Excitation of *g*-modes of solar-type stars by an orbiting companion

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The possibility of exciting the *g*-modes of a solar-type star as a consequence of the gravitational interaction with a close companion (a planet or a brown dwarf) is studied by a perturbative approach. The amplitude of the emitted gravitational wave is computed and compared with the quadrupole emission of the system, showing that in some cases it can be considerably larger. The effects of radiation reaction are considered to evaluate the time scale of the emission process, and a Roche lobe analysis is used to establish the region where the companion can orbit without being disrupted by tidal interactions with the star.

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I. INTRODUCTION

In recent years a significant number of systems composed of a solar-type star and one or more orbiting companions has been discovered in our neighborhood. According to the *Extrasolar Planets Catalog* [1], at the time we write this paper 58 extrasolar planetary systems have been confirmed, and this number is destinated to grow in the near future (see [2] for a recent review). 46 of the observed orbiting objects are planets with masses ranging between $[0.16\text{-}11]M_J$, where M_J is Jupiter's mass; 12 are bigger, up to 60 M_J, and include super-planets and brown dwarfs. In the following, we shall indicate all the orbiting objects as ''planets.''

The newly discovered systems exhibit some unexpected features; 22 planets move on elliptic orbits with eccentricities $e>0.3$, larger than the largest eccentricity encountered in the solar system (Mercury, $e=0.2$), and many of them are incredibly close to the central star, much closer than expected. For instance the planet orbiting 51 Pegasi, discovered in 1995 [3,4], has a period $P=4.23$ days and orbits its star at a distance $a=0.05$ AU. More than two thirds of the planets discovered up to now are orbiting their host star much closer than Mercury orbits the Sun ($P=88$ days, $a=0.39$ AU), and even closer than 51 Pegasi's planet. These orbital properties challenge the existing theories on planet formation and evolution, and suggest an interesting possibility: there may exist planets on orbital radii sufficiently small to excite some *g*-modes of the star by tidal interaction. This issue has been discussed by Terquem *et al.* [5] in connection with the short orbital period of 51 Pegasi's planet. They calculated the dynamical tides raised by the planet on the star, and the dissipation time scales due to turbulent viscosity in the convective zone and to radiative damping in the radiative core. In this paper we shall approach the problem of the excitation of the *g*-modes from a different point of view: by using a perturbative approach, justified by the fact that the mass of the central star is much bigger than that of the planet, we shall compute the gravitational signal emitted by a system in which the planet moves on a circular orbit of radius R_0, close to that which would correspond to the resonant excitation of a *g*-mode, R_{res}. We shall extend the work presented in a previous paper ([6], to be referred to hereafter as paper I), were we have studied the quadrupole emission due to the orbital motion of the observed planetary systems, and started to explore the possibility for a planet to excite some stellar modes. First of all, in Sec. II we shall verify by a Roche-lobe analysis whether the planet can move on the orbit $R=R_0$ without being disrupted by tidal interactions. In Sec. III, using the quadrupole formalism we shall compute the characteristic amplitude of the gravitational signal, h_Q, emitted by the system because of the orbital motion of the planet on R_0. As usual, the two bodies will be treated as pointlike masses, and the effects of the planet on the internal structure of the star will be neglected. These effects will be considered in Sec. IV, where we shall assume that the planet induces a small perturbation on the gravitational field of the star and on its thermodynamical structure. We shall integrate the perturbed Einstein's equations coupled to the hydrodynamical equations, having the stress energy tensor of the planet as a source, and compute the characteristic amplitude of the emitted gravitational signal, h_R, which will be compared with h_Q. In this way we will be able to evaluate the enhancement in the radiation emitted by the system as a consequence of the *g*-modes excitation. In Sec. V we shall discuss how long can a planet move on an orbit close to resonance before radiation reaction effects lead the planet off resonance. Conclusions will be drawn in Sec. VI.

II. WHICH STELLAR MODES CAN BE EXCITED BY AN ORBITING PLANET

In this section we shall verify, by a Roche-lobe analysis, if a planet can orbit on a radius such that the keplerian orbital frequency, $\omega_k=\sqrt{G(M_\star+M_p)/R_0^3}$, is half the frequency of a given mode of the star ω_i. As discussed in [5,7,8], in this case the mode would be excited by the dynamical tides raised by the planet. We shall consider as a model a polytropic star, $p=K\epsilon^{1+1/n}$ with $n=3$, $\epsilon_0/p_0=5.53\times10^5$ and adiabatic exponent $\gamma=5/3$. Choosing the central density $\epsilon_0=76$ g/cm^3, this model gives a star with the same mass and radius as the Sun. In paper I we showed that if one considers a polytropic star with $n=2$ and mass equal to that of the Sun, the *p*-modes and the fundamental mode cannot be excited by a planet, because it would be tidally disrupted. The same holds for the more appropriate, though still simplified, $n=3$ model considered in this paper, therefore we shall deal

TABLE I. The minimum mean density, $\rho_{min}/\rho_{\odot}$, that a planet should have in order to be allowed to move on an orbit which corresponds to the excitation of a g-mode, without being disrupted by tidal interactions. The values of $\rho_{min}/\rho_{\odot}$ are tabulated for three planets with mass equal to that of the Earth (M_E), of Jupiter (M_J) and of a brown dwarf with $M_{BD}=40\ M_J$.

	$\rho_{min}/\rho_{\odot}$									
	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
M_E	12.5	7.21	4.64	3.24	2.38	1.83	1.45	1.17	0.97	0.82
M_J	13.0	7.49	4.82	3.37	2.48	1.90	1.51	1.22	1.01	0.85
M_{BD}	-	-	-	3.68	2.71	2.08	1.65	1.34	1.11	0.93

only with the excitation of the g-modes.

We shall consider three companions for the central star: two planets with the mass of the Earth and of Jupiter, and a brown dwarf of 40 Jovian masses, M_E, M_J and M_{BD}, respectively. Smaller planets produce gravitational signals that are too small to be interesting. Let us assume that a planet orbits the star at a constant orbital radius R^i_{res}, in a position to excite the mode g_i. Following the procedure described in paper I it is possible to determine, by a Roche lobe analysis, what is the maximum radius the planet can have in order not to overflow its Roche lobe. Since we have fixed the mass of the planet, this constraint also determines the minimum value of the planet's mean density, ρ_{min}, compatible with the excitation of that mode. The results of this analysis are shown in Table I, where we tabulate the value of ρ_{min}, expressed in units of the mean density of the Sun, for the three considered companions. It should be stressed that in the Roche lobe analysis the ratio $\rho_{min}/\rho_{\star}$, for each given mode, depends only on the ratio between the mass of the planet and that of the star. From Table I we see that, since $\rho_{Earth}/\rho_{\odot}=3.9$ and $\rho_J/\rho_{\odot}=0.9$, a planet like the Earth can orbit sufficiently close to the star to excite g-modes of order higher or equal to $n=4$, whereas a planet like Jupiter can excite only the mode g_{10} or higher.

According to the brown dwarf model (''model G'') by Burrows and Liebert [9] an evolved, 40 M_J brown dwarf has a radius $R_{BD}=5.9\times10^4$ km, and a corresponding mean density $\rho_{BD}=88\ \mathrm{g\,cm}^{-3}$; consequently $\rho_{BD}/\rho_{\odot}=64$, and this value is high enough to allow a brown dwarf companion to excite all the g-modes of the central star. However, we also need to take into account the destabilizing mechanism of mass accretion *from* the central star. This imposes a further constraint, and this is the reason why the slots corresponding to the excitation of the g-modes lower than g_4 in the last row of Table I are empty.

III. THE QUADRUPOLE EMISSION OF A BINARY SYSTEM

In this section we shall use the quadrupole formalism to compute the gravitational signal emitted by a binary system. We shall assume that a planet orbits its host star in circular motion, and that the radius of the orbit corresponds to the excitation of one of the g-modes allowed by the Roche lobe analysis discussed in Sec. II. As usual, both the star and the planet will be treated as pointlike masses, with no reference to their internal structure. From the reduced quadrupole moment

$$Q_{kl}=\mu\left(X^kX^l-\frac{1}{3}\delta^k_l|\mathbf{X}|^2\right), \tag{3.1}$$

where $\mu=M_pM_\star/(M_p+M_\star)$ and $\mathbf{X}=\mathbf{x}_\star-\mathbf{x}_p$ is the relative position vector, the nonvanishing TT-components of the emitted wave can easily be computed

$$h^{TT}_{ij}(t,\mathbf{x})=\frac{2}{r}\left(P_{ik}P_{jl}-\frac{1}{2}P_{ij}P_{kl}\right)\ddot{Q}_{kl}(\tau)\bigg|_{\tau=t-r/c} \tag{3.2}$$

where $P_{ik}=\delta^i_k-n_in_k$ is the projector onto the 2-sphere $r=\mathrm{const}$, and $\mathbf{n}$ is the radial unit vector. It should be stressed that Eq. (3.2) allows to compute the radiative contribution due solely to the *orbital motion* of the system. For a circular orbit the wave is emitted at twice the keplerian frequency [10], and from the two independent wave components $h_\pm(2\omega_k,r,\vartheta,\varphi)$ we compute the characteristic amplitude [11]

$$\begin{aligned}h_Q(2\omega_k,r)&=\sqrt{\frac{2}{3}}\sqrt{\frac{1}{4\pi}\int d\Omega[|h_+|^2+|h_-|^2]}\\&=\frac{M_p}{r}4\sqrt{\frac{2}{15}}(\omega_kR_0)^2,\end{aligned} \tag{3.3}$$

where the factor $\sqrt{2/3}$ takes into account the average over orientation, and Ω is the solid angle. This amplitude will be compared with that determined by the perturbative approach. In Table II we give the values of h_Q for the three companions orbiting the sun-like star which we use as a model. The orbital radius is chosen to correspond to the excitation of a permitted g-mode (cfr. Table I). The planetary systems are assumed to be located at a distance of 10 pc from Earth. In the last row we give the frequency (in μHz) of the g-modes, which is also the frequency at which waves are emitted.

IV. THE GRAVITATIONAL SIGNAL EMITTED BY A PERTURBED STAR

In this section we shall briefly introduce the equations of general relativity which describe the perturbations of a star excited by a planet moving on a circular orbit. Unlike the quadrupole formalism discussed in Sec. III, the perturbative approach takes into account the internal structure of the star and the modification induced on it by the interaction with the planet, and it allows to evidentiate the contribution to the emitted wave due to the excitation of the stellar modes. When the orbit is circular, only the polar, nonaxisymmetric perturbations are excited, and the metric appropriate for their description can be written in the following form [12]:

TABLE II. The amplitude of the gravitational signal emitted when the three companions considered in Table I move on a circular orbit of radius R^i_{res}, such that the condition of resonant excitation of a g-mode is satisfied, is computed by the quadrupole formalism for the modes allowed by the Roche-lobe analysis. The planetary systems are assumed to be at a distance of 10 pc from Earth. In the last line the emission frequencies ν_{GW} are given.

	$h_Q(R^i_{res})$						
	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
M_E	3.0×10^{-26}	2.7×10^{-26}	2.5×10^{-26}	2.3×10^{-26}	2.2×10^{-26}	2.0×10^{-26}	1.9×10^{-26}
M_{BD}	3.9×10^{-22}	3.5×10^{-22}	3.2×10^{-22}	3.0×10^{-22}	2.8×10^{-22}	2.6×10^{-22}	2.4×10^{-22}
M_J	-	-	-	-	-	-	6.1×10^{-24}
$\nu_{GW}(\mu\mathrm{Hz})$	112.5	96.6	84.7	75.4	67.9	61.8	56.7

$$ds^2=e^{2\nu(r)}dt^2-e^{2\mu_2(r)}dr^2-r^2d\vartheta^2-r^2\sin^2\vartheta d\varphi^2+2\sum_{lm}\int_{-\infty}^{+\infty}d\omega e^{-i\omega t}\Bigg\{[e^{2\nu}N_{lm}dt^2-e^{2\mu_2}L_{lm}dr^2]Y_{lm}-r^2\Bigg[T_{lm}+V_{lm}\frac{\partial^2}{\partial\vartheta^2}\Bigg]Y_{lm}d\vartheta^2-r^2\sin^2\vartheta\Bigg[T_{lm}+V_{lm}\Bigg(\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}+\cot\vartheta\frac{\partial}{\partial\vartheta}\Bigg)\Bigg]Y_{lm}d\varphi^2-2r^2V_{lm}\Bigg[\frac{\partial^2}{\partial\varphi\partial\vartheta}-\frac{\partial}{\partial\varphi}\cot\vartheta\Bigg]Y_{lm}d\vartheta d\varphi\Bigg\}, \tag{4.1}$$

where the functions $[N_{lm}(\omega,r),T_{lm}(\omega,r),V_{lm}(\omega,r),L_{lm}(\omega,r)]$ describe the radial part of the perturbed metric, and $Y_{lm}(\vartheta,\varphi)$ are the scalar spherical harmonics. The metric functions $\nu(r)$ and $\mu_2(r)$ describe the unperturbed spacetime and are determined by numerically integrating the TOV equations for hydrostatic equilibrium for the chosen equation of state. The functions $[N,T,V,L]$ have to be found by solving the perturbed Einstein equations outside the star and inside, where they couple to the hydrodynamical equations. Inside the star, after separating the variables and assuming a nonbarotropic equation of state, the perturbed equations can be reduced to the following set, from which the hydrodynamical variables have been eliminated [13]:

$$\begin{aligned}
&X_{,r,r}+\left(\frac{2}{r}+\nu_{,r}-\mu_{2,r}\right)X_{,r}+\frac{n}{r^2}e^{2\mu_2}(N+L)+\omega^2e^{2(\mu_2-\nu)}X=0,\\
&(r^2G),r=n\nu_{,r}(N-L)+\frac{n}{r}(e^{2\mu_2}-1)(N+L)+r(\nu_{,r}-\mu_{2,r})X_{,r}+\omega^2e^{2(\mu_2-\nu)}rX,\\
&-\nu_{,r}N_{,r}=-G+\nu_{,r}[X_{,r}+\nu_{,r}(N-L)]+\frac{1}{r^2}(e^{2\mu_2}-1)(N-rX_{,r}-r^2G)\\
&\qquad-e^{2\mu_2}(\epsilon+p)N+\frac{1}{2}\omega^2e^{2(\mu_2-\nu)}\left\{N+L+\frac{r^2}{n}G+\frac{1}{n}[rX_{,r}+(2n+1)X]\right\},\\
&L_{,r}(1-D)+L\left[\left(\frac{2}{r}-\nu_{,r}\right)-\left(\frac{1}{r}+\nu_{,r}\right)D\right]+X_{,r}+X\left(\frac{1}{r}-\nu_{,r}\right)+DN_{,r}\\
&\qquad+N\left(D\nu_{,r}-\frac{D}{r}-F\right)+\left(\frac{1}{r}+E\nu_{,r}\right)\left[N-L+\frac{r^2}{n}G+\frac{1}{n}(rX_{,r}+X)\right]=0,
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
&D=1-\frac{\omega^2e^{-2\nu}}{2[\omega^2e^{-2\nu}+e^{-2\mu_2}\nu_{,r}(\epsilon_{,r}-Qp_{,r})/(\epsilon+p)]},\\
&E=D(Q-1)-Q,
\end{aligned} \tag{4.3}$$

$$F=\frac{\epsilon_{,r}-Qp_{,r}}{\omega^2 e^{-2\nu}+e^{-2\mu_2}\nu_{,r}(\epsilon_{,r}-Qp_{,r})/(\epsilon+p)}.$$

In Eqs. (4.2) the harmonic indices (l,m) have been suppressed, we have replaced the function V by $X=nV$, with $n=(l-1)(l+2)/2$, and the function T by

$$G=\nu_{,r}\left[\frac{n+1}{n}X-T\right]_{,r}+\frac{1}{r^2}(e^{2\mu_2}-1)[n(N+T)+N]+\frac{\nu_{,r}}{r}(N+L)-e^{2\mu_2}(\epsilon+p)N+\frac{1}{2}\omega^2 e^{2(\mu_2-\nu)}\left[L-T+\frac{2n+1}{n}X\right].$$

We have also defined $Q=(\epsilon+p)/\gamma p$, where γ is the adiabatic exponent

$$\gamma=\frac{(\epsilon+p)}{p}\left(\frac{\partial p}{\partial \epsilon}\right)_{entropy=const},$$

and ϵ and p are the energy density and the pressure of the fluid composing the star. Since in this paper we apply the relativistic theory to nonbarotropic stars, we choose, as in Ref. [13], $\gamma=\text{const}=5/3$. The system of Eqs. (4.2) can be solved numerically by integrating the two independent solutions which satisfy the regularity condition at the center, and superimposing them in such a way that the perturbation of the pressure vanishes at the boundary, as discussed in [14].

Outside the star, ϵ and p vanish, and a source term given by the stress-energy tensor of the moving planet must be added on the right-hand side of the Einstein equations. Eqs. (4.2) reduce to the Zerilli equation [15] for the Schwarzschild perturbations

$$\left\{\frac{d^2}{dr_*^2}+\omega^2-\frac{2(r-2M_\star)[n^2(n+1)r^3+3M_\star n^2r^2+9M_\star^2 nr+9M_\star^3]}{r^4(nr+3M_\star)^2}\right\}Z_{lm}(\omega,r)=S_{lm}(\omega,r), \tag{4.4}$$

where $r_*=\int_0^r e^{-\nu+\mu_2}dr$, and $M_\star$ is the mass of the star. The Zerilli function matches continuously with the solution of Eqs. (4.2) in the interior, i.e., at the surface of the star

$$Z_{lm}(\omega,R)=\frac{R}{nR+3M_\star}\left(\frac{3M_\star}{n}X_{lm}(\omega,R)-RL_{lm}(\omega,R)\right), \tag{4.5}$$

and similarly for its first derivative (see Ref. [14] for details). The source term, $S_{lm}(\omega,r)$, is derived from the stress-energy tensor of the planet, considered as a pointlike particle moving on a circular orbit around the star in the equatorial plane

$$T^{\mu\nu}=\frac{M_p}{\sqrt{-g}}\frac{dT}{d\tau}\frac{dz^\mu}{dt}\frac{dz^\nu}{dt}\delta(r-R_0)\delta(\theta-\pi/2)\delta(\varphi-\omega_k t). \tag{4.6}$$

From the geodesic equations $dT/d\tau=E(1-2M_\star/R_0)^{-1}$, where the planet's energy per unit rest mass is

$$E=\left(1-\frac{2M_\star}{R_0}\right)\left(1-\frac{3M_\star}{R_0}\right)^{-1/2}. \tag{4.7}$$

In terms of these quantities the source term can be written as

$$S_{lm}(\omega,r)=M_p\left\{\frac{8\pi(r-2M_\star)(12M_\star^2+r^2n^2+3nrM_\star-6rM_\star)}{\omega\sqrt{n+1}\,r(nr+3M_\star)^2}B_{lm}^{(0)}-\frac{8\pi(r-2M_\star)^2}{\omega\sqrt{n+1}(nr+3M_\star)}B_{lm,r}^{(0)}+\frac{8\pi\sqrt{2}(r-2M_\star)}{\sqrt{n(n+1)}}F_{lm}\right\} \tag{4.8}$$

where

$$B_{lm}^{(0)}(\omega,r)=\sqrt{\frac{2}{l(l+1)}}\frac{m\omega_k}{\sqrt{1-3M_\star/R_0}}\frac{r-2M_\star}{r^2}P_{lm}(\pi/2)\,\delta(r-R_0)\,\delta(\omega-m\omega_k),$$

$$F_{lm}(\omega,r)=\frac{[l(l+1)-2m^2]}{\sqrt{2(l-1)l(l+1)(l+2)}}\frac{M_\star/R_0^3}{\sqrt{1-3M_\star/R_0}}P_{lm}(\pi/2)\,\delta(r-R_0)\,\delta(\omega-m\omega_k). \tag{4.9}$$

In order to find the amplitude of the emitted wave we follow the same approach used by Kojima [16], who first calculated the gravitational signal emitted by a particle in circular motion around a star. In particular, he considered the excitation of the fundamental mode of a neutron star by an orbiting particle, showing that a sharp resonance occurs if the frequency of the *f*-mode is twice the orbital frequency, and that the characteristic wave amplitude emitted at that frequency can be up to 100 times larger than that evaluated by the quadrupole formula. As discussed in Sec. II, if the central star is of solar-type only the g-modes can be excited by a planet, and therefore we shall focus on these modes.

The solution of Eq. (4.4) can be constructed by the Green's-function technique as follows. We first integrate the interior equations (4.2) from $r=0$ to $r=R$, where we compute the Zerilli function $Z_{lm}(R)$ as given in Eq. (4.5), and its first derivative. We then construct, by numerical integration, a solution of the homogenous Zerilli equation, Z_{lm}^1, which satisfies the boundary condition $Z_{lm}^1(R)=Z_{lm}(R)$; at radial infinity Z^1 behaves as a superposition of ingoing and outgoing waves, $Z_{lm}^1\sim A_{lm}^{in}e^{-i\omega r_*}+A_{lm}^{out}e^{+i\omega r_*}$. Let us consider a second solution of the homogenous Zerilli equation, Z_{lm}^2, which behaves as a pure outgoing wave at infinity, i.e. $Z_{lm}^2\sim e^{+i\omega r_*}$. The general solution of the non-homogeneous Zerilli equation (4.4), which satisfies the physical requirement of pure outgoing radiation at infinity, $Z_{lm}(r\to\infty)\sim Z_{lm}^{out}(\omega)e^{+i\omega r_*}$, and the matching condition at the surface, can be written in terms of these two functions

$$Z_{lm}=\frac{1}{W_{lm}}\left\{Z_{lm}^2\int_R^{r_*}Z_{lm}^1S_{lm}dr_*-Z_{lm}^1\int_\infty^{r_*}Z_{lm}^2S_{lm}dr_*\right\} \tag{4.10}$$

where W_{lm} is the Wronskian of the two solutions

$$W_{lm}\equiv(Z_{lm}^1Z_{lm,r_*}^2-Z_{lm}^2Z_{lm,r_*}^1)=2i\omega A_{lm}^{in}. \tag{4.11}$$

The amplitude of the Zerilli function at infinity therefore is

$$Z_{lm}^{out}(\omega)=\frac{1}{2i\omega A_{lm}^{in}}\int_R^\infty Z_{lm}^1S_{lm}dr_*. \tag{4.12}$$

Because of the form of the source term (4.8),(4.9), this amplitude can be written as

$$Z_{lm}^{out}\equiv M_p\bar{Z}_{lm}^{out}(m\omega_k,R_0)\,\delta(\omega-m\omega_k).$$

It should be noted that if we restrict our analysis to $l=2$, the source is nonzero only for $m=\pm2$. Finally, the amplitude of the outgoing gravitational radiation at infinity can be computed by using the relation between Z^{out} and the two wave components $\bar{h}_\pm$ in the radiative gauge [15] as follows:

$$[\bar{h}_+(2\omega_k,r)+i\bar{h}_-(2\omega_k,r)]=-\frac{M_p}{r}\sum_m 2\sqrt{n(n+1)}\bar{Z}_{2m}^{out}(m\omega_k,R_0)\,{}_2Y_{2m}(\vartheta,\varphi), \tag{4.13}$$

where ${}_2Y_{lm}$ is the $s=2$ spin-weighted spherical harmonic. As in Sec. III, we shall define the relativistic characteristic amplitude, $h_R^{l=2}$, to be compared with h_Q, by

$$h_R^{l=2}\equiv\sqrt{\frac{2}{3}}\sqrt{\frac{1}{4\pi}\int d\Omega[|\bar{h}_+|^2+|\bar{h}_-|^2]}=\frac{M_p}{r}\frac{2\sqrt{n(n+1)}|\bar{Z}_{22}^{out}(2\omega_k,R_0)|}{\sqrt{3\pi}}, \tag{4.14}$$

where we have used the orthonormality of the spin-weighted harmonics, and, in summing over *m*, the symmetry property $|\bar{Z}_{22}^{out}|^2=|\bar{Z}_{2,-2}^{out}|^2$.

We have computed $h_R^{l=2}$ assuming that the planet moves on a circular orbit of radius $R_0=R_{res}^i+\Delta R$, where R_{res}^i is the orbit corresponding to the excitation of the mode g_i, for the modes allowed by the Roche lobe analysis. We find that,

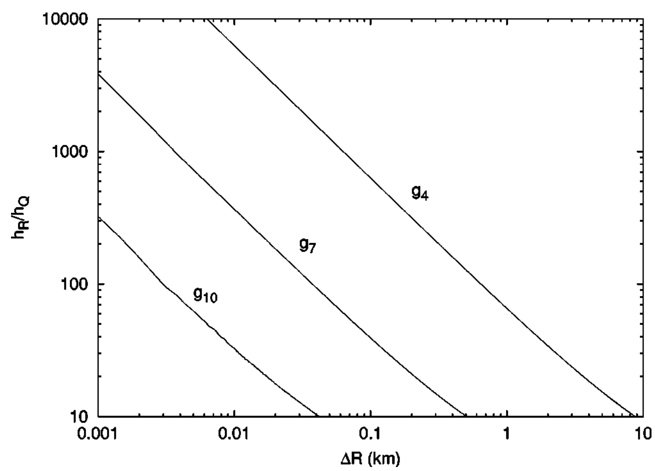

FIG. 1. The logarithm of the ratio $h_R^{l=2}(R_{res}^i+\Delta R)/h_Q(R_{res}^i)$ is plotted as a function of ΔR for the modes g_4, g_7, and g_{10} (see text).

as the planet approaches the resonant orbit, $h_R^{l=2}$ grows very sharply. It is instructive to plot the ratio $h_R^{l=2}(R_0)/h_Q(R^i_{res})$ as a function of ΔR, to see how much the amplitude of the emitted wave grows with respect to the quadrupole emission, because of the excitation of a g-mode. In Fig. 1 the logarithm of this ratio is plotted for the modes g_4, g_7, and g_{10} as a function of $\log \Delta R$, showing a power-law behavior nearly independent of the order of the mode. It should be stressed that $h_R^{l=2}(R_0)/h_Q(R^i_{res})$ is independent of the mass of the planet, but depends, of course, on the selected stellar model. From Fig. 1 we deduce that, in principle, as the planet approaches a resonant orbit the amplitude of the emitted wave may become significantly higher than that emitted because of the orbital motion (h_Q). Thus a relevant question to answer is how long can a planet move on an orbit close to a resonance, before radiation reaction effects move it off resonance. This issue will be discussed in the following section.

V. THE EFFECT OF RADIATION REACTION

The loss of energy in gravitational waves causes a shrinking of the orbit of a planetary system, and the efficiency of this process increases as the planet approaches a resonant orbit. We shall now compute the time a planet takes to move from an orbit of radius $R_0 = R^i_{res} + \Delta R$, where the amplification factor h_R/h_Q has some assigned value, to the resonant orbit R^i_{res}, because of radiation reaction effects. This time scale will indicate whether a planet can stay in the resonant region long enough to be possibly observed. On the assumption that the time scale over which the orbital radius evolves is much longer than the orbital period (adiabatic approximation), the orbital shrinking can be computed from the energy conservation law

$$M_p\left\langle \frac{dE}{dt}\right\rangle + \left\langle \frac{dE_{GW}}{dt}\right\rangle = 0, \tag{5.1}$$

where E is the energy per unit mass of the planet as given in Eq. (4.7), and $\langle dE_{GW}/dt\rangle$ is the energy emitted in gravitational waves, which can be computed in terms of the wave amplitude (4.12) as follows (cf. e.g. [17]):

$$\left\langle \frac{dE_{GW}}{dt}\right\rangle = \lim_{T\to\infty}\frac{E_{GW}}{T} = \lim_{T\to\infty}\frac{1}{T}\int \frac{dE_{GW}}{d\omega}d\omega$$
$$= \sum_{lm} \frac{(l-1)l(l+1)(l+2)}{32\pi} \times (m\omega_k M_p)^2 |\bar{Z}^{out}_{lm}(m\omega_k)|^2. \tag{5.2}$$

Since $\langle dE/dt\rangle = \langle dR_0/dt\rangle / \langle dR_0/dE\rangle$, using Eq. (4.7) and Eq. (5.2), Eq. (5.1) gives

$$\left\langle \frac{dR_0}{dt}\right\rangle = -\frac{2R_0^2}{M_p M_\star}\frac{(1-3M_\star/R_0)^{3/2}}{(1-6M_\star/R_0)}\left\langle \frac{dE_{GW}}{dt}\right\rangle, \tag{5.3}$$

from which the time needed for the planet to reach the resonant orbit can be computed

$$\Delta T = -\frac{M_p M_\star}{2}\int_{R^i_{res}+\Delta R}^{R^i_{res}} \frac{1}{\left\langle \dfrac{dE_{GW}}{dt}\right\rangle}\frac{(1-6M_\star/R_0)}{(1-3M_\star/R_0)^{3/2}}\frac{dR_0}{R_0^2}. \tag{5.4}$$

TABLE III. In column 2 we give the frequency, ν_{GW}, of the wave emitted when a companion moves around the host star on an orbit resonant with a g-mode allowed by the Roche lobe analysis (see text). Because of resonant effects, the amplitude of the wave is amplified by a factor greater than A (column 3) when the companion spans a radial region of thickness ΔR (column 4), which is the same for all planets. In the last three columns we give the time interval ΔT needed for the three companions to span the region ΔR and reach the resonance, because of radiation reaction effects.

Mode	ν_{GW} (μHz)	A	ΔR(m)	ΔT_E(yrs)	T_{BD}(yrs)	T_J(yrs)
g_4	112.5	10	8740	8.7×10^6	6.9×10^2	-
		50	1326	4.7×10^4	3.7	-
g_5	96.6	10	3166	4.3×10^6	3.4×10^2	-
		50	481	2.3×10^4	1.8	-
g_6	84.7	10	1234	2.2×10^6	1.7×10^2	-
		50	187	1.2×10^4	9.2×10^{-1}	-
g_7	75.4	10	506	1.1×10^6	89	-
		50	77	6.0×10^3	4.7×10^{-1}	-
g_8	67.9	10	216	5.9×10^5	47	-
		50	33	3.1×10^3	2.4×10^{-1}	-
g_9	61.8	10	94	3.1×10^5	24	-
		50	14	1.7×10^3	1.3×10^{-1}	-
g_{10}	56.7	10	42	1.7×10^5	13	5.2×10^2
		50	6	9.2×10^2	7.2×10^{-2}	2.9

It should be noted that since $\langle dE_{GW}/dt\rangle$ is proportional to M_p^2, ΔT is longer for smaller planets. We have computed ΔT for the three companions M_E, M_J and M_{BD}, by the following steps:

For each g-mode allowed by the Roche lobe analysis we find the radius of the resonant orbit R^i_{res}.

We assume that each companion orbits the star at a distance $R_0 = R^i_{res} + \Delta R$ such that the amplification factor $A = h_R(R_0)/h_Q(R^i_{res})$ has an assigned value, and compute the corresponding energy radiated in gravitational waves, $\langle dE_{GW}/dt\rangle$.

We compute ΔT, which tells us how long can the companion orbit the star in the resonant region between R_0 and R^i_{res}, emitting a wave of amplitude higher than $A\ \ h_Q(R^i_{res})$. The results are summarized in Table III. These data have to be used together with those in Table II as follows. Consider for instance a planet like the Earth, orbiting its star on an orbit resonant with the mode g_4. According to the quadrupole formalism, which does not consider changes in the internal structure of the star and therefore does not include the resonant contributions to the emitted radiation, it would emit a signal of amplitude $h_Q(g_4) = 3\times10^{-26}$, at a frequency $\nu_{GW} = 1.1\times10^{-4}$ Hz (Table II, first and last row, respectively). The data of Table III, which include the resonant contribution, indicate that before reaching the resonant orbit

R^4_{res}, the Earth-like planet would orbit in a region of thickness $\Delta R = 8.7$ km slowly spiralling in, emitting waves with amplitude $h_R > 10 h_Q(g_4) = 3 \times 10^{-25}$, for a time interval of 8.7×10^6 years, and that while spanning the smaller radial region $\Delta R = 1.3$ km, the emitted wave would reach an amplitude $h_R > 50 h_Q(g_4) = 1.5 \times 10^{-24}$, for a time interval of 4.7×10^4 years.

A Jovian planet, on the other hand, could only excite modes of order $n = 10$ or higher, which would correspond to a resonant frequency of $\nu_{GW} = 5.7 \times 10^{-5}$ Hz and a gravitational wave amplitude greater than 6×10^{-23} for 520 years ($\Delta R = 42$ m), or greater than 3×10^{-22} for ~ 3 years ($\Delta R = 6$ m). From these data we see that the higher the order of the mode, the more difficult it is to excite it, because the region where the resonant effects become significant gets narrower and the planet transits through it for a shorter time.

Much more interesting are the data for a brown dwarf companion. In this case, for instance, the region which would correspond to the resonant excitation of the mode g_4 ($\Delta R = 8.7$ km), with a wave amplitude greater than 3.9×10^{-21}, would be spanned in 690 years, whereas the emitted wave would have an amplitude greater than $\sim 2 \times 10^{-20}$ ($\Delta R = 1.3$ km) over a time interval of ~ 4 years.

VI. CONCLUDING REMARKS

The study carried out in this paper has been motivated by the recent discovery of several extrasolar planetary systems very close to Earth, with companions orbiting the host star much closer than previously expected. This discovery suggests the possibility that a companion may be found orbiting at such a close distance to excite one of the g-modes of the star, and a Roche-lobe analysis indicates that this may be possible, especially if the companion is a brown dwarf. The frequency of a g-mode in a sun-like star is of the order of a few digits in 10^{-4} Hz, and would be in the bandwith of the Laser Interferometer Space Antenna (LISA). Thus, our aim was to understand how much this resonant process may enhance the emission of gravitational waves of the system with respect to the radiation it emits because of the time varying quadrupole moment associated to the orbital motion. This enhancement has been evaluated by integrating the equations that describe the perturbation the companion induces on the star, and computing the amplitude of the emitted wave when the planet moves close to an orbit corresponding to the resonant excitation of a g-mode. The amplitude increases sharply as the orbit approaches the resonant one; however, in order to get a chance to observe one of these signals emitted by a system in our vicinity, it is also needed that it stays above a certain threshold long enough. For this reason we have also computed the effects of radiation reaction on the shrinking of the orbit and the associated time scales. The main information we extract from our study is that only close brown dwarf companions may produce signals of some relevance; a brown dwarf of 40 Jovian masses, orbiting a star located at a distance of 10 pc from Earth, if close to an orbit resonant with the mode g_4, may emit radiation of amplitude greater than $\sim 2 \times 10^{-20}$ at a frequency of the order of $\sim 10^{-4}$ Hz, spanning a region $\Delta R = 1.3$ km over a time interval of ~ 4 years. Of course the amplitude of the wave increases with the mass of the brown dwarf and decreases with the distance, and it will be interesting to see whether astronomical observations will identify any such system in our vicinity in the future.

As a last remark, we would like to mention that things would not be significantly different had we considered more realistic models of solar-type stars, such as those discussed in [18,19]; they predict oscillation frequencies that are a little higher than those of our simple polytropic model; however, our frequencies are at least qualitatively correct to give order-of-magnitude estimates for the effects of resonant excitation of the stellar modes and for the excitation time scales.

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