Wahlquist-Newman solution

Marc Mars*
Albert Einstein Institut, Am Mühlenberg 1, D-14476 Golm, Germany
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Based on a geometrical property which holds both for the Kerr metric and for the Wahlquist metric we argue that the Kerr metric is a vacuum subcase of the Wahlquist perfect-fluid solution. The Kerr-Newman metric is a physically preferred charged generalization of the Kerr metric. We discuss which geometric property makes this metric so special and claim that a charged generalization of the Wahlquist metric satisfying a similar property should exist. This is the Wahlquist-Newman metric, which we present explicitly in this paper. This family of metrics has eight essential parameters and contains the Kerr–Newman–de Sitter and the Wahlquist metrics, as well as the whole Plebański limit of the rotating *C* metric, as particular cases. We describe the basic geometric properties of the Wahlquist-Newman metric, including the electromagnetic field and its sources, the static limit of the family and the extension of the spacetime across the horizon.

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I. INTRODUCTION

Among the few explicitly known stationary (non-static) and axially symmetric perfect-fluid spacetimes, the Wahlquist family [1] enjoys a privileged position. First, it is the oldest known solution and it remains, in some sense, the simplest one. More importantly, it has interesting physical properties (see [2] and references therein) which have made this metric a good candidate to describe the interior of an isolated rotating body in equilibrium. This view has been recently challenged in [3], where the matching conditions between the Wahlquist metric and a vacuum, asymptotically flat spacetime are claimed to be incompatible in a perturbative sense. This suggests that the Wahlquist metric does not describe the interior or a rotating body in vacuum. In order to make this result conclusive it would be of interest to develop a proper theoretical analysis of the perturbative approach to the matching conditions.

In any case, the fundamental properties which make the Wahlquist metric so special are of geometrical nature. Indeed, this metric is known to be uniquely characterized among stationary, rigidly rotating, perfect-fluid spacetimes by any of the following, seemingly unrelated, properties (see [4] for a discussion):

(1) The Simon tensor vanishes [5].

(2) The spacetime admits a Killing tensor of type [(11)(11)] [6].

(3) The spacetime is axially symmetric, the Weyl tensor is Petrov type D and the equation of state of the perfect fluid is $\rho+3p=\text{const}$ [7].

For the purposes of this paper, characterization (1) will be the most relevant one. The Simon tensor [8] was put forward in order to obtain a unique characterization of the Kerr metric [9]. More precisely, the Kerr spacetime is the only strictly stationary (i.e. with a Killing vector which is timelike everywhere), vacuum and asymptotically flat spacetime for which the Simon tensor vanishes. This fact, combined with (1), shows that there may exist a close relationship between the Wahlquist metric and the Kerr metric. However, no such relationship has been found so far. One of the aims of this paper is to show that the Kerr metric can be obtained as a particular, vacuum, subcase of the Wahlquist metric. In fact, we will also show that the Kerr–de Sitter metric [10], which is vacuum with a cosmological constant, belongs to the Wahlquist family in the limit $\rho+p=0$.

The existence of physically privileged charged generalizations of the Kerr and Kerr–de Sitter metrics, namely the Kerr–Newman [11] and Kerr–Newman–de Sitter spacetimes [10], leads us to consider whether a similar, privileged, charged generalization of the Wahlquist metric exists. To analyze such a question we should first make precise the meaning of the term ‘‘privileged.’’ As we shall see, the Kerr, the Kerr–de Sitter and the Kerr–Newman–de Sitter metrics have very special geometric properties which relate the Weyl tensor, the Killing vector and the electromagnetic field (when one is present). Moreover, these conditions turn out to be fulfilled also by the Wahlquist metric. Thus, there exists a geometrically clear sense in which a privileged charged generalization of the Wahlquist metric might exist. We call it Wahlquist-Newman metric, first because it contains both Wahlquist and Kerr–Newman–de Sitter as particular cases and also in order to emphasize the very special geometrical properties fulfilled by this spacetime. The main objective of this paper is to obtain the explicit form of this metric. It turns out that the Wahlquist-Newman family contains eight arbitrary parameters. It represents a rigidly rotating perfect fluid, which may be charged or not, together with an electromagnetic field. The sources of the electromagnetic field are the perfect fluid (when this is charged) and/or a singularity of the spacetime. The latter corresponds to the singular source hidden behind the event horizon in the Kerr-Newman spacetime.

The paper is organized as follows. In Sec. II, we recall the relationship between the vanishing of the Simon tensor and the Weyl tensor and we discuss which geometrical properties make the Kerr, Kerr-Newman, Kerr–Newman–de Sitter and Wahlquist metrics so special. In Sec. III, we rewrite the Wahlquist metric in such a way that the Kerr–de Sitter met-

*Also at Laboratori de Física Matemàtica, Societat Catalana de Física, IEC, Barcelona, Spain.

ric (and the Kerr metric) are contained as particular subcases. In Sec. IV, we present the Wahlquist-Newman metric and we describe its fundamental properties. First, we stress that the geometrical properties described in Sec. II also hold for this metric. Then, we give the explicit expressions for the energy-density, the pressure and the fluid velocity of the perfect fluid. The electromagnetic field and its charge current are also written down and the number of essential parameters in the family is discussed. We also show that the particular case in which the perfect fluid vanishes corresponds to the well-known Plebański metric [12], which is an important limiting case of the rotating C metric [13]. This shows, in particular, that the Wahlquist-Newman spacetime contains the Kerr–Newman–de Sitter metric as a particular case. In Sec. V, we analyze the static limit of the Wahlquist-Newman metric. To do that, we rewrite the metric in a suitable coordinate system which admits an explicit static limit and which, in addition, allows for an extension of the Wahlquist-Newman spacetime across its horizon (although the metric represents a perfect fluid, it does have a regular horizon, as we shall see). Finally, we include an Appendix where Einstein-Maxwell's equations under the assumptions of this paper are solved.

II. GEOMETRIC PROPERTIES OF THE WAHLQUIST AND THE KERR–NEWMAN–de SITTER METRICS

The Kerr metric and the Wahlquist metrics share the property that the Simon tensor [8] vanishes identically. The geometrical meaning of the vanishing of the Simon tensor in vacuum has been recently clarified in [14]. The fundamental underlying property is a close relationship between the Weyl tensor and the stationary Killing vector. Properties of the Weyl tensor can be quite naturally described using the language of self-dual two forms, which are two-forms $\boldsymbol{\mathcal{X}}$ satisfying $\boldsymbol{\mathcal{X}}^{\star}=-i\boldsymbol{\mathcal{X}}$ where $\star$ denotes the Hodge dual with respect to the volume form $\eta_{\alpha\beta\gamma\delta}$. From the Weyl tensor $C_{\alpha\beta\gamma\delta}$ and the stationary Killing vector $\vec{\xi}$ we can write down two canonical self-dual objects, the self-dual Weyl tensor $\mathcal{C}_{\nu\mu\alpha\beta}\equiv C_{\nu\mu\alpha\beta}+(i/2)\eta_{\alpha\beta\rho\sigma}C_{\nu\mu}{}^{\rho\sigma}$ and the so-called Killing form $\mathcal{F}_{\alpha\beta}\equiv\nabla_{\alpha}\xi_{\beta}+(i/2)\eta_{\alpha\beta\gamma\delta}\nabla^{\gamma}\xi^{\delta}$. It is natural to ask which spacetimes have the property that the self-dual Weyl tensor and the Killing form are related to each other. The simplest relationship between these two objects which respects all the symmetries of the self-dual Weyl tensor is [the object $\mathcal{I}_{\alpha\beta\gamma\delta}\equiv(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}+i\eta_{\alpha\beta\gamma\delta})/4$ is the canonical metric in the space of two-forms]

$$\mathcal{C}_{\alpha\beta\gamma\delta}=L\left(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta}-\frac{1}{3}\mathcal{I}_{\alpha\beta\gamma\delta}\boldsymbol{\mathcal{F}}^{2}\right), \tag{1}$$

where L is a complex, scalar function and $\boldsymbol{\mathcal{F}}^{2}\equiv\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$. It turns out [14] that the vanishing of the Simon tensor in vacuum is equivalent to Eq. (1). We know that the Simon tensor vanishes for the Wahlquist metric. So, we can ask whether Eq. (1) holds also for the Wahlquist spacetime. A straightforward calculation shows that this is indeed the case. Actually, it can be seen that the original assumptions made by Wahlquist in order to find his spacetime, although written in another formalism (see [2]), can be rewritten so that they consist of condition (1) plus axial symmetry. Thus, with hindsight, Kramer's uniqueness result [5] of the Wahlquist metric is equivalent to dropping the condition of axial symmetry from Wahlquist's original assumptions.

TABLE I. Relationships between the metrics discussed in this paper.

Non-charged metrics		**Charged counterparts**
Kerr	→	Kerr-Newman
↓		↓
Kerr-de Sitter	→	Kerr-Newman-de Sitter
?↓?		?↓?
Wahlquist	?→?	Wahlquist-Newman?

Following the discussion in the Introduction, we can ask whether condition (1) is also fulfilled by Kerr–de Sitter, Kerr-Newman and Kerr–Newman–de Sitter. The answer is yes, as a simple calculation shows. However, the Kerr-Newman and the Kerr–Newman–de Sitter spacetimes contain, in addition, an electromagnetic field. So we should analyze whether this field fits nicely into the geometrical relation (1). This is very important for our purposes because it will determine what makes these charged spacetimes so special, and it will indicate how the charged generalization of Wahlquist metric should be defined. Let us call the electromagnetic field as $K_{\alpha\beta}$. This two-form defines canonically a self-dual two-form according to $\mathcal{K}_{\alpha\beta}\equiv K_{\alpha\beta}+iK^{\star}_{\alpha\beta}$. It can be easily checked that in Kerr-Newman and Kerr–Newman–de Sitter the self-dual electromagnetic field *is proportional* to the Killing form, i.e. $\mathcal{K}_{\alpha\beta}\propto\mathcal{F}_{\alpha\beta}$. This is the most natural relationship one could think of between these two objects. Thus, all these metrics do have very special geometrical properties.

This discussion above indicates two things. First, that the Wahlquist metric is likely to contain the Kerr–de Sitter metric (and hence the Kerr metric) as a particular subcase and, second, that a charged generalization of Wahlquist should also exist satisfying the following properties: (1) It contains both Wahlquist and Kerr–Newman–de Sitter as subcases, (2) it satisfies the relationship (1) between the Weyl tensor and the Killing form and (3) its self-dual electromagnetic field is proportional to the Killing form. Its energy-momentum tensor should contain both an electromagnetic field part and a perfect-fluid part.

Table I shows graphically the interrelationships between these metrics. Single arrows indicate well-established and natural generalizations and arrows between question marks indicate plausible relations between metrics. In particular, it becomes apparent that some metric, the Wahlquist-Newman metric, should fill the lower, right corner of this table.

III. KERR–de SITTER LIMIT IN THE WAHLQUIST FAMILY

Let us start by writing down the line-element of the Wahlquist family as it appears in [2]. This is actually a generalization (by adding a discrete parameter) of the original Wahlquist metric as given in [1] and was originally given by

Senovilla in [7] (see [15] for a discussion on the different published versions of the Wahlquist metric and their interrelationships). The Wahlquist line element is

$$ds^2=-\frac{1}{\Phi^2}(dt-Ad\theta)^2+r^2d\theta^2+\frac{g}{\mu_0}\left(\frac{du^2}{h_1}+\frac{dv^2}{h_2}\right), \tag{2}$$

where

$$h_1(u)=h_0+\epsilon_0\cos(2u)+(u+u_0)\sin(2u),$$

$$h_2(v)=h_0-\epsilon_0\cosh(2v)+(v+v_0)\sinh(2v),$$

$$g=\cos(2u)+\cosh(2v),\quad \frac{1}{\Phi^2}=\frac{h_1-h_2}{\kappa g},$$

$$r^2=4r_0^2\Phi^2h_1h_2,$$

$$A=-2\kappa r_0\cosh(v_A)+\frac{2\kappa r_0[h_2\cos(2u)+h_1\cosh(2v)]}{h_1-h_2}.$$

All symbols with zero subscripts, as well as κ and v_A, are arbitrary constants. The energy-momentum of this spacetime is a rigidly rotating perfect fluid (i.e. its velocity vector is proportional to the Killing vector $\vec{\xi}=\partial_t$). The energy-density ρ and pressure p are $\rho=\mu_0(1-\kappa/\Phi^2)$ and $p=\mu_0(3(\kappa/\Phi^2)-1)$, so that the equation of state is $\rho+3p=2\mu_0$. We want to rewrite this metric in such a way that the Kerr metric is included as a particular case. We first rescale u and v as follows: $u=\beta y+\pi/2$, $v=\beta z$, where β is any non-zero constant. The function g transforms into $g=\cosh(2\beta z)-\cos(2\beta y)$. The constant β is superfluous as long as it remains non-zero, but it may be that the limit $\beta\to 0$ gives another metric, perhaps the Kerr metric we are looking for. In order to work out this idea, we should make $\beta\to 0$ meaningful. This requires some redefinitions of constants. We start by defining

$$Q(y,z)\equiv\frac{\cosh(2\beta z)-\cos(2\beta y)}{2\beta^2}, \tag{3}$$

which is regular at $\beta=0$. The 2×2 block spanned by $\{u,v\}$ in Eq. (2) takes the form $Q(U(z)^{-1}dz^2+V(y)^{-1}dy^2)$, where $U=\mu_0h_2/(2\beta^4)$ and $V=\mu_0h_1/(2\beta^4)$. The constants must be redefined so that U and V are regular at $\beta=0$. Furthermore μ_0 should remain finite and non-zero (because of the relation $\rho+3p=2\mu_0$, which is non-zero in the Kerr–de Sitter metric). In addition, the number of parameters should not be reduced in the limit $\beta\to 0$. All this is achieved by the following redefinition of constants

$$\mu_0\text{invariant},\quad \frac{h_0\mu_0}{2\beta^4}=Q_0+\frac{\nu_0}{2\beta^2}+\frac{\mu_0}{2\beta^4},\quad \frac{\mu_0\epsilon_0}{\beta^2}=\nu_0+\frac{\mu_0}{\beta^2},$$

$$\frac{\mu_0v_0}{\beta^3}=a_1,\quad \frac{\mu_0(u_0+\pi/2)}{\beta^3}=-a_2, \tag{4}$$

which brings U and V into the form

$$U=Q_0-\frac{\mu_0}{2\beta^2}\left[\frac{\cosh(2\beta z)-1}{\beta^2}-\frac{z\sinh(2\beta z)}{\beta}\right]+\nu_0\frac{1-\cosh(2\beta z)}{2\beta^2}+a_1\frac{\sinh(2\beta z)}{2\beta},$$

$$V=Q_0+\frac{\mu_0}{2\beta^2}\left[\frac{1-\cos(2\beta y)}{\beta^2}-\frac{y\sin(2\beta y)}{\beta}\right]+\nu_0\frac{1-\cos(2\beta y)}{2\beta^2}+a_2\frac{\sin(2\beta y)}{2\beta}. \tag{5}$$

We should now analyze the $\{t,\theta\}$ block. The constants κ, r_0 and v_A correspond to the freedom of performing linear coordinate changes in t and θ. Since the coordinates should remain adapted to the Killing vector $\vec{\xi}$ (which is privileged both for the Wahlquist and for the Kerr metrics), we consider changes of the type $\tau=b_1(t+b_2\theta),\sigma=b_3\theta$. Let us choose

$$b_1=\frac{\beta}{\sqrt{\mu_0\kappa}},\quad b_2=2\kappa r_0(\cosh(v_A)-1),$$

$$b_3=4\beta^3r_0\sqrt{\frac{\kappa}{\mu_0}},$$

which bring the Wahlquist line element (2) into the form

$$ds^2=-\lambda\left(d\tau-\frac{v_1V+v_2U}{V-U}d\sigma\right)^2+\frac{UV}{\lambda}d\sigma^2+(v_1+v_2)\left(\frac{dy^2}{V}+\frac{dz^2}{U}\right), \tag{6}$$

where U and V are given by Eq. (5), v_1, v_2 read

$$v_1=\frac{\cosh(2\beta z)-1}{2\beta^2},\quad v_2=\frac{1-\cos(2\beta y)}{2\beta^2}, \tag{7}$$

and $\lambda=(V-U)/(v_1+v_2)$. All metric functions in Eq. (6) are independently regular at $\beta=0$. The structure of this line element is very similar to the one given by Senovilla in [7], the only difference being the choice of parameters. It is not difficult to obtain the redefinitions which bring Senovilla's form into Eq. (6). Thus, a regular limit $\beta=0$ could also have been obtained starting from that line element. We preferred to start from Eq. (2) in order to deal only with essential parameters.

The explicit form of the metric (6) when $\beta=0$ is (after trivially reorganizing the block $\{\tau,\sigma\}$)

$$ds^2=-\frac{\hat{V}}{y^2+z^2}(d\tau-z^2d\sigma)^2+\frac{\hat{U}}{y^2+z^2}(d\tau+y^2d\sigma)^2+(y^2+z^2)\left(\frac{dy^2}{\hat{V}}+\frac{dz^2}{\hat{U}}\right), \tag{8}$$

where

$$\hat{U}=Q_0+\frac{\mu_0}{3}z^4-\nu_0 z^2+a_1 z, \hat{V}=Q_0+\frac{\mu_0}{3}y^4+\nu_0 y^2+a_2 y,$$

$$\hat{\lambda}=\nu_0+a_2\frac{y}{y^2+z^2}-a_1\frac{z}{y^2+z^2}+\frac{\mu_0}{3}(y^2-z^2).$$

This is the uncharged subcase of the Plebański metric [12], which is an important limiting case of the Plebański-Demiański metric, also called rotating C metric [13]. The constant a_1 is closely related to the NUT parameter and a_2 is related to the mass parameter. A particular case of this metric is obtained by setting $Q_0=a^2$, $\nu_0=1-a^2\Lambda/3$, $a_1=0$ and redefining $a_2\to -2M$ and $\mu_0\to-\Lambda$. After the coordinate changes

$$y=r, \quad z=a\cos\theta, \quad a\sigma=\frac{-\phi}{1+\frac{1}{3}\Lambda a^2}, \quad \tau=\frac{t-a\phi}{1+\frac{1}{3}\Lambda a^2}, \tag{9}$$

we obtain the Kerr–de Sitter metric [10] in Boyer-Lindquist coordinates,

$$ds^2=\rho^{-2}[-\Delta_r\boldsymbol{\alpha}_0^2+\Delta_\theta\sin^2\theta\boldsymbol{\alpha}_1^2]+\rho^2\left(\frac{dr^2}{\Delta_r}+\frac{d\theta^2}{\Delta_\theta}\right), \tag{10}$$

where $\rho^2=r^2+a^2\cos^2\theta$, $\Delta_r=(a^2+r^2)(1-\frac{1}{3}\Lambda r^2)-2Mr$ and $\Delta_\theta=1+\frac{1}{3}\Lambda a^2\cos^2\theta$. The one-forms $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$ are

$$\boldsymbol{\alpha}_0=\frac{1}{1+\frac{1}{3}\Lambda a^2}(dt-a\sin^2\theta d\phi),$$

$$\boldsymbol{\alpha}_1=\frac{1}{1+\frac{1}{3}\Lambda a^2}[a dt-(a^2+r^2)d\phi].$$

Of course, by setting $a=0$ in this metric we get the Schwarzschild–de Sitter metric and the particular case $\Lambda=0$ is the Kerr metric. Thus, the Wahlquist metric *does* contain Kerr–de Sitter (and Kerr) as a particular case, as we wanted to prove.

IV. THE WAHLQUIST-NEWMAN FAMILY OF METRICS

The line-element of the Kerr–Newman–de Sitter spacetime can be obtained from Eq. (10) just by modifying the function Δ_r with an additive constant, i.e.,

$$\Delta_r=(a^2+r^2)\left(1-\frac{1}{3}\Delta r^2\right)-2Mr+q^2,$$

the constant q being directly related to the electric charge of the black hole (and hence to the electromagnetic field). By analogy, we assume as a working hypothesis that the Wahlquist-Newman metric we are seeking can be obtained by modifying the functions on the block $\{dy,dz\}$ of the metric (6). The reason why we must allow both functions $V(y)$ and $U(z)$ to be changed instead of only one (as in the Kerr–Newman–de Sitter case) will become clear later. So, let us assume that the Wahlquist-Newman metric can be written in the form

$$ds^2=-\frac{V_1}{v_1+v_2}(d\tau-v_1 d\sigma)^2+\frac{U_1}{v_1+v_2}(d\tau+v_2 d\sigma)^2 +(v_1+v_2)\left(\frac{dy^2}{V_1}+\frac{dz^2}{U_1}\right), \tag{11}$$

where $V_1(y)$ and $U_1(z)$ are unknown functions and $v_1(z)$, $v_2(y)$ are given by Eq. (7). We want to solve the Einstein-Maxwell field equations for an energy-momentum tensor $T_{\mu\nu}$ consisting of two parts: a perfect-fluid component $T^{pf}_{\mu\nu}$ with the fluid velocity being proportional to the stationary Killing vector $\vec{\xi}=\partial_\tau$ and an electromagnetic part $T^{em}_{\mu\nu}$. If we denote by $K_{\mu\nu}$ the electromagnetic field and by $\mathcal{K}_{\mu\nu}$ its self dual part, we want to impose $\boldsymbol{\mathcal{K}}\propto\boldsymbol{\mathcal{F}}$, so that the fundamental geometric property satisfied by the Kerr-Newman metric is preserved. In Kerr-Newman, the electromagnetic field is source-free (more precisely, the source of the electromagnetic field is located at the singularity inside the black hole). This is most reasonable because there is no matter to support electric charge. In our case, however, there is a perfect fluid which may perfectly be charged. So, we admit a charge current $\vec{j}$ proportional to the fluid velocity $\vec{u}$. Hence, Maxwell's equations read

$$d\boldsymbol{K}=0, \quad d\star\boldsymbol{K}=4\pi\star\boldsymbol{j}, \quad \boldsymbol{j}=C\boldsymbol{\xi}, \tag{12}$$

where C is a scalar function. Our aim is to solve Einstein-Maxwell's field equations under these assumptions. Although the calculations are not very difficult, some manipulations are required. The details are given in the Appendix. The solution reads as follows:

$$U_1(z)=Q_0-2\alpha_i^2+a_1\frac{\sinh(2\beta z)}{2\beta} +\frac{\gamma z}{4}\left[(8\alpha_i-2\gamma z)\cosh(2\beta z)+3\gamma\frac{\sinh(2\beta z)}{\beta}\right] +(\nu_0+2\beta^2\alpha_r^2+2\beta^2\alpha_i^2)\frac{1-\cosh(2\beta z)}{2\beta^2} -\frac{\mu_0}{2\beta^2}\left[\frac{\cosh(2\beta z)-1}{\beta^2}-\frac{z\sinh(2\beta z)}{\beta}\right], \tag{13}$$

$$V_1(y)=Q_0+2\alpha_r^2+a_2\frac{\sin(2\beta y)}{2\beta} +\frac{\gamma y}{4}\left[(2\gamma y-8\alpha_r)\cos(2\beta y)-3\gamma\frac{\sin(2\beta y)}{\beta}\right] +(\nu_0-2\beta^2\alpha_r^2-2\beta^2\alpha_i^2)\frac{1-\cos(2\beta y)}{2\beta^2}$$

$$+\frac{\mu_0}{2\beta^2}\left[\frac{1-\cos(2\beta y)}{\beta^2}-\frac{y\sin(2\beta y)}{\beta}\right], \qquad (14)$$

where β, Q_0, μ_0, a_1, a_2, ν_0, α_r, α_i and γ are arbitrary constants. These symbols have been chosen so that the uncharged subcase (i.e. the Wahlquist metric) can be directly obtained just by setting $\gamma=\alpha_r=\alpha_i=0$. Thus, the Wahlquist-Newman family of metrics contains three more essential parameters than the Wahlquist family. It is worth pointing out that Kerr–Newman–de Sitter has only one additional parameter with respect to the Kerr–de Sitter metric (i.e. the charge of the black hole). The difference comes from the fact that, in our case, a non-vanishing charge current $\vec{j}$ is allowed. The electromagnetic field $\boldsymbol{K}$ of the Wahlquist-Newman spacetime is

$$\boldsymbol{K}=X_r\theta^0\wedge dy-X_i\theta^1\wedge dz,$$

where $\theta^0=d\tau-v_1d\sigma$, $\theta^1=d\tau+v_2d\sigma$, and the functions X_r and X_i are the real and imaginary parts of the complex function $X=X_r+iX_i$ given by

$$X=\frac{1}{(v_1+v_2)^2}\left[\frac{1-\cos(2\beta y)\cosh(2\beta z)}{\beta^2}-i\frac{\sin(2\beta y)\sinh(2\beta z)}{\beta^2}\right]\times\left[\alpha_r-\frac{\gamma y}{2}+\frac{\gamma}{4}\frac{\sin(2\beta y)}{\beta}\cosh(2\beta z)+i\left(\alpha_i-\frac{\gamma z}{2}+\frac{\gamma}{4}\frac{\sinh(2\beta z)}{\beta}\cos(2\beta y)\right)\right].$$

The charge current of this electromagnetic field is

$$\vec{j}=\frac{\gamma\beta^2}{2\pi}\frac{\partial}{\partial\tau}. \qquad (15)$$

Thus, the constant γ is directly related to the charge of the particles in the fluid. Notice that the value $\gamma=0$ (i.e. uncharged particles) is perfectly possible. In that case, the source of the electromagnetic field lies in the singularity $v_1+v_2=0\Leftrightarrow z=0, y=n\pi/\beta, n\in\mathbb{Z}$, analogously as in the Kerr–Newman–de Sitter metric. When $\gamma=0$, the electromagnetic field is described by the two constants α_r and α_i but only the combination $\alpha_r^2+\alpha_i^2$ appears in the metric. This reflects the well-known electromagnetic duality symmetry of the source-free Einstein-Maxwell field equations. Thus, for uncharged particles the Wahlquist-Newman family adds only one parameter to the Wahlquist family, exactly the same as in the Kerr–Newman–de Sitter case.

Regarding the perfect fluid, its velocity is, by assumption, proportional to ∂_τ so only the energy-density ρ and pressure p remain to be given. They can be directly obtained from the expressions

$$\frac{\rho+3p}{2}=\mu_0+\beta^2\gamma^2+\beta\gamma\times\frac{(2\alpha_r-\gamma y)\sin(2\beta y)+(2\alpha_i-\gamma z)\sinh(2\beta z)}{v_1+v_2},$$

$$\rho+p=2\beta^2\lambda, \qquad (16)$$

where $\lambda=(V_1-U_1)(v_1+v_2)=-\xi^\alpha\xi_\alpha$ is minus the squared norm of the Killing vector. When the particles are uncharged ($\gamma=0$) the perfect fluid satisfies $\rho+3p=2\mu_0$ as in the Wahlquist family. When $\gamma\neq0$, there is no functional relation between ρ and p and therefore no barotropic equation of state. Thus, the presence of an electric charge in the particles seems to change the thermodynamic properties of the perfect fluid (this cannot be made certain until a proper thermodynamic analysis is done).

From Eq. (14) we observe that both functions V_1 and U_1 have a smooth limit $\beta\rightarrow0$ (the integration constants were chosen carefully so that this property holds). The expressions for the density and pressure (16) shows that $\beta=0$ corresponds to having no perfect fluid but rather a cosmological constant with value $\Lambda=-\mu_0$. The electromagnetic field in this case is source-free, as it should be because no matter is present. The explicit form for v_1, v_2, U_1 and V_1 in the limit $\beta=0$ is, after redefining a_1, a_2 and ν_0 so that the constant γ disappears (no trace of γ can be left in this case because the charge current vanishes)

$$v_1=z^2,\quad v_2=y^2,\quad U_1=Q_0-2\alpha_i^2+a_1z-\nu_0z^2+\frac{\mu_0}{3}z^4,$$

$$V_1=Q_0+2\alpha_r^2+a_2y+\nu_0y^2+\frac{\mu_0}{3}y^4. \qquad (17)$$

The metric (11) with the functions (17) is the Plebański limit of the rotating C metric, as expected, and therefore it contains the Kerr–Newman–de Sitter metric as a particular case.

Hence, metric (11) contains both the Wahlquist and Kerr–Newman–de Sitter metrics. Furthermore, a simple calculation shows that the geometric relationship (1) is also satisfied by this metric. Since the self-dual electromagnetic field is proportional to the Killing form of $\vec{\xi}$ by construction, we conclude that Eq. (11) is the Wahlquist-Newman metric we are seeking (this completes Table I). This family of metrics contains eight arbitrary parameters (or nine if we count β).

Finally, we can now see why both functions U_1 and V_1 had to be modified instead of only one as in Kerr–Newman–de Sitter. In the cosmological constant case, both functions get modified by the inclusion of an electromagnetic field [see Eq. (17)]. However, a redefinition of Q_0 can be used to compensate one of the changes. In the perfect-fluid case, the modifications are more complicated and cannot be reabsorbed by redefinitions of constants.

V. EXTENSION OF THE WAHLQUIST-NEWMAN SOLUTION AND STATIC LIMIT

The metric as written in Eq. (11) does not have an obvious static limit. Analyzing whether such a limit exists is relevant because the Wahlquist metric has an interesting, spherically symmetric static limit, namely the Whittaker solution [16] which represents an isolated fluid ball in equilibrium. Moreover, the static limit of the Plebański metric has interesting subcases, like the fundamental Schwarzschild–de Sitter–Reissner–Nordström metric or the so-called rotating topological black holes (see e.g. [17]). Thus, it is reasonable to expect that the static limit of the Wahlquist-Newman spacetime may also have interesting properties. We devote this section to find this limit.

To do that, the coordinate system in Eq. (11) must clearly be changed. We choose a coordinate system which, in addition, extends the metric (11) across its Killing horizon, which is contained within the set of points where the Killing vectors ∂_η and ∂_σ span a null two-plane. Notice, that this can only happen at points where $\lambda \leqslant 0$. From the perfect-fluid interpretation of Eq. (11) this would seem to be impossible. However, the energy-momentum tensor of Eq. (11) is regular at the points where $\lambda=0$, i.e. at the ergospheres of the Killing vector $\vec{\xi}$. Indeed, the electromagnetic field is easily seen to be regular there and even though the velocity of the perfect fluid becomes singular where $\lambda=0$, the combination $(\rho+p)u_\alpha u_\beta = \lambda^{-1}(\rho+p)\xi_\alpha\xi_\beta = 2\beta^2\xi_\alpha\xi_\beta$ is finite. Obviously the perfect-fluid interpretation breaks down at the ergospheres of $\vec{\xi}$ but still the spacetime is regular. This indicates that horizons may also be present in the Wahlquist-Newman spacetime. In order to find them, we should evaluate $N=(\partial_\tau,\partial_\tau)(\partial_\sigma,\partial_\sigma)-(\partial_\tau,\partial_\sigma)^2$ where (,) means scalar product with the metric (11). A simple calculation gives $N=V_1U_1$. Thus, N vanishes at the points where either V_1 or U_1 vanish. It is not clear *a priori* whether we should try to extend the metric across the hypersurface $V_1=0$ or across the hypersurface $U_1=0$. We know from Eq. (9) that the coordinate y is radial and z angular, at least in the limit $\beta=0$ without electromagnetic field. Therefore, we choose to extend the spacetime across $V_1(y)=0$. Let us choose the region $V_1(y)>0$ and define the following coordinate transformation:

$$v=\tau+\int \frac{v_2}{V_1}dy, \varphi=-\sigma+\int \frac{1}{V_1}dy.$$

It is easy to check that the metric can be cast into the form

$$ds^2=-\lambda(dv+v_1d\varphi)^2+2(dy-U_1d\varphi)(dv+v_1d\varphi) +Q\left(\frac{dz^2}{U_1}+U_1d\varphi^2\right), \quad (18)$$

where $Q\equiv v_1+v_2$. This metric is regular at $V_1(y)=0$ and can therefore be extended. It is straightforward to check that the hypersurface $y=y_0$ with $V_1(y_0)=0$ is null and that the Killing vector $-v_2(y_0)\partial_\tau+\partial_\sigma$ is also null and tangent to this hypersurface. Thus, $y=y_0$ is a Killing horizon. Extensions of spacetimes are not unique in general. The extension we have performed, however, is uniquely determined by the geometric condition (1) which still holds in the extended spacetime. Thus, this is the natural extension of the Wahlquist-Newman metric from the geometrical point of view. It must be emphasized, however, that this extension may not be the most relevant from the physical point of view because the extended region contains, in addition to an electromagnetic field, a charged tachyonic fluid, which is rather unphysical.

We can now try to determine the static limit of Eq. (18). From Kerr–de Sitter, we know that some limit $z\to$const will be involved. So, we should avoid using z as a coordinate. We accomplish this as follows. Let us consider a connected two-dimensional manifold S endowed with the metric

$$h=\frac{1}{U_1(z)}dz^2+U_1(z)d\varphi^2, \quad (19)$$

and volume form $\eta_h=dz\wedge d\varphi$. Denote by $\star_h$ the Hodge dual in (S,h,η_h). We obviously have $\star_h dz=U_1d\varphi$ and $d\star_h dz=(dU_1/dz)\eta_h$. Furthermore, the one-form $\boldsymbol{\omega}=-v_1(z)d\varphi$ on S satisfies $d\boldsymbol{\omega}=-(dv_1/dz)\eta_h$. The scalar curvature of the metric (19) is easily computed to be $R(h)=-d^2U_1/dz^2$. With these definitions, the metric (18) can be written as

$$ds^2=-\lambda(dv-\boldsymbol{\omega})^2+2(dy-\star_h dz)(dv-\boldsymbol{\omega})+Qh. \quad (20)$$

In this metric, z need not be a coordinate any longer and can be regarded just as a real function defined on S. The functions Q and λ depend on the spacetime point only through the values of y and z at that point. $\vec{\xi}=\partial_v$ is static if

$$\boldsymbol{\xi}\wedge d\boldsymbol{\xi}=-V_1\lambda_{,z}dv\wedge\eta_h+dy\wedge[Q\lambda_{,z}\eta_h+(\lambda_{,z}dz+\lambda_{,y}\star_h dz)\wedge(\boldsymbol{\omega}-dv)]=0,$$

which holds if and only if $\lambda_{,z}=0$ and $\star_h dz=0$. Thus, $z=z_0=$const and $\lambda_{,z}|_{z=z_0}=0$. From $\lambda=(V_1-U_1)/Q$ and Eqs. (7), (14) this can only happen iff $z_0=0$ and $a_1=-2\gamma\alpha_i$. In that case, the one-form $\boldsymbol{\omega}$ and the scalar curvature of h are

$$d\boldsymbol{\omega}=-\left(\left.\frac{dv_1}{dz}\right|_{z=0}\right)\eta_h=0,$$

$$R(h)=-\left.\frac{d^2U_1}{dz^2}\right|_{z=0}=2[\nu_0+2\beta^2(\alpha_r^2+\alpha_i^2)-\gamma^2].$$

Thus, $\boldsymbol{\omega}$ is locally exact and can be reabsorbed into the coordinate v. Since h is of constant curvature, there exist coordinates x_1 and x_2 such that

$$h=B^2[dx_1^2+\Sigma(x_1,\epsilon)dx_2^2],$$

where $\Sigma(-1,\ x_1)=\sinh(x_1)$, $\Sigma(0,\ x_1)=x_1$ and $\Sigma(1,\ x_1)=\sin(x_1)$ and $B\in\mathbb{R}$ satisfies

$$\epsilon B^{-2} = \nu_0 + 2\beta^2(\alpha_r^2 + \alpha_i^2) - \gamma^2. \tag{21}$$

Inserting this into Eq. (20) we find that the static limit of the Wahlquist-Newman metric is

$$ds^2 = -\tilde{\lambda} dv^2 + 2dydv + \frac{1-\cos(2\beta y)}{2\beta^2} \times B^2 (dx_1^2 + \Sigma(x_1,\epsilon) dx_2^2), \tag{22}$$

where $\tilde{\lambda} \equiv \lambda(y,0)$ reads explicitly

$$\tilde{\lambda} = (\nu_0 - 2\beta^2\alpha_r^2 - 2\beta^2\alpha_i^2) + \frac{2\beta^2}{1-\cos(2\beta y)}\Bigg\{ 2(\alpha_r^2+\alpha_i^2) + a_2 \frac{\sin(2\beta y)}{2\beta} + \frac{\mu_0}{2\beta^2}\left[\frac{1-\cos(2\beta y)}{\beta^2} - \frac{y\sin(2\beta y)}{\beta}\right] + \frac{\gamma y}{4}\left[(2\gamma y - 8\alpha_r)\cos(2\beta y) - 3\gamma\frac{\sin(2\beta y)}{\beta}\right]\Bigg\}.$$

We call this metric Whittaker-Reissner-Nordström metric. Its energy-momentum tensor is (in the region $\lambda > 0$) the sum of a perfect-fluid and an electromagnetic field. The density and pressure of the perfect fluid can be read off from Eq. (16) after inserting $z=0$. The electromagnetic field can be obtained by performing the coordinate changes we made to get the static limit. The result is

$$K = \frac{2\beta^2}{1-\cos(2\beta y)}\left[2\alpha_r + \gamma\left(\frac{\sin(2\beta y)}{2\beta} - y\right)\right] \times dv \wedge dy - 2\alpha_i \eta_h,$$

where the two-form η_h is $\eta_h = B^2\Sigma(x_1,\epsilon) dx_1 \wedge dx_2$. Its charge current is still given by Eq. (15). The metric (22) is static and spherically symmetric as long as $[\nu_0 + 2\beta^2(\alpha_r^2 + \alpha_i^2) - \gamma^2] > 0$. When this expression is zero or negative, the spacetime is plane symmetric and "hyperbolic" symmetric respectively. When the electromagnetic field vanishes and $\nu_0 > 0$ the metric is the spherically symmetric perfect-fluid found by Whittaker [16]. The limit $\beta \to 0$ gives the de Sitter–Reissner–Nordström metric (when $\epsilon = 1$) or its hyperbolic or plane counterparts. Another physically relevant subcase of Whittaker-Reissner-Nordström is $\alpha_r = \alpha_i = a_2 = 0$ and $\nu_0 > \gamma^2$. This represents a charged fluid ball in equilibrium, with no singularities inside.

A thorough investigation of the geometry of the Wahlquist-Newman and Whittaker-Reissner-Nordström spacetimes would be of interest. Also, studying the physical applications of this geometrically privileged metrics should be a matter of further investigation.

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APPENDIX

In this appendix we solve the Einstein-Maxwell equations under the assumptions described in Sec. IV. Let us start by introducing an orthogonal tetrad

$$\theta^0 = d\tau - v_1 d\sigma, \quad \theta^1 = d\tau + v_2 d\sigma, \quad \theta^2 = dy, \quad \theta^3 = dz, \tag{A1}$$

so that the metric (11) takes the form

$$ds^2 = -\frac{V_1}{v_1+v_2}(\theta^0)^2 + \frac{U_1}{v_1+v_2}(\theta^1)^2 + \frac{v_1+v_2}{V_1}(\theta^2)^2 + \frac{v_1+v_2}{U_1}(\theta^3)^2.$$

We take the volume form $\eta = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$. Lowering the indices to $\vec{\xi} = \partial_\tau$ we find

$$\boldsymbol{\xi} = -\frac{V_1}{v_1+v_2}\theta^0 + \frac{U_1}{v_1+v_2}\theta^1.$$

In order to impose $\mathcal{K}_{\alpha\beta} \propto \mathcal{F}_{\alpha\beta}$, we need to evaluate the Killing form $\boldsymbol{\mathcal{F}}$ associated to $\vec{\xi}$. After a simple computation we obtain

$$\boldsymbol{\mathcal{F}} = \frac{1}{2}\left[\frac{V_{1,y} + iU_{1,z}}{Q} + \frac{V_1 - U_1}{Q^2}(iv_{1,z} - v_{2,y})\right] \times (\theta^0 \wedge \theta^2 + i\theta^1 \wedge \theta^3), \tag{A2}$$

where $Q = v_1 + v_2$. Thus, two of the three linearly independent (complex) coefficients of the Killing form $\boldsymbol{\mathcal{F}}$ are identically zero. Since the fluid velocity $\vec{u} \propto \vec{\xi}$, the perfect-fluid part of the energy-momentum tensor reads

$$T^{pf} = \left(D\frac{V_1^2}{Q^2} - p\frac{V_1}{Q}\right)(\theta^0)^2 - 2D\frac{V_1U_1}{Q^2}\theta^0\theta^1 + \left(D\frac{U_1^2}{Q^2} + p\frac{U_1}{Q}\right)(\theta^1)^2 + pQ\left(\frac{(\theta^2)^2}{V_1} + \frac{(\theta^3)^2}{U_1}\right),$$

where p is the pressure and the density ρ is obtained from the scalar D by $\rho + p = Q^{-1}D(U_1 - V_1)$. The electromagnetic field $\boldsymbol{K}$ is required to satisfy

$$\boldsymbol{\mathcal{K}} = X[\theta^0 \wedge \theta^2 + i\theta^1 \wedge \theta^3],$$

where X is a complex scalar function. Hence the electromagnetic energy-momentum tensor $T^{em}_{\mu\nu} = (1/4)\mathcal{K}_{\mu\alpha}\bar{\mathcal{K}}_\nu{}^\alpha$ takes the diagonal form

$$T^{em} = \frac{1}{2}X\bar{X}\left[\frac{V_1}{Q}(\theta^0)^2 + \frac{U_1}{Q}(\theta^1)^2 - \frac{Q}{V_1}(\theta^2)^2 + \frac{Q}{U_1}(\theta^3)^2\right].$$

Using units in which $8\pi G=c=1$ and denoting by $G_{\alpha\beta}$ the Einstein tensor of Eq. (11), the Einstein equations $G_{\mu\nu}=T^{em}_{\mu\nu}+T^{pf}_{\mu\nu}$ become

$$G_{00}+\frac{V_1}{U_1}G_{01}+\frac{V_1^2}{Q^2}G_{22}=0,\quad G_{11}+\frac{U_1}{V_1}G_{01}-\frac{U_1^2}{Q^2}G_{33}=0, \tag{A3}$$

$$D=-\frac{G_{01}Q^2}{V_1U_1},\quad p=\frac{V_1G_{22}}{2Q}+\frac{U_1G_{33}}{2Q}, \tag{A4}$$

$$X\overline{X}=\frac{U_1G_{33}}{Q}-\frac{V_1G_{22}}{Q}. \tag{A5}$$

The two equations (A3) are identically satisfied by the metric (11). Actually, it can be proven that allowing $v_1(y)$ and $v_2(z)$ to be arbitrary, the two equations (A3) force them to be Eq. (7). Thus, our assumption that v_1 and v_2 remain unchanged implies no loss of generality. The two equations in Eq. (A4) can be regarded as defining expressions for ρ and p (we do not impose any equation of state for the perfect fluid *a priori*). Equation (A5) needs to be solved in combination with the Maxwell's equations (12), which we now analyze. The electromagnetic field is required to be Lie constant along the Killing vector fields ∂_τ and ∂_σ. Thus $X=X(y,z)$. In our case, it is simpler to solve Maxwell's equations by looking for an electromagnetic potential $\boldsymbol{A}$ satisfying $d\boldsymbol{A}=\boldsymbol{K}$. Since $\boldsymbol{A}$ can be chosen to be Lie constant along the Killing vectors, we can write

$$\boldsymbol{A}=A_0(y,z)\theta^0+A_1(y,z)\theta^1+A_2(y,z)\theta^2+A_3(y,z)\theta^3,$$

so that its exterior derivative takes the form

$$\begin{aligned} d\boldsymbol{A}=&\left[-\partial_yA_0+\frac{\sin(2\beta y)}{\beta}\frac{A_1}{Q}\right]\theta^0\wedge\theta^2\\ &-\left[\partial_zA_0+\frac{\sinh(2\beta z)}{\beta}\frac{A_0}{Q}\right]\theta^0\wedge\theta^3\\ &-\left[\partial_yA_1+\frac{\sin(2\beta y)}{\beta}\frac{A_1}{Q}\right]\theta^1\wedge\theta^2\\ &+\left[-\partial_zA_1+\frac{\sinh(2\beta z)}{\beta}\frac{A_0}{Q}\right]\theta^1\wedge\theta^3\\ &+[\partial_yA_3-\partial_zA_2]\theta^2\wedge\theta^3. \end{aligned} \tag{A6}$$

Decomposing X into its real and imaginary parts $X=X_r+iX_i$, the electromagnetic field reads $\boldsymbol{K}=X_r\theta^0\wedge\theta^2-X_i\theta^1\wedge\theta^3$. Imposing now $d\boldsymbol{A}=\boldsymbol{K}$ we obtain, first of all, that the coefficient in $\theta^2\wedge\theta^3$ must vanish. Thus $A_2\theta^2+A_3\theta^3$ is closed and can be redefined away by a gauge transformation. So, we can put $A_2=A_3=0$. The vanishing of the coefficients in $\theta^0\wedge\theta^3$ and $\theta^1\wedge\theta^2$ in (A6) implies $A_0=\tilde{A}_0(y)/Q$ and $A_1=\tilde{A}_1(z)/Q$. The remaining components of $d\boldsymbol{A}=\boldsymbol{K}$ give expressions for X_r and X_i in terms of $\tilde{A}_0$ and $\tilde{A}_1$ and their derivatives. A convenient way of writing them is

$$X_r=-\partial_y\left[\frac{\tilde{A}_0+\tilde{A}_1}{v_1+v_2}\right],\quad X_i=\partial_z\left[\frac{\tilde{A}_0+\tilde{A}_1}{v_1+v_2}\right]. \tag{A7}$$

We turn now into the equation $d\star\boldsymbol{K}=4\pi\star\boldsymbol{j}$, which after using the form of $\boldsymbol{K}$ and $\boldsymbol{j}$ reads

$$\begin{aligned} &\left(X_{i,z}+X_i\frac{v_{1,z}}{Q}+X_r\frac{v_{2,y}}{Q}\right)\theta^0\wedge\theta^2\wedge\theta^3\\ &\quad+\left(-X_{r,y}-X_i\frac{v_{1,z}}{Q}-X_r\frac{v_{2,y}}{Q}\right)\theta^1\wedge\theta^2\wedge\theta^3\\ &=4\pi C(\theta^1\wedge\theta^2\wedge\theta^3-\theta^0\wedge\theta^2\wedge\theta^3). \end{aligned} \tag{A8}$$

This implies $X_{r,y}-X_{i,z}=0$, or using Eq. (A7),

$$(\partial_{yy}+\partial_{zz})\left[\frac{\tilde{A}_0+\tilde{A}_1}{Q}\right]=0. \tag{A9}$$

Defining the complex variable $\zeta=y+iz$, the general solution of Eq. (A9) is $\tilde{A}_0+\tilde{A}_1=Q\cdot[g(\zeta)+\overline{g(\zeta)}]$, where g is a holomorphic function of ζ. In terms of ζ, the function $Q=v_1+v_2$ becomes simply $Q=\beta^{-2}\sin(\beta\zeta)\sin(\beta\overline{\zeta})$. It remains to impose that $\tilde{A}_0$ and $\tilde{A}_1$ depend only on y and z respectively, or equivalently $(\partial_{\zeta\zeta}-\partial_{\overline{\zeta\zeta}})(\tilde{A}_0+\tilde{A}_1)=0$. This implies the following equation for g:

$$g_{,\zeta\zeta}+2\beta\frac{\cos(\beta\zeta)}{\sin(\beta\zeta)}g_{\zeta}=\overline{g}_{,\overline{\zeta\zeta}}+2\beta\frac{\cos(\beta\overline{\zeta})}{\sin(\beta\overline{\zeta})}\overline{g}_{\overline{\zeta}}.$$

Thus, there exits a real constant $\beta^2\gamma$ such that each term of this equation equals $\beta^2\gamma$. The resulting ordinary differential equation (ODE) can be integrated once to give (after choosing the integration constant so that the limit $\beta\to0$ exists)

$$g_{\zeta}=\frac{\beta^2}{\sin^2(\beta\zeta)}\left[-\alpha+\gamma\left(\frac{\zeta}{2}-\frac{\sin(\beta\zeta)\cos(\beta\zeta)}{2\beta}\right)\right], \tag{A10}$$

where α is an arbitrary complex constant. From this expression we could easily integrate $g(\zeta)$ and obtain $\boldsymbol{A}$. However, to obtain $\boldsymbol{K}$ we only need to determine X,

$$X=X_r+iX_i=\left(-\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)\left(\frac{\tilde{A}_0+\tilde{A}_1}{Q}\right)=-2\frac{\partial}{\partial\zeta}(g(\zeta)+\overline{g(\zeta)})=-2g_{,\zeta}.$$

The scalar C can now be read off from Eq. (A8), the result being $C=\gamma\beta^2/2\pi$. We can now solve the Einstein field equation (A5). First, we need to evaluate $X\overline{X}$. Decomposing α into its real and imaginary parts as $\alpha=\alpha_r+i\alpha_i$, and using Eq. (A10) we get $X\overline{X}=4Q^{-1}Y\overline{Y}$, where

$$Y=-\alpha_r+\frac{\gamma y}{2}-\frac{\gamma}{4}\frac{\sin(2\beta y)}{\beta}\cosh(2\beta z)+i\left(-\alpha_i+\frac{\gamma z}{2}-\frac{\gamma}{4}\frac{\sinh(2\beta z)}{\beta}\cos(2\beta y)\right).$$

Einstein's equation (A5) reads, after dropping a factor $Q=v_1+v_2$,

$$(v_1+v_2)(U_1G_{33}-V_1G_{22})-4Y\overline{Y}=0, \qquad \text{(A11)}$$

which is a rather long equation involving the functions V_1, U_1 and their derivatives. Since they are functions of different variables, a reasonable strategy is to try and separate this equation. This can be accomplished after taking the partial derivative of Eq. (A11) with respect to y and z. The resulting expression separates nicely into the form

$$\frac{\beta}{\sin(2\beta y)}[V_{1,yyy}+4\beta^2V_{1,y}+8\beta^2\gamma\cos(2\beta y)(\gamma y-2\alpha_r)]=\frac{\beta}{\sinh(2\beta z)}[U_{1,zzz}-4\beta^2U_{1,z}+8\beta^2\gamma\cosh(2\beta z)\times(\gamma z-2\alpha_i)]=4\mu_0,$$

where μ_0 is the separation constant. Thus, we are faced with two linear, third order ordinary differential equations. Their solution can be explicitly written down in the following form, after choosing carefully the integration constants so that the limit $\beta\rightarrow 0$ exists:

$$U_1=L_1+a_1\frac{\sinh(2\beta z)}{2\beta}+\frac{\gamma z}{4}\left[(8\alpha_i-2\gamma z)\cosh(2\beta z)+3\gamma\frac{\sinh(2\beta z)}{\beta}\right]+S_1\frac{\cosh(2\beta z)-1}{2\beta^2}-\frac{\mu_0}{2\beta^2}\left[\frac{\cosh(2\beta z)-1}{\beta^2}-\frac{z\sinh(2\beta z)}{\beta}\right],$$

$$V_1=L_0+a_2\frac{\sin(2\beta y)}{2\beta}+\frac{\gamma y}{4}\left[(2\gamma y-8\alpha_r)\cos(2\beta y)-3\gamma\frac{\sin(2\beta y)}{\beta}\right]+S_0\frac{1-\cos(2\beta y)}{2\beta^2}+\frac{\mu_0}{2\beta^2}\left[\frac{1-\cos(2\beta y)}{\beta^2}-\frac{y\sin(2\beta y)}{\beta}\right],$$

where L_0, L_1, S_0, S_1, a_1 and a_2 are integration constants. Inserting these expressions back into the Einstein equation (A11), we find that the equation is satisfied if and only if $S_1+S_0=-4\beta^2(\alpha_r^2+\alpha_i^2)$ and $L_1=L_0-2(\alpha_r^2+\alpha_i^2)$. By redefining S_0, S_1 and L_0 in a trivial way we obtain the form for U_1 and V_1 given in Eq. (14).

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