

## Casimir effect of the graviton and the entropy bound

Feng-Li Lin\*

School of Physics & Center for Theoretical Physics, Seoul National University, Seoul 151-742, Korea

(Received 19 October 2000; published 22 February 2001)

In this paper we calculate the Casimir effect of free thermal gravitons in Einstein's universe and discuss how it changes the entropy bound condition proposed recently by Verlinde as a higher dimensional generalization of Cardy's formula for conformal field theories (CFT). We find that the graviton's Casimir effect is necessary in order not to violate Verlinde's bound for weakly coupled CFT. We also comment on the implication of this new Cardy formula to the thermodynamics of a black  $p$ -brane.

DOI: 10.1103/PhysRevD.63.064026

PACS number(s): 04.62.+v

### I. INTRODUCTION

Verlinde in his recent paper [1] proposed that the entropy of a  $D$ -dimensional conformal field theory (CFT) is given by the generalized Cardy formula

$$S = \frac{2\pi R}{D-1} \sqrt{E_c(2E - E_c)}, \quad (1)$$

where  $R$  is linear size of the system and

$$E_c \equiv DE - (D-1)TS \quad (2)$$

is the Casimir energy that corresponds to the subleading term in the high  $T$  (temperature) expansion of the total energy  $E$ .

For a given total energy  $E$  this formula automatically leads to Bekenstein's entropy bound [2] of the macroscopic system with limited gravity,

$$S \leq S_B \equiv \frac{2\pi}{D-1} ER; \quad (3)$$

the bound is saturated when  $E_c = E$ .

Verlinde has shown that Cardy's formula is exact for the strongly coupled CFTs by using their holographic dual description [1,3]. Moreover, he showed that Cardy's formula of Eq. (1) holds even for strong gravitational system such as the early universe with the help of a newly proposed cosmological principle that states that *the Casimir energy itself is not sufficient to form a universe-size black hole*. Furthermore, the Cardy formula coincides exactly with the Friedman equation when the above energy bound is saturated; the resultant entropy, which is called Hubble bound [4,1] obeys the area law as expected from the holographic nature of gravity theory [5,6].

The Cardy formula is also checked for weakly coupled CFTs in [7] by Kutasov and Larsen. They find that the formula is in general not exact, which results in a violation of Bekenstein bound in the low energy density.

As shown in [7] the partition function of a free  $D=4$  CFT in Einstein universe can be decomposed into the product of the basic partition functions as the following:

$$Z_{CFT}^{(4)} = [Z_b^{(4)}]^A [Z_b^{(2)}]^B, \quad (4)$$

where

$$A = n_S + 2n_V + \frac{7}{4}n_F, \quad (5)$$

$$B = -(2n_V + \frac{1}{4}n_F), \quad (6)$$

for a theory with  $n_S$  scalars,  $n_F$  Weyl fermions, and  $n_V$  Maxwell fields. For example,  $A = 15N$ ,  $B = -3N$  for  $\mathcal{N}=4$   $U(N)$  super-Yang-Mills (SYM) theory.

The basic partition functions are defined by

$$Z_b^{(d)} = \prod_{n=0}^{\infty} \left( \frac{1}{1-q^{n+1}} \right)^{(n+1)^{d-2}}, \quad (7)$$

where  $q = e^{-2\pi\delta}$  and  $\delta = 1/(2\pi RT)$  with  $R$  the radius of  $S^3$  and  $T$  the temperature. For references the explicit expression of  $\ln Z_b^{(2,4)}$  in the high  $T$  expansion are

$$\ln Z_b^{(2)} = 2\pi \frac{1}{24} (\delta^{-1} - \delta) + \frac{1}{2} + \ln \delta + O(e^{-2\pi\delta}), \quad (8)$$

$$\ln Z_b^{(4)} = 2\pi \frac{1}{240} \left( \frac{1}{3} \delta^{-3} + \delta \right) + O(e^{-2\pi\delta}). \quad (9)$$

The absence of the higher polynomial terms in  $\delta$  is due to the modular invariance of CFT on  $S^1 \times S^3$  [8], that is

$$I_b^{(d)} \left( \frac{1}{\delta} \right) = (-1)^{d/2} I_b^{(d)}(\delta), \quad (10)$$

where

$$I_b^{(d)}(\delta) \equiv -\delta^{d/2} \frac{\partial}{\partial \delta} \ln Z_b^{(d)}. \quad (11)$$

As will be shown there is no modular invariance for graviton partition sum.

From Eqs. (4), (8), and (9), we can derive the free energy  $F = -T \ln Z_{CFT}^{(4)}$ , and the result is

\*Email address: linfl@phya.snu.ac.kr

$$-FR = \frac{A}{720}\delta^{-4} + \frac{B}{24}\delta^{-2} + \left(\frac{A}{240} - \frac{B}{24}\right) + O(e^{-2\pi\delta}). \quad (12)$$

We note that the leading term ( $\sim \delta^{-4}$  for small  $\delta$ ) is coming from  $\ln Z_b^{(4)}$  and the subleading term ( $\sim \delta^{-2}$ ) from  $\ln Z_b^{(2)}$ , which is the leading Casimir effect. This result is exactly the same as the more familiar one [9] derived from the path integral using zeta-function regularization for CFT on general curved background  $\mathcal{M}$ :

$$\frac{F}{V} = -\frac{\pi^2 T^4}{90} A a_0(\mathcal{M}) - \frac{T^2}{24} B a_1(\mathcal{M}) + \dots, \quad (13)$$

where  $a_k(\mathcal{M})$  are the well-known ‘‘Hamidew’’ coefficients [9]. For  $\mathcal{M}=S^1 \times S^3$ ,  $a_0=1$ ,  $a_1=2/R^2$ .

One can then deduce  $E=F+TS$  and  $S$  from  $F$  in the way for a canonical ensemble and  $E_c$  from Eq. (2).<sup>1</sup> It is easy to see [7] that Cardy’s formula of Eq. (1) is not exact; and for the entropy to be bounded by the formula requires

$$\frac{A}{-B} \leq \frac{5}{2}, \quad (14)$$

where the equality holds when the bound is saturated. For  $\mathcal{N}=4$ ,  $U(N)$  SYM,  $A/-B=5$  for all  $N$  and thus the bound is violated. In general we could arbitrarily adjust the matter content to satisfy the above entropy bound condition, but in this paper we will consider only  $\mathcal{N}=4$  SYM and see how graviton’s Casimir effect changes the entropy bound condition for SYM.

Moreover, the authors of [7] observe that if Eq. (14) does not hold, then the Bekenstein bound of Eq. (3) will be violated when  $ER \leq A/(9 \times 720)$ ; however this condition can be translated into  $\delta \geq (3)^{3/4}$  by using the explicit  $\delta$ -dependence of  $ER = (A/240)\delta^{-4} + O(\delta^{-2})$ , which implies the high  $T$  (small  $\delta$ ) expansion of free energy in Eq. (12) is no longer valid. It deserves more study of the low temperature thermodynamics on the Bekenstein bound.

On the other hand, in the high  $T$  regime, where Eq. (12) works and the Bekenstein bound is not violated, the curvature effect becomes important because  $\delta = 1/2 \pi RT \ll 1$ , the thermal energy becomes larger than the characteristic planckian energy, which is inversely proportional to  $R$ . It is then natural to incorporate the contribution of thermal gravitons and gravitinos to the total partition function  $Z^{(4)} = Z_G^{(4)} Z_{CFT}^{(4)}$ , where  $Z_G^{(4)}$  is the partition function due to gravitons and gravitinos.

<sup>1</sup>For completeness, the explicit expressions are  $S = 2\pi[(A/180)\delta^{-3} + (B/12)\delta^{-1}] + O(e^{-2\pi\delta})$ , and  $E_c R = -(B/12)\delta^{-2} + (-A/60 + B/6) + O(e^{-2\pi\delta})$ . Note that, the leading term in  $E_c$  is zero for conformal scalars but positive for fermions and gauge fields, and also for supergravitons as shown later; however, the  $\delta$ -independent piece is negative in general, which is the usual Casimir energy at zero temperature.

## II. CASIMIR EFFECT OF GRAVITON

In the following we will calculate  $Z_G^{(4)}$  and discuss how it changes the condition on the entropy bound. The usual way to calculate the partition function or the effective action of a field theory on a fixed background is by evaluating the path integral up to one-loop [9]. However, in [7] a more efficient way for the CFT on  $S^1 \times S^3$  is to classify the operator content by the representations of  $SO(4) \simeq SU(2) \times SU(2)$ , the isometry group of  $S^3$ , and to calculate the partition sum from it. For example, a conformal scalar and its higher descendants are represented by  $(n/2, n/2)$  of  $SO(4)$  with degeneracy  $(n+1)^2$  and conformal weight  $\Delta = n+1$  for  $n=1, 2, 3, \dots$ , and the resulting partition sum is

$$Z_S^{(4)} = \prod_{n=0}^{\infty} \left( \frac{1}{1-q^{n+1}} \right)^{(n+1)^2} = Z_b^{(4)}. \quad (15)$$

This method of enumerating the operator content has the advantage of automatically taking care of the constraints such as equations of motion, Bianchi identities, etc.

Similarly, the Maxwell field and its descendants are represented by  $(n/2, (n+2)/2) + \text{H.c.}$  with degeneracy  $2(n+1)(n+3)$  and conformal weight  $\Delta = n+2$ ,<sup>2</sup> and the resulting partition sum is

$$Z_V^{(4)} = \prod_{n=0}^{\infty} \left( \frac{1}{1-q^{n+1}} \right)^{2n(n+2)} = [Z_b^{(4)}]^2 [Z_b^{(2)}]^{-2}. \quad (16)$$

Note that the leading term is just twice the one for the scalar as expected for massless photon; however, this is not the case for the leading Casimir effect.

Generalizing the above counting to graviton, the contribution to the partition sum is due to the spin 2 representations  $(n/2, (n+4)/2) + \text{H.c.}$  with degeneracy  $2(n+1)(n+5)$  and conformal weight  $\Delta = n+3$ . The scaling dimension of  $\delta g_{\mu\nu} = g_{\mu\nu} - g_{\mu\nu}^{(B)}$  is one, and from the requirement of general covariance and conformal invariance, the lowest operator should be the Weyl tensor ( $\sim \partial\partial\delta g$ ), which has ten independent components [10] and scaling dimension 3, this agrees with the above counting for  $n=0$ .

The resulting partition sum for graviton is

$$Z_g^{(4)} = \prod_{n=0}^{\infty} \left( \frac{1}{1-q^{n+2}} \right)^{2n(n+4)}, \quad (17)$$

which cannot be decomposed into the basic partition functions of Eq. (7). Instead we should evaluate the following new basic partition functions

$$Z_b'^{(d)} = \prod_{n=0}^{\infty} \left( \frac{1}{1-q^{n+2}} \right)^{(n+2)^{d-2}}. \quad (18)$$

<sup>2</sup>The primary operator is not  $A_\mu$  of scaling dimension one but the field strength  $F_{\mu\nu}$  of scaling dimension 2 because the first is not gauge invariant but the latter is.

We generalize the method in [8,7] to calculate  $Z_b^{(d)}$  by the following expansion:

$$-\frac{\partial \ln Z_b^{(d)}}{\partial \delta} = 2\pi \sum_{n=0}^{\infty} (n+2)^{d-1} \sum_{k=1}^{\infty} e^{-2\pi\delta(n+2)k}, \quad (19)$$

and using the Mellin representation

$$e^{-x} = \frac{1}{2\pi i} \int_C x^{-z} \Gamma(z) dz, \quad (20)$$

where the contour  $C$  is along the imaginary axis with  $\text{Re}(z) > 0$  large, we arrive

$$\begin{aligned} -\frac{\partial}{\partial \delta} \ln Z_b^{(d)} &= \frac{1}{2\pi i} \int_C (2\pi)^{1-z} \delta^{-z} \zeta(z+1-d) \zeta(z) \Gamma(z) dz \\ &\quad - \frac{1}{2\pi i} \int_C (2\pi)^{1-z} \delta^{-z} \zeta(z) \Gamma(z) dz. \end{aligned} \quad (21)$$

It is easy to see that the first term is just the same as  $-(\partial/\partial\delta)\ln Z_b^{(d)}$ , and the integrand of the second term has the poles at  $z=1, 0, -1, -3, \dots$ . The resulting expressions of  $Z_b^{(d)}$  in the expansion of  $\delta$  are

$$\ln Z_b^{(2)} = \ln Z_b^{(2)} + \ln \delta + O(\delta) = 2\pi \frac{1}{24} \delta^{-1} + \frac{3}{2} \ln \delta + O(\delta), \quad (22)$$

$$\ln Z_b^{(4)} = \ln Z_b^{(4)} + \ln \delta + O(\delta) = 2\pi \frac{1}{720} \delta^{-3} + \frac{1}{2} \ln \delta + O(\delta). \quad (23)$$

We see that the high order terms exist because there is no modular invariance property for the new partition sums; however, the leading terms here are still the same as those in  $Z_b^{(d)}$ . Note that the leading terms are the only relevant terms in determining the entropy bound condition.

The graviton partition sum  $Z_g^{(4)}$  can be decomposed into

$$Z_g^{(4)} = [Z_b^{(4)}]^2 [Z_b^{(2)}]^{-8} \simeq [Z_b^{(4)}]^2 [Z_b^{(2)}]^{-8}, \quad (24)$$

where “ $\simeq$ ” means having the same leading and subleading terms. Note that the leading term is just twice of the one for scalar as expected. The resulting  $Z_g^{(4)}$  also implies that the theory consisting of only free conformal thermal graviton will not violate the entropy bound given by the Cardy formula of Eq. (1) because it has  $(A/-B)_g = \frac{1}{4} < \frac{5}{2}$ .

Similarly, we can calculate the gravitino's partition sum by identifying its descendants as described by the representation  $2(n, (n+3)/2)$  of  $\text{SO}(4)$  with degeneracy  $2(n+1)(n+4)$  and conformal weight  $\Delta = n + \frac{5}{2}$ . The resulting partition sum for gravitino is

$$Z_{gf}^{(4)} = \prod_{n=0}^{\infty} (1+q^{n+3/2})^{2n(n+3)} = [Z_f^{(4)}]^2 [Z_f^{(2)}]^{-9/2}, \quad (25)$$

where the basic fermionic partition functions are defined as

$$Z_f^{(d)} = \prod_{n=0}^{\infty} (1+q^{n+3/2})^{(n+3/2)^{d-2}}. \quad (26)$$

Using the identity

$$\sum_{n=0}^{\infty} (n+\frac{3}{2})^{-z} = (2^z-1)\zeta(z)-2^z \quad (27)$$

and the Mellin representation, it can be shown that the leading term in  $Z_f^{(d)}$  is the same as in the basic partition function for a Weyl fermion

$$Z_f^{(d)} = \prod_{n=0}^{\infty} (1+q^{n+1/2})^{(n+1/2)^{d-2}} = e^{(1-1/2^{d-1})} Z_b^{(d)}. \quad (28)$$

We then arrive at

$$Z_{gf}^{(4)} \simeq [Z_b^{(4)}]^{7/4} [Z_b^{(2)}]^{-9/4}, \quad (29)$$

note that the leading term is the same as the one for a Weyl fermion.

Combining the contributions of graviton and gravitino together we find that the total partition function of the on-shell supergravity theory is

$$Z_G^{(4)} = Z_g^{(4)} [Z_{gf}^{(4)}]^{\mathcal{N}} \simeq [Z_b^{(4)}]^{2+(7/4)\mathcal{N}} [Z_b^{(2)}]^{-8-(9/4)\mathcal{N}}, \quad (30)$$

where  $\mathcal{N}$  is the number of supersymmetries. We see that the entropy bound condition of Eq. (14) is not violated because  $(A/-B)_{\text{supgra}} = (2+7\mathcal{N}/4)/(8+9\mathcal{N}/4) \leq 5/2$ .

Now we could combine the contribution of  $\mathcal{N}=4$   $\text{SU}(N)$  SYM theory and the thermal supergraviton together, it yields

$$Z^{(4)} = Z_{CFT}^{(4)} Z_G^{(4)} = [Z_b^{(4)}]^{2+(7/4)\mathcal{N}+15N} [Z_b^{(2)}]^{-8-(9/4)\mathcal{N}-3N}, \quad (31)$$

we see that the entropy bound condition becomes

$$\frac{A}{-B} = \frac{9+15N}{17+3N} \leq \frac{5}{2}, \quad (32)$$

which leads to a constraint on the rank of the gauge group

$$N \leq \frac{67}{15}. \quad (33)$$

By generalizing to more physical matter contents such as the standard model, one may find a deep connection between entropy bound and why there are only three colors in nature.

### III. BLACK $p$ -BRANE AND CARDY'S FORMULA

The  $D=2$  Cardy formula of Eq. (1) is the same as the one derived from the saddle point approximation of the formula for density of states [11,12]

$$\rho(\Delta) = \int d\delta e^{2\pi\delta\Delta} [Z_b^{(2)}(\delta)]^c \simeq e^{2\pi\sqrt{(c/6)(\Delta - c/24)}}, \quad (34)$$

where  $c$  is the central charge;  $Z_b^{(2)}$  is given by Eq. (8) in which the higher order terms is suppressed by the modular invariance property and thus the saddle point approximation can be simplified. The entropy  $S = \ln \rho(\Delta)$  agrees with the Cardy formula of Eq. (1) by identifying  $E_c R = c/12$  and  $ER = \Delta$ . For  $D > 2$ , there is no such simple square-root Cardy formula.

On the other hand, the square-root behavior of  $D = 2$  Cardy formula has been shown to be obeyed by the near-horizon classical gravitational dynamics for the  $D > 2$  systems such as the black holes [13–16], de Sitter universe [17,18], and the apparent horizons [19]. These cases imply that the near-horizon physics is associated with a  $D = 2$  CFT, and exemplify the holographic nature of strong-gravity regime. It is then curious to see if the Cardy formula of Eq. (1) can be an indication of holographic nature of strong gravitational system. It is easy to check that the Cardy formula is satisfied trivially by the Schwarzschild black hole but not by the Reissner-Nordstrom black hole with  $E_c \equiv E - TS - \mu Q$  where  $\mu$  and  $Q$  are the chemical potential and the corresponding charge. As shown below, a nontrivial example that fits the Cardy formula is the black  $p$ -brane ( $p < 7$ ) described by

$$ds_{10}^2 = [H_p(r)]^{-1/2} [-f(r)dt^2 + d^2x_{\parallel}] + [H_p(r)]^{1/2} \times \left( \frac{dr^2}{f(r)} + r^2 d\Omega_{8-p}^2 \right), \quad (35)$$

$$C_{012\dots p}(r) = \sqrt{1 + \frac{r_0^{7-p} H_p(r) - 1}{L^{7-p} H_p(r)}}, \quad (36)$$

$$H_p(r) = 1 + \frac{L^{7-p}}{r^{7-p}}, \quad f(r) = 1 - \frac{r_0^{7-p}}{r^{7-p}}, \quad (37)$$

where the parameters  $L$  and  $r_0$  are the anti-de Sitter throat size and the position of the horizon, respectively. They play analogous roles of the size and temperature as those in CFT.

From the metric and the  $RR$  potential  $C_{p+1}$ , we can derive the mass  $M$ ,  $RR$  charge  $Q$ , chemical potential  $\mu$ , temperature  $T$ , and entropy  $S$  for the system [20]:

$$M = \frac{\Omega_{8-p} V_p}{2\kappa_{10}^2} [(8-p)r_0^{7-p} + (7-p)L^{7-p}], \quad (38)$$

$$Q = \frac{(7-p)\Omega_{8-p}}{2\kappa_{10}^2 T^p} L^{(7-p)/2} \sqrt{r_0^{7-p} + L^{7-p}}, \quad (39)$$

$$\mu = V_p T_p \frac{L^{(7-p)/2}}{\sqrt{r_0^{7-p} + L^{7-p}}}, \quad T = \frac{7-p}{4\pi} \frac{r_0^{(5-p)/2}}{\sqrt{r_0^{7-p} + L^{7-p}}}, \quad (40)$$

$$S = \frac{4\pi\Omega_{8-p} V_p}{2\kappa_{10}^2} r_0^{(9-p)/2} \sqrt{r_0^{7-p} + L^{7-p}}, \quad (41)$$

where  $V_p$  is the spatial volume of the  $p$ -brane, and  $T_p$  is the brane tension. These thermodynamical quantities satisfy the first law:  $dM = TdS + \mu dQ$ ; however, in its integral form there is an excess, which can be identified as the Casimir energy in [1]

$$E_c = 2(M - TS - \mu Q). \quad (42)$$

It is then straightforward to see that

$$S = \frac{2\pi r_0}{\sqrt{7-p}} \sqrt{E_c(2M - E_c)}, \quad (43)$$

which is in the same form of Cardy's formula of Eq. (1) but with different overall factors. It is interesting to see if this formula bears any microscopic interpretation from  $D$ -brane and string theory as done for black string [21]. It also deserves further study of the thermodynamics of the holographic dual theory corresponding to the black  $p$ -brane and examine the validity of Eq. (43) from the dual point of view.

#### IV. CONCLUSION

In Verlinde's proposal [1] the Cardy formula of Eq. (1) is expected to be exact for CFT, but it turns out that this is true only for strongly coupled theory but not for weakly coupled one [7]. On the other hand, this formula unifies the Bekenstein bound and Hubble bound in the cosmology context. As long as the theory does not violate the entropy bound given by Cardy's formula, it would have no problem to satisfy both Bekenstein bound for weak-gravity regime and Hubble bound for strong-gravity regime along the evolution of the closed universe. From these facts it is more natural to think Verlinde's proposal as a universal entropy bound but not an exact entropy formula for CFT. We have seen that the pure perturbative effect of gravity will not violate the bound by Cardy's formula. This can be viewed as a self-consistent test for the Bekenstein entropy bound though that its validity has been under debate [22] since it was proposed twenty years ago. Moreover, when combining with CFT's contribution, we see that graviton's Casimir effect is necessary for the CFT to satisfy the entropy bound condition that yields a constraint on the rank of the gauge group of the CFT. We also see that there is an intriguing resemblance of Cardy's formula in black  $p$ -brane's thermodynamics, which deserves further study for its physical implication.

#### ACKNOWLEDGMENTS

The author would like to thank Pei-Ming Ho, Choony Lee, Miao Li, and Soo-Jong Rey for helpful discussions, also for the warm hospitality of NTU string theory group where this work was finalized. This work was supported by BK-21 Initiative in Physics (SNU - Project 2), KRF International Collaboration Grant, and KOSEF Interdisciplinary Research Grant 98-07-02-07-01-5.

- [1] E. Verlinde, “On the holographic principle in a Radiation dominated universe,” hep-th/0008140.
- [2] J.D. Bekenstein, Phys. Rev. D **23**, 287 (1981).
- [3] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).
- [4] G. Veneziano, Phys. Lett. B **452**, 22 (1999).
- [5] G. ’t Hooft, in Salamfest 1993:0284, gr-qc/9310026; L. Susskind, J. Math. Phys. **36**, 6377 (1995).
- [6] W. Fischler and L. Susskind, hep-th/9806039.
- [7] D. Kutasov and F. Larsen, “Partition sums and entropy bounds in weakly coupled CFT,” hep-th/0009244.
- [8] J.L. Cardy, Nucl. Phys. **B366**, 403 (1991).
- [9] C.P. Burgess, N.R. Constable, and R.C. Myers, J. High Energy Phys. **08**, 017 (1999); B.S. DeWitt, in *Relativity, Groups and Topology*, edited by B.S. DeWitt and C. DeWitt (Gordon and Breach, New York, 1965); P.B. Gilkey, J. Diff. Geom. **10**, 601 (1975); Adv. Math. **15**, 334 (1975).
- [10] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
- [11] J.L. Cardy, Nucl. Phys. **B270**, 186 (1986).
- [12] S. Carlip, Class. Quantum Grav. **15**, 3609 (1988).
- [13] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
- [14] A. Strominger, J. High Energy Phys. **02**, 009 (1998).
- [15] S. Carlip, Phys. Rev. Lett. **82**, 2828 (1999).
- [16] S.N. Solodukhin, Phys. Lett. B **454**, 213 (1999).
- [17] J. Maldacena and A. Strominger, J. High Energy Phys. **02**, 014 (1998).
- [18] F.L. Lin and Y.S. Wu, Phys. Lett. B **453**, 222 (1999).
- [19] R. Brustein, Phys. Rev. Lett. **86**, 576 (2001).
- [20] E. Kiritsis and T.R. Taylor, “Thermodynamics of D-brane probes,” hep-th/9906048.
- [21] G. Horowitz and A. Strominger, Phys. Rev. Lett. **77**, 2368 (1996).
- [22] D.N. Page, “Defining entropy bound,” hep-th/0007238; “Subsystem entropy exceeding Bekenstein’s bound,” hep-th/0007237; J.D. Bekenstein, “On Page’s examples challenging the entropy bound,” gr-qc/0006003; D.N. Page, “Huge violations of Bekenstein’s entropy bound,” gr-qc/0005111.