiated which lead to two well-defined and consistent schemes of regulating and computing: mode regularization (MR) [3,4,7] and time slicing (TS) [8,7]. Recently, following the suggestion in Ref. [9] of using dimensional regularization a third way of properly defining the path integrals has been developed in [10]: the dimensional regularization scheme (DR). While the counterterms required in MR and TS are noncovariant, they happen to be covariant in the DR scheme [10,11,9].

It is the purpose of this paper to describe in details this new scheme, test further its consistency and show the technical advantage of having to deal with a manifestly covariant action in performing higher loop calculations for $0+1$ nonlinear sigma models on a finite time interval (i.e., quantum mechanics on curved spaces with a finite propagation time). We apply the DR regulated path integral to compute the trace anomaly of a conformal scalar in six dimensions (for a review on trace anomalies see [12]). The correct full trace anomaly for such a scalar (and also other six dimensional conformal free fields) has only recently been calculated in [13] by using the heat kernel results of Gilkey [14]. With a DR path integral calculation we are going to reproduce the complete expression of this anomaly.

The paper is structured as follows. In Sec. II we review the DR scheme, in Sec. III we use it to calculate the trace anomaly for a conformal scalar field in 6D and in Sec. IV we present our conclusions. Finally, in Appendix A we report a list of structures and integrals employed in the main text: since the complete calculation is somewhat lengthy, it is useful for comparison purposes and future reference to record intermediate results.

II. DIMENSIONAL REGULARIZATION OF THE PATH INTEGRAL

First, we briefly review the quantization with path integrals of the motion of a (nonrelativistic and unit mass) particle on a curved space with metric $g_{ij}(x)$ and scalar potential $V(x)$. The model is described by the Euclidean action[2]

*Email address: bastianelli@bo.infn.it
†Email address: olindo@insti.physics.sunysb.edu

[1]One exception is the description of the phase-space path integral in the book of Sakita [6]: it contains noncovariant counterterms and the Feynman rules given there can be used to compute to any desired loop order since no ambiguous product of distributions is ever to be found in the loop expansion [7]. The same cannot be said for the configuration space version: ambiguities are present there and must be resolved with a consistent scheme for multiplying distributions, which is what a regularization scheme provides.

[2]We perform a Wick rotation on the time variable and work con-

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$$S[x^i]=\int_{t_i}^{t_f} dt\ \left[\frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j+V(x)\right]. \tag{1}$$

In canonical quantization one must choose an ordering consistent with reparametrization invariance to produce a quantum Hamiltonian $H=-\frac{1}{2}\nabla^2+\alpha R+V$ (our curvature conventions are found in Appendix A). The value of the parameter α depends on the particular order chosen [15] and conventionally can be taken to vanish with the agreement of reintroducing the coupling to R through the potential V.

Using path integrals the canonical ordering ambiguities re-emerge as the need of specifying a regularization scheme. The 1D sigma model in Eq. (1) contains double derivative interactions which make Feynman graphs superficially divergent at one and two loops. However, the nontrivial path integral measure can be exponentiated using ghost fields: their effect is to make finite the sum of the Feynman graphs but a regularization scheme is still necessary to render finite each individual divergent graph. Different regularization schemes require different counterterms to reproduce a quantum Hamiltonian with $\alpha=0$. In mode regularization and time slicing such counterterms are noncovariant. In the DR scheme the counterterm is covariant and equal to $V_{DR}=R/8$, as demonstrated in [10,11].

Now, let us describe the DR scheme which we are going to apply in the next section. First of all, we find it convenient to use a rescaled time parameter τ by defining $t=\beta\tau+t_f$ and $\beta=t_f-t_i$, so that τ will take values on the finite interval $I\equiv[-1,0]$. Then, we introduce bosonic a^i and fermionic b^i,c^i ghosts to exponentiate the nontrivial part of the path integral measure: integrating them back will formally reproduce the $\Pi\sqrt{\det g_{ij}}$ factor of the measure. Finally, we introduce D extra infinite regulating dimensions $\mathbf{t}=(t^1,\ldots,t^D)$ with the prescription that one will take the limit $D\to 0$ at the very end of all calculations. Denoting $t^\mu\equiv(\tau,\mathbf{t})$ with $\mu=0,1,\ldots,D$ and $d^{D+1}t=d\tau d^D\mathbf{t}$, the action in $D+1$ dimensions reads

$$S[x,a,b,c]=\frac{1}{\beta}\int_\Omega d^{D+1}t\left[\frac{1}{2}g_{ij}(x)(\partial_\mu x^i\partial_\mu x^j+a^ia^j+b^ic^j)+\beta^2V(x)+\beta^2V_{DR}(x)\right] \tag{2}$$

where $V_{DR}=R/8$ is the counterterm in dimensional regularization and $\Omega=I\times R^D$ is the region of integration containing the finite interval I.

The perturbative expansion can be generated by first decomposing the paths $x^i(\tau)$ into a classical part $x^i_{cl}(\tau)$ satisfying the boundary conditions and quantum fluctuations $q^i(\tau)$ which vanish at the boundary (the ghost fields are taken to vanish at the boundary as well) and then decomposing the Lagrangian into a free part plus interactions. The latter step is achieved by Taylor expanding the metric and the potential around a fixed point, which we choose to be the final point x_f. Thus, the propagators are recognized to be

$$\begin{aligned}\langle x^i(t)x^j(s)\rangle&=-\beta\ g^{ij}(x_f)\ \Delta(t,s)\\ \langle a^i(t)a^j(s)\rangle&=\beta\ g^{ij}(x_f)\ \Delta_{gh}(t,s),\\ \langle b^i(t)c^j(s)\rangle&=-2\beta\ g^{ij}(x_f)\ \Delta_{gh}(t,s)\end{aligned} \tag{3}$$

with

$$\Delta(t,s)=\int\frac{d^D\mathbf{k}}{(2\pi)^D}\times\sum_{n=1}^{\infty}\frac{-2}{(\pi n)^2+\mathbf{k}^2}\sin(\pi n\tau)\sin(\pi n\sigma)e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})} \tag{4}$$

$$\Delta_{gh}(t,s)=\int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_{n=1}^{\infty}2\sin(\pi n\tau)\sin(\pi n\sigma)e^{i\mathbf{k}\cdot(\mathbf{t}-\mathbf{s})}=\delta^{D+1}(t,s)=\delta(\tau,\sigma)\delta^D(\mathbf{t}-\mathbf{s}) \tag{5}$$

where

$$\delta(\tau,\sigma)=\sum_{n=1}^{\infty}2\sin(\pi n\tau)\sin(\pi n\sigma) \tag{6}$$

is the Dirac delta on the space of functions vanishing at $\tau,\sigma=-1,0$. Of course, the function $\Delta(t,s)$ satisfies the Green equation

$$\partial_\mu^2\Delta(t,s)=\Delta_{gh}(t,s)=\delta^{D+1}(s,t). \tag{7}$$

The $D\to 0$ limits of these propagators are the usual ones

$$\Delta(\tau,\sigma)=\tau(\sigma+1)\theta(\tau-\sigma)+\sigma(\tau+1)\theta(\sigma-\tau) \tag{8}$$

$$\Delta_{gh}(\tau,\sigma)=\ddot{\Delta}(\tau,\sigma)=\delta(\tau,\sigma) \tag{9}$$

where dots on the left or right side denote derivatives with respect to the first or second variable, respectively. However, such limits can be used only after one has cast the integrands corresponding to the various Feynman diagrams in an unambiguous from by making use of the manipulations allowed by the regularization scheme. In particular, in DR one can use partial integration: it is always allowed in the extra D dimension because of momentum conservation, while it can be performed along the finite time interval whenever there is an explicit function which vanishes at the boundary [for example the propagator of the coordinates $\Delta(t,s)$]. Along the way one may find terms of the form $\partial_\mu^2\Delta(t,s)$ which according to Eq. (7) gives Dirac delta functions. The latter can be safely used at the regulated level, i.e., in $D+1$ dimensions. By performing such partial integrations one tries to arrive at an unambiguous form of the integrals which can be safely and easily calculated even after the limit $D\to 0$ is taken.

An explicit example will suffice to describe how the above rules are concretely used:

sistently in the Euclidean framework. We also set $\hbar=1$.

$$
\begin{aligned}
&\int_{-1}^{0} d\tau \int_{-1}^{0} d\sigma \ (\dot{\Delta}) \ (\Delta^{\cdot}) \ (\dot{\Delta}^{\cdot}) \\
&\quad \to \int d^{D+1}t \int d^{D+1}s \ (_{\mu}\Delta) \ (\Delta_{\nu}) \ (_{\mu}\Delta_{\nu}) \\
&\quad = \int d^{D+1}t \int d^{D+1}s \ (_{\mu}\Delta) \ _{\mu}\left(\frac{1}{2}(\Delta_{\nu})^2\right) \\
&\quad = -\frac{1}{2}\int d^{D+1}t \int d^{D+1}s \ (_{\mu\mu}\Delta) \ (\Delta_{\nu})^2 \\
&\quad = -\frac{1}{2}\int d^{D+1}t \int d^{D+1}s \ \delta^{D+1}(t,s) \ (\Delta_{\nu})^2 \\
&\quad = -\frac{1}{2}\int d^{D+1}t \ (\Delta_{\nu})^2|_{t} \to -\frac{1}{2}\int_{-1}^{0} d\tau \ (\Delta^{\cdot})^2|_{\tau} \\
&\quad = -\frac{1}{24}
\end{aligned}
$$

where the symbol $|_{\tau}$ means that one should set $\sigma=\tau$.

Thus, we see that the rules of computing in DR are quite similar to those used in MR, the only diversity being in the different options allowed in partial integrations. In DR the rule for contracting which indices with which indices follows directly from the regulated action in Eq. (2) and only certain partial integrations are allowed in $D+1$ dimensions. In MR one regulates by cutting off all mode sums at a large mode N and then performs partial integrations: all derivatives are now of the same nature and different options of partial integrations arise. This explains the origin of different counterterms for these two regularizations.

III. TRACE ANOMALIES FOR A CONFORMAL SCALAR IN 6D

As described in [3,4], one-loop trace anomalies can be obtained by computing a certain Fujikawa Jacobian suitably regulated and represented as a quantum mechanical path integral with periodic boundary conditions

$$
\begin{aligned}
\int d^6x \ \sqrt{g} \ \sigma(x)\langle T^{a}{}_{a}(x)\rangle &= \lim_{\beta\to 0} \mathrm{Tr}[\sigma \ \mathrm{e}^{-\beta H}] \\
&= \lim_{\beta\to 0} \int_{PBC} \mathcal{D}x \ \sigma(x) \ \mathrm{e}^{-S[x]},
\end{aligned}
\tag{10}
$$

where on the left hand side $T^{a}{}_{a}$ denotes the trace of the stress tensor for a 6D conformal scalar and $\sigma(x)$ is an arbitrary function describing an infinitesimal Weyl variation. In the first equality the infinitesimal part of the Fujikawa Jacobian has been regulated with the conformal scalar field kinetic operator $H=-\frac{1}{2}\nabla^2-\frac{1}{10}R$. The limit $\beta\to 0$ should be taken after removing divergent terms in β (which is what the renormalization of the scalar field QFT will do), and so it picks up just the β independent term. Finally on the right hand side the trace is given a representation as a path integral corresponding to a model with Hamiltonian H and with periodic boundary conditions. The latter can be obtained using the quantum mechanics described in the previous section with a scalar potential $V=-\frac{1}{10}R$.

Thus we start computing the terms in the loop expansion of the path integral described in Sec. II. It will soon be clear that it is enough to compute up to order β^3, i.e., up to 4 loops [β can be taken as the loop counting parameter, as evident form Eq. (2)]. We use reparametrization invariance and choose Riemann normal coordinates centered at the point x_0^i representing the boundary conditions at $\tau=-1,0$, and which will be integrated over to recreate the full periodic boundary conditions on the right hand side of Eq. (10).

The expansion of the metric in Riemann normal coordinates is well known. For our case, since the action including the counterterm is manifestly covariant, that expansion can be easily generated by the method described for this context in [3]. One obtains the following terms needed in our approximation:

$$
\begin{aligned}
g_{mn}(x)dx^m dx^n = \Bigg[& g_{mn} + \frac{1}{3}R_{mabn}x^a x^b + \frac{1}{3!}\nabla_i R_{mabn}x^a x^b x^i + \frac{6}{5!}\left(\nabla_i\nabla_j R_{mabn} + \frac{8}{9}R_{mab}{}^{\alpha}R_{\alpha ijn}\right)x^a x^b x^i x^j \\
& + \frac{1}{5!}\frac{4}{3}(\nabla_i\nabla_j\nabla_k R_{mabn} + 4R_{mab}{}^{\alpha}\nabla_i R_{\alpha jkn})x^a x^b x^i x^j x^k + \frac{10}{7!}\Bigg(\nabla_i\nabla_j\nabla_k\nabla_l R_{mabn} + \frac{34}{5}R_{mij}{}^{\alpha}\nabla_k\nabla_l R_{\alpha abn} \\
& + \frac{11}{2}\nabla_i R_{mab}{}^{\alpha}\nabla_j R_{\alpha kln} + \frac{8}{5}R_{mab}{}^{\alpha}R_{\alpha ij}{}^{\beta}R_{\beta kln}\Bigg)x^a x^b x^i x^j x^k x^l + \cdots\Bigg]dx^m dx^n
\end{aligned}
$$

$$
V(x) = V + (\nabla_i V)x^i + \frac{1}{2}(\nabla_i\nabla_j V)x^i x^j + \frac{1}{3!}(\nabla_i\nabla_j\nabla_k V)x^i x^j x^k + \frac{1}{4!}(\nabla_i\nabla_j\nabla_k\nabla_l V)x^i x^j x^k x^l + \cdots \tag{11}
$$

where all tensorial quantities on the right hand sides are evaluated at the origin of the coordinate system. Notice that for the MR and TS regularization schemes the counterterms are noncovariant and their expansions cannot be generated so easily: obtaining the vertices from those counterterms would require a tedious computation.

Plugging the above expansions in the action (2) and noticing that the factor β^2 raises by two the loop order for each vertex coming from the potential or the counterterm, we compute[3]

$$\langle x_0,\beta|x_0,0\rangle = \int \mathcal{D}x e^{-S} = A \ \langle e^{-S_{int}}\rangle$$
$$= A \ \exp\left[-\langle S_4\rangle - \langle S_6\rangle - \langle S_8\rangle + \frac{1}{2}\langle S_4^2\rangle_c + \frac{1}{2}\langle S_5^2\rangle_c + \langle S_4 S_6\rangle_c - \frac{1}{6}\langle S_4^3\rangle_c + O(\beta^4)\right] \tag{12}$$

where the subscript ''c'' stands for connected diagrams only and where $A=(2\pi\beta)^{-D/2}$ gives the correct normalization of the path integral measure. Because of this normalization we see that for $D=6$ the β-independent term is obtained by picking up the β^3 contributions from the expansion of the exponential on the right hand side of Eq. (12).

The terms up to 3 loops are easily computed in DR by using the detailed expressions reported in [16]: one just needs to compute the integrals reported there using the DR rules. Including for simplicity the counterterm inside the potential V, we obtain

$$\langle S_4\rangle = -\beta\left[\frac{1}{24}R - V\right] \tag{13}$$

$$\langle S_6\rangle = -\frac{\beta^2}{12}\left[\frac{1}{40}\nabla^2 R + \frac{1}{90}R_{mn}^2 + \frac{1}{60}R_{mnab}^2 - \nabla^2 V\right] \tag{14}$$

$$\langle S_4^2\rangle_c = -\frac{\beta^2}{72}\left[\frac{1}{3}R_{mn}^2\right]. \tag{15}$$

To achieve notational simplicity in the remaining 4-loop terms we use the basis of curvature invariants given in Appendix A and compute the terms reported there. We obtain

$$\langle S_8\rangle = -\frac{\beta^3}{7!}\left[\frac{17}{15}K_4 - \frac{16}{15}K_5 + \frac{8}{5}K_6 + \frac{5}{12}K_7 - \frac{8}{3}K_8\right.$$
$$\left. + \frac{11}{10}K_{10} + \frac{3}{2}K_{11} - \frac{19}{20}K_{12} + \frac{149}{48}K_{13} - \frac{25}{8}K_{14} + \frac{11}{16}K_{15} - \frac{5}{24}K_{16} + \frac{3}{8}K_{17}\right]$$
$$- \frac{\beta^3}{6!}[2R^{mn}\nabla_m\nabla_n V + \nabla^m R\nabla_m V - 3\nabla^4 V] \tag{16}$$

$$\langle S_5^2\rangle_c = -\frac{\beta^3}{6!}\left[\frac{23}{24}K_{13} - \frac{3}{4}K_{14} - \frac{1}{8}K_{15} - \frac{5}{48}K_{16} + 5\nabla^m R\nabla_m V - 60(\nabla_m V)^2\right] \tag{17}$$

$$\langle S_4 S_6\rangle_c = -\frac{\beta^3}{6!}\left[\frac{13}{45}K_4 - \frac{1}{5}K_5 + \frac{2}{15}K_6 + \frac{3}{10}K_{10} - \frac{1}{10}K_{12}\right] \tag{18}$$

$$\langle S_4^3\rangle_c = -\frac{\beta^3}{6!}\left[\frac{2}{3}K_4 + \frac{1}{6}K_7 + \frac{4}{3}K_8\right]. \tag{19}$$

Inserting all these values into Eq. (12) we get

[3]Since $x^i(-1)=x^i(0)\equiv x_0$ the classical field is $x_{cl}(\tau)=0$; hence all diagrams with external fields vanish. We indicate with S_n the interaction terms containing n fields when originating from the expansion of the metric and $n-4$ fields when originating from the scalar potential.

$$\langle x_0,\beta|x_0,0\rangle=\frac{1}{(2\pi\beta)^3}\exp\Bigg[\beta\left(\frac{1}{24}R-V\right)+\beta^2\left(\frac{1}{720}(R^2_{mnab}-R^2_{mn})+\frac{1}{480}\nabla^2R-\frac{1}{12}\nabla^2V\right)$$
$$+\frac{\beta^3}{8!}\Big(-\frac{8}{9}K_4+\frac{8}{3}K_5+\frac{16}{3}K_6+\frac{44}{9}K_7-\frac{80}{9}K_8-8K_{10}+12K_{11}-2K_{12}$$
$$-2K_{13}-4K_{14}+9K_{15}+\frac{5}{4}K_{16}+3K_{17}\Big)$$
$$+\frac{\beta^3}{6!}\Big(2R^{mn}\nabla_m\nabla_nV-\frac{3}{2}\nabla^mR\nabla_mV+30(\nabla_mV)^2-3\nabla^4V\Big)+O(\beta^4)\Bigg]. \tag{20}$$

Now, using the value $V=\frac{1}{8}R-\frac{1}{10}R=\frac{1}{40}R$ to take into account the counterterm V_{DR} and the conformal coupling of the scalar field, we compare with Eq. (10) and obtain the corresponding trace anomaly

$$\langle T^a{}_a\rangle=\frac{1}{(2\pi)^3}\frac{1}{8!}\Big[\frac{7}{225}K_1-\frac{14}{15}K_2+\frac{14}{15}K_3-\frac{8}{9}K_4+\frac{8}{3}K_5$$
$$+\frac{16}{3}K_6+\frac{44}{9}K_7-\frac{80}{9}K_8-8K_{10}+12K_{11}+\frac{4}{5}K_{12}$$
$$-2K_{13}-4K_{14}+9K_{15}+\frac{1}{5}K_{16}-\frac{6}{5}K_{17}\Big]. \tag{21}$$

It agrees with the one given in [13], where it was shown that it can be cast also as

$$\langle T^a{}_a\rangle=\frac{1}{(2\pi)^3}\frac{1}{8!}\Big[-\frac{5}{72}E_6-\frac{28}{3}I_1+\frac{5}{3}I_2+2I_3$$
$$+\text{trivial anomalies}\Big] \tag{22}$$

with the topological Euler density given by

$$E_6=-\epsilon_{m_1n_1m_2n_2m_3n_3}\epsilon^{a_1b_1a_2b_2a_3b_3}R^{m_1n_1}{}_{a_1b_1}$$
$$\times R^{m_2n_2}{}_{a_2b_2}R^{m_3n_3}{}_{a_3b_3} \tag{23}$$

and the three Weyl invariants

$$I_1=C_{amnb}C^{mijn}C_i{}^{ab}{}_j, \tag{24}$$

$$I_2=C_{ab}{}^{mn}C_{mn}{}^{ij}C_{ij}{}^{ab}, \tag{25}$$

$$I_3=C_{mabc}\left(\nabla^2\delta^m_n+4R^m_n-\frac{6}{5}R\,\delta^m_n\right)C^{nabc}+\text{trivial anomalies}, \tag{26}$$

whereas the coefficients of the trivial anomalies are unimportant since they can be changed by the variation of local counterterms. The structure of trivial anomalies has been fully analyzed in [17]. It is interesting to note, after inspecting the results in [17], that the coefficients of K_1, K_2 and K_3 never appear in the trivial anomalies. At the same time they are produced in the previous calculation by disconnected diagrams. Thus one may fix three of the four true anomalies by a simpler lower loop calculation, while the remaining independent fourth nontrivial anomaly, which can be taken as the one corresponding to E_6, could be fixed by an independent calculation on the simplified geometry of a maximally symmetric space.

IV. CONCLUSIONS

We have used the recently developed dimensional regularization scheme for quantum mechanical path integrals [10] to compute the trace anomaly for a scalar field in six dimensions. The identification of the full anomaly required a complete 4-loop quantum mechanical computation. Technically, the covariance of the counterterm V_{DR} allows a more efficient identification of the corresponding vertices than in the MR and TS regularization schemes.

We noticed that the coefficients of three of the four nontrivial anomalies could as well be obtained by a simpler 3-loop calculation. One may speculate that such a fact may happen also for $D=8$ trace anomalies: there one would need to compute the quantum mechanics up to 5-loops, but it could happen that all nontrivial anomalies but one could be fixed by a simpler 4-loop calculation (presented in this paper) and the remaining one by a calculation on a simplified geometry. To concretely check this conjecture, one would need a cohomological analysis to identify the structure of all trivial and nontrivial anomalies, as the one given in Ref. [17] for the six dimensional case. However, such an analysis is not available in the literature yet.

One could couple the nonlinear sigma model to non-Abelian gauge potentials to obtain the trace anomalies of other six dimensional conformal fields [13]. In such an extension the main new complication is related to the time ordering prescription to be used for achieving gauge covariance, as employed in [4], which forces to compute different DR integrals for different ordering of the vertices. An approach which could guarantee non-Abelian covariance in a more straightforward manner would clearly be welcome. It may be related to the extra ghost fields used in [8].

While we have justified our anomaly computation by

viewing it as the calculation of a certain Fujikawa Jacobian, conceptually it can be thought of as performed in the first quantized approach of the scalar particle theory (see the discussion in [19]). Given that interpretation, it would be interesting to investigate if such worldline path integrals in curved space could be useful to simplify computations of scattering amplitudes and effective actions of perturbative QFT coupled to gravity, as it happens in the flat space case [20].

APPENDIX A

We use the convention $[\nabla_a,\nabla_b]V^c=R_{ab}{}^c{}_dV^d$, $R_{ab}=R_{ac}{}^c{}_b$. It is useful for notational purposes to introduce a basis of curvature invariants cubic in the curvature

$$\begin{aligned}
&K_1=R^3 && K_2=RR_{ab}^2 && K_3=RR_{abmn}^2\\
&K_4=R_a{}^mR_m{}^iR_i{}^a && K_5=R_{ab}R_{mn}R^{mabn} && K_6=R_{ab}R^{amnl}R^b{}_{mnl}\\
&K_7=R_{ab}{}^{mn}R_{mn}{}^{ij}R_{ij}{}^{ab} && K_8=R_{amnb}R^{mijn}R_i{}^{ab}{}_j && K_9=R\nabla^2R\\
&K_{10}=R_{ab}\nabla^2R^{ab} && K_{11}=R_{abmn}\nabla^2R^{abmn} && K_{12}=R^{ab}\nabla_a\nabla_bR\\
&K_{13}=(\nabla_aR_{mn})^2 && K_{14}=\nabla_aR_{bm}\nabla^bR^{am} && K_{15}=(\nabla_iR_{abmn})^2\\
&K_{16}=(\nabla_aR)^2 && K_{17}=\nabla^4R. &&
\end{aligned} \tag{A1}$$

It differs from the basis used in [17,18] only in the definition of K_{16}: the one used above enters more naturally in our calculations.

In the main text contributions of order β^3 to the effective action come from the terms listed below. In the list of integrals we use the following conventions. The limits of integration are $[-1,0]$ for all variables. For 3-dimensional integrals the first group of propagator in round brackets depends on (τ,σ), the second on (σ,ρ) and the third on (ρ,τ), with this precise order, while terms at coinciding points are explicitly indicated. For 2-dimensional integrals the propagators at non-coinciding points depend on (τ,σ), while for 1-dimensional integrals all terms are obviously taken at coinciding points. We use the shorthand notation $\overline{\dot\Delta\dot}=\dot\Delta\dot+\ddot\Delta$. The DR regularization is immediate and we quote the DR values.

$\underline{\langle S_4^3\rangle_c}$

$$\langle S_4^3\rangle_c=A_0+A_1+A_2+A_3 \tag{A2}$$

$$\begin{aligned}
A_0=\frac{\beta^3}{9}\Bigg[&\left(\frac14K_7+2K_8\right)\left(I_1^{A_0}-I_2^{A_0}-I_3^{A_0}+2I_4^{A_0}+2I_5^{A_0}-2I_6^{A_0}+\frac13I_{11}^{A_0}\right)\\
&-\left(\frac72K_7+K_8\right)\left(\frac13I_7^{A_0}+I_9^{A_0}\right)+\left(\frac{13}{4}K_7-K_8\right)\left(I_8^{A_0}+\frac13I_{10}^{A_0}\right)\Bigg]
\end{aligned} \tag{A3}$$

$$\begin{aligned}
A_1=\frac{\beta^3}{6}\ &K_6(I_1^{A_1}-I_2^{A_1}-2I_3^{A_1}+2I_4^{A_1}+I_5^{A_1}-I_6^{A_1}+I_7^{A_1}-I_8^{A_1}-2I_9^{A_1}+2I_{10}^{A_1}\\
&+I_{11}^{A_1}-I_{12}^{A_1}-2I_{13}^{A_1}+2I_{14}^{A_1}+2I_{15}^{A_1}-2I_{16}^{A_1}+2I_{17}^{A_1}-2I_{18}^{A_1}-2I_{19}^{A_1}+2I_{20}^{A_1})
\end{aligned} \tag{A4}$$

$$\begin{aligned}
A_2=\frac{\beta^3}{9}K_5[&I_1^{A_2}-I_2^{A_2}+I_3^{A_2}-I_4^{A_2}+4I_5^{A_2}-I_6^{A_2}-2I_7^{A_2}-I_8^{A_2}+I_9^{A_2}+I_{10}^{A_2}+2(-I_{11}^{A_2}-I_{12}^{A_2}-I_{13}^{A_2}\\
&+I_{14}^{A_2}+I_{15}^{A_2}-I_{16}^{A_2}-I_{17}^{A_2}+I_{18}^{A_2}+I_{19}^{A_2})]
\end{aligned} \tag{A5}$$

$$A_3=\frac{\beta^3}{27}K_4(I_1^{A_3}-6I_2^{A_3}+3I_3^{A_3}+3I_4^{A_3}-6I_5^{A_3}-6I_6^{A_3}+3I_7^{A_3}+3I_8^{A_3}+6I_9^{A_3}$$
$$+I_{10}^{A_3}-6I_{11}^{A_3}+3I_{12}^{A_3}+3I_{13}^{A_3}+6I_{14}^{A_3}-2I_{15}^{A_3}-6I_{16}^{A_3}). \tag{A6}$$

Integrals in A_0:

$$I_1^{A_0}=\int\int\int(\Delta\dot{}^2)(\Delta^2)(\dot{}\Delta\dot{}^2-\ddot{}\Delta^2)=-\frac{13}{3780}\qquad I_2^{A_0}=\int\int\int(\Delta\ \Delta\dot{})(\dot{}\Delta\ \Delta)(\dot{}\Delta\dot{}^2-\ddot{}\Delta^2)=-\frac{1}{15120}$$
$$I_3^{A_0}=\int\int\int(\dot{}\Delta\dot{}\ \Delta\dot{})(\Delta^2)(\dot{}\Delta\dot{}\ \dot{}\Delta)=\frac{13}{7560}\qquad I_4^{A_0}=\int\int\int(\dot{}\Delta\dot{}\ \Delta)(\dot{}\Delta\ \Delta)(\dot{}\Delta\dot{}\ \dot{}\Delta)=-\frac{13}{15120}$$
$$I_5^{A_0}=\int\int\int(\dot{}\Delta\ \Delta\dot{})(\dot{}\Delta\ \Delta)(\dot{}\Delta\dot{}\ \dot{}\Delta)=-\frac{1}{3780}\qquad I_6^{A_0}=\int\int\int(\dot{}\Delta\ \Delta)(\dot{}\Delta^2)(\dot{}\Delta\dot{}\ \dot{}\Delta)=\frac{1}{1890}$$
$$I_7^{A_0}=\int\int\int[(\Delta\ \dot{}\Delta\dot{})(\Delta\ \dot{}\Delta\dot{})(\Delta\ \dot{}\Delta\dot{})+(\Delta\ \ddot{}\Delta)(\Delta\ \ddot{}\Delta)(\Delta\ \ddot{}\Delta)]=-\frac{43}{15120}$$
$$I_8^{A_0}=\int\int\int(\dot{}\Delta\ \Delta\dot{})(\Delta\ \dot{}\Delta\dot{})(\Delta\ \dot{}\Delta\dot{})=\frac{11}{15120}\qquad I_9^{A_0}=\int\int\int(\dot{}\Delta\ \Delta\dot{})(\dot{}\Delta\ \Delta\dot{})(\Delta\ \dot{}\Delta\dot{})=\frac{1}{756}$$
$$I_{10}^{A_0}=\int\int\int(\dot{}\Delta\ \Delta\dot{})(\dot{}\Delta\ \Delta\dot{})(\dot{}\Delta\ \Delta\dot{})=-\frac{1}{945}\qquad I_{11}^{A_0}=\int\int\int(\dot{}\Delta^2)(\dot{}\Delta^2)(\dot{}\Delta^2)=\frac{1}{756}. \tag{A7}$$

Integrals in A_1:

$$I_1^{A_1}=\int\int\int(\dot{\overline{\Delta\dot{}}})|_\tau(\Delta\dot{})(\dot{}\Delta\dot{}\ \Delta^2)(\dot{}\Delta)=-\frac{13}{3780}\qquad I_2^{A_1}=\int\int\int(\dot{\overline{\Delta\dot{}}})|_\tau(\Delta\dot{})(\dot{}\Delta\ \Delta\ \Delta\dot{})(\dot{}\Delta)=\frac{1}{1890}$$
$$I_3^{A_1}=\int\int\int(\dot{\overline{\Delta\dot{}}})|_\tau(\Delta\dot{})(\dot{}\Delta\dot{}\ \Delta\ \Delta\dot{})(\Delta)=-\frac{1}{945}\qquad I_4^{A_1}=\int\int\int(\dot{\overline{\Delta\dot{}}})|_\tau(\Delta\dot{})(\dot{}\Delta\ \Delta\dot{}^2)(\Delta)=-\frac{1}{3780}$$
$$I_5^{A_1}=\int\int\int(\dot{\overline{\Delta\dot{}}})|_\tau(\Delta)(\Delta[\dot{}\Delta\dot{}^2-\ddot{}\Delta^2])(\Delta)=\frac{1}{432}\qquad I_6^{A_1}=\int\int\int(\dot{\overline{\Delta\dot{}}})|_\tau(\Delta)(\dot{}\Delta\dot{}\ \Delta\dot{}\ \dot{}\Delta)(\Delta)=-\frac{1}{15120}$$
$$I_7^{A_1}=\int\int\int(\Delta)|_\tau[(\dot{}\Delta\dot{})(\Delta^2\ \dot{}\Delta\dot{})(\dot{}\Delta\dot{})+(\ddot{}\Delta)(\Delta^2\ \ddot{}\Delta)(\ddot{}\Delta)]=-\frac{1}{216}$$
$$I_8^{A_1}=\int\int\int(\Delta)|_\tau(\dot{}\Delta\dot{})(\dot{}\Delta\ \Delta\ \Delta\dot{})(\dot{}\Delta\dot{})=-\frac{1}{15120}$$
$$I_9^{A_1}=\int\int\int(\Delta)|_\tau(\dot{}\Delta\dot{})(\dot{}\Delta\dot{}\ \Delta\ \Delta\dot{})(\Delta\dot{})=-\frac{1}{2160}\qquad I_{10}^{A_1}=\int\int\int(\Delta)|_\tau(\dot{}\Delta\dot{})(\dot{}\Delta\ \Delta\dot{}^2)(\Delta\dot{})=\frac{2}{945}$$
$$I_{11}^{A_1}=\int\int\int(\Delta)|_\tau(\dot{}\Delta)(\Delta[\dot{}\Delta\dot{}^2-\ddot{}\Delta^2])(\Delta\dot{})=-\frac{19}{15120}\qquad I_{12}^{A_1}=\int\int\int(\Delta)|_\tau(\dot{}\Delta)(\dot{}\Delta\dot{}\ \dot{}\Delta\ \Delta\dot{})(\Delta\dot{})=-\frac{1}{1512}$$
$$I_{13}^{A_1}=\int\int\int(\dot{}\Delta)|_\tau(\dot{}\Delta\dot{})(\dot{}\Delta\dot{}\ \Delta^2)(\dot{}\Delta)=\frac{13}{7560}\qquad I_{14}^{A_1}=\int\int\int(\dot{}\Delta)|_\tau(\dot{}\Delta\dot{})(\dot{}\Delta\ \Delta\ \Delta\dot{})(\dot{}\Delta)=-\frac{1}{3780}$$
$$I_{15}^{A_1}=\int\int\int(\dot{}\Delta)|_\tau(\dot{}\Delta\dot{})(\dot{}\Delta\dot{}\ \Delta\dot{}\ \Delta)(\Delta)=\frac{17}{15120}\qquad I_{16}^{A_1}=\int\int\int(\dot{}\Delta)|_\tau(\dot{}\Delta\dot{})(\dot{}\Delta\ \Delta\dot{}^2)(\Delta)=\frac{1}{7560}$$
$$I_{17}^{A_1}=\int\int\int(\dot{}\Delta)|_\tau(\dot{}\Delta)(\dot{}\Delta\dot{}\ \dot{}\Delta\ \Delta)(\dot{}\Delta)=-\frac{1}{15120}\qquad I_{18}^{A_1}=\int\int\int(\dot{}\Delta)|_\tau(\dot{}\Delta)(\dot{}\Delta^2\ \Delta\dot{})(\dot{}\Delta)=\frac{1}{7560}$$
$$I_{19}^{A_1}=\int\int\int(\dot{}\Delta)|_\tau(\dot{}\Delta)(\Delta[\dot{}\Delta\dot{}^2-\ddot{}\Delta^2])(\Delta)=-\frac{1}{864}\qquad I_{20}^{A_1}=\int\int\int(\dot{}\Delta)|_\tau(\dot{}\Delta)(\dot{}\Delta\dot{}\ \dot{}\Delta\ \Delta\dot{})(\Delta)=\frac{1}{30240}. \tag{A8}$$

Integrals in A_2:

$$I_1^{A_2}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\overline{\dot{\Delta}})|_\sigma(1)(\Delta^2)(\dot{\Delta}^2)=\frac{1}{756} \qquad I_2^{A_2}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\overline{\dot{\Delta}})|_\sigma(1)(\Delta\ \dot{\Delta})(\dot{\Delta}\ \Delta)=\frac{1}{1890}$$

$$I_3^{A_2}=\int\int\int(\Delta)|_\tau(\Delta)|_\sigma(1)(\dot{\Delta}^2)(\dot{\Delta}^{\cdot 2}-\ddot{\Delta}^2)=-\frac{11}{1890} \qquad I_4^{A_2}=\int\int\int(\Delta)|_\tau(\Delta)|_\sigma(1)(\dot{\Delta}\ \dot{\Delta})(\dot{\Delta}\ \dot{\Delta})=\frac{2}{945}$$

$$I_5^{A_2}=\int\int\int(\dot{\Delta})|_\tau(\dot{\Delta})|_\sigma(1)(\dot{\Delta}\ \Delta)(\dot{\Delta}\ \dot{\Delta})=\frac{1}{3024} \qquad I_6^{A_2}=\int\int\int(\dot{\Delta})|_\tau(\dot{\Delta})|_\sigma(1)(\dot{\Delta}\ \dot{\Delta})(\dot{\Delta}\ \dot{\Delta})=\frac{1}{30240}$$

$$I_7^{A_2}=\int\int\int(\dot{\Delta})|_\tau(\dot{\Delta})|_\sigma(1)(\Delta\ \dot{\Delta})(\dot{\Delta}\ \dot{\Delta})=-\frac{1}{6048} \qquad I_8^{A_2}=\int\int\int(\dot{\Delta})|_\tau(\dot{\Delta})|_\sigma(1)(\Delta\ \dot{\Delta})(\Delta\ \dot{\Delta})=\frac{5}{6048}$$

$$I_9^{A_2}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\Delta)|_\sigma(1)(\dot{\Delta}^2)(\dot{\Delta}^2)=-\frac{1}{378} \qquad I_{10}^{A_2}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\Delta)|_\sigma(1)(\dot{\Delta}^{\cdot 2}-\ddot{\Delta}^2)(\Delta^2)=\frac{11}{3780}$$

$$I_{11}^{A_2}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\Delta)|_\sigma(1)(\dot{\Delta}\ \dot{\Delta})(\dot{\Delta}\ \Delta)=-\frac{1}{945} \qquad I_{12}^{A_2}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\dot{\Delta})|_\sigma(1)(\dot{\Delta}\Delta)(\dot{\Delta}^2)=-\frac{1}{1512}$$

$$I_{13}^{A_2}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\dot{\Delta})|_\sigma(1)(\dot{\Delta}\ \dot{\Delta})(\Delta^2)=-\frac{1}{1512} \qquad I_{14}^{A_2}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\dot{\Delta})|_\sigma(1)(\Delta\ \dot{\Delta})(\dot{\Delta}\ \Delta)=-\frac{1}{1512}$$

$$I_{15}^{A_2}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\dot{\Delta})|_\sigma(1)(\dot{\Delta}\ \dot{\Delta})(\dot{\Delta}\ \Delta)=\frac{1}{7560} \qquad I_{16}^{A_2}=\int\int\int(\Delta)|_\tau(\dot{\Delta})|_\sigma(1)(\dot{\Delta}\Delta)(\dot{\Delta}^{\cdot 2}-\ddot{\Delta}^2)=-\frac{11}{7560}$$

$$I_{17}^{A_2}=\int\int\int(\Delta)|_\tau(\dot{\Delta})|_\sigma(1)(\dot{\Delta}\ \dot{\Delta})(\dot{\Delta}^{\cdot 2})=\frac{1}{756} \qquad I_{18}^{A_2}=\int\int\int(\Delta)|_\tau(\dot{\Delta})|_\sigma(1)(\Delta\ \dot{\Delta})(\dot{\Delta}\ \dot{\Delta})=\frac{1}{756}$$

$$I_{19}^{A_2}=\int\int\int(\Delta)|_\tau(\dot{\Delta})|_\sigma(1)(\dot{\Delta}\ \dot{\Delta})(\dot{\Delta}\ \dot{\Delta})=-\frac{1}{3780}. \tag{A9}$$

Integrals in A_3:

$$I_1^{A_3}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\overline{\dot{\Delta}})|_\sigma(\overline{\dot{\Delta}})|_\rho(\Delta)(\Delta)(\Delta)=-\frac{1}{945} \qquad I_2^{A_3}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\overline{\dot{\Delta}})|_\sigma(\dot{\Delta})|_\rho(\Delta)(\dot{\Delta})(\Delta)=\frac{1}{1890}$$

$$I_3^{A_3}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\overline{\dot{\Delta}})|_\sigma(\Delta)|_\rho(\Delta)(\dot{\Delta})(\dot{\Delta})=\frac{1}{756} \qquad I_4^{A_3}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\Delta)|_\sigma(\Delta)|_\rho(\dot{\Delta})(\dot{\Delta})(\dot{\Delta})=-\frac{29}{7560}$$

$$I_5^{A_3}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\dot{\Delta})|_\sigma(\Delta)|_\rho(\Delta)(\dot{\Delta})(\dot{\Delta})=-\frac{13}{15120} \qquad I_6^{A_3}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\dot{\Delta})|_\sigma(\Delta)|_\rho(\dot{\Delta})(\dot{\Delta})(\dot{\Delta})=-\frac{1}{2160}$$

$$I_7^{A_3}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\dot{\Delta})|_\sigma(\dot{\Delta})|_\rho(\dot{\Delta})(\Delta)(\dot{\Delta})=-\frac{11}{30240} \qquad I_8^{A_3}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\dot{\Delta})|_\sigma(\dot{\Delta})|_\rho(\Delta)(\dot{\Delta})(\dot{\Delta})(\Delta)=-\frac{11}{30240}$$

$$I_9^{A_3}=\int\int\int(\overline{\dot{\Delta}})|_\tau(\dot{\Delta})|_\sigma(\dot{\Delta})|_\rho(\dot{\Delta})(\dot{\Delta})(\Delta)=-\frac{1}{6048}$$

$$I_{10}^{A_3}=\int\int\int(\Delta)|_\tau(\Delta)|_\sigma(\Delta)|_\rho[(\dot{\Delta})(\dot{\Delta})(\dot{\Delta})+(\ddot{\Delta})(\ddot{\Delta})(\ddot{\Delta})]=-\frac{71}{7560}$$

$$I_{11}^{A_3}=\int\int\int(\Delta)|_\tau(\dot{\Delta})|_\sigma(\Delta)|_\rho(\dot{\Delta})(\dot{\Delta})(\dot{\Delta})=\frac{29}{15120} \qquad I_{12}^{A_3}=\int\int\int(\Delta)|_\tau(\dot{\Delta})|_\sigma(\dot{\Delta})|_\rho(\dot{\Delta})(\dot{\Delta})(\dot{\Delta})=\frac{19}{30240}$$

$$I_{13}^{A_3}=\int\int\int(\Delta)|_\tau(\dot{\Delta})|_\sigma(\dot{\Delta})|_\rho(\dot{\Delta})(\Delta)(\dot{\Delta})=\frac{31}{30240} \qquad I_{14}^{A_3}=\int\int\int(\Delta)|_\tau(\dot{\Delta})|_\sigma(\dot{\Delta})|_\rho(\dot{\Delta})(\Delta)(\dot{\Delta})=-\frac{1}{6048}$$

$$I_{15}^{A_3}=\int\int\int(\dot{\Delta})|_\tau(\dot{\Delta})|_\sigma(\dot{\Delta})|_\rho(\dot{\Delta})(\Delta)(\dot{\Delta})=-\frac{1}{60480} \qquad I_{16}^{A_3}=\int\int\int(\dot{\Delta})|_\tau(\dot{\Delta})|_\sigma(\dot{\Delta})|_\rho(\Delta)(\dot{\Delta})(\dot{\Delta})=\frac{11}{60480}.$$

(A10)

$\underline{\langle S_5^2\rangle_c}$

$$\langle S_5^2\rangle_c = -\frac{\beta^3}{144}[K_{15}(6I_1^{B_0}-12I_2^{B_0}+6I_3^{B_0})+K_{16}(4I_1^{B_2}-8I_2^{B_2}+4I_3^{B_2})+K_{13}(2I_1^{B_1}-8I_2^{B_1}+4I_3^{B_1}+22I_4^{B_1}-20I_5^{B_1}-48I_6^{B_1}+40I_7^{B_1}$$
$$+24I_8^{B_1}-16I_9^{B_1})+K_{14}(4I_1^{B_1}-16I_2^{B_1}+8I_3^{B_1}-20I_4^{B_1}+24I_5^{B_1}+32I_6^{B_1}-48I_7^{B_1}-16I_8^{B_1}+32I_9^{B_1})]$$
$$-\frac{\beta^3}{3}[3(\nabla_a V)^2 I_1^{B_3}+\nabla_k R\nabla^k V(I_2^{B_3}-I_3^{B_3})]. \tag{A11}$$

Integrals:

$$I_1^{B_0}=\int\int \Delta^3(\dot{\Delta}^2-\ddot{\Delta}^2)=0 \qquad I_2^{B_0}=\int\int \Delta^2\ \dot{\Delta}\ \dot{\Delta}\ \dot{\Delta}=\frac{1}{840}$$
$$I_3^{B_0}=\int\int \Delta(\dot{\Delta})^2(\dot{\Delta})^2=-\frac{1}{560} \qquad I_1^{B_1}=\int\int \Delta^3=-\frac{1}{560}$$
$$I_2^{B_1}=\int\int \dot{\Delta}|_\sigma \Delta^2\dot{\Delta}=\frac{1}{1680} \qquad I_3^{B_1}=\int\int \Delta|_\sigma\ \Delta\ (\dot{\Delta})^2=\frac{1}{336}$$
$$I_4^{B_1}=\int\int \Delta|_\tau\ \Delta|_\sigma\ \Delta(\dot{\Delta}^2-\ddot{\Delta}^2)=\frac{17}{2520} \qquad I_5^{B_1}=\int\int \Delta|_\tau\ \Delta|_\sigma\ \dot{\Delta}\ \dot{\Delta}\ \dot{\Delta}=-\frac{1}{2520}$$
$$I_6^{B_1}=\int\int \Delta|_\tau\ \dot{\Delta}|_\sigma\ \dot{\Delta}\ \dot{\Delta}\ \Delta=-\frac{1}{720} \qquad I_7^{B_1}=\int\int \Delta|_\tau\ \dot{\Delta}|_\sigma\ (\dot{\Delta})^2\dot{\Delta}=-\frac{1}{5040}$$
$$I_8^{B_1}=\int\int \dot{\Delta}|_\tau\ \dot{\Delta}|_\sigma\ \Delta^2\ \dot{\Delta}\dot{=}-\frac{1}{1008} \qquad I_9^{B_1}=\int\int \dot{\Delta}|_\tau\ \dot{\Delta}|_\sigma\ \dot{\Delta}\Delta\dot{\Delta}=\frac{1}{5040}$$
$$I_1^{B_2}=\int\int \Delta|_\tau\ \Delta|_\sigma\ \Delta=-\frac{17}{5040} \qquad I_2^{B_2}=\int\int \Delta|_\tau(\dot{\Delta}|_\sigma)^2\Delta=\frac{1}{1260}$$
$$I_3^{B_2}=\int\int (\dot{\Delta}|_\tau)^2(\dot{\Delta}|_\sigma)^2\Delta=-\frac{1}{4032} \qquad I_1^{B_3}=\int\int \Delta=-\frac{1}{12}$$
$$I_2^{B_3}=\int\int \Delta|_\tau\ \Delta=\frac{1}{60} \qquad I_3^{B_3}=\int\int (\dot{\Delta}|_\tau)^2\Delta=-\frac{1}{240}. \tag{A12}$$

$\underline{\langle S_4S_6\rangle_c}$

$$\langle S_4S_6\rangle_c=-\frac{\beta^3}{5!}\Big\{(4K_6+2K_7-8K_8+5K_{11})(I_2^{C_1}+I_3^{C_1}-2I_4^{C_1})+(-2K_4+2K_5+K_{10}+3K_{12})(I_1^{C_2}+I_2^{C_2}-2I_3^{C_2})$$
$$+(2K_4-2K_5+3K_{10}-K_{12})(I_4^{C_2}+I_5^{C_2}-2I_6^{C_2})+(-4K_4+4K_5-6K_{10}+2K_{12})(I_7^{C_2}+I_8^{C_2}-I_9^{C_2}-I_{10}^{C_2})$$
$$+(4K_4-4K_5+2K_{10}-4K_{12})(I_{11}^{C_2}+I_{12}^{C_2}-2I_{13}^{C_2})+\frac{4}{9}[(3K_6-7K_7-2K_8)(I_2^{C_1}+I_3^{C_1}-2I_4^{C_1})$$
$$+(4K_5+6K_6)(I_1^{C_2}+I_2^{C_2}-2I_3^{C_2})+(2K_4+3K_6)(I_4^{C_2}+I_5^{C_2}-2I_6^{C_2})+(-4K_4-8K_5-6K_{10})(I_7^{C_2}+I_8^{C_2}-I_9^{C_2}-I_{10}^{C_2})$$
$$+(2K_4-4K_5-3K_6)(I_{11}^{C_2}+I_{12}^{C_2}-2I_{13}^{C_2})]\Big\}-\frac{\beta^3}{6}R^{mn}\nabla_n\nabla_m V(I_1^{C_3}+I_2^{C_3}-2I_3^{C_3}). \tag{A13}$$

Integrals:

$$I_1^{C_1}=\int\int (\overline{\dot{\Delta}})|_\tau\ \Delta^2(\dot{\Delta})^2=\frac{1}{420} \qquad I_2^{C_1}=\int\int \Delta|_\tau\ (\dot{\Delta})^2(\dot{\Delta})^2=-\frac{1}{420}$$

$$I_3^{C_1}=\int\int \Delta|_\tau\ \Delta^2(\dot{\Delta}\dot{\Delta}^2-\ddot{\Delta}^2)=\frac{1}{280} \qquad I_4^{C_1}=\int\int \Delta|_\tau\ \Delta\ \dot{\Delta}\ \dot{\Delta}\dot{\Delta}\ \dot{\Delta}=\frac{1}{1680}$$

$$I_5^{C_1}=\int\int \dot{\Delta}|_\tau\ \Delta\ \dot{\Delta}(\dot{\Delta})^2=0 \qquad I_6^{C_1}=\int\int \dot{\Delta}|_\tau\ \Delta^2\ \dot{\Delta}\dot{\Delta}\ \dot{\Delta}=-\frac{1}{840}$$

$$I_1^{C_2}=\int\int (\Delta\ \overline{\dot{\Delta}\dot{\Delta}})|_\tau\ (\overline{\dot{\Delta}\dot{\Delta}})|_\sigma\ \Delta^2=-\frac{1}{420} \qquad I_2^{C_2}=\int\int (\Delta\ \overline{\dot{\Delta}\dot{\Delta}})|_\tau\ \Delta|_\sigma\ (\dot{\Delta})^2=\frac{1}{210}$$

$$I_3^{C_2}=\int\int (\Delta\ \overline{\dot{\Delta}\dot{\Delta}})|_\tau\ \dot{\Delta}|_\sigma\ \Delta\dot{\Delta}=\frac{1}{840} \qquad I_4^{C_2}=\int\int (\Delta^2)|_\tau\ (\overline{\dot{\Delta}\dot{\Delta}})|_\sigma\ (\dot{\Delta})^2=\frac{1}{252}$$

$$I_5^{C_2}=\int\int (\Delta^2)|_\tau\ \Delta|_\sigma\ (\dot{\Delta}\dot{\Delta}^2-\ddot{\Delta}^2)=\frac{11}{1260} \qquad I_6^{C_2}=\int\int (\Delta^2)|_\tau\ \dot{\Delta}|_\sigma\ \dot{\Delta}\ \dot{\Delta}\dot{\Delta}=-\frac{1}{504}$$

$$I_7^{C_2}=\int\int (\Delta\ \dot{\Delta})|_\tau\ (\overline{\dot{\Delta}\dot{\Delta}})|_\sigma\Delta\Delta=\frac{1}{1260} \qquad I_8^{C_2}=\int\int (\Delta\ \dot{\Delta})|_\tau\ \Delta|_\sigma\ \dot{\Delta}\ \dot{\Delta}\dot{\Delta}=-\frac{1}{630}$$

$$I_9^{C_2}=\int\int (\Delta\ \dot{\Delta})|_\tau\ \dot{\Delta}|_\sigma\ \Delta\ \dot{\Delta}\dot{\Delta}=-\frac{1}{1008} \qquad I_{10}^{C_2}=\int\int (\Delta\ \dot{\Delta})|_\tau\ \dot{\Delta}|_\sigma\ \dot{\Delta}\Delta\dot{\Delta}=\frac{1}{5040}$$

$$I_{11}^{C_2}=\int\int (\dot{\Delta}^2)|_\tau\ (\overline{\dot{\Delta}\dot{\Delta}})|_\sigma\ \Delta^2=\frac{1}{2520} \qquad I_{12}^{C_2}=\int\int (\dot{\Delta}^2)|_\tau\ \Delta|_\sigma\ (\dot{\Delta})^2=-\frac{1}{1260}$$

$$I_{13}^{C_2}=\int\int (\dot{\Delta}^2)|_\tau\ \dot{\Delta}|_\sigma\ \Delta\dot{\Delta}=-\frac{1}{5040} \qquad I_1^{C_3}=\int\int (\overline{\dot{\Delta}\dot{\Delta}})|_\tau\ \Delta^2=\frac{1}{90}$$

$$I_2^{C_3}=\int\int \Delta|_\tau\ \dot{\Delta}^2=-\frac{1}{45} \qquad I_3^{C_3}=\int\int \dot{\Delta}|_\tau\ \dot{\Delta}=-\frac{1}{180}. \tag{A14}$$

$\underline{\langle S_8\rangle}$

$$\langle S_8\rangle=\frac{\beta^3}{7!}(I_1^D-I_2^D)\Big[4(2K_4+9K_6-7K_7-2K_8)+\frac{55}{2}(11K_{13}-10K_{14}+3K_{15})+34(2K_4-2K_5+4K_6+2K_7-8K_8+3K_{10}$$
$$+5K_{11}-K_{12})+5(12K_4-12K_5+4K_6+2K_7-8K_8+6K_{10}+2K_{11}-16K_{12}+14K_{13}-20K_{14}$$
$$-5K_{16}+9K_{17})\Big]-\frac{\beta^3}{24}I_3^D(2R^{mn}\nabla_m\nabla_n V+\nabla^m R\nabla_m V-3\nabla^4 V). \tag{A15}$$

Integrals:

$$I_1^D=\int \Delta^3(\dot{\Delta}\dot{\Delta}+\Delta\ddot{\Delta})=-\frac{1}{140}; \quad I_2^D=\int \Delta^2\dot{\Delta}^2=\frac{1}{840}; \quad I_3^D=\int \Delta^2=\frac{1}{30}. \tag{A16}$$

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