Nonvanishing magnetic flux through the slightly charged Kerr black hole

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(Received 15 November 2000; published 26 February 2001)

In association with the Blandford-Znajek mechanism for rotational energy extraction from Kerr black holes, it is of some interest to explore how much magnetic flux can actually penetrate the horizon at least in idealized situations. For the completely uncharged Kerr hole case, it has been known for some time that the magnetic flux gets entirely expelled when the hole is maximally rotating. In the mean time, it is also known that when the rotating hole is immersed in an originally uniform magnetic field surrounded by an ionized interstellar medium (plasma), which is a more realistic situation, the hole accretes a certain amount of electric charge. In the present work, it is demonstrated that, as a result of this accretion charge small enough not to disturb the geometry, the magnetic flux through this slightly charged Kerr hole depends not only on the hole's angular momentum but on the hole's charge as well, such that it never vanishes for any value of the hole's angular momentum.

DOI: 10.1103/PhysRevD.63.064037 PACS number(s): 04.70.−s, 04.40.Nr, 97.60.Lf

I. INTRODUCTION

Among various models attempting to provide a consistent account for the spectrum of quasars, radio galaxies or the gamma ray bursters in which strong magnetic field is believed to be anchored in the central black hole, perhaps the mechanism proposed by Blandford and Znajek [1] might be the most attractive and natural one to theoretical physicists. Despite the skepticism generally held by astronomers and the majority of astrophysicists, the Blandford-Znajek mechanism for the extraction of rotational energy from rotating black holes has remained an issue of great interest in the theoretical astrophysics community due to its concreteness in the formulation and plausibility in operational nature. Being so, there have been continuous research activities to ask and answer questions relevant to the environment set by the Blandford-Znajek mechanism such as the stationary, axisymmetric magnetosphere in which the rotating hole is surrounded by a strong magnetic field and plasma. Among such questions, one of the most interesting thought experiments that can be explored in an analytic manner at least in idealized situations is the issue of how much of the magnetic field can actually penetrate the rotating hole's horizon. Or more precisely, through one-half of the surface of the horizon as the total flux across the whole horizon should be zero. The right answer to this question can indeed be crucial in order for the Blandford-Znajek mechanism and its generalized versions [2] to work at all as they all rely on the picture in which magnetic fields (particularly, their poloidal components) penetrating the hole's horizon and the surrounding accretion disk, transmit the rotational energy of the hole to distant relativistic particles and fields. Interestingly but rather to our dismay, it has been found [3,10] that this magnetic flux intersecting the horizon decreases as the angular momentum of the hole increases and reaches zero when the hole is maximally rotating. Considering that larger amount of rotational energy can be extracted from more rapidly rotating black holes, this result, even if we take into account the idealized setup employed, is quite adverse to the operational aspect of the Blandford-Znajek mechanism. Because of this negative implication of the result, there have also been some other works employing more careful treatments to turn the conclusion around and hence save the mechanism. The work performed by Dokuchaev [4] employing the Ernst-Wild solution [5] to the coupled Einstein-Maxwell equations belongs to this category. Starting from the exact Ernst-Wild solution, he showed that the magnetic flux through the hole becomes independent of the hole's angular momentum when the hole builds up equilibrium (Wald) charge. In the present work, we choose to deal with the issue more directly and demonstrate in a transparent manner that in a more realistic situation when there are charges around (but small enough not to disturb the background vacuum geometry), the magnetic flux penetrating the horizon depends not only on the angular momentum but on the amount of charge accreted on the hole as well and can never vanish no matter how fast the hole spins. In order to find the electromagnetic field around a rotating black hole and hence eventually the magnetic flux through one-half of the horizon, we look for the solution to the source-free Maxwell equations in the background of the stationary, axisymmetric vacuum spacetime, the Kerr geometry. And perhaps the most concise and elegant way to obtain this solution which occurs when a stationary, axisymmetric black hole is placed in an asymptotically uniform magnetic field would be the simple algorithm suggested long ago by Wald [6] which is based upon the realization [7] that the Killing vectors owned by a given vacuum space time generate solutions of source-free Maxwell equations in the background of that space time. Here, although ''vacuum'' situation repre

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sents the case when both the Einstein and Maxwell equations have no source terms, the solution-generating method given by Wald actually allows us to construct the solution for the electromagnetic test field that occurs when the hole is immersed in an originally uniform magnetic field surrounded by plasma (i.e., ionized particles) and hence eventually gets slightly charged via accretion. We already announced that we shall consider the case when the charge accreted on the hole is small enough not to distort the background Kerr geometry. Certainly, it needs to be justified that this is indeed what can actually happen. This will be done at the end of Sec. II. Since we shall basically employ this algorithm proposed by Wald, the present work can be thought of as having its basis on the ''first'' approximation of the Einstein-Maxwell system in which only the Maxwell equation is being solved in the background of Kerr space time assuming that neither the field nor the amount of charge is strong enough to distort the background geometry. In this sense, any attempt to deal with the problem by directly solving the coupled Einstein-Maxwell equations, such as the work by Dokuchaev employing the Ernst-Wild solution to the coupled system, can be regarded as being more fundamental though less practical in many respects. Later, we shall notice that our result for the expression for the magnetic flux through the hole can indeed be deduced as a leading approximation to that appeared in the work by Dokuchaev although it has not been realized there.

II. SOLUTION GENERATING METHOD BY WALD

A. Wald field

From the general properties of Killing fields [7]—*a Killing vector in a vacuum space time generates a solution of Maxwell's equations in the background of that vacuum space time*—long ago, Wald [6] constructed a stationary, axisymmetric solution of Maxwell's equations in Kerr black hole space time. To be a little more concrete, Wald's construction is based on the following two statements.

(A) The axial Killing vector $\psi^{\mu}=(\partial/\partial\phi)^{\mu}$ generates a stationary, axisymmetric test electromagnetic field which asymptotically approaches a uniform magnetic field, has no magnetic monopole moment and has charge $=4J, F_{\psi}=d\psi$, where d denotes the exterior derivative and J is the angular momentum of a Kerr black hole;

(B) the time translational Killing vector $\xi^{\mu}=(\partial/\partial t)^{\mu}$ generates a stationary, axisymmetric test electromagnetic field which vanishes asymptotically, has no magnetic monopole moment and has charge $=-2m, F_{\xi}=d\xi$, where m is the mass of the Kerr hole.

Equipped with this preparation, we are now interested in obtaining the solution for the electromagnetic test field F that occurs when a stationary, axisymmetric black hole (i.e., Kerr hole) is placed in an originally uniform magnetic field of strength B_0 aligned along the symmetry axis of the black hole. And for now we consider the case when the electric charge is absent. Obviously, then the solution can be readily written down by referring to the statements (A) and (B) above as follows:

$$F=\frac{1}{2}B_0\left[F_{\psi}-\left(\frac{4J}{-2m}\right)F_{\xi}\right]=\frac{1}{2}B_0\left[d\psi+\frac{2J}{m}d\xi\right]. \quad (1)$$

Then using $F=\frac{1}{2}F_{\mu\nu}dx^{\mu}\wedge dx^{\nu}, \psi=\psi_{\mu}dx^{\mu}, \xi=\xi_{\nu}dx^{\nu}$, with

$$\xi_{\mu}=g_{\mu\nu}\xi^{\nu}=g_{\mu\nu}\delta^{\nu}_{t}=g_{\mu t}, \quad (2)$$
$$\psi_{\mu}=g_{\mu\nu}\psi^{\nu}=g_{\mu\nu}\delta^{\nu}_{\phi}=g_{\mu\phi}$$

and for the Kerr metric given in Boyer-Lindquist coordinates [6,14] [or see Eq. (A12) in Appendix A], the solution above can be written in a concrete form as

$$\begin{aligned}F=\frac{1}{2}B_0\Bigg[&-\frac{2ma}{\Sigma^2}(r^2-a^2\cos^2\theta)(1+\cos^2\theta)(dr\wedge dt)\\&-\frac{4mra}{\Sigma^2}(r^2-a^2)\sin\theta\cos\theta(d\theta\wedge dt)\\&+\left\{2r\sin^2\theta+\frac{2ma^2\sin^2\theta}{\Sigma^2}(r^2-a^2\cos^2\theta)(1+\cos^2\theta)\right\}\\&\times(dr\wedge d\phi)+\Bigg\{2(r^2+a^2)+\frac{4mra^2}{\Sigma^2}[\Sigma\sin^2\theta-(r^2+a^2)\\&\times(1+\cos^2\theta)]\sin\theta\cos\theta\Bigg\}(d\theta\wedge d\phi)\Bigg]. \quad (3)\end{aligned}$$

B. Wald charge

We now turn to the issue of *charge accretion* onto the Kerr black hole immersed in a magnetic field surrounded by an ionized interstellar medium (''plasma''). We shall essentially follow the argument given in [6] and to do so, we first need to know the physical components of electric and magnetic fields. This can be achieved by projecting the Maxwell field tensor given above in Eq. (3) onto a tetrad frame. An appropriate tetrad frame here is that suggested by Carter [6,8] whose physical properties are discussed in detail in Appendix A. Now using the (dual of) Carter's orthonormal tetrad $e_A=\{e_0,e_1,e_2,e_3\}$ [8],

$$\begin{aligned}e_0&=\frac{(r^2+a^2)}{(\Sigma\Delta)^{1/2}}\frac{\partial}{\partial t}+\frac{a}{(\Sigma\Delta)^{1/2}}\frac{\partial}{\partial\phi}=e_0^t\partial_t+e_0^{\phi}\partial_{\phi},\\e_1&=\left(\frac{\Delta}{\Sigma}\right)^{1/2}\frac{\partial}{\partial r}=e_1^r\partial_r,\\e_2&=\frac{1}{\Sigma^{1/2}}\frac{\partial}{\partial\theta}=e_2^{\theta}\partial_{\theta},\\e_3&=\frac{a\sin\theta}{\Sigma^{1/2}}\frac{\partial}{\partial t}+\frac{1}{\Sigma^{1/2}\sin\theta}\frac{\partial}{\partial\phi}=e_3^t\partial_t+e_3^{\phi}\partial_{\phi},\end{aligned} \quad (4)$$

where $\Sigma=r^2+a^2\cos^2\theta$ and $\Delta=r^2+a^2-2mr$ with $a=J/m$ being the angular momentum per unit mass and the Carter tetrad components of the Maxwell field strength can be computed as

$$F=B_0\left[\frac{ar\sin^2\theta}{\Sigma}-\frac{ma}{\Sigma^2}(r^2-a^2\cos^2\theta)(1+\cos^2\theta)\right](e^1\wedge e^0)+B_0\frac{a\Delta^{1/2}}{\Sigma}\sin\theta\cos\theta(e^2\wedge e^0)+B_0\frac{r\Delta^{1/2}}{\Sigma}\sin\theta(e^1\wedge e^3)+B_0\frac{\cos\theta}{\Sigma}\left[(r^2+a^2)-\frac{2mra^2}{\Sigma}(1+\cos^2\theta)\right](e^2\wedge e^3). \tag{5}$$

Here, consider particularly the radial component of the electric field (as observed by a local observer in this Carter tetrad frame),

$$E_{\hat{r}}=E_1=F_{10}=B_0\left[\frac{ar\sin^2\theta}{\Sigma}-\frac{ma}{\Sigma^2}(r^2-a^2\cos^2\theta)\times(1+\cos^2\theta)\right],$$

which, along the symmetry axis ($\theta=0,\pi$) of the Kerr hole, becomes

$$E_{\hat{r}}(\theta=0,\pi)=-B_0\frac{2ma(r^2-a^2)}{\Sigma^2}.$$

Note that $E_{\hat{r}}(\theta=0,\pi)<0$ for $B_0>0$ and $E_{\hat{r}}(\theta=0,\pi)>0$ for $B_0<0$ meaning that it is radially *inward/outward* if the hole's axis of rotation and the external magnetic field are *parallel/antiparallel*. Put differently, this implies that if the spin of the hole and the magnetic field are *parallel*, then *positively charged* particles on the symmetry axis of the hole will be pulled into the hole, whereas if the spin of the hole and the magnetic field are *antiparallel*, *negatively charged* particles on the symmetry axis of the hole will be pulled into the hole. In this manner a rotating black hole will ''selectively'' accrete charged particles until it builds up ''equilibrium'' net charge. Then the next natural question to ask would be, how the equilibrium net charge can be determined. To answer this question, we resort to the ''injection energy'' argument originally due to Carter [8]. Recall first that the energy of a charged particle in a stationary space time with the time translational isometry generated by the Killing field $\xi^\mu=(\partial/\partial t)^\mu$ in the presence of a stationary electromagnetic field is given by

$$\varepsilon=-p_\alpha\xi^\alpha=-g_{\alpha\beta}p^\alpha\xi^\beta$$

with $p^\mu=\tilde{m}u^\mu-eA^\mu$ being the four-momentum of the charged particle with mass and charge $\tilde{m}$ and e, respectively. Now if we lower the charged particle down the symmetry axis into the Kerr hole, the change in electrostatic energy of the particle will be

$$\delta\varepsilon=\varepsilon_{final}-\varepsilon_{initial}=eA_\alpha\xi^\alpha|_{horizon}-eA_\alpha\xi^\alpha|_\infty. \tag{6}$$

Now, if $\delta\varepsilon<0\rightarrow$ it will be energetically favorable for the hole to accrete particles with this charge, whereas if $\delta\varepsilon>0$ $\rightarrow$ the black hole will accrete particles with opposite charge. In either case, the Kerr hole will selectively accrete charges until A^μ is changed sufficiently that the electrostatic ''injection energy'' $\delta\varepsilon$ is reduced to zero. We are then ready to determine, by this injection-energy argument due to Carter, what the equilibrium net charge accreted onto the hole would be. In the discussion of Wald field given above, we restricted ourselves only to the case of solutions to Maxwell equation in the background of *uncharged* stationary, axisymmetric black hole space time and it was given by Eq. (1). Now we need the solution when the stationary, axisymmetric black hole is slightly charged via charge accretion process described above. Then according to the statement (B) in the discussion of Wald field given earlier, there can be at the most one more perturbation of a stationary, axisymmetric vacuum black hole that corresponds to adding a charge Q to the hole and it is nothing but to linearly superpose the solution $(-Q/2m)F_\xi=(-Q/2m)d\xi$ to the solution given in Eq. (1) to get

$$F=\frac{1}{2}B_0\left[d\psi+\frac{2J}{m}d\xi\right]-\frac{Q}{2m}d\xi, \tag{7}$$

which, in terms of the gauge potental, amounts to

$$A_\mu=\frac{1}{2}B_0\left(\psi_\mu+\frac{2J}{m}\xi_\mu\right)-\frac{Q}{2m}\xi_\mu. \tag{8}$$

Then the electrostatic-injection energy can be computed as

$$\delta\varepsilon=eA_\alpha\xi^\alpha|_{horizon}-eA_\alpha\xi^\alpha|_\infty=e\left(\frac{B_0J}{m}-\frac{Q}{2m}\right). \tag{9}$$

Thus one may conclude that a rotating hole in a uniform magnetic field will accrete charge until the gauge potential evolves to a value at which $\delta\varepsilon=0$ yielding the equilibrium net charge as $Q=2B_0J$. This amount of charge is called ''Wald charge.''

Note that we announced from the beginning that we shall consider the case when the charge accreted on the hole is small enough not to distort the background Kerr geometry. Now we provide the rationale that this is indeed what can actually happen. To do so, we first assume that the typical value of the charge on the hole is the equilibrium Wald charge $Q=2B_0J$ just described. Then using the fact that $J=ma\leqslant m^2$, the charge-to-mass ratio of a hole has an upper bound

$$\frac{Q}{m}=2B_0\left(\frac{J}{m}\right)\leqslant 2B_0m=2\left(\frac{B_0}{10^{15}}\right)\left(\frac{m}{m_\odot}\right)10^{-5}$$

where in the last equality we have converted B_0 and m from geometrized units to solar-mass units and Gauss [17]. Thus for the two typical examples: (i) the binary system in our galaxy with mass $\sim 5\sim 10$ $m_\odot$ in the surrounding magnetic field of strength $\sim 10^{14}$ G, $Q/m \sim 10^{-5} \ll 1$, and (ii) the active galactic nuclei with mass $\sim 10^{6}-10^{9}$ $m_{\odot}$ in the magnetic field of strength $\sim 10^{4}$ G, $Q/m \sim 10^{-7} \ll 1$. As we can see in these two cases, the charge-to-mass ratios of the rotating black hole in examples (i) and (ii) are small enough not to disturb the geometry itself. Thus we can safely employ the solution-generating method suggested by Wald to construct the solution to the Maxwell equations when some amounts of charges are around to which we now turn.

III. STATIONARY, AXISYMMETRIC MAXWELL FIELD AROUND A "SLIGHTLY CHARGED" KERR HOLE

As discussed in the previous Sec. II B, the stationary, axisymmetric solution to the Maxwell equation in the background of a Kerr hole with charge Q accreted in an originally uniform magnetic field can be constructed as

$$F = \frac{1}{2}B_0\left[F_{\psi} - \left(\frac{4J}{-2m}\right)F_{\xi}\right] + \left(\frac{Q}{-2m}\right)F_{\xi}$$
$$= \frac{1}{2}B_0\left[d\psi + \frac{2J}{m}\left(1 - \frac{Q}{2B_0J}\right)d\xi\right]. \tag{10}$$

Again for the Kerr metric given in Boyer-Lindquist coordinates, the solution above can be written as

$$F = \frac{1}{2}B_0\Bigg[-\frac{2ma}{\Sigma^2}(r^2 - a^2\cos^2\theta)\left\{\left(1 - \frac{Q}{B_0J}\right) + \cos^2\theta\right\}$$
$$\times(dr\wedge dt) - \frac{4mra}{\Sigma^2}\left\{r^2 - a^2\left(1 - \frac{Q}{B_0J}\right)\right\}$$
$$\times\sin\theta\cos\theta(d\theta\wedge dt) + \Bigg\{2r\sin^2\theta + \frac{2ma^2\sin^2\theta}{\Sigma^2}$$
$$\times(r^2 - a^2\cos^2\theta)\left(1 - \frac{Q}{B_0J} + \cos^2\theta\right)\Bigg\}(dr\wedge d\phi)$$
$$+\Bigg\{2(r^2 + a^2) + \frac{4mra^2}{\Sigma^2}\bigg[\Sigma\sin^2\theta - (r^2 + a^2)$$
$$\times\left(1 - \frac{Q}{B_0J} + \cos^2\theta\right)\bigg]\sin\theta\cos\theta\Bigg\}(d\theta\wedge d\phi)\Bigg]. \tag{11}$$

Obviously, in order to have some insight into the nature of this solution to the Maxwell equations, one may wish to obtain physical components of electric field and magnetic induction. And this can only be achieved by projecting the Maxwell field tensor above onto an appropriate tetrad frame as mentioned earlier. To be a little more concrete, in Boyer-Lindquist coordinates, one can think of two orthonormal tetrads, first, the familiar zero-angular-momentum-observer (ZAMO) tetrad frame [9] and next, rather unfamiliar Carter tetrad frame [8]. The ZAMO tetrad is well known and widely employed in various anayses in the literature. ZAMO is a fiducial observer following timelike geodesic orthogonal to spacelike hypersurfaces. And this implies that its four-velocity is just the timelike ZAMO tetrad $u^{\mu} = e_0^{\mu}$ given below in Eq. (12). The Carter tetrad, however, seems less known than ZAMO despite its physical and technical advantages over ZAMO. Thus we provide some of the basics of Carter tetrad in Appendix A. In the following, we just give the components of Maxwell fields projected on these two tetrad frames without getting into the details of the nature of the tetrad frames themselves. Upon projecting the Maxwell field tensor components on a given tetrad frame, the physical components of electric and magnetic fields can be read off as $E_i = F_{i0} = F_{\mu\nu}(e_i^{\mu}e_0^{\nu})$ and $B_i = \epsilon_{ijk}F^{jk}/2 = \epsilon_{ijk}F^{\mu\nu}(e_j^{\mu}e_k^{\nu})/2$, respectively.

a. Computation of the ZAMO tetrad components of the Maxwell fields. From the dual to the ZAMO tetrad, $e^A = (e_0 = e_{(t)}, e_1 = e_{(r)}, e_2 = e_{(\theta)}, e_3 = e_{(\phi)})$,

$$e_0 = \left(\frac{A}{\Sigma\Delta}\right)^{1/2}\left(\partial_t + \frac{2mra}{A}\partial_\phi\right) = e_0^t\partial_t + e_0^\phi\partial_\phi,$$
$$e_1 = \left(\frac{\Delta}{\Sigma}\right)^{1/2}\partial_r = e_1^r\partial_r, \tag{12}$$
$$e_2 = \Sigma^{-1/2}\partial_\theta = e_2^\theta\partial_\theta,$$
$$e_3 = \left(\frac{\Sigma}{A}\right)^{1/2}\frac{1}{\sin\theta}\partial_\phi = e_3^\phi\partial_\phi$$

where $A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$, we can now read off the ZAMO tetrad components of $F_{\mu\nu}$

$$F_{10} = F_{\mu\nu}e_1^{\mu}e_0^{\nu}$$
$$= \frac{B_0ma}{(A\Sigma\Delta)^{1/2}\Sigma^2}\Bigg[2r^2\Sigma^2\sin^2\theta + (r^2 - a^2\cos^2\theta)$$
$$\times\left(1 - \frac{Q}{B_0J} + \cos^2\theta\right)\left\{2mra^2\sin^2\theta - \left(\frac{\Delta}{\Sigma}\right)^{1/2}A\right\}\Bigg],$$

$$F_{20} = F_{\mu\nu}e_2^{\mu}e_0^{\nu}$$
$$= \frac{B_0mra}{(A\Sigma\Delta)^{1/2}\Sigma^2}\Bigg[2(r^2 + a^2)\Sigma^2 + 4mra^2$$
$$\times\left\{\Sigma\sin^2\theta - (r^2 + a^2)\left(1 - \frac{Q}{B_0J} + \cos^2\theta\right)\right\}$$
$$-2\frac{A}{\Sigma^{1/2}}\left\{r^2 - a^2\left(1 - \frac{Q}{B_0J}\right)\right\}\Bigg]\sin\theta\cos\theta,$$

$$F_{30} = F_{\mu\nu}e_3^{\mu}e_0^{\nu} = 0, \tag{13}$$

$$F_{12} = F_{\mu\nu}e_1^{\mu}e_2^{\nu} = 0,$$

$$F_{13}=F_{\mu\nu}e_1^{\mu}e_3^{\nu}$$

$$=B_0\left(\frac{\Delta}{A}\right)^{1/2}\frac{\sin\theta}{\Sigma^2}\Bigg[(r^2+a^2)r\Sigma+a^2 \times\Bigg\{2r(r^2-a^2)\cos^2\theta-(r-m)(r^2-a^2\cos^2\theta)\times(1+\cos^2\theta)-\frac{Qm}{B_0J}(r^2-a^2\cos^2\theta)\Bigg\}\Bigg],$$

$$F_{23}=F_{\mu\nu}e_2^{\mu}e_3^{\nu}$$

$$=B_0\frac{1}{A^{1/2}}\frac{\cos\theta}{\Sigma^2}\Bigg[(r^2+a^2)\Bigg\{(r^2-a^2)(r^2-a^2\cos^2\theta)+2a^2r(r-m)(1+\cos^2\theta)+2a^2r\frac{Qm}{B_0J}\Bigg\}-a^2\Delta\Sigma\sin^2\theta\Bigg].$$

b. Computation of the Carter tetrad components of the Maxwell fields. From the dual to the Carter tetrad, $e_A=(e_0,e_1,e_2,e_3)$, given earlier in Eq. (4),

$$F_{10}=F_{\mu\nu}e_1^{\mu}e_0^{\nu}=B_0\left[\frac{ar\sin^2\theta}{\Sigma}-\frac{ma}{\Sigma^2}(r^2-a^2\cos^2\theta)\times\left(1-\frac{Q}{B_0J}+\cos^2\theta\right)\right],$$

$$F_{20}=F_{\mu\nu}e_2^{\mu}e_0^{\nu}=B_0\frac{a\Delta^{1/2}}{\Sigma}\sin\theta\cos\theta,$$

$$F_{30}=F_{\mu\nu}e_3^{\mu}e_0^{\nu}=0, \tag{14}$$

$$F_{12}=F_{\mu\nu}e_1^{\mu}e_2^{\nu}=0,$$

$$F_{13}=F_{\mu\nu}e_1^{\mu}e_3^{\nu}=B_0\frac{r\Delta^{1/2}}{\Sigma}\sin\theta,$$

$$F_{23}=F_{\mu\nu}e_2^{\mu}e_3^{\nu}=B_0\frac{\cos\theta}{\Sigma}\left[(r^2+a^2)-\frac{2mra^2}{\Sigma}\times\left(1-\frac{Q}{B_0J}+\cos^2\theta\right)\right].$$

Therefore, in the Carter tetrad frame,

$$F=B_0\left[\frac{ar\sin^2\theta}{\Sigma}-\frac{ma}{\Sigma^2}(r^2-a^2\cos^2\theta)\left(1-\frac{Q}{B_0J}+\cos^2\theta\right)\right]\times(e^1\wedge e^0)+B_0\frac{a\Delta^{1/2}}{\Sigma}\sin\theta\cos\theta(e^2\wedge e^0)+B_0\frac{r\Delta^{1/2}}{\Sigma}\sin\theta(e^1\wedge e^3)+B_0\frac{\cos\theta}{\Sigma}\left[(r^2+a^2)-\frac{2mra^2}{\Sigma}\left(1-\frac{Q}{B_0J}+\cos^2\theta\right)\right](e^2\wedge e^3). \tag{15}$$

As is pointed out in Appendix A, these Carter tetrad components of the Maxwell field tensor take much simpler forms than the ZAMO tetrad components given in Eq. (13) and hence are easier to deal with.

IV. THE MAGNETIC FLUX ACROSS ONE-HALF OF EVENT HORIZON

With the asymptotically uniform stationary, axisymmetric magnetic field, which is aligned with the spin axis of a ‘‘slightly charged’’ Kerr hole given above, we now would like to compute the flux of the magnetic field across one-half of the surface of the event horizon that occurs at points where

$$\Delta(r_+)=r_+^2+a^2-2mr_+=0 \quad \text{or} \quad r_+=m+\sqrt{m^2-a^2} \tag{16}$$

The physical motivation that underlies this study is to have some insight into the question of how much of the electromagnetic field can actually penetrate into the hole's horizon—at least in idealized situations. We now begin by considering two vectors lying on the horizon that, for instance in the Boyer-Lindquist coordinates, are given by

$$dx_1^{\alpha}=(0,0,d\theta,0),\quad dx_2^{\alpha}=(0,0,0,d\phi). \tag{17}$$

Then in terms of the second rank tensor constructed from these two vectors [10],

$$d\sigma^{\alpha\beta}=\frac{1}{2}(dx_1^{\alpha}dx_2^{\beta}-dx_1^{\beta}dx_2^{\alpha}). \tag{18}$$

Now one can define the invariant-surface element of the horizon as

$$ds=(2d\sigma_{\alpha\beta}\sigma^{\alpha\beta})^{1/2}\big|_{r_+} \tag{19}$$

$$=(g_{\theta\theta}g_{\phi\phi})^{1/2}\big|_{r_+}d\theta d\phi$$

$$=(r_+^2+a^2)\sin\theta d\theta d\phi.$$

Next, since the tensor $d\sigma^{\alpha\beta}$ is associated with the invariant-surface element of any (not necessarily closed) two-surface, the flux of electric field and magnetic field across any two-surfaces can be given, respectively, by

$$\Phi_E = \int \tilde{F}_{\alpha\beta} d\sigma^{\alpha\beta} = \int \frac{1}{2}\epsilon^{\gamma\delta}_{\alpha\beta} F_{\gamma\delta} d\sigma^{\alpha\beta},$$

$$\Phi_B = \int F_{\alpha\beta} d\sigma^{\alpha\beta}. \tag{20}$$

In particular, the flux of magnetic field across one-half of the horizon of the Kerr hole is

$$\Phi_B = \int_{r=r_+} F_{\alpha\beta} d\sigma^{\alpha\beta} = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta F_{\theta\phi}|_{r_+}. \tag{21}$$

Then using $F_{\theta\phi}$ evaluated on the horizon

$$F_{\theta\phi}|_{r_+} = B_0 \frac{\sin\theta\cos\theta}{(r_+^2 + a^2\cos^2\theta)^2}(r_+^2+a^2)^2\left[r_+^2 - a^2\left(1 - \frac{Q}{B_0 J}\right)\right] \tag{22}$$

and the result of integration

$$\int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \frac{\sin\theta\cos\theta}{(r_+^2+a^2\cos^2\theta)^2} = \frac{\pi}{r_+^2(r_+^2+a^2)}, \tag{23}$$

we finally get

$$\Phi_B = B_0 \pi r_+^2 \left(1 + \frac{a^2}{r_+^2}\right)\left[1 - \frac{a^2}{r_+^2}\left(1 - \frac{Q}{B_0 J}\right)\right]. \tag{24}$$

Note that thus far we assumed the case when the spin of the hole and the magnetic field are parallel (i.e., $B_0 > 0$ for $J > 0$) and hence the positively charged particles ($Q > 0$) are being accreted on the hole via Carter's injection energy argument discussed in Sec. II, i.e., in calculating the magnetic flux we started with the expression for the solution $F = (B_0/2)[d\psi + (2J/m)d\xi] - (Q/2m)d\xi$ given earlier. If, instead, we consider the other case when the hole's rotation axis and the magnetic field are antiparallel (i.e., $B_0 < 0$ for $J > 0$), the negatively charged particles ($Q < 0$) would be accreted on the hole and thus this time we should start with the solution $F = (-B_0/2)[d\psi + (2J/m)d\xi] + (Q/2m)d\xi$ that has overall sign just *opposite* to that in *parallel/positive charge* case. In other words, B_0 and Q always have the same sign. This indicates that if we started out with the *antiparallel/negative charge* case, we would end up with the expression for the magnetic flux having *opposite overall* sign to that in Eq. (24) above. Therefore, the general expression for the *magnitude* of the magnetic field should take the form

$$\Phi_B = |B_0| \pi r_+^2 \left(1 + \frac{a^2}{r_+^2}\right)\left[1 - \frac{a^2}{r_+^2}\left(1 - \frac{|Q|}{|B_0| J}\right)\right] \tag{25}$$

where now Φ_B is to be understood as denoting the *absolute* value of the flux. Indeed this expression for the magnetic flux through the slightly charged Kerr hole is a new result that has never been realized in the literature and as such needs close analysis in more detail.

(i) In the absence of the accretion charge, i.e., $Q = 0$, the magnetic flux above correctly reduces to that obtained long ago by King, Lasota, and Kundt [3],

$$\Phi_B^{KLK} = |B_0| \pi r_+^2 \left(1 - \frac{a^4}{r_+^4}\right). \tag{26}$$

(ii) With the nonvanishing accretion charge having value in the range $0 < |Q| < 2|B_0|J$, the total magnetic flux through the hole can *never* become zero. And this property holds true even when the hole is maximally rotating, i.e., $a \to m, r_+ \to m$ in which case the total flux becomes

$$\Phi_B = 2|B_0| \pi r_+^2 \left(\frac{|Q|}{|B_0| J}\right) = 2\pi r_+^2 \left(\frac{|Q|}{m^2}\right) = 2\pi |Q|, \tag{27}$$

which, interestingly, is independent of m, J, and B_0 and has dependence only on the hole's charge Q. Actually, this is the point of central importance we would like to make in the present work. The physical interpretation of this characteristic can be briefly stated as follows. When the spin of the hole and the asymptotically uniform magnetic field are parallel (antiparallel), the hole selectively accretes positive (negative) charges as we have discussed in the earlier section following the injection energy argument proposed by Carter and they, in turn, generate magnetic fields additive to the existing ones. Thus unlike the uncharged Kerr hole case, the magnetic flux through a slightly charged Kerr hole can never become zero. This effect manifests itself in a particularly interesting manner when the accreted charge reaches its maximum value, the Wald charge.

(iii) For the Wald charge value, $|Q| = 2|B_0|J$,

$$\Phi_B = |B_0| \pi r_+^2 \left(1 + \frac{a^2}{r_+^2}\right)^2 = |B_0| 4\pi m^2, \tag{28}$$

which is exactly the standard flux across a Schwarzschild black hole. This point is particularly interesting since the Wald charge, i.e., the amount of equilibrium net charge accreted on a Kerr hole restores the magnetic flux to the value precisely the same as that of an uncharged, nonrotating Schwarzschild hole. This point also has been noticed in the work by Dokuchaev [4] and more recently by van Putten [11]. Particularly in Dokuchaev's work, it is just this point that led him to conclude that the magnetic flux through the hole becomes independent of the hole's angular momentum when the hole has the equilibrium Wald charge.

(iv) It is also interesting to note that our result for the magnetic flux through the slightly charged Kerr hole given in Eq. (24) above can indeed be derived as the leading approximation to the result obtained from the analysis [4] based on the exact Ernst-Wild solution [5] to the coupled Einstein-Maxwell equations, although it was not realized there in the work by Dokuchaev.

V. THE MAGNETIC FLUX THROUGH A KERR HOLE IN AN OBLIQUE CONFIGURATION CASE

Thus far we have considered the case with symmetric geometry in which the stationary, axisymmetric magnetic field is precisely aligned with Kerr hole's axis of rotation. It would, however, be of some interest to explore more general case when the asymptotically uniform, stationary magnetic field happens to be ''oblique,'' i.e., aligned at some angle to the hole's axis of rotation. Then, of course, the natural question to be addressed is to see how much of such fields actually can penetrate the horizon, and we now turn to this issue.

Indeed the case of uncharged Kerr hole has been studied long ago by Bicak and Janis and hence in this section, we would like to explore what happens when the Kerr hole is again ''slightly charged,'' i.e., $Q\neq 0$, along the same line of analysis as employed in the previous section. We now start with the solution given by Bicak and Janis [10]. The electromagnetic field that is generated when an uncharged Kerr hole is placed in an originally uniform magnetic field, the direction of which does not coincide with the hole's axis of rotation has been given by Bicak and Janis and will here be denoted by F^{BJ} and A^{BJ} for the Maxwell field strength and the associated gauge potential, respectively. They are given in Appendix B. And in their solution, it is assumed that asymptotically, the field is decomposed into two components, B_0 and B_1. B_0 being in the direction of the *z* axis (i.e., the hole's rotation axis) and B_1 being chosen to lie, without loss of generality, along the *x* axis. Next, since the source-free Maxwell equation is a linear differential equation, it would admit any linear combination of particular solutions as another solution. Once again, therefore, we invoke the solution generating method due to Wald. In particular, in order to construct the solution in the presence of some charge, we recall that, according to the statement (B) in the discussion of Wald field given earlier, there can be one more perturbation of a stationary, axisymmetric vacuum black hole that corresponds to adding a charge Q to the hole and it is nothing but to linearly superpose the solution $(-Q/2m)F_{\xi}=(-Q/2m)d\xi$ to the existing solution. Evidently, therefore,

$$F=F^{BJ}+\frac{-Q}{2m}d\xi \quad \text{or} \quad A_{\mu}=A^{BJ}_{\mu}+\frac{-Q}{2m}\xi_{\mu} \tag{29}$$

constitutes a legitimate solution of Maxwell equations representing the electromagnetic field around a slightly charged Kerr hole with charge Q that is asymptotically uniform but is not aligned with the hole's axis of rotation. In fact, this seems to be the only way available to construct the solution in the presence of some charge and it is worthy to note that since the particular solution being added, $(-Q/2m)F_{\xi}=(-Q/2m)d\xi$ represents an axisymmetric test electromagnetic field that *vanishes asymptotically*, this new solution given in Eq. (28) above would not change the asymptotic behavior of the Bicak and Janis solution at all. In Boyer-Lindquist coordinates, the solution we are after can thus be explicitly written down as

$$\begin{aligned}
A_t&=A^{BJ}_t+\frac{Q}{2m}\left[\frac{\Delta-a^2\sin^2\theta}{\Sigma}\right],\\
A_r&=A^{BJ}_r,\\
A_{\theta}&=A^{BJ}_{\theta},\\
A_{\phi}&=A^{BJ}_{\phi}+\frac{Q}{2m}\left[\frac{a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}\right]
\end{aligned} \tag{30}$$

for the gauge potential and

$$\begin{aligned}
F_{rt}=&-B_0\frac{ma}{\Sigma^2}(r^2-a^2\cos^2\theta)\left(1-\frac{Q}{B_0J}+\cos^2\theta\right)\\
&-B_1\frac{mar}{\Sigma^2\Delta}\sin\theta\cos\theta[\{r^3-2mr^2+ra^2(1+\sin^2\theta)\\
&+2ma^2\cos^2\theta\}\cos\psi-a\{r^2-4mr+a^2\\
&\times(1+\sin^2\theta)\}\sin\psi],\\
F_{\theta t}=&-B_0\frac{2mar}{\Sigma^2}\sin\theta\cos\theta\left\{r^2-a^2\left(1-\frac{Q}{B_0J}\right)\right\}\\
&-B_1\frac{ma}{\Sigma^2}(r^2\cos 2\theta+a^2\cos^2\theta)(a\sin\psi-r\cos\psi),\\
F_{\phi t}=&F^{BJ}_{\phi t},\\
F_{r\theta}=&F^{BJ}_{r\theta},\\
F_{r\phi}=&B_0\frac{\sin^2\theta}{\Sigma^2}\bigg[r\Sigma^2+ma^2(r^2-a^2\cos^2\theta)\\
&\times\left(1-\frac{Q}{B_0J}+\cos^2\theta\right)\bigg]-B_1\frac{\sin\theta\cos\theta}{\Delta}\\
&\times[(r\Delta-ma^2)\cos\psi-a(\Delta+mr)\sin\psi]\\
&+B_1\frac{ma^2r}{\Sigma^2\Delta}\sin^3\theta\cos\theta[\{r^3-2mr^2+ra^2(1+\sin^2\theta)\\
&+2ma^2\cos^2\theta\}\cos\psi-a\{r^2-4mr+a^2\\
&\times(1+\sin^2\theta)\}\sin\psi],
\end{aligned} \tag{31}$$

$$
\begin{aligned}
F_{\theta\phi}=B_0\frac{\sin\theta\cos\theta}{\Sigma^2}\bigg[&(r^2+a^2)\bigg\{(r^2-a^2)(r^2-a^2\cos^2\theta)\\
&+2a^2r(r-m)(1+\cos^2\theta)+2a^2r\frac{Qm}{B_0J}\bigg\}\\
&-a^2\Delta\Sigma\sin^2\theta\bigg]+B_1r\sin^2\theta[(r-m)\cos\psi-a\sin\psi]\\
&+B_1\frac{m\sin^2\theta}{\Sigma^2}\\
&\times[(r^2+a^2)(r^2-a^2\cos^2\theta)-\Sigma a^2\cos^2\theta]\\
&\times(r\cos\psi-a\sin\psi)
\end{aligned}
$$

for the Maxwell field strength and here ψ has been defined in terms of the azimuthal angle coordinate ϕ as

$$\psi=\phi+\frac{a}{r_+-r_-}\ln\left(\frac{r-r_+}{r-r_-}\right). \tag{32}$$

This solution in the presence of the accretion charge Q on the Kerr hole reduces to the Bicak-Janis solution in the absence of the charge given in Appendix B for $Q=0$ as it should. Then in order to calculate the flux of this magnetic field strength through the horizon of a Kerr hole, we, as usual, need the component $F_{\theta\phi}$ evaluated on the horizon that is

$$
\begin{aligned}
F_{\theta\phi}|_{r=r_+}=B_0&\frac{\sin\theta\cos\theta}{(r_+^2+a^2\cos^2\theta)^2}(r_+^2+a^2)^2\\
&\times\left\{r_+^2-a^2\left(1-\frac{Q}{B_0J}\right)\right\}+B_1r_+\sin^2\theta\\
&\times\{(r_+-m)\cos\phi-a\sin\phi\}\\
&+B_1\frac{m\sin^2\theta}{(r_+^2+a^2\cos^2\theta)^2}\{(r_+^2+a^2)\\
&\times(r_+^2-2a^2\cos^2\theta)+a^4\sin^2\theta\cos^2\theta\}\\
&\times(r_+\cos\phi-a\sin\phi),
\end{aligned}\tag{33}
$$

where we used

$$\psi(r_+)=\phi+\frac{a}{r_+-r_-}\ln\left(\frac{r_+-r_+}{r_+-r_-}\right)=\phi-\infty=\phi. \tag{34}$$

Now we would like to compute the flux of magnetic field across a generally oriented one-half portion of Kerr hole's horizon, which shall henceforth be called, ''generally located hemisphere.'' Basically, our goal is the same as before and it is the evaluation of the magnetic flux across any two-surface as given by Eq. (19) with the invariant-surface element of a Kerr hole's horizon $d\sigma^{\alpha\beta}$ given in Eqs. (17) and (18). However, since we now have to deal with the invariant-magnetic flux across the *generally located hemisphere*, we need a more careful following analysis that is similar to the one encountered when determining principal axis of spinning rigid body in classical mechanics. As depicted in Fig. 1, we set a coordinate system in which the hole's axis of rotation coincides with the z axis and the equatorial plane is represented by x-y plane. In addition, since we are considering the general case when the asymptotically uniform, stationary magnetic field is ''oblique,'' i.e., not aligned with the hole's rotation axis, we assume asymptotically and without any loss of generality that the field can be decomposed into the z component B_0 and the x component B_1. Now, in order to characterize the position of the generally located hemisphere, we consider rotating the (x,y,z) axes by an angle β around the z axis and then next rotating the resulting (ω',y',z) axes by an angle α around the y' axis, to get (x',y',z') axes with now the z' axis representing a kind of ''principal axis'' for the generally located hemisphere. Then the series of coordinate transformations $(x,y,z)\to(\omega',y',z)\to(x',y',z')$ obviously involves two stages of $SO(3)$ rotations and if we denote the spherical polar angle coordinates for (x,y,z) system and (x',y',z') system by (θ,ϕ) and (θ',ϕ'), respectively, they are related by equations

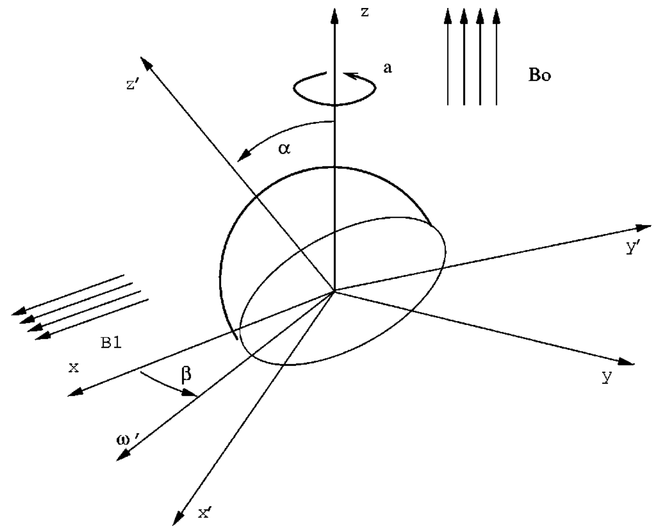

FIG. 1. A generally located hemisphere on some part of the rotating hole's horizon across which the magnetic flux of an asymptotically uniform magnetic field is to be evaluated. Two angles (α,β) completely specify the location of the hemisphere and the hole rotates around the z axis. Asymptotically, the magnetic field is decomposed into two components: B_0 along the z axis, the hole's axis of rotation, and B_1 along the x axis that is perpendicular to the hole's rotation axis.

$$
\begin{aligned}
&\sin\theta'\cos\phi'=\sin\theta\cos\alpha\cos(\phi-\beta)-\cos\theta\sin\alpha,\\
&\sin\theta'\sin\phi'=\sin\theta\sin(\phi-\beta),\\
&\cos\theta'=\sin\theta\sin\alpha\cos(\phi-\beta)+\cos\theta\cos\alpha.
\end{aligned}\tag{35}
$$

In the final ''principal axes'' coordinate system, the integration over the polar angle coordinate θ' should be done from 0 to $\pi/2$. Thus we need to determine the value of the original polar angle θ that corresponds to $\theta' = \pi/2$. This can easily be achieved by plugging $\theta' = \pi/2$ in the last equation above that yields

$$\theta = \Theta(\phi;\alpha,\beta) \equiv \frac{\pi}{2} + \tan^{-1}[\tan\alpha\cos(\phi-\beta)]. \tag{36}$$

Thus the integration over the generally located hemisphere in the new angle coordinates $0 \leq \phi' \leq 2\pi, 0 \leq \theta' \leq \pi/2$ can be translated into that in the original angle coordinates as

$$0 \leq \phi \leq 2\pi, \quad 0 \leq \theta \leq \Theta(\phi;\alpha,\beta), \tag{37}$$

and hence finally the magnetic flux across a generally oriented one-half portion of Kerr hole's horizon reads

$$\Phi_B = \int_0^{2\pi} d\phi \int_0^{\Theta(\phi;\alpha,\beta)} d\theta F_{\theta\phi}|_{r=r_+}. \tag{38}$$

Therefore, the magnetic flux across the ''generally located hemisphere'' on slightly charged Kerr hole's horizon is given by

$$\begin{aligned}\Phi_B &= \int_0^{2\pi} d\phi \int_0^{\Theta(\phi;\alpha,\beta)} d\theta [F_{\theta\phi}(B_1=0)|_{r=r_+} \\ &\quad + F_{\theta\phi}(B_0=0)|_{r=r_+}] \\ &= \Phi(B_1=0) + \Phi(B_0=0),\end{aligned} \tag{39}$$

where as given earlier

$$F_{\theta\phi}(B_1=0)|_{r_+} = B_0 \frac{\sin\theta\cos\theta}{(r_+^2+a^2\cos^2\theta)^2}(r_+^2+a^2)^2 \times\left\{r_+^2 - a^2\left(1-\frac{Q}{B_0 J}\right)\right\},$$

$$\begin{aligned}F_{\theta\phi}(B_0=0)|_{r_+} = B_1 &\frac{\sin^2\theta}{(r_+^2+a^2\cos^2\theta)^2} \\ &\times [r_+(r_+^2+a^2\cos^2\theta)^2\{(r_+-m) \\ &\times\cos\phi - a\sin\phi\} + m\{(r_+^2+a^2) \\ &\times(r_+^2-2a^2\cos^2\theta)+a^4\sin^2\theta\cos^2\theta\} \\ &\times(r_+\cos\phi - a\sin\phi)].\end{aligned}$$

Just as the case we considered in the preceding section when the asymptotically uniform magnetic field is aligned with Kerr hole's rotation axis, the effect of adding to the Kerr hole some charge small enough not to disturb its geometry or equivalently adding a particular solution represented by the Wald field $F = (-Q/2m)d\xi$ changes the value of $\Phi(B_1=0)$ above via the shift in $F_{\theta\phi}(B_1=0)|_{r_+}$ as given. Since $F_{\theta\phi}(B_0=0)|_{r_+}$ and hence its contribution to the total flux $\Phi(B_0=0)$ remains unchanged upon adding the Wald field, the actual value of $\Phi(B_0=0)$ is precisely as given originally by Bicak and Janis [10]. Thus our task here simply reduces to the calculation of $\Phi(B_1=0)$, namely,

$$\Phi(B_1=0) = B_0(r_+^2+a^2)^2\left\{r_+^2 - a^2\left(1-\frac{Q}{B_0 J}\right)\right\} I_0 \tag{40}$$

with

$$I_0 \equiv \int_0^{2\pi} d\phi \int^{\Theta} d\theta \frac{\sin\theta\cos\theta}{(r_+^2+a^2\cos^2\theta)^2}. \tag{41}$$

Note, however, this contribution to the total flux across the generally located hemisphere is invariant under the rotation about the hole's spin axis (which is chosen to be the z axis). Thus in order to go to the principal axis for the generally located hemisphere, one only needs to perform a single step of $SO(3)$ rotation $(x,y,z) \rightarrow (x',y',z')$ that yields the relations between polar angles

$$\begin{aligned}\sin\theta'\cos\phi' &= \sin\theta\cos\alpha\cos\phi - \cos\theta\sin\alpha, \\ \sin\theta'\sin\phi' &= \sin\theta\sin\phi, \\ \cos\theta' &= \sin\theta\sin\alpha\cos\phi + \cos\theta\cos\alpha.\end{aligned} \tag{42}$$

This, in turn, implies that the integration over the generally located hemisphere actually amounts to the ranges

$$0 \leq \phi \leq 2\pi, \quad 0 \leq \theta \leq \Theta(\phi;\alpha) \tag{43}$$

with

$$\Theta(\phi;\alpha) \equiv \frac{\pi}{2} + \tan^{-1}(\tan\alpha\cos\phi). \tag{44}$$

Then the result of actual computation reads

$$\begin{aligned}I_0 &= \frac{1}{2r_+^2(r_+^2+a^2)}\int_0^{2\pi} d\phi \frac{1}{(1+\rho_+^2\tan^2\alpha\cos^2\phi)} \\ &= \frac{\pi}{r_+^2(r_+^2+a^2)}(1+\rho_+^2\tan^2\alpha)^{-1/2}\end{aligned} \tag{45}$$

and hence

$$\begin{aligned}\Phi(B_1=0) = B_0\pi r_+^2 &\left(1+\frac{a^2}{r_+^2}\right)\left\{1-\frac{a^2}{r_+^2}\left(1-\frac{Q}{B_0 J}\right)\right\} \\ &\times(1+\rho_+^2\tan^2\alpha)^{-1/2},\end{aligned} \tag{46}$$

where $\rho_+^2 \equiv (1+a^2/r_+^2)$. Finally, putting this result of ours, $\Phi(B_1=0)$ together with the contribution coming from the B_1 component of the asymptotically uniform magnetic field $\Phi(B_0=0)$, we arrive at the total magnetic flux across the generally located hemisphere on slightly charged Kerr hole's horizon

$$\Phi_B = \Phi(B_1=0) + \Phi(B_0=0) \tag{47}$$

$$= B_0 \pi r_+^2 \left(1+\frac{a^2}{r_+^2}\right)\left\{1-\frac{a^2}{r_+^2}\left(1-\frac{Q}{B_0 J}\right)\right\}$$

$$\times (1+\rho_+^2 \tan^2\alpha)^{-1/2} + 2B_1[r_+\{(r_+-m)$$

$$\times \cos\beta - a\sin\beta\} I_1 + m(r_+\cos\beta - a\sin\beta)$$

$$\times (2\rho_+\overline{I_1} - I_1 - a^2 I_2)]$$

where

$$I_1 = \frac{\pi}{2}\tan\alpha(1+\tan^2\alpha)^{-1/2},$$

$$\overline{I_1} = \frac{\pi}{2}\rho_+ \tan\alpha(1+\rho_+^2\tan^2\alpha)^{-1/2}, \tag{48}$$

$$I_2 = \frac{\pi}{a^2\tan\alpha}\{(1+\rho_+^2\tan^2\alpha)^{1/2} - (1+\tan^2\alpha)^{1/2}\}.$$

Finally, some discussions on interesting observations are in order.

(i) In the absence of the accretion charge, i.e., $Q=0$, the total magnetic flux above correctly reduces to that obtained by Bicak and Janis as it should.

(ii) For $\alpha=0$, namely, over the hemisphere that is symmetrically located around the hole's rotation axis, we have

$$\Phi_B = |B_0| \pi r_+^2 \left(1+\frac{a^2}{r_+^2}\right)\left[1-\frac{a^2}{r_+^2}\left(1-\frac{|Q|}{|B_0|J}\right)\right],$$

i.e., one recovers the result in Eq. (25) for the case where the asymptotically uniform magnetic field is aligned with the hole's spin axis we studied earlier. Obviously this was expected since with this orientation of the hemisphere, the contribution to the total flux coming from the B_1 component (which is perpendicular to the hole's spin axis) is zero. Here, the points worthy to note are essentially the same as before. First, with the nonvanishing accretion charge $Q\neq 0$, the total magnetic flux through the hole can *never* go to zero. In particular, when the hole is maximally rotating, the total flux becomes $\Phi_B = 2\pi|Q|$ that is independent of m, J, and B_0 and depends only on the hole's charge. Lastly, when the accreted charge takes the particular value $|Q| = 2|B_0|J$, the total flux gets maximized and it is precisely the standard flux across a Schwarzschild black hole, $\Phi_B = |B_0| 4\pi m^2$ as pointed out earlier.

(iii) For $\alpha=\pi/2$, i.e., when the principal axis for the hemisphere is perpendicular to the hole's rotation axis, we have

$$\Phi_B = B_1 \pi [r_+^2 \cos\beta - (r_+ + m) a \sin\beta]. \tag{49}$$

The fact that the total flux depends only on the B_1 component is also expected since with this orientation of the hemisphere, the B_0 component contribution to the total flux is obviously zero. Next, the angle β determines, asymptotically, the angle between the principal axis for the hemisphere and the axis x which is along the B_1 component of the field. As such, the total flux gets maximum for $\beta=0$ when $\Phi_B = B_1 \pi r_+^2$ and gets minimum for $\beta=\beta_0$, with $\tan\beta_0 = [r_+^2/a(r_+ + m)]$, when $\Phi_B = 0$. For extreme Kerr hole, $a = m = r_+$ and hence $\tan\beta_0 = 1/2$ or $\beta_0 = 27°$. Moreover, since we may assume $-\pi/2 \leq \beta \leq \pi/2$, $\cos\beta_0$ is always positive and hence $\Phi_B = 0$ occurs only if $a\sin\beta_0 > 0$. Thus if $a>0$, then $\beta_0 > 0$ and this confirms our intuition that the field lines are bent near the hole in the same direction in which the hole rotates since the rotating hole drags field lines along.

(iv) Now in this case when the rotating hole takes small amount of accretion charge, one might wonder how come only the part of the magnetic flux $\Phi(B_1=0)$ coming from the B_0 component of the field gets affected with the other part $\Phi(B_0=0)$ coming from the B_1 component of the field remaining unchanged. In fact, it has been anticipated from the way we constructed the solution to the Maxwell equations in the presence of some charge. Notice that we employed the solution generating scheme by Wald in which the particular solution $(-Q/2m)F_\xi = (-Q/2m)d\xi$ has been superposed to another particular solution, i.e., the Bicak-Janis solution F^{BJ}. And this solution, $F = (-Q/2m)d\xi$ represents axisymmetric electromagnetic test field aligned precisely with the hole's rotation axis. Besides, this particular solution vanishes asymptotically without affecting the asymptotic behavior of the Bicak-Janis solution. Therefore, only the B_0 component of the Bicak-Janis solution has been modified locally and thus the $\Phi(B_1=0)$ part of the magnetic flux gets affected as a consequence.

(v) Lastly, we point out that in this generally oblique geometry case, the charge on the hole Q has been regarded as being arbitrary. Indeed, the application of Carter's injection energy argument to the determination of equilibrium charge $Q=2B_0J$ we discussed earlier was quite straightforward for the case when the field and the hole's spin are exactly aligned. Of course, it can be attributed to the simple structure of the solution to the Maxwell equations given in terms of the exterior derivatives of Killing fields as can be seen in Eqs. (7) and (8). This kind of advantage, however, does not seem to be available in the present oblique geometry case as the solution given in Eq. (30) or (31) no longer possesses such a privileged structure. As a result, the determination of the equilibrium charge value is unlikely to be successful here although such equilibrium charge is still expected to exist in principle.

VI. CONCLUDING REMARKS

In the present work, based on the solution-generating method given by Wald, it has been demonstrated in a transparent manner that in a more realistic situation when there are charges around (but small enough not to disturb the background vacuum geometry), the magnetic flux penetrating the horizon depends not only on the angular momentum but on the amount of charge accreted on the hole as well and can never vanish no matter how fast the hole spins. The points worthy of note can be summarized as follows. First we start with the case when the asymptotically uniform magnetic field and the hole's spin are precisely aligned with each other. Then by the argument given by Wald concerning the charge accretion process, the hole gradually accretes the charge until it reaches the equilibrium value $|Q|=2|B_0|J$. Thus with the nonvanishing accretion charge having value in the range $0<|Q|<2|B_0|J$, the total magnetic flux through the hole can *never* go to zero. In particular, when the hole is maximally rotating, the total flux becomes $\Phi_B=2\pi|Q|$ that is independent of m, J, and B_0 and depends only on the hole's charge. Lastly, when the accreted charge reaches its maximum value $|Q|=2|B_0|J$, the total flux also gets maximized and it is precisely the standard flux across a Schwarzschild black hole, $\Phi_B=|B_0|4\pi m^2$ as pointed out earlier. Next, the physical interpretation of this characteristic can be briefly stated as follows. When the spin of the hole and the asymtotically uniform magnetic field are parallel (antiparallel), the hole selectively accretes positive (negative) charge as we have discussed in the earlier section following the injection energy argument proposed by Carter, and they, in turn, generate magnetic fields additive to the existing ones. Thus, unlike the uncharged Kerr hole case, the magnetic flux through a slightly charged Kerr hole can never go to zero. We have mainly considered the case with symmetric geometry in which the stationary, axisymmetric magnetic field is precisely aligned with Kerr hole's axis of rotation. It would, however, be of some interest to explore more general case when the asymptotically uniform, stationary magnetic field happens to be ''oblique,'' i.e., aligned at some angle to the hole's axis of rotation. Then, of course, the natural question to be addressed is to see how much of such fields actually can penetrate the horizon. Thus we also explored what would happen when the Kerr hole is again ''slightly charged'' along the same line of analysis as the one employed in the previous symmetric geometry case. In this oblique geometry case, however, although we could write down the solution to the Maxwell equations and evaluate the magnetic flux through the hole in a quite straightforward manner using basically the solution generating scheme by Wald, the determination of equilibrium charge value was not attempted due to technical barriers. This issue might be worth persuing and we hope we can come back to it in a future work. It is our belief that the result of this work, obtained in a more realistic situation when some amount of charges are around, eventually lends support to the operational nature of the Blandford-Znajek mechanism and puts it on a firmer ground at least on the theoretical side.

ACKNOWLEDGMENTS

This work was supported in part by Brain Korea 21 project. C.H.L. and H.K.L. are also supported in part by the interdisciplinary research program of the KOSEF, Grant No. 1999-2-003-5.

APPENDIX A: THE CARTER TETRAD

Generally speaking, in order to represent a given background geometry, one needs to first choose a coordinate system in which the metric is to be given and next, in order to obtain physical components of a tensor (such as the electric and magnetic field values), one has to select a tetrad frame (in a given coordinate system) to which the tensor components are to be projected. For the Kerr background space time, here we choose the Boyer-Lindquist coordinates that can be viewed as the generalization of Schwarzschild coordinates to the stationary, axisymmetric case. Turning to the choice of tetrad frame, there are largely two types of tetrad frames: orthonormal tetrad and null tetrad. As is well known, the orthonormal tetrad is a set of four mutually orthogonal unit vectors at each point in a given space time that give the directions of the four axes of locally Minkowskian coordinate system

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=\eta_{AB}e^A e^B$$
$$=-(e^0)^2+(e^1)^2+(e^2)^2+(e^3)^2 \tag{A1}$$

where $e^A=e^A_\mu dx^\mu$. Every physical observer with four-velocity u^μ has associated with him an orthonormal frame in which the basis vectors are the (reciprocal of) orthonormal tetrad $e_A=\{e_0=u,e_1,e_2,e_3\}$. And corresponding to this is a null tetrad $Z_A=\{l,n,m,\bar{m}\}$ defined by

$$e_0=\frac{1}{\sqrt{2}}(l+n), \qquad e_1=\frac{1}{\sqrt{2}}(l-n), \tag{A2}$$

$$e_2=\frac{1}{\sqrt{2}}(m+\bar{m}), \qquad e_2=\frac{1}{\sqrt{2}i}(m-\bar{m})$$

satisfying the orthogonality relation

$$-l^\mu n_\mu=1=m^\mu\bar{m}_\mu \tag{A3}$$

with all other contractions being zero and

$$g^{\mu\nu}=-l^\mu n^\nu-n^\mu l^\nu+m^\mu\bar{m}^\nu+\bar{m}^\mu m^\nu. \tag{A4}$$

Conversely, given a nonsingular null tetrad, there is a corresponding physical observer. The tetrad vectors then can be used to obtain, from tensors in arbitrary coordinate system, their physical (i.e., finite and nonzero) components measured by an observer in this locally flat tetrad frame. And the rules for calculating the physical components of a tensor, say, $T_{\mu\nu}$ in the orthonormal frame and in the null frame are given, respectively, by

$$T_{AB}=T_{\mu\nu}(e^\mu_A e^\nu_B), \qquad T_{lm}=T_{\mu\nu}(l^\mu m^\nu), \quad \text{etc.} \tag{A5}$$

where e^μ_A is the inverse of the tetrad vectors e^A_μ in that

$e^{\mu}_{A}e^{A}_{\nu}=\delta^{\mu}_{\nu}$ and $e^{\mu}_{A}e^{B}_{\mu}=\delta^{B}_{A}$. As just stated, all that is required of the ''correct'' boundary conditions for electric and magnetic fields at the horizon is that the physical field's components in the neighborhood of an event horizon should have ''nonspecial'' values. Or put another way, a physically well-behaved observer at the horizon should see the fields as having finite and nonzero values. One such choice of well-behaved tetrad frame has been suggested long ago by Carter [8]. The construction of Carter's orthonormal tetrad starts from Kinnersley's null tetrad [12]. In Boyer-Lindquist coordinates $x^{\mu}=(t,r,\theta,\tilde{\phi})$, its contravariant and covariant components are given by

$$l^{\mu}=\left(\frac{(r^2+a^2)}{\Delta},\quad 1,\quad 0,\quad \frac{a}{\Delta}\right),$$

$$n^{\mu}=\left(\frac{(r^2+a^2)}{2\Sigma},\quad \frac{-\Delta}{2\Sigma},\quad 0,\quad \frac{a}{2\Sigma}\right),\tag{A6}$$

$$m^{\mu}=\frac{1}{\sqrt{2}(r+ia\cos\theta)}\times\left(ia\sin\theta,\quad 0,\quad 1,\quad \frac{i}{\sin\theta}\right)$$

and

$$l_{\mu}=g_{\mu\nu}l^{\nu}=\left(1,\quad \frac{-\Sigma}{\Delta},\quad 0,\quad a\sin^2\theta\right),$$

$$n_{\mu}=g_{\mu\nu}n^{\nu}=\left(\frac{\Delta}{2\Sigma},\quad \frac{1}{2},\quad 0,\quad \frac{-a\Delta}{2\Sigma}\sin^2\theta\right),\tag{A7}$$

$$m_{\mu}=g_{\mu\nu}m^{\nu}=\frac{1}{\sqrt{2}(r+ia\cos\theta)}\times[ia\sin\theta,\quad 0,\quad -\Sigma,\quad -i(r^2+a^2)\sin\theta].$$

where, as before, $\Sigma=r^2+a^2\cos^2\theta$ and $\Delta=r^2+a^2-2mr$ with m and a being the ADM mass and the angular momentum per unit mass of the hole respectively. This Kinnersley's null tetrad has been chosen so that l^{μ} and n^{μ} lie along the two principal repeated null directions of the Weyl tensor. Kinnersley's null tetrad has proved very useful for separating and solving the equations governing scalar, electromagnetic, and gravitational perturbations of Kerr geometry [13]. However, the associated orthonormal tetrad suffers from two disadvantages. It is singular on the horizon and an observer at rest in it has nonzero radial velocity. This last point is caused by the asymmetric normalization of l^{μ} and n^{μ}, and means that the corresponding orthonormal tetrad is unnatural in that it frequently hides interesting features of the fields. For these reasons and others, one obtains another null tetrad and the associated orthonormal tetrad (Carter tetrad) by ''null rotating'' the Kinnersley's null tetrad. Thus at this point, it seems relevant to recall some of the basics of null rotation. Notice that the orthogonality relations for null tetrad given in Eq. (A4) remain invariant under the six-parameter group of homogeneous Lorentz transformations at each point of space time. And this Lorentz group can be decomposed into three Abelian subgroups:

$$\text{(I)}\quad l\to l,\qquad m\to m+al,\qquad n\to n+a\bar{m}+\bar{a}m+a\bar{a}l,$$

$$\text{(II)}\quad n\to n,\qquad m\to m+bn,\qquad l\to l+b\bar{m}+\bar{b}m+b\bar{b}n,\tag{A8}$$

$$\text{(III)}\quad l\to\Lambda l,\qquad n\to\Lambda^{-1}n,\qquad m\to e^{i\theta}m,$$

where a and b are complex numbers and Λ and θ are real. Each of these group transformations is called a ''null rotation'' [14] and here we particularly consider the null rotation (III). Under this null rotation (III), the corresponding orthonormal tetrad e_A is boosted in the $e_1=e_{\hat{r}}$ direction with three-velocity $(\Lambda^2-1)/(\Lambda^2+1)$ and spatially rotated about $e_1=e_{\hat{r}}$ through the angle θ. Indeed this action is precisely what we need. In order to get a null tetrad well behaved at the horizon, we need to boost it by an amount that becomes suitably infinite on the horizon. Thus we perform the null rotation (III) on the Kinnersley's null tetrad with $\Lambda=(\Delta/2\Sigma)^{1/2}$ and $e^{i\theta}=\Sigma^{1/2}/(r-ia\cos\theta)$ to obtain the following nonsingular null tetrad on the horizon:

$$l'^{\mu}=\left(\frac{(r^2+a^2)}{(2\Sigma\Delta)^{1/2}},\quad \left(\frac{\Delta}{2\Sigma}\right)^{1/2},\quad 0,\quad \frac{a}{(2\Sigma\Delta)^{1/2}}\right),$$

$$n'^{\mu}=\left(\frac{(r^2+a^2)}{(2\Sigma\Delta)^{1/2}},\quad -\left(\frac{\Delta}{2\Sigma}\right)^{1/2},\quad 0,\quad \frac{a}{(2\Sigma\Delta)^{1/2}}\right),\tag{A9}$$

$$m'^{\mu}=\frac{1}{\sqrt{2}\Sigma^{1/2}}\left(ia\sin\theta,\quad 0,\quad 1,\quad \frac{i}{\sin\theta}\right).$$

Then the associated orthonormal tetrad is

$$e^{\mu}_{0}=\left(\frac{(r^2+a^2)}{(\Sigma\Delta)^{1/2}},\quad 0,\quad 0,\quad \frac{a}{(\Sigma\Delta)^{1/2}}\right),$$

$$e^{\mu}_{1}=\left(0,\quad \left(\frac{\Delta}{\Sigma}\right)^{1/2},\quad 0,\quad 0\right),\tag{A10}$$

$$e^{\mu}_{2}=\left(0,\quad 0,\quad \frac{1}{\Sigma^{1/2}},\quad 0\right)$$

$$e^{\mu}_{3}=\left(\frac{a\sin\theta}{\Sigma^{1/2}},\quad 0,\quad 0,\quad \frac{1}{\Sigma^{1/2}\sin\theta}\right)$$

and its dual is given by Carter as

$$e^0_\mu = \left[\left(\frac{\Delta}{\Sigma}\right)^{1/2}, \quad 0, \quad 0, \quad -\left(\frac{\Delta}{\Sigma}\right)^{1/2} a\sin^2\theta\right],$$
$$e^1_\mu = \left[0, \quad \left(\frac{\Sigma}{\Delta}\right)^{1/2}, \quad 0, \quad 0\right], \tag{A11}$$
$$e^2_\mu = (0, \quad 0, \quad \Sigma^{1/2}, \quad 0)$$
$$e^3_\mu = \left(\frac{-a\sin\theta}{\Sigma^{1/2}}, \quad 0, \quad 0, \quad \frac{(r^2+a^2)}{\Sigma^{1/2}}\sin\theta\right).$$

Now, in order to have some insight into the nature of this Carter's orthonormal tetrad in Boyer-Lindquist coordinates, we first rewrite the metric of Kerr geometry as implied by this Carter's dual tetrad:

$$ds^2 = \eta_{AB} e^A e^B \tag{A12}$$
$$= -\frac{\Delta}{\Sigma}[dt - a\sin^2\theta d\phi]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}[(r^2+a^2)d\phi - a dt]^2.$$

Then one can immediately realize that an observer at rest in this Carter frame travels around the hole at $(r=\text{const}, \theta=\text{const})$ with the angular velocity, $\Omega^c = a/(r^2+a^2)$ which is independent of θ. Certainly, this is in contrast to what happens in ZAMO (or LNRF) tetrad frame in which case a ZAMO observer travels around the hole with the angular velocity, $\Omega = a[(r^2+a^2)-\Delta]/[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]$ that has dependence on the polar angle θ. Indeed, the physical significance of the Carter tetrad frame is that observers at rest in it see principal null congruence photons moving with purely radial velocities. This leads one to suppose that Maxwell equations and their solutions should take relatively simple forms in this Carter tetrad frame as first pointed out by Znajek [16] and it is indeed the case [17].

APPENDIX B: THE BICAK-JANIS SOLUTION

Here, we provide the electromagnetic field that is generated when a Kerr black hole is placed in an originally uniform magnetic field with its direction not coinciding with the rotation axis of the hole. This solution has been given by Bicak and Janis [10]. As is well known, in Newman-Penrose formalism [14], the three independent complex null tetrad components of the Maxwell field strength tensor are given by

$$\phi_0 = F_{\mu\nu} l^\mu m^\nu,$$
$$\phi_1 = \frac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu), \tag{B1}$$
$$\phi_2 = F_{\mu\nu}\bar{m}^\mu n^\nu.$$

The explicit form of ϕ_0, ϕ_1, and ϕ_2 as a solution of Maxwell equations is given in the earlier work of Bicak and Dvorak [15] and it contains parameters B_0^x, B_0^y, and B_0^z denoting the components of the oblique magnetic field in asymptotically Minkowskian coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. Without any loss of generality, however, one can put $B_0^y = 0$ and denote by $B_0^x \equiv B_1$, the field component perpendicular to the rotation axis and by $B_0^z \equiv B_0$, the field component aligned along the rotation axis. Now given the solutions ϕ_0, ϕ_1, ϕ_2, we would like to have the expression in terms of Maxwell field strength $F_{\mu\nu}$, say, in Boyer-Lindquist coordinates. This can be achieved first by inverting the above expression

$$F_{\mu\nu} = 2(\phi_1 + \bar{\phi}_1) n_{[\mu} l_{\nu]} + 2(\phi_1 - \bar{\phi}_1) m_{[\mu}\bar{m}_{\nu]} + 2\phi_2 l_{[\mu} m_{\nu]} + 2\bar{\phi}_2 l_{[\mu}\bar{m}_{\nu]} + 2\phi_0 \bar{m}_{[\mu} n_{\nu]} + 2\bar{\phi}_0 m_{[\mu} n_{\nu]} \tag{B2}$$

and evaluating this with the standard Kinnersley's null tetrad (i.e., the covariant components) given earlier in Appendix A. The result is

$$\begin{aligned}
F_{rt} &= -B_0 \frac{ma}{\Sigma^2}(r^2 - a^2\cos^2\theta)(1+\cos^2\theta) \\
&\quad - B_1 \frac{mar}{\Sigma^2\Delta}\sin\theta\cos\theta[\{r^3 - 2mr^2 + ra^2(1+\sin^2\theta) + 2ma^2\cos^2\theta\}\cos\psi - a\{r^2 - 4mr + a^2 \\
&\quad \times (1+\sin^2\theta)\}\sin\psi], \\
F_{\theta t} &= -B_0 \frac{2mar}{\Sigma^2}\sin\theta\cos\theta(r^2 - a^2) \\
&\quad - B_1 \frac{ma}{\Sigma^2}(r^2\cos 2\theta + a^2\cos^2\theta)(a\sin\psi - r\cos\psi), \\
F_{\phi t} &= -B_1 \frac{ma}{\Sigma}\sin\theta\cos\theta(r\sin\psi + a\cos\psi), \\
F_{r\theta} &= -B_1 \frac{1}{\Delta}[\Delta(r\sin\psi + a\cos\psi) \\
&\quad + a\{(mr - a^2\sin^2\theta)\cos\psi \\
&\quad - a(r\sin^2\theta + m\cos^2\theta)\sin\psi\}], \\
F_{r\phi} &= B_0 r\sin^2\theta - B_1 \frac{\sin\theta\cos\theta}{\Delta} \\
&\quad \times[(r\Delta - ma^2)\cos\psi - a(\Delta + mr)\sin\psi] \\
&\quad - a\sin^2\theta F_{rt}, \\
F_{\theta\phi} &= B_0\Delta\sin\theta\cos\theta + B_1[(r^2\sin^2\theta + mr\cos 2\theta)\cos\psi \\
&\quad - a(r\sin^2\theta + m\cos^2\theta)\sin\psi] - \frac{(r^2+a^2)}{a}F_{\theta t}
\end{aligned} \tag{B3}$$

for the Maxwell field strength tensor and

$$A_t = B_0 \frac{a}{\Sigma}[-\Sigma + mr(1+\cos^2\theta)] + B_1 \frac{ma}{\Sigma}\sin\theta\cos\theta(r\cos\psi - a\sin\psi),$$

$$A_r = -B_1(r-m)\sin\theta\cos\theta\sin\psi, \tag{B4}$$

$$A_\theta = -B_1[a(r\sin^2\theta + m\cos^2\theta)\cos\psi + (r^2\cos^2\theta - mr\cos 2\theta + a^2\cos 2\theta)\sin\psi],$$

$$A_\phi = B_0 \frac{\sin^2\theta}{2\Sigma}[\Sigma(r^2+a^2) - 2a^2 mr(1+\cos^2\theta)] - B_1 \frac{\sin\theta\cos\theta}{\Sigma}[\Sigma\Delta\cos\psi + m(r^2+a^2)(r\cos\psi - a\sin\psi)]$$

for the corresponding gauge potential. To summarize, this is the solution in the absence of the charge on the hole, which is to be compared with the solution given in the text in the presence of the accretion charge on the Kerr black hole Q.

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