

## Chiral fermions, orbifolds, scalars, and fat branes

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(Received 29 September 2000; published 23 February 2001)

We note that orbifold boundary conditions that produce chiral fermion zero modes in compactified higher dimensional theories may distort scalar field vacuum expectation values, giving rise to nontrivial dependence on the extra dimensions. We illustrate this in a simple five-dimensional model, which has chiral fermion zero modes stuck to fat branes. The model could provide a simple and explicit realization of the separation of quarks and leptons in the fifth dimension. We discuss the Kaluza-Klein expansion in some detail. We find that there are, in general, non-zero-mode states stuck to the brane, just as the chiral zero mode is. We see explicitly the transition from the states dominated by the internal structure of the fat brane to those dominated by the compactification.

DOI: 10.1103/PhysRevD.63.064027

PACS number(s): 04.50.+h

### I. CHIRAL FERMIONS IN FIVE DIMENSIONS

A field theory in a space of more than four space-time dimensions may be relevant to the description of the real world if the extra dimensions are compactified [1,2]. It may even be relevant if the extra dimensions are infinite and if the gravitational interactions distort the extra dimensions in an appropriate way [3]. We will ignore the gravitational interactions. Our starting point will be an effective description, approximately valid at long distances, of a theory with compactified extra dimensions. We will focus on the chiral orbifold boundary conditions that seem necessary to obtain chiral fermion zero modes from the compactified extra dimensions [4,5]. The basic point is simple and generic. If the orbifold boundary conditions force a scalar field to be odd at an orbifold fixed point, the vacuum expectation value (VEV) must vanish on the fixed point. If the potential is such that the field develops a VEV in the interior, a nontrivial shape must result for the VEV. We will describe in detail a simple model in which the orbifold boundary condition clashes with the tendency of a scalar field to develop a constant VEV. The result is a nontrivial model of a fat brane that supports chiral fermion zero modes in a larger compactified space. We will be able to analyze the Kaluza-Klein (KK) expansion for this system in quantitative detail. We will find that there are, in general, non-zero-mode states stuck to the brane, just as the chiral zero mode is. We will see explicitly the transition from these states dominated by the internal structure of the fat brane to those dominated by the compactification.

Our starting point is a simple example of a chiral orbifold boundary condition equivalent to a model discussed in Ref. [5]. Consider a free massless fermion field in five dimensions

in which the extra dimension  $x_5$  is in the interval  $[0, L]$ . The Lagrangian is<sup>1</sup>

$$\mathcal{L} = \bar{\psi}(i\partial - \gamma_5\partial_5)\psi. \quad (1.1)$$

The field  $\psi$  has four components and at the Lagrangian level the theory appears vectorlike. However, we will impose boundary conditions on the field that are periodic up to a  $Z_2$  symmetry of the Lagrangian so that the extra dimension becomes an orbifold.<sup>2</sup> In the process, we will introduce some chiral structure. The Lagrangian (1.1) is invariant under the transformation<sup>3</sup>

$$\psi(x, x_5) \rightarrow \Psi(x, x_5) \equiv \gamma_5 \psi(x, L - x_5). \quad (1.2)$$

With this  $Z_2$  symmetry in hand, we can impose modified periodic conditions on our fermion field in the following form:

$$\psi(x, x_5) = \Psi(x, L + x_5) = \psi(x, 2L + x_5). \quad (1.3)$$

<sup>1</sup>Note that we are ignoring the possibility of interactions on the boundaries at  $x_5=0$  and  $L$ . This is dangerous even in an effective-field theory treatment because interactions in the bulk may induce interactions on the boundaries. However, we believe that our conclusions are unaffected, and we will return to the general issue in a separate paper. We are grateful to Misha Voloshin for discussions on this issue.

<sup>2</sup>Cumrun Vafa has emphasized to us the differences between our construction and a string theory orbifold. He notes that because we miss the “winding” modes that are stuck to the fixed points, our procedure may be quite dangerous, possibly leading to nonunitary theories. We do not see how such disasters can occur in the effective-field theory approach we take in this paper. However, the reader should be warned that our examples may be difficult to reproduce in a more fundamental scheme such as string theory.

<sup>3</sup>Note that the masslessness of the fermion is important—a constant mass term would not be invariant under Eq. (1.2). However, there is a singular limit of the model we discuss that corresponds to a mass term that is piecewise continuous with a discontinuity on the orbifold boundary.

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That is, the field is periodic in  $x_5$  with a period  $2L$ , but if  $x_5$  is translated by an odd multiple of  $L$ , one gets not  $\psi$  but the transformed field  $\Psi$ . It is through this boundary condition that chirality enters into the theory [5]. Specifically, Eq. (1.3) implies the following behaviors near  $x_5=0$  and  $L$ :

$$\begin{aligned}\psi(x, -x_5) &= \Psi(x, L - x_5) = \gamma_5 \psi(x, x_5), \\ \psi(x, L + x_5) &= \Psi(x, x_5) = \gamma_5 \psi(x, L - x_5).\end{aligned}\quad (1.4)$$

Equation (1.4) shows that the points  $x_5=0$  and  $L$  are fixed-points of the orbifold boundary conditions. If we decompose  $\psi$  into chiral components  $\psi_{\pm}$ , where

$$\psi = \psi_+ + \psi_-, \quad \gamma_5 \psi_{\pm} = \pm \psi_{\pm}, \quad (1.5)$$

then Eq. (1.4) is equivalent to having the chiral fields defined on a circle,  $x_5 \in [0, 2L)$  with  $2L$  identified with 0, but with the chiral components  $\psi_{\pm}$  required to be, respectively, symmetric and antisymmetric at the fixed points  $x_5=0$  and  $x_5=L$ , so this is an  $S_1/Z_2$  orbifold [6].

Obviously, this simple model has a chiral fermion zero mode,

$$\psi_+(x, x_5) = \psi(x), \quad \psi_-(x, x_5) = 0, \quad (1.6)$$

independent of the extra dimension. All the non-zero-modes come in chiral pairs, as they must. In this simple case, we can find them explicitly with ease. In general, the modes of mass  $M$  have the form

$$\begin{aligned}\psi_{M+}(x, x_5) &= \psi_{M+}(x) \xi_{M+}(x_5), \\ \psi_{M-}(x, x_5) &= \psi_{M-}(x) \xi_{M-}(x_5),\end{aligned}\quad (1.7)$$

where

$$-\partial_5 \xi_{M-} = M \xi_{M+}, \quad \partial_5 \xi_{M+} = M \xi_{M-} \quad (1.8)$$

and  $\xi_{M\pm}(x_5)$  are, respectively, symmetric and antisymmetric at the points  $x_5=0$  and  $x_5=L$ . For nonzero  $M$ , we can change the sign of  $M$  by simply changing the sign of  $\xi_{M-}$ . Solving Eq. (1.8) gives

$$\xi_{M+}(x_5) = k \cos n \pi x_5 / L, \quad \xi_{M-}(x_5) = -k \sin n \pi x_5 / L, \quad (1.9)$$

where

$$M = n \pi / L \quad (1.10)$$

and  $k$  is a normalization factor.

This simple model has a chiral zero mode that is uniformly spread over the compact extra dimension. In Sec. II, we show that when we add a scalar field to the model in a simple and obvious way, we produce zero modes that are concentrated near the orbifold fixed point. The reason is that we add a potential that produces a nonzero VEV for the scalar field that breaks the symmetry between the two orbifold fixed points. Furthermore, the orbifold boundary conditions make it impossible for the VEV to be constant. The generic result is a pair of fat branes with a highly nontrivial

structure in the fifth dimension whose consequences we explore in the rest of the paper. In Sec. III, we discuss KK expansion for the scalar field and a fermion field. Using techniques borrowed from supersymmetric quantum mechanics [7], we construct many of the KK modes in detail and identify qualitative features that depend on the nontrivial structure in the extra dimension.

## II. SCALARS AND THEIR VEVs

The elaborated model lives in the same five dimensional space as the previous model, and involves a single additional real scalar field  $\phi$ . The Lagrangian is

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(i \not{\partial} - \gamma_5 \not{\partial}_5 - f \phi) \psi + \frac{1}{2} \partial^\mu \phi \partial_\mu \phi \\ &\quad - \frac{1}{2} \partial_5 \phi \partial_5 \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2,\end{aligned}\quad (2.1)$$

where the couplings  $f$  and  $\lambda$  are real. The Lagrangian (2.1) is invariant under the transformation

$$\begin{aligned}\phi(x, x_5) &\rightarrow \Phi(x, x_5) \equiv -\phi(x, L - x_5), \\ \psi(x, x_5) &\rightarrow \Psi(x, x_5) \equiv \gamma_5 \psi(x, L - x_5).\end{aligned}\quad (2.2)$$

Now, as before, we can require modified periodic boundary conditions

$$\begin{aligned}\psi(x, -x_5) &= \Psi(x, L - x_5) = \gamma_5 \psi(x, x_5), \\ \psi(x, L + x_5) &= \Psi(x, x_5) = \gamma_5 \psi(x, L - x_5),\end{aligned}\quad (2.3)$$

$$\begin{aligned}\phi(x, -x_5) &= \Phi(x, L - x_5) = -\phi(x, x_5), \\ \phi(x, L + x_5) &= \Phi(x, x_5) = -\phi(x, L - x_5).\end{aligned}\quad (2.4)$$

The boundary conditions (2.4) require that the scalar field vanish on the orbifold fixed points at  $x_5=0$  and  $L$ . However, if  $v^2 > 0$  in Eq. (2.1), the scalar field wants to develop a vacuum expectation value. The result is that if  $\lambda v^2$  is sufficiently large, there is a minimum energy configuration in which

$$\langle \phi(x, x_5) \rangle = h(x_5), \quad (2.5)$$

where the real function  $h(x_5)$  satisfies

$$h(0) = h(L) = 0, \quad h(x_5) = h(L - x_5) > 0 \quad \text{for } 0 < x_5 < L. \quad (2.6)$$

There is another solution with

$$\langle \phi(x, x_5) \rangle = -h(x_5), \quad (2.7)$$

related to Eq. (2.5) by the symmetry transformation (2.2).

Now the fermion modes are given by Eq. (1.7) where the  $\xi$ 's satisfy

$$\begin{aligned}\{-\partial_5 + fh(x_5)\} \xi_{M-} &= M \xi_{M+}, \\ \{\partial_5 + fh(x_5)\} \xi_{M+} &= M \xi_{M-},\end{aligned}\quad (2.8)$$

with  $\xi_{M\pm}(x_5)$ , respectively, symmetric and antisymmetric at the points  $x_5=0$  and  $x_5=L$ .

The non-zero-modes of Eq. (2.8) come in pairs as always. For every mode with  $M=\mu\neq 0$ , we can always find a mode with  $M=-\mu$  by changing the sign of  $\xi_{M-}$ . However, there is a chiral zero mode that must have  $\xi_{0-}=0$  because the boundary conditions and the differential equations for  $M=0$  cannot be satisfied simultaneously for nonzero  $\xi_{0-}$ . The zero mode looks like

$$\xi_{0+}(x_5)=ke^{-s(x_5)}, \quad \xi_{0-}(x_5)=0, \quad (2.9)$$

for

$$s(x_5)=f\int_0^{x_5} dy h(y). \quad (2.10)$$

Note that the boundary conditions (2.3) are automatically satisfied because  $h(x_5)$  vanishes on the fixed points at  $x_5=0$  and  $x_5=L$ . If instead, we tried to find a nonzero solution for  $\xi_{0-}$ , we would have to choose the normalization constant  $k$  equal to zero to satisfy the boundary conditions, so no nontrivial solution is possible.

If  $fh(x_5)$  is positive, the zero mode in Eq. (2.9) is concentrated at  $x_5=0$ . If it is negative, the zero mode is concentrated at  $x_5=L$ . If there are several fermions with couplings of different signs, those with positive couplings will be concentrated at  $x_5=0$  while those with negative couplings will be concentrated at  $x_5=L$ . Thus this could give a very simple explicit realization of the idea of [8,9] that if quarks and leptons are localized on different branes, the proton can be stabilized.

In the free-fermion example of Sec. I, the limit  $L\rightarrow\infty$  is singular because the zero mode is not normalizable in the limit. However, in the model of Eqs. (2.1) with  $fh(x_5)>0$  and with the normalizable zero mode stuck to the orbifold fixed point at the origin, we can take  $L\rightarrow\infty$  without doing violence to the physics. In fact, the theory simplifies in this limit. This simple  $L=\infty$  limit is not particularly interesting phenomenologically. If we were to couple gauge fields to the fermions, as we must certainly do to get a realistic model, taking  $L\rightarrow\infty$  would send the effective gauge coupling to the fermion zero modes to zero (because the gauge fields would be spread over the whole space and the four-dimensional gauge coupling would go to zero like  $1/\sqrt{L}$ ). However, the  $L=\infty$  theory is a very interesting toy model because we can do the KK expansion explicitly. Thus we will discuss the  $L=\infty$  theory to help us understand the more interesting case of finite but large  $L$ .

In the  $L=\infty$  theory, the orbifold is a halfline, which is the real line modded out by  $Z_2$ :

$$\phi(x, x_5)\rightarrow-\phi(x, -x_5), \quad \psi(x, x_5)\rightarrow\gamma_5\psi(x, -x_5). \quad (2.11)$$

We will see in the next section that the KK modes in the  $L=\infty$  model can be found analytically [10].

### III. FAT BRANES

In this section, we examine the model of Sec. II in more quantitative detail. Because the shape of the fermion zero mode in Eq. (2.9) defines a particular ‘‘fat brane,’’ it may be interesting to identify effects that depend on the finite extent of the zero mode. One such effect arises when we integrate out the scalars. We will get four-fermion operators with calculable coefficients. These and other such effects depend on the structure of the KK modes. Here we discuss the KK expansion. We will see that we can find the form of the modes explicitly in the limit  $L\rightarrow\infty$ . That in turn will allow us to write an excellent approximation to many of the modes for large finite  $L$ .

We are interested in the Lagrangian as a function of the shifted field,

$$\tilde{\phi}(x, x_5)\equiv\phi(x, x_5)-h(x_5). \quad (3.1)$$

The function  $h(x_5)$  is the value of  $\varphi(x_5)$  that minimizes

$$\int_0^L dx_5 \left[ \frac{1}{2} \partial_5 \varphi(x_5) \partial_5 \varphi(x_5) + \frac{\lambda}{4} \{ \varphi(x_5)^2 - v^2 \}^2 \right] \quad (3.2)$$

subject to the boundary condition

$$\varphi(0)=\varphi(L)=0. \quad (3.3)$$

The function satisfies

$$\partial_5^2 h(x_5) = -\lambda \{ v^2 - h(x_5)^2 \} h(x_5). \quad (3.4)$$

Evidently, there is a tension between the boundary conditions that force the field to vanish on the orbifold fixed points and the potential, which wants to produce a VEV in the interior.

Before we discuss the form of  $h(x_5)$ , let us consider the constraints on the parameters  $L$ ,  $f$ ,  $\lambda$ , and  $v$  in the effective-field theory. It is important to note that the various dimensional parameters in the effective low-energy theory are not *a priori* related. All come down to us from some more fundamental theory at shorter distances, and each of the effective-theory parameters must satisfy a constraint in order that the effective theory makes sense. But they need not be related to each other. This will be important to us because we will find a region in the parameter space in which the calculation is particularly simple and transparent. The constraints from effective-field theory are simply that the dimensional parameters are small (or large) compared to the fundamental scale to the appropriate power. Thus if the fundamental scale is  $M_P$ , the length  $L$  is much greater than  $1/M_P$ , the Yukawa coupling  $f$  is much smaller than  $1/\sqrt{M_P}$ , the self-coupling  $\lambda$  is much smaller than  $1/M_P$ , and the mass  $\lambda v^2$  is much smaller than  $M_P^2$ . This is summarized in Eq. (3.5):

$$L \gg \frac{1}{M_P}, \quad f \ll \frac{1}{\sqrt{M_P}}, \quad \lambda \ll \frac{1}{M_P}, \quad \lambda v^2 \ll M_P^2. \quad (3.5)$$

But, for example, the dimensionless quantity  $\lambda v^2 L^2$  is not constrained.

For generic values of the parameters  $\lambda$ ,  $v$ ,  $L$ , etc., it is difficult to study the model analytically. However, in the limit

$$L^2 \gg \frac{1}{\lambda v^2}, \quad (3.6)$$

it is relatively easy to construct approximate solutions. In this case, the solution for  $h(x_5)$  can be approximated by a series of well separated kinks.

First consider  $L = \infty$ . Then a solution to Eq. (3.4) is a single kink given by

$$h(x_5) = v \tanh\left(\frac{\lambda}{2}\right)^{1/2} v x_5. \quad (3.7)$$

It is convenient to choose units in which

$$2\lambda v^2 = 1 \quad (3.8)$$

because  $2\lambda v^2$  is the mass parameter that determines the physical size of the kink. In these units, Eq. (3.6) becomes simply  $L \gg 1$  and Eq. (3.7) is

$$h(x_5) = v \tanh \frac{x_5}{2}. \quad (3.9)$$

For large finite  $L$ , we can construct approximate solutions by putting together kinks at  $x_5 = 0$  and  $x_5 = L$ . In the interval  $[0, L]$ , such a solution can be accurately approximated by

$$h(x_5) \approx v \tanh \frac{x_5}{2} \tanh \frac{L - x_5}{2} + \mathcal{O}(e^{-L}). \quad (3.10)$$

The VEV,  $h(x_5)$ , is odd about each of the orbifold fixed points and can be continued to all values of  $x_5$  subject to the orbifold boundary conditions.

Fluctuations of  $\phi(x)$  about  $\langle \phi(x) \rangle = h(x_5)$  can be studied using a KK expansion. We write

$$\phi(x) = h(x_5) + \tilde{\phi}(x) = h(x_5) + \sum_n \phi_n(x^\mu) f_n(x_5), \quad (3.11)$$

where  $\phi_n(x^\mu)$  depends only on the four coordinates of the noncompact space. We normalize the  $f_n$  to unity.

$$\int dx_5 f_n^2(x_5) = 1. \quad (3.12)$$

Substituting Eq. (3.11) into the action for the scalars from Eq. (2.1) and expanding to quadratic order in  $\phi_n$ , we find

$$S = \int d^4x \sum_n \left\{ \frac{1}{2} (\partial \phi_n)^2 - \frac{1}{2} \left[ \int dx_5 f_n \{ -f_n'' + m^2(x_5) f_n \} \phi_n^2 \right] \right\}, \quad (3.13)$$

where

$$m^2(x_5) = \left. \frac{\partial^2 V(\phi)}{\partial \phi^2} \right|_{\phi=h(x_5)}. \quad (3.14)$$

For infinite  $L$

$$m^2(x_5) = \left\{ 1 - \frac{3}{2} \operatorname{sech}^2 \frac{x_5}{2} \right\} \quad (3.15)$$

while for large finite  $L$

$$m^2(x_5) \approx \left\{ 1 - \frac{3}{2} \operatorname{sech}^2 \frac{x_5}{2} - \frac{3}{2} \operatorname{sech}^2 \frac{L - x_5}{2} \right\}. \quad (3.16)$$

If the KK modes  $f_n$  are chosen to satisfy the equivalent Schrödinger eigenvalue problem

$$-f_n'' + m^2(x_5) f_n = m_n^2 f_n, \quad (3.17)$$

then Eq. (3.13) reduces to the action for an infinite number of four-dimensional scalar particles of mass  $m_1, m_2$ , etc.

We can understand the KK spectrum by considering the case of an infinite extra dimension. The Schrödinger equation in this limit becomes

$$-f_n'' + \left( 1 - \frac{3}{2} \operatorname{sech}^2 \frac{x_5}{2} \right) f_n = m_n^2 f_n. \quad (3.18)$$

On the finite line, this Schrödinger equation possesses two bound states, with  $m^2 = 0$  and  $m^2 = \frac{3}{4}$ , with wave functions

$$f_0(x_5) \propto \operatorname{sech}^2 \frac{x_5}{2}, \quad f_1(x_5) \propto \sinh \frac{x_5}{2} \operatorname{sech}^2 \frac{x_5}{2}. \quad (3.19)$$

As expected, the ground state is even about  $x_5 = 0$  and the excited state is odd. What is going on here is that the zero mode is associated with the translational symmetry of the infinite case. There is a zero mode because in the theory on the infinite line, the kink is free to sit anywhere. In our theory, however, translation invariance in the  $x_5$  direction is broken by the boundary conditions. This kink is stuck to the orbifold and there is no scalar zero mode. This is consistent because the solution  $f_0$  is ruled out by our boundary condition that  $\phi$  be odd in  $x_5$  at the origin. Thus  $f_1$  is the only discrete KK mode for  $L = \infty$ . Evidently, since it exists for  $L = \infty$ , it is associated with the fat brane rather than the compactification. The remaining solutions are continuum states with  $m^2 \geq 1$  (again in units with  $2\lambda v^2 = 1$ ). The continuum state with  $m^2 = k^2 + 1$  has the form

$$f_{2k}(x_5) \propto \left( \frac{1}{2} - k^2 - \frac{3}{4} \operatorname{sech}^2 \frac{x_5}{2} \right) \sin kx_5 - \frac{3k}{2} \tanh \frac{x_5}{2} \cos kx_5. \quad (3.20)$$

All these results are derived in detail in Appendix A.

Returning to the case of a finite extra dimension, we expect the solutions to look like solutions to the infinite problem near the orbifold fixed points. Because the normalized solution  $f_1(x_5)$  goes to zero as  $x_5 \rightarrow \infty$ , we get approximate solutions  $m^2 \approx 3/4 (= 3\lambda v^2/2)$  for large  $L$  by taking linear combinations of copies of this mode centered at  $x_5 = 0$  and

$x_5=L$ . There will be two such states, corresponding to the ‘‘plus’’ and ‘‘minus’’ linear combinations of the wave functions centered at  $x_5=0$  and  $x_5=L$ ,

$$f_1(x_5) \pm f_1(L-x_5). \quad (3.21)$$

In addition, we expect the usual KK ‘‘continuum’’ states, with masses above  $2\lambda v^2$  spaced by  $\Delta m \simeq \pi/L$ . Approximate solutions for these can be obtained from Eq. (3.20) by taking

$$f_{2k}(x_5) \text{ for } x_5 < L/2 \quad (3.22)$$

and either

$$f_{2k}(L-x_5) \text{ for } x_5 > L/2 \text{ for } k \text{ such that } f'_{2k}(L/2)=0, \quad (3.23)$$

or

$$-f_{2k}(L-x_5) \text{ for } x_5 > L/2 \text{ for } k \text{ such that } f_{2k}(L/2)=0. \quad (3.24)$$

The matching conditions at  $x_5=L/2$  then approximately determine the allowed  $k$ 's.

For the fermions, the modes of mass  $M$  must satisfy

$$a\xi_{M+} = M\xi_{M-}, \quad a^\dagger\xi_{M-} = M\xi_{M+}, \quad (3.25)$$

where

$$a = \partial_5 + fh(x_5), \quad a^\dagger = -\partial_5 + fh(x_5). \quad (3.26)$$

As with the scalars, we can make exact statements about these modes in the case  $L=\infty$ , and reliable approximate statements for large but finite  $L$ . We will simply state results here. Some details are in appendix B and more will appear in [11].

For  $L=\infty$ , Eqs. (3.25) and (3.26) become

$$a_w\xi_{M+} = M\xi_{M-}, \quad a_w^\dagger\xi_{M-} = M\xi_{M+}, \quad (3.27)$$

where

$$a_w = \partial_5 + w \tanh \frac{x_5}{2}, \quad a_w^\dagger = -\partial_5 + w \tanh \frac{x_5}{2} \quad (3.28)$$

with

$$w = fv > 0. \quad (3.29)$$

The condition (3.29) is necessary to ensure that the normalizable fermion zero mode is stuck to the orbifold at  $x_5=0$  so that  $L$  can be taken to infinity.<sup>4</sup>

There is always a normalizable chiral zero mode stuck to the brane given by Eqs. (2.9) and (2.10), which in this case becomes simply

$$\xi_{M+}(x_5) \propto g_{0,w}(x_5) \equiv \text{sech}^{2w} \frac{x_5}{2}, \quad \xi_{M-}(x_5) = 0. \quad (3.30)$$

However, we find that for  $j-1 < w \leq j$  for positive integer  $j$ , there are  $j-1$  massive modes stuck to the brane. Like the zero mode, these states are associated with the fat brane itself and not the compactification. They have the form

$$\xi_{M+}(x_5) \propto g_{l,w}(x_5), \quad \xi_{M-}(x_5) \propto a_w \xi_{M+}(x_5), \quad (3.31)$$

with

$$M^2 = 2wl - l^2. \quad (3.32)$$

The function  $g_{l,w}$  is obtained by acting with  $2l$   $a^\dagger$ 's with decreasing  $w$  values on  $g_{0,w-l}$ ,

$$g_{l,w} = \overbrace{a_w^\dagger a_{w-1/2}^\dagger \cdots a_{w-l+1}^\dagger a_{w-l+1/2}^\dagger}^{l \text{ pairs}} g_{0,w-l}(x_5) \quad (3.33)$$

for  $l=1$  to  $j-1$ .

There are a few things worth noticing about these solutions.

(1) All the functions  $g_{l,w}(x_5)$  are even for  $x_5 \rightarrow -x_5$ , so the boundary condition at  $x_5=0$  is satisfied.

(2)  $g_{l,w}(x_5)$  goes to zero like  $e^{-(w-l)x_5}$  as  $x_5 \rightarrow \infty$ . The function inherits this behavior from  $g_{0,w-l}$ . The  $a^\dagger$ 's acting on it do not affect the leading exponential behavior. This is one reason why we cannot go beyond  $l=j-1$  in Eq. (3.33)—the resulting functions would grow at infinity and would not be normalizable.

(3)  $\xi_{M-}(x_5)$  is also proportional to the product in Eq. (3.33) with the initial  $a_w^\dagger$  removed because this state is an eigenstate of  $a_w a_w^\dagger$ .

For  $M^2 > w^2$ , we find continuum solutions. These cannot be written in elementary closed form except for integer or half-integer  $w$ . But they can be found in terms of hypergeometric functions [10,11].

Returning to the finite case, one might worry that because of the asymmetry between  $x_5=0$  where the fermions are bound and  $x_5=L$  where they are repelled, it might be difficult to find modes corresponding to the massive normalized states in Eq. (3.33) that satisfy the boundary conditions at large  $L$ . The boundary condition is automatic for the zero mode but not for the massive modes. Fortunately, there is a simple way to construct approximate eigenfunctions for large  $L$ .

One way to find the normalizable chiral zero mode for finite  $L$  is to use Eqs. (2.9) and (2.10) with our approximate form for  $h(x_5)$  [Eq. (3.10)]. But another way is to think of dividing the orbifold into two regions as we did for the scalar modes,  $x_5 < L/2$  dominated by the fixed point at  $x_5=0$ , and  $x_5 > L/2$  dominated by the fixed point at  $x_5=L$ . Near  $x_5=L$ , the solution looks like a non-normalizable zero mode, just the inverse of  $g_{0,w}$ :

<sup>4</sup>Note also that (as Misha Voloshin pointed out to us) if  $f$  is large, radiative corrections may be important in the calculation of the VEV of  $\phi$ . We ignore this issue in this paper.

$$\xi_{M^+}(x_5) \propto \tilde{g}_{0,w}(x_5) \equiv \cosh^{2w} \frac{L-x_5}{2}, \quad \xi_{M^-}(x_5) = 0. \quad (3.34)$$

Now as  $x_5$  decreases from  $x_5=L$  toward  $x_5=L/2$ ,  $\tilde{g}_{0,w}$  increases exponentially, and at  $x_5=L/2$  it can be matched to an excellent approximation onto  $g_{0,w}(x_5)$ , which is exponentially falling at the same rate.

For the  $j-1$  massive modes of Eqs. (3.31)–(3.33), a similar strategy can be applied. There are non-normalizable solutions that are analogous to the normalizable modes at the other end of the orbifold:

$$\xi_{M^+}(x_5) \propto \tilde{g}_{l,w}(x_5), \quad \xi_{M^-}(x_5) \propto a_w \xi_{M^+}(x_5), \quad (3.35)$$

$$\tilde{g}_{l,w} = \overbrace{a_w^\dagger a_{w-1/2}^\dagger \cdots a_{w-l+1}^\dagger a_{w-l+1/2}^\dagger}^{l \text{ pairs}} \tilde{g}_{0,w-l}(x_5) \quad (3.36)$$

for  $l=1$  to  $j-1$ . These satisfy the boundary condition at  $x_5=L$  and match smoothly onto Eqs. (3.31)–(3.33) at  $x_5=L/2$ . Note that the  $\tilde{g}$ 's never vanish. Thus all the nodes in these wave functions are near the fixed point at  $x_5=0$ , as expected.

We do not know a similarly simple approximation to match the continuum modes from the two sides of the orbifold.

#### IV. ZERO MODES NEAR THE FIXED POINT

The fermion zero modes described in Sec. II are all concentrated on one of the orbifold fixed points, at  $x_5=0$  or  $L$ . In this section, we note that by elaborating the model slightly, we can produce zero modes that are maximized near but not on the fixed points. Consider the following Lagrangian:<sup>5</sup>

$$\mathcal{L} = \bar{\psi} [i \not{\partial} - \gamma_5 \partial_5 - f \{1 - a(\partial_\mu \partial^\mu - \partial_5^2)\} \phi] \psi + \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} \partial_5 \phi \partial_5 \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2. \quad (4.1)$$

We have added to Eq. (2.1) only the single term proportional to the new parameter  $a$ . Now, however, the zero mode is given by Eqs. (2.9) and (2.10) with a different  $h(x_5)$  that includes the effect of the new term. For large  $L$ , we can write

$$h(x_5) \approx f v \left( \tanh \frac{x_5}{2} + a \partial_5^2 \tanh \frac{x_5}{2} \right) \left( \tanh \frac{L-x_5}{2} + a \partial_5^2 \tanh \frac{L-x_5}{2} \right) \quad (4.2)$$

<sup>5</sup>This Lagrangian was suggested to us by Martin Schmaltz to replace a more complicated scheme that we used to get the same final result.

for constants  $f$  and  $a$ . This is interesting because for  $a > 2$ ,  $h(x_5)$  changes sign at

$$\tilde{x}_5 = 2 \operatorname{arctanh} \sqrt{1 - 2/a} \quad (4.3)$$

and at  $x_5=L-\tilde{x}_5$ . For  $f > 0$ , this describes a fermion concentrated at  $\tilde{x}_5$  or  $L-\tilde{x}_5$  depending on the sign of  $f v$ .

Note that for reasonable values of  $a$ ,  $\tilde{x}_5$  is of order one. Thus the zero mode does not stray very far from the orbifold fixed point. What we would expect in a model with finite  $L$  and several fermions is that the fermions would fall into four sets:

for  $f > 0$  and  $a < 2$  the zero mode is concentrated at  $x_5=0$ ,

for  $f > 0$  and  $a > 2$  the zero mode is concentrated at  $x_5=\tilde{x}_5$  near  $x_5=0$ ,

for  $f < 0$  and  $a < 2$  the zero mode is concentrated at  $x_5=L$ ,

for  $f < 0$  and  $a > 2$  the zero mode is concentrated at  $x_5=L-\tilde{x}_5$  near  $x_5=L$ . (4.4)

One annoying thing about this is that the higher derivative coupling we have added has a higher dimension than the ordinary Yukawa coupling, and therefore we might expect the parameter  $a$  to be small—of order  $\lambda v^2/M_p^2$ . We can consistently take  $a$  of order one in our units only if  $f$  is small. This could be a problem in model building.

#### V. CONCLUDING QUESTIONS

We have shown in a very explicit example how scalar VEVs and orbifold boundary conditions combine to produce nontrivial structure in the extra dimensions. This behavior is generic, and we expect behavior of this kind to appear in other explicit models of fat branes. The most interesting general result is that the KK expansion may produce two kinds of massive modes—those truly associated with the compactification and those stuck to the fat brane. Let us close with a couple of very different questions.

(i) Do the KK states that are normalizable in the  $L=\infty$  limit play a special role or are they simply the lightest of the KK excitations? It seems likely to us that the answer is the former. These states are stuck to the fat brane and are thus very different from the KK states associated with compactification. Particularly intriguing is the nearly degenerate pair of scalars in Eq. (3.21). These may be an important source of communication between fermions localized on the two different branes.

(ii) Does this kind of construction (which in some ways resembles the Kaplan idea [12]) help in any way with the difficulties of putting chiral fermions on the lattice? We do not think so. It seems that this structure makes it impossible

to decouple the doublers associated with the zero mode, but the question is interesting and may be worth pursuing further.

### ACKNOWLEDGMENTS

We would like to thank Nima Arkani-Hamed, Bogdan Dobrescu, Lisa Randall, Martin Schmaltz, John Terning, Misha Voloshin, and Koichi Yamawaki for helpful comments. H.G. is grateful to the Aspen Center for Physics for providing a splendid work environment while this paper was being revised. Our research was supported in part by the National Science Foundation under Grant No. NSF-PHY/98-02709.

### APPENDIX A: SUSY QUANTUM MECHANICS

Here we give more details about the solution for the scalar modes in the  $L=\infty$  model. We use techniques from supersymmetric (SUSY) quantum mechanics [7]. In solving for the scalar KK modes in the kink background, one needs the eigenfunctions of the Schrödinger equation (3.18). In the case of an infinite extra dimension, we can solve this equation exactly using supersymmetric quantum mechanics. First we define two sets of “raising” and “lowering” operators

$$a_i^\dagger = -\partial_5 + g_i(x_5), \quad a_i = \partial_5 + g_i(x_5), \quad (\text{A1})$$

where

$$g_1 = \frac{1}{2} \tanh \frac{x_5}{2}, \quad g_2 = \tanh \frac{x_5}{2}. \quad (\text{A2})$$

Because

$$\partial_5 \tanh \frac{x_5}{2} = \frac{1}{2} \left( 1 - \tanh^2 \frac{x_5}{2} \right), \quad (\text{A3})$$

it is straightforward to verify that

$$\begin{aligned} a_1 a_1^\dagger &= -\partial_5^2 + \frac{1}{4} \left( 1 - \tanh^2 \frac{x_5}{2} \right) + \frac{1}{4} \tanh^2 \frac{x_5}{2} \\ &= -\partial_5^2 + \frac{1}{4}, \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} a_1^\dagger a_1 &= -\partial_5^2 - \frac{1}{4} \left( 1 - \tanh^2 \frac{x_5}{2} \right) + \frac{1}{4} \tanh^2 \frac{x_5}{2} \\ &= -\partial_5^2 - \frac{1}{4} + \frac{1}{2} \tanh^2 \frac{x_5}{2}, \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} a_2 a_2^\dagger &= -\partial_5^2 + \frac{1}{2} \left( 1 - \tanh^2 \frac{x_5}{2} \right) + \tanh^2 \frac{x_5}{2} \\ &= -\partial_5^2 + \frac{1}{2} + \frac{1}{2} \tanh^2 \frac{x_5}{2}, \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} a_2^\dagger a_2 &= -\partial_5^2 - \frac{1}{2} \left( 1 - \tanh^2 \frac{x_5}{2} \right) + \tanh^2 \frac{x_5}{2} \\ &= -\partial_5^2 - \frac{1}{2} + \frac{3}{2} \tanh^2 \frac{x_5}{2}, \end{aligned} \quad (\text{A7})$$

Thus

$$a_1^\dagger a_1 = a_2 a_2^\dagger - \frac{3}{4} = -\partial_5^2 + \left( \frac{1}{4} - \frac{1}{2} \operatorname{sech}^2 \frac{x_5}{2} \right), \quad (\text{A8})$$

while

$$a_2^\dagger a_2 = -\partial_5^2 + \left( 1 - \frac{3}{2} \operatorname{sech}^2 \frac{x_5}{2} \right). \quad (\text{A9})$$

We see that  $a_2^\dagger a_2$  is the Hamiltonian we wish to diagonalize. To construct solutions to this Hamiltonian, we first observe that it is trivial to find the eigenfunctions of Eq. (A4): these are just plane waves. Furthermore, given a plane wave with wave number  $k$  obeying

$$a_1 a_1^\dagger \chi_{2k} = (k^2 + 1/4) \chi_{2k}, \quad (\text{A10})$$

we can construct an eigenfunction of Eq. (A8) by applying  $a_1^\dagger$  to both sides:

$$a_1^\dagger a_1 (a_1^\dagger \chi_{2k}) = (k^2 + 1/4) (a_1^\dagger \chi_{2k}). \quad (\text{A11})$$

So we conclude that  $a_1^\dagger a_1$  has all of the plane wave eigenstates of  $a_1 a_1^\dagger$  plus an additional zero-energy bound state, which is obtained by solving  $a_1 \chi_1 = 0$ :

$$a_1 \chi_1 = 0 \rightarrow \chi_1 \propto \operatorname{sech} \frac{x_5}{2}. \quad (\text{A12})$$

Furthermore, since Eq. (A8) can also be expressed in terms of  $a_2 a_2^\dagger$ , we can use the solutions of Eq. (A8) to find solutions of the final Hamiltonian (A9). Indeed, for each eigenvalue  $\kappa$  of  $a_1^\dagger a_1$ , we have

$$a_2 a_2^\dagger f = (a_1^\dagger a_1 + 3/4) f = (\kappa^2 + 3/4) f. \quad (\text{A13})$$

Thus

$$a_2 a_2^\dagger \chi_1 = \frac{3}{4} \chi_1, \quad a_2 a_2^\dagger a_1^\dagger \chi_{2k} = (k^2 + 1) a_1^\dagger \chi_{2k}. \quad (\text{A14})$$

Multiplying both sides by  $a_2^\dagger$  yields eigenfunctions of our original Hamiltonian. So the spectrum of  $a_2^\dagger a_2$  consists of all eigenvalues of  $a_1^\dagger a_1$  (shifted by  $\frac{3}{4}$ ) plus a zero-energy bound state obtained from  $a_2 f_0 = 0$ :

$$a_2 f_0 = 0 \rightarrow f_0 \propto \operatorname{sech}^2 \frac{x_5}{2}. \quad (\text{A15})$$

However, this zero mode is even and therefore, does not satisfy the boundary conditions. Thus, the allowed eigenstates are  $f_1$  and the odd plane waves

$$f_1 \propto a_2^\dagger \chi_1 \propto a_2^\dagger \operatorname{sech} \frac{x_5}{2} = \frac{3}{2} \tanh \frac{x_5}{2} \operatorname{sech} \frac{x_5}{2}, \quad \text{with } m^2 = \frac{3}{4}, \quad (\text{A16})$$

and

$$f_{2k} \propto a_2^\dagger a_1^\dagger \chi_{2k} \propto \left( \frac{1}{2} - k^2 - \frac{3}{4} \operatorname{sech}^2 \frac{x_5}{2} \right) \sin kx_5 - \frac{3k}{2} \tanh \frac{x_5}{2} \cos kx_5, \quad (\text{A17})$$

with  $m^2 = k^2 + 1$ .

### APPENDIX B: FERMIONS ON FAT BRANES

Here we will sketch the proof of Eqs. (3.31)–(3.33). From Eqs. (3.27) and (3.28) it follows that  $\xi^-$  is an eigenfunction of

$$a_w a_w^\dagger = -\partial_5^2 + \frac{w}{2} \left( 1 - \tanh^2 \frac{x_5}{2} \right) + w^2 \tanh^2 \frac{x_5}{2} \quad (\text{B1})$$

and  $\xi_{M+}$  is an eigenfunction of

$$a_w^\dagger a_w = -\partial_5^2 - \frac{w}{2} \left( 1 - \tanh^2 \frac{x_5}{2} \right) + w^2 \tanh^2 \frac{x_5}{2}. \quad (\text{B2})$$

We will use a Dirac notation for the eigenfunctions, denoting an eigenstate of Eq. (B2) with eigenvalue  $E$  by  $|E, w\rangle$ .

Now it is obvious that

$$a_{w-\alpha}^\dagger a_{w-\alpha} = -\partial_5^2 - \frac{w-\alpha}{2} \left( 1 - \tanh^2 \frac{x_5}{2} \right) + (w-\alpha)^2 \tanh^2 \frac{x_5}{2} \quad (\text{B3})$$

and thus

$$a_w a_w^\dagger - a_{w-\alpha}^\dagger a_{w-\alpha} = + \frac{2w-\alpha}{2} \left( 1 - \tanh^2 \frac{x_5}{2} \right) + (2w\alpha - \alpha^2) \tanh^2 \frac{x_5}{2}. \quad (\text{B4})$$

For  $\alpha = \frac{1}{2}$ , the terms proportional to  $\tanh^2$  cancel in Eq. (B4) and we have

$$a_w a_w^\dagger = a_{w-1/2}^\dagger a_{w-1/2} + w - \frac{1}{4}. \quad (\text{B5})$$

Thus if  $|E - w + \frac{1}{4}, w - \frac{1}{2}\rangle$  is an eigenstate of  $a_{w-1/2}^\dagger a_{w-1/2}$  with eigenvalue  $E - u + \frac{1}{4}$ , then  $a_w^\dagger |E - w + \frac{1}{4}, w - \frac{1}{2}\rangle$  is an eigenstate of  $a_w^\dagger a_w$  with eigenvalue  $E$ . That is, so long as the eigenstate  $|E - w + \frac{1}{4}, w - \frac{1}{2}\rangle$  exists, we can write

$$|E, w\rangle \propto a_w^\dagger |E - w + \frac{1}{4}, w - \frac{1}{2}\rangle. \quad (\text{B6})$$

Applying the same argument again shows that if the eigenstate  $|E - 2w + 1, w - 1\rangle$  exists, we can write

$$|E - w + \frac{1}{4}, w - \frac{1}{2}\rangle \propto a_{w-1/2}^\dagger |E - 2w + 1, w - 1\rangle, \quad (\text{B7})$$

and therefore,

$$|E, w\rangle \propto a_w^\dagger |E - w + \frac{1}{4}, w - \frac{1}{2}\rangle \propto a_w^\dagger a_{w-1/2}^\dagger |E - 2w + 1, w - 1\rangle. \quad (\text{B8})$$

If  $|E, w\rangle$  is to be an eigenstate, this process must terminate in a chiral zero mode. Conversely, we get all the normalizable modes by acting on the zero modes by pairs of  $a$ 's as in Eq. (B8). This is the basis of Eq. (3.33).

### APPENDIX C: ANOTHER SIMPLE LIMIT

If  $L$  is not large, the approximations discussed in Sec. III fail badly. In general, we then have to resort to numerical techniques. Here we discuss one way of approaching the problem and identify another simple limit. In order to incorporate the effects of the boundary conditions, we could expand the  $\tilde{\phi}$  field in a set of basis functions in  $x_5$ . In this case the obvious ones are

$$\xi_n(x_5) \equiv \left( \frac{2}{L} \right)^{1/2} \sin \frac{n\pi x_5}{L}. \quad (\text{C1})$$

This will allow us to formulate the problem in general, and also make it easy to solve it exactly in a particular limit.

Now we can formulate the problem in general by expanding  $h(x_5)$  and  $\tilde{\phi}(x, x_5)$  in terms of the basis functions (C1). We can truncate this expansion for some large  $n$  and solve the finite problem. The coefficients in the expansion of  $h(x_5)$  can be determined by minimizing Eq. (3.2). However, a close look suggests that there is a limit of the theory in which the calculation is much simpler. One can immediately see that the expectation value  $h(x_5)$  goes to zero as

$$\lambda v^2 L^2 \rightarrow \pi^2 \quad (\text{C2})$$

from above. For smaller values of  $\lambda v^2 L^2$ , there is no vacuum expectation value. This suggests taking

$$\lambda v^2 L^2 = \pi^2 + \epsilon \quad (\text{C3})$$

for small  $\epsilon$ . When we do that, we find that we can calculate the coefficients in  $h(x_5)$  as a power series in  $\epsilon$ . The first terms are

$$h(x_5) = \left( \frac{2}{3\lambda L} \right)^{1/2} \epsilon^{1/2} \xi_1(x_5) + \frac{1}{24\pi^2} \left( \frac{2}{3\lambda L} \right)^{1/2} \epsilon^{3/2} \xi_3(x_5) + O(\epsilon^{5/2}). \quad (\text{C4})$$

Furthermore, doing the KK expansion, we find that the mass squared of the  $\tilde{\phi}_1$  mode is  $2\epsilon/L^2$ , while all the other mass squares scale with  $\pi^2/L^2$ , not suppressed by  $\epsilon$ . To leading order, the mass squared of the mode  $\tilde{\phi}_n$  for  $n > 1$  is  $(n^2 - 1)\pi^2/L^2$ .



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