Exact solutions of a charged wormhole

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In this paper, the back reaction to the traversable Lorentzian wormhole spacetime by a scalar field or electric charge is considered to find the exact solutions. The charges play the role of the additional matter to the static wormhole which is already constructed by the exotic matter. The stability conditions for the wormhole with a scalar field and electric charge are found from the positiveness and flareness for the wormhole shape function.

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I. INTRODUCTION

To make a Lorentzian wormhole traversable, one usually uses exotic matter, which violates the well-known energy conditions according to the need of the geometrical structure [1,2]. Even though various efforts were tried to avoid using the exotic matter, all were in vain. Among the models, a wormhole in the inflating cosmological model still required exotic matter to be traversable and to maintain its shape [3]. It is known that the vacuum energy of the inflating wormhole does not change the sign of the exoticity function. However, a traversable wormhole in the Friedmann-Robertson-Walker (FRW) cosmological model did not necessarily require exotic matter at very early times [4]. This result meant that there was an exotic period in the very early universe. As another example, when the wormhole was modified by the generally coupled, massive, classical scalar field without any geometrical modification, the exotic property was compensated with the scalar field [5]. For these or other reasons, the problem about the maintenance of the wormhole by other fields relating to the exotic property has also been interesting to us.

There are two ways to generalize or modify the Lorentzian traversable wormhole spacetime. (From now on "wormhole" will be simply used as the meaning of the "Lorentzian traversable wormhole" unless there is confusion.) One way is the generalization of the wormhole by an alternative theory, for example, Brans-Dicke theory [6]. The other way is the generalization by adding the extra matter.

As an example of the second generalization, the exact solutions of the wormhole with classical, minimally coupled, massless scalar field, and electric charge are discussed in this paper. The back reactions of the scalar field and the electric charge on wormhole spacetime are also found to see the stabilities of the wormhole. We found the stability conditions for the wormhole with scalar field and electric charge from the positiveness and flareness for the wormhole shape function.

Similar works about the scalar field effect on wormholes have been done by several authors. Taylor and Hiscock [7] examined whether the stress energy of quantized fields in fact will have an appropriate form to support a wormhole geometry. They do not attempt to solve the semiclassical Einstein equations. They found that the stress energy tensor of the quantized scalar field is not even qualitatively of the correct form to support the wormhole. To maintain a wormhole classically, Vollick [8] found the effect of coupling a scalar field to matter which satisfies the weak energy condition. Kim and Kim [5] found the solutions of the wormhole with various scalar fields which share the role of exotic matter with other matter. In this case, the stability of the wormhole by the scalar field was not discussed.

II. GENERALIZATION OF WORMHOLE BY EXTRA MATTER

Let the Einstein equation for the simplest normal (usual) wormhole spacetime be

$$G^{(0)}_{\mu\nu} = 8\pi T^{(0)}_{\mu\nu}. \tag{1}$$

The left hand side $G^{(0)}_{\mu\nu}$ is the wormhole geometry and the right hand side $T^{(0)}_{\mu\nu}$ is the exotic matter violating the known energy conditions. The exotic matter is basically required to construct the wormhole.

If the additional matter $T^{(1)}_{\mu\nu}$ is added to the right hand side and the corresponding back reaction $G^{(1)}_{\mu\nu}$ is added to the left hand side, the Einstein equation becomes

$$G^{(0)}_{\mu\nu} + G^{(1)}_{\mu\nu} = 8\pi [T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu}]. \tag{2}$$

The sum of the matters of the right hand side satisfies the conservation law naturally. However, there is no guarantee on the exoticity of the total matter. When the effect on the geometry of wormhole $G^{(1)}_{\mu\nu}$ is large enough to dominate any structure, the additional matter might prevent from sustaining the wormhole.

Apparently, the structure of FRW cosmological model with wormhole [4] is similar to Eq. (2), but not exactly same. In this model, $G^{(1)}_{\mu\nu}$ is the cosmological term which plays the role of the background spacetime and $T^{(1)}_{\mu\nu}$ is the cosmic part of the matter. They are not an additional term as above.

There might be an interaction term $T^{(\mathrm{int})}_{\mu\nu}$ between $T^{(0)}_{\mu\nu}$ and $T^{(1)}_{\mu\nu}$ such as Ref. [8]. The model is the maintaining wormhole by the coupling a scalar field to matter that is not exotic,

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but the coupling is exotic. There also might be the interaction term $G_{\mu\nu}^{(\mathrm{int})}$ in geometry. In some case, $G_{\mu\nu}^{(\mathrm{int})}$ may be joined in Eq. (2) without $T_{\mu\nu}^{(\mathrm{int})}$, so that the equation can have the form as $G_{\mu\nu}^{(0)}+G_{\mu\nu}^{(1)}+G_{\mu\nu}^{(\mathrm{int})}=8\pi(T_{\mu\nu}^{(0)}+T_{\mu\nu}^{(1)})$. This example is the electrically charged wormhole as we will see later.

III. EXACT SOLUTIONS OF WORMHOLE

A. Static wormhole

The spacetime of the static wormhole without any charge is given by

$$ds^2=-e^{2\Lambda(r)}dt^2+\frac{dr^2}{1-b(r)/r}+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{3}$$

The arbitrary functions $\Lambda(r)$ and $b(r)$ are defined as the lapse and wormhole shape functions, respectively. The shape of the wormhole is determined by $b(r)$. There can be two requirements about the wormhole function $b(r)$ for the wormhole in order to be maintained. They are positiveness and flareness conditions. As $r\rightarrow\infty$, $b(r)$ approaches $2M$ which is defined as the mass of wormhole [9]. Therefore $b(r)$ should be defined as the positive function. And the condition $r>b(r)$ which means the existence of the minimum radius also support the positiveness of $b(r)$. Another condition comes from the flare-out condition of the shape of the wormhole. When the proper distance $l\in(-\infty,+\infty)$ is defined as $dl=dr/(1-b/r)$, the condition should be

$$\frac{d^2l}{dr^2}>0 \quad \text{or} \quad \frac{b-b'r}{b^2}>0. \tag{4}$$

The Einstein equation Eq. (1) for the metric Eq. (3) is given as

$$\frac{b'}{8\pi r^2}=\rho^{(0)}, \tag{5}$$

$$\frac{b}{8\pi r^3}-\frac{1}{4\pi}\left(1-\frac{b}{r}\right)\frac{\Lambda'}{r}=\tau^{(0)}, \tag{6}$$

$$\frac{1}{8\pi}\left(1-\frac{b}{r}\right)\left(\Lambda''-\frac{b'r-b}{2r(r-b)}\Lambda'+\Lambda'^2+\frac{\Lambda'}{r}-\frac{b'r-b}{2r^2(r-b)}\right)=P^{(0)}, \tag{7}$$

where a prime denotes the differentiation with respect to r. Assuming a spherically symmetric spacetime, one finds the components of $T_{\hat{\mu}\hat{\nu}}^{(0)}$ in orthonormal coordinates

$$T_{\hat{t}\hat{t}}^{(0)}=\rho^{(0)}(r), \quad T_{\hat{r}\hat{r}}^{(0)}=-\tau^{(0)}(r), \quad T_{\hat{\theta}\hat{\theta}}^{(0)}=P^{(0)}(r), \tag{8}$$

where $\rho^{(0)}(r)$, $\tau^{(0)}(r)$, and $P^{(0)}(r)$ are the mass energy density, radial tension per unit area, and lateral pressure, respectively, as measured by an observer at fixed r,θ,ϕ.

B. Wormhole with scalar field

Consider the simplest case of a static Lorentzian wormhole with a minimally coupled massless scalar field. The additional matter Lagrangian due to the scalar field is given by

$$\mathcal{L}=\frac{1}{2}\sqrt{-g}g^{\mu\nu}\varphi_{;\mu}\varphi_{;\nu} \tag{9}$$

and the equation of motion for φ by

$$\Box\varphi=0. \tag{10}$$

The stress-energy tensor for φ is obtained from Eq. (9) as

$$T_{\mu\nu}^{(1)}=\varphi_{;\mu}\varphi_{;\nu}-\frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\varphi_{;\rho}\varphi_{;\sigma}. \tag{11}$$

Since not only the scalar field φ but also the matter $T_{\mu\nu}$ are assumed to depend only on r similar to the static wormhole geometry, the radiation by this scalar field is not introduced here. The components of $T_{\mu\nu}^{(1)}$ in the static wormhole metric with $\Lambda=0$ have the form

$$T_{tt}^{(1)}=\frac{1}{2}\left(1-\frac{b}{r}\right)\varphi'^2, \tag{12}$$

$$T_{rr}^{(1)}=\frac{1}{2}\varphi'^2, \tag{13}$$

$$T_{\theta\theta}^{(1)}=-\frac{1}{2}r^2\left(1-\frac{b}{r}\right)\varphi'^2, \tag{14}$$

$$T_{\phi\phi}^{(1)}=-\frac{1}{2}r^2\left(1-\frac{b}{r}\right)\varphi'^2\sin^2\theta. \tag{15}$$

In this spacetime, the field equation Eq. (10) of φ becomes

$$\frac{\varphi''}{\varphi'}+\frac{1}{2}\frac{(1-b/r)'}{(1-b/r)}+\frac{2}{r}=0 \quad \text{or} \quad r^4\varphi'^2\left(1-\frac{b}{r}\right)=\text{const}. \tag{16}$$

The integration constant will be represented as $\alpha/4\pi$ for some positive α.

If we add $G_{\mu\nu}^{(1)}$ as an additional geometry to $G_{\mu\nu}^{(0)}$, the effect by scalar field, the Einstein equation Eq. (2) for $\Lambda=0$ is changed from Eqs. (5)–(7) into

$$\frac{b'}{8\pi r^2}+\frac{1}{8\pi}\frac{\alpha}{r^4}=\rho^{(0)}+\frac{1}{2}\varphi'^2\left(1-\frac{b}{r}\right), \tag{17}$$

$$\frac{b}{8\pi r^3}-\frac{1}{8\pi}\frac{\alpha}{r^4}=\tau^{(0)}-\frac{1}{2}\varphi'^2\left(1-\frac{b}{r}\right), \tag{18}$$

$$\frac{b-b'r}{16\pi r^3}-\frac{1}{8\pi}\frac{\alpha}{r^4}=P^{(0)}-\frac{1}{2}\varphi'^2\left(1-\frac{b}{r}\right), \tag{19}$$

when the interaction between the matter and additional scalar fields is neglected. The term α/r^4 is added to the left hand side, because the field equation Eq. (16) for φ shows that $\varphi'^2(1-b/r)\propto r^{-4}$.

If we put $b_{\rm eff}=b-\alpha/r$ instead of b and $T^{\rm eff}_{\mu\nu}=T^{(0)}_{\mu\nu}+T^{(1)}_{\mu\nu}$ instead of $T^{(0)}_{\mu\nu}$ into Eqs. (5)–(7) with $\Lambda=0$, then the effective equations will satisfy self-consistently and has the form of Eqs. (17)–(19). Thus the effect by the scalar field on the wormhole is simply represented as the change of the wormhole function b into $(b-\alpha/r)$ without any interaction term in the left hand side. Since b is proportional to $r^{1/(1+2\beta)}$, with the proper parameter β of equation of state [4], the shape of the effective wormhole will vary with the value of parameters β and α via the additional factor $-\alpha/r$. While β is given as the equation of state by the choice of the appropriate matter, α depends on the changing rate of the scalar field φ. Since α plays the role of the scalar charge, the metric of the wormhole spacetime with the scalar field should be

$$ds^2=-dt^2+\left(1-\frac{b(r)}{r}+\frac{\alpha}{r^2}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{20}$$

in the case where $\Lambda=0$. No horizon occurs in this spacetime, since the components are always positive.

For dimensional reasons, the wormhole function has the form

$$b=b_0^{2\beta/(2\beta+1)}r^{1/(2\beta+1)}, \tag{21}$$

where β should be less than $-\frac{1}{2}$ so that the exponent of r can be negative to satisfy the flareness condition.

The positiveness and flareness conditions for the effective wormhole can be written as

$$b_{\rm eff}>0, \tag{22}$$

$$\frac{b_{\rm eff}-b'_{\rm eff}r}{b^2_{\rm eff}}>0. \tag{23}$$

If we rewrite these in terms of $b(r)$,

$$b-\frac{\alpha}{r}>0, \tag{24}$$

$$\gamma b-\frac{\alpha}{r}>0, \tag{25}$$

where $\gamma=\beta/(2\beta+1)$.

When $-1<\beta<-\frac{1}{2}$, the flareness condition is included in the positiveness condition, since $\gamma>1$. That is, if only $b>\alpha/r$, the flareness condition Eq. (25) is satisfied automatically. In this case, the power of $b(r)$ is less than -1 and the function $b(r)$ vanishes more quickly than the second term, in the far region. Thus it gives the negative region for $b_{\rm eff}$ at large r, even though it has positive regions near the throat when $b_0^2>\alpha$. At the throat, the effective wormhole shape function becomes $(1/b_0)(b_0^2-\alpha)$, which shows that the size of neck is reduced by the scalar field. The region of the positive $b_{\rm eff}$ is $b_0<r<r_0$, where $r_0=\alpha^{(2\beta+1)/(2\beta+2)}/b_0^{\beta/(2\beta+1)}$. If $b_0^2<\alpha$, $b_{\rm eff}$ is negative at all r, which is not suitable for a wormhole.

If only $\beta\leqslant-1$ and $b_0^2>\alpha$, the wormhole is safe, because $b_{\rm eff}$ is positive at all r. When $b_0^2<\alpha$, there also be a region $(r<r_0)$ of negative b so that the scalar field effect will change the wormhole structure into others, since the scalar field dominates the exotic matter. Thus the addition of the minimally coupled, massless scalar field does not guarantee the structure of wormhole.

Now we shall examine the special case of this back reaction problem, for instance, $\beta=-1$ which is $b=b_0^2/r$. In this case, the solution of the scalar field is given as [5]

$$\varphi=\varphi_0\left[1-\cos^{-1}\left(\frac{b_0}{r}\right)\right]. \tag{26}$$

Thus the proportional constant α becomes

$$\alpha=4\pi\varphi_0^2b_0^2, \tag{27}$$

where b_0 is the minimum size of the wormhole and φ_0 is the value of $\varphi(r)$ at $r=b_0$. Therefore, $\varphi_0^2<1/4\pi$ is the condition that is required for maintaining the wormhole under the addition of the scalar field. In this choice of $\beta=-1$, there is no r_0 at which the sign of $b_{\rm eff}$ changes.

We can also apply the result to the other form of $b(r)$, which means the exotic matter distribution in the restricted region only, ''absurdly benign'' wormhole [2]

$$b(r)=\begin{cases} b_0[1-(r-b_0)/a_0]^2, \Phi(r)=0 & \text{for } b_0\leqslant r\leqslant b_0+a_0, \\ b=\Phi=0 & \text{for } r\geqslant b_0+a_0 \end{cases} \tag{28}$$

In this case, since the second term $-\alpha/r$ in the effective shape function extends to over the region $r \geqslant b_0 + a_0$, there will be a negative $b_{\rm eff}$ within this range of r, which is not safe for wormhole formation.

C. Wormhole with electric charge

In the case of electric charge, we can follow the same procedures as the case of scalar field. However, the simplification with $\Lambda = 0$ is not adequate here, because the self-consistent solution cannot be found in this case. Thus the work on finding extra terms in geometry will be very complicated one, compared with the scalar field case. We would rather start from assuming the probable spacetime metric and check later whether it will be correct or not. We can presumably set the wormhole spacetime with electric charge Q as

$$ds^2 = -\left(1 + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{b(r)}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{29}$$

This spacetime is the combination of Morris-Thorne (MT) type spherically symmetric static wormhole and Reissner-Nordström (RN) spacetime. When $Q = 0$, the spacetime means the MT wormhole and when $b = 0$, it becomes the RN black hole with zero mass. The metric Eq. (29) should be checked whether it satisfies the Einstein's equation self-consistently or not.

The Einstein-Maxwell equation becomes

$$\frac{b'}{r^2} + \frac{Q^2}{r^4} = 8\pi(\rho^{(0)} + \rho^{(1)}), \tag{30}$$

$$\frac{b}{r^3} - \frac{Q^2}{r^4} - 2\left(1 - \frac{b}{r} + \frac{Q^2}{r^2}\right)\left(-\frac{Q^2}{r^2(r^2+Q^2)}\right) = 8\pi(\tau^{(0)} + \tau^{(1)}), \tag{31}$$

$$\begin{aligned}\left(1 - \frac{b}{r} + \frac{Q^2}{r^2}\right)\Bigg[\frac{Q^2(3r^2+Q^2)}{r^2(r^2+Q^2)^2} - \left(\frac{b'r - b + \frac{2Q^2}{r}}{2(r^2 - br + Q^2)}\right) \\ \times\left(-\frac{Q^2}{r(r^2+Q^2)^2}\right) + \left(-\frac{Q^2}{r(r^2+Q^2)^2}\right)^2 \\ - \frac{Q^2}{r^2(r^2+Q^2)^2} - \frac{b'r - b + \frac{2Q^2}{r}}{2r(r^2 - br + Q^2)}\Bigg] \\ = 8\pi(P^{(0)} + P^{(1)}).\end{aligned} \tag{32}$$

The matter terms are

$$\rho^{(1)} = \tau^{(1)} = P^{(1)} = \frac{Q^2}{8\pi r^4}, \tag{33}$$

since $T^{(1)}_{\mu\nu} = (1/4\pi)(F_{\mu\lambda}F^{\lambda}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma})$ and the electromagnetic strength tensor $F_{\mu\nu}$ is given as

$$F_{\mu\nu} = E(r)\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{34}$$

by the spherical symmetry. The electric field is given as

$$E = \frac{Q}{r^2}\sqrt{g_{tt}g_{rr}} \tag{35}$$

by Maxwell's equation. There are the interacting coupling terms Qb in geometry, even though no interaction terms in matter part at all. The reason is that the zero-tidal force is not assumed in this case unlike the scalar field case.

If

$$b \rightarrow b_{\rm eff} = b - \frac{Q^2}{r},$$

the effective wormhole shape function $b_{\rm eff}$ and the total matter $T^{\rm eff}_{\mu\nu}$ self-consistently satisfy e equations similar to the scalar field case. Therefore, the metric Eq. (29) can be seen as the spacetime of wormhole with electric charge Q.

When $b = b_0^{2\beta/(2\beta+1)} r^{1/(2\beta+1)}$ as in the case of the scalar field, there will be the same sign relations of $b_{\rm eff}$ as shown in Table I, except that α is replaced by Q^2. Here, r_0 which changes the sign of the wormhole becomes

$$r_0 = Q^{(2\beta+1)/(\beta+1)} b_0^{\beta/(2\beta+1)}$$

In the special case of $\beta = -1$, $b = b_0^2/r$, $Q^2 < b_0^2$ is the condition that is required for maintaining the wormhole under the addition of the electric charge. When someone will try to prevent from formation of a wormhole with throat of 1 m radius, he should prepare the charge over than 3×10^{16} Coulombs. This amount of charge is too huge to exist as a single kind of charge.

TABLE I. The sign of the effective wormhole function $b_{\rm eff}(r)$.

Value of β	$b_0^2 > Q^2$ or α [a]	$b_0^2 < Q^2$ or α [a]
$\beta < -1$	$+$	$-+$ [b]
$\beta = -1$	$+$	$-$
$-1 < \beta < -\frac{1}{2}$	$+-$ [b]	$-$

[a] Q^2 in the case of electric charge and α in the case of scalar field.
[b] The signs change at r_0.

IV. DISCUSSION

Here we studied the back reactions to the static wormhole by the scalar field and electric charge and found self-consistent solutions. The charges are considered to be a static case similar to a wormhole, so there is no radiation by the fields. We found exact solutions of the wormhole with extra fields such as scalar field and electric charge. The conditions of the wormhole shape through the requirement of positiveness and flareness were also found.

As we see in Eqs. (20) and (29), no horizon occurs in these wormhole spacetimes with charges, because of the positiveness for the metric components. This means that the addition of charge might change the wormhole but will not change the spacetime seriously.

In this paper, the interaction between the extra field and the original matter is neglected. If the interaction exists and it is large, it might change the whole geometry drastically. If it is very small, it does not seem to change the main structure of the wormhole. A detailed discussion on these interactions will be in a separate paper.

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