Black hole solutions in Euler-Heisenberg theory

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We construct static and spherically symmetric black hole solutions in the Einstein-Euler-Heisenberg (EEH) system which is considered as an effective action of a superstring theory. We consider electrically charged, magnetically charged, and dyon solutions. We can solve analytically for the magnetically charged case. We find that they have some remarkable properties about causality and black hole thermodynamics depending on the coupling constant of the EH theory a and b, though they have a central singularity as in the Schwarzschild black hole. We restrict $a>0$ because it is natural if we think of EH theory as a low-energy limit of the Born-Infeld (BI) theory. (i) For the magnetically charged case, whether or not the extreme solution exists depends on the critical parameter $a=a_{\rm crit}$. For $a\leqslant a_{\rm crit}$, there is an extreme solution as in the Reissner-Nortström (RN) solution. The main difference from the RN solution is that there appear solutions below the horizon radius of the extreme solution and they exist till $r_H\to 0$. Moreover, for $a>a_{\rm crit}$, there is no extreme solution. For arbitrary a, the temperature diverges in the $r_H\to 0$ limit. (ii) For the electrically charged case, the inner horizon appears under some critical mass M_0 and the extreme solution always exists. The lower limit of the horizon radius decreases when the coupling constant a increases. (iii) For the dyon case, we expect a variety of properties because of the term $b(\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma})^2$ which is peculiar to the EH theory. But their properties are mainly decided by the combination of the parameters $a+8b$. We show that solutions have similar properties to the magnetically charged case in the $r_H\to 0$ limit for $a+8b\leqslant 0$. For $a+8b>0$, it depends on the parameters a,b.

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I. INTRODUCTION

Recently, much attention has been paid to Born-Infeld (BI) type of actions after its recognition as an effective theory of superstring theory [1]. Moreover, since they describe the action of the brane, their importance has been increasing [2]. In this context, there have been some studies that investigated black hole solutions in the Einstein-BI type actions [3]. Actually a new nonlinear electromagnetism was proposed, which produces a nonsingular exact black hole solution satisfying the weak energy condition [4,5], and has distinct properties from Bardeen black holes [6]. But there remain subjects which should be manifested such as thermodynamical properties. Here, we concentrate on the EH action which was first proposed in 1936 [7]. Though not so much attention has been payed to it compared with the BI action, the EH action well approximates the supersymmetric system of minimally coupled spin-1/2, -0 particles for appropriate parameters [8]. From the experimental aspect, this is a more accurate classical approximation of QED than Maxwell's theory when fields have high intensity [9].

We investigate the black hole solutions in the Einstein-Euler-Heisenberg (EEH) system from following aspects: (i) The electric-magnetic duality, (ii) the black hole thermodynamics, (iii) the causality and stability of the black holes. As for the electric-magnetic duality, it is already pointed out that though BI action preserves this, EH action breaks it at the higher order of the electromagnetic field [10]. Thus black hole solution with electric charge or magnetic charge will have clear differences which should be clarified. Moreover, the dyon solution may have specific properties which can not be seen in the electrically charged or magnetically charged cases. The thermodynamical properties of black holes are one of the main topics in superstring theory after the discoveries of the microscopic origin of the black hole entropy [11] and the holographic principle [12]. It is worth noting that BI type action also plays an important role in AdS conformal field theory (CFT) correspondence [13]. Causality for the black hole in the EEH system has already been investigated by Oliveira. But this is restricted to the black hole with electric charge, and the physical implications such as the stability of the black hole are not discussed. The stability of the black hole can be interpreted from thermodynamical properties established in Ref. [14] and we refer to this by calling it a turning point method. Using thermodynamical variables, we can easily apply catastrophe theory to hairly black holes and this is consistent with linear perturbation analysis and the turning point method [15]. We are interested in the thermodynamical properties of black holes in this system and the relation between causality and the stability of black holes.

This paper is organized as follows. In Sec. II, we introduce basic *Ansätze* and the field equations in the EEH system. In Sec. III, we investigate the thermodynamical properties of black holes with electric charge or magnetic charge. In Sec. IV, we investigate those of dyonic ones. In Sec. V, we summarize the results and comment on future work. Throughout this paper we use units $c=\hbar=1$. Notations and definitions such as Christoffel symbols and curvature follow Misner-Thorne-Wheeler [16].

II. BASIC EQUATIONS

We take the following EEH action:

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{16\pi}\left(\frac{R}{G}-P+aP^2+bQ^2\right)\right], \qquad (2.1)$$

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where R is the scalar curvature, $P\equiv F^{\mu\nu}F_{\mu\nu}$, $Q\equiv\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ and $\epsilon^{\mu\nu\rho\sigma}$ is a completely antisymmetric unit tensor, which yields

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma}=-4!. \tag{2.2}$$

In Ref. [7], this corresponds to the weak field approximation and the coupling constants are written as $a=he^4/(360\pi^2m^4)$, $b=7he^4/(1440\pi^2m^4)$, where h, e, and m are the Planck constant, electron charge, and electron mass, respectively. From the present point of view, they should be related to the inverse string tension α' which restrict $a>0$ because of its correspondence to the BI action in the low-energy limit. As has been pointed out in Ref. [1], constructing a gravitational counterpart of the BI action is very difficult. Here, we adopt the Einstein-Hilbert action as a first approximation. We can derive Einstein equations as

$$\begin{aligned}G_{\mu\nu}=&\tfrac{1}{2}g_{\mu\nu}(-P+aP^2+bQ^2)+2F_{\mu\lambda}F^{\lambda}_{\nu}\\&-4aP(F_{\mu\lambda}F^{\lambda}_{\nu})-8bQ(\epsilon_{\mu\zeta\eta\vartheta}F^{\zeta\eta}F^{\vartheta}_{\nu}).\end{aligned} \tag{2.3}$$

We consider the metric of static and spherically symmetric,

$$ds^2=-f(r)e^{-2\delta(r)}dt^2+f(r)^{-1}dr^2+r^2d\Omega^2, \tag{2.4}$$

where $f(r)\equiv1-2Gm(r)/r$. We introduce the gauge potential A_μ, as

$$A_\mu=[A(r),0,0,Q_m\cos\theta]. \tag{2.5}$$

Then, the Einstein equations are

$$-\frac{2Gm'}{r^2}=-F_e-F_m-F_{dy}, \tag{2.6}$$

$$-\left(\frac{2Gm'}{r^2}+\frac{2}{r}\delta' f\right)=-F_e-F_m-F_{dy}, \tag{2.7}$$

where $'$ represents d/dr. We used the abbreviations as

$$F_e\equiv e^{2\delta}(A')^2+6ae^{4\delta}(A')^4, \tag{2.8}$$

$$F_m\equiv\frac{Q_m^2}{r^4}-2a\frac{Q_m^4}{r^8}, \tag{2.9}$$

$$F_{dy}\equiv(96b-4a)e^{2\delta}(A')^2\frac{Q_m^2}{r^4}. \tag{2.10}$$

Subtracting Eq. (2.6) from Eq. (2.7) yields $d\delta/dr=0$. We require asymptotically flatness for the solution:

$$A(r)\to-\frac{Q_e}{r},\quad \delta(r)\to0,\quad Gm(r)\to\text{const}, \tag{2.11}$$

as $r\to\infty$. Thus we obtain $\delta=0$. So we have only one independent Einstein equation as

$$Gm'=\frac{r^2}{2}(F_e+F_m+F_{dy}). \tag{2.12}$$

The field equation is

$$4ar^2(A')^3+A'z(r)=Q_e, \tag{2.13}$$

where

$$z(r)\equiv r^2-4(a+8b)\frac{Q_m^2}{r^2}. \tag{2.14}$$

This is third order algebraic equation for A' except for $Q_e=0$. For regularity at the horizon r_H, we require

$$Gm(r_H)=\tfrac{1}{2}r_H,\quad A(r_H)<\infty. \tag{2.15}$$

III. BLACK HOLE SOLUTIONS WITH ELECTRIC OR MAGNETIC CHARGE

In this section, we show the properties of black hole solutions with magnetic or electric charge. First, we point out that the zeroth and the first law of black hole thermodynamics can be applicable even for nonlinear matter terms which violate dominant energy condition though Smarr's formula cannot [17].

A. Magnetically charged case

In the case $Q_e\equiv0$, we can solve equations analytically. In this case, there remains only F_m part in Eq. (2.12). Note that Gm' can be negative which makes an intrinsic difference from the Reissner-Nordström (RN) solution. We can integrate Eq. (2.12):

$$Gm=GM-\frac{Q_m^2}{2r}+a\frac{Q_m^4}{5r^5}, \tag{3.1}$$

where M is the gravitational mass of the black hole. Thus, the horizon radius r_H must satisfy

$$h(r_H)\equiv r_H^6-2GMr_H^5+Q_m^2r_H^4-\tfrac{2}{5}aQ_m^4=0. \tag{3.2}$$

Since $h(0)<0$ and $h(\infty)\to\infty$, the solution which satisfies $h(r_H)=0$ for $r_H>0$ always exists.

From Eq. (3.2),

$$\frac{dh}{dr_H}=2r_H^3(3r_H^2-5GMr_H+2Q_m^2). \tag{3.3}$$

So we can classify the number of the horizon as follows. For $L\equiv(5GM)^2-24Q_m^2>0$, if $h([5GM+\sqrt{L}]/6)<0$ and $h([5GM-\sqrt{L}]/6)>0$, there are three positive solutions, which means that there are one outer horizon and two inner horizons. If $h([5GM+\sqrt{L}]/6)=0$ or $h([5GM-\sqrt{L}]/6)=0$, there are two horizons. In other cases, there is only one horizon.

We also evaluate that what condition would be required to exist an extreme solution. $Gm'=1/2$ leads to

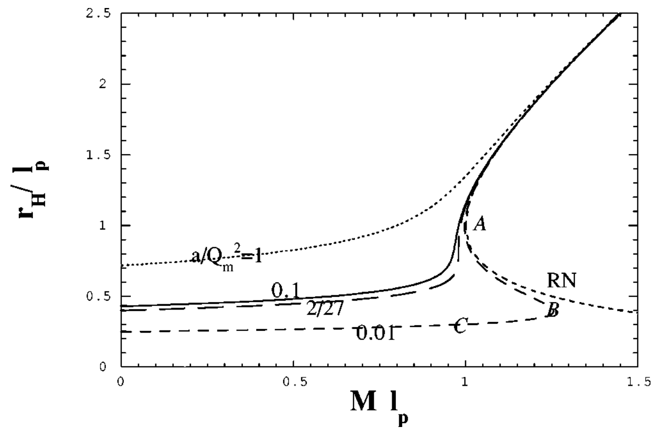

FIG. 1. M-r_H relation for $Q_m/l_p=1$ and $a/Q_m^2=0$, 0.01, 2/27, 0.1, 1. The points A correspond to the extreme solutions. The lines between A to B, B to C correspond to outer inner horizon and inner inner horizon, respectively. We can see that $a_{\mathrm{crit}}=2Q_m^2/27$ divides the properties qualitatively.

$$K(r)\equiv r^6-Q_m^2r^4+2aQ_m^4=0. \tag{3.4}$$

Thus

$$K'=2r^3(3r^2-2Q_m^2). \tag{3.5}$$

So $r=\sqrt{2/3}Q_m$ is a local minimum of K for $r>0$. $K(\sqrt{2/3}Q_m)=0$ leads to $a=(2/27)Q_m^2\equiv a_{\mathrm{crit}}$, which means that there is not an extreme solution for $a>a_{\mathrm{crit}}$.

We first show the relation between the gravitational mass M and the horizon r_H for $Q_m/l_p=1$ and $a/Q_m^2=0$, 0.01, 2/27, 0.1, 1 (Fig. 1). l_p is the Planck length. For $a/Q_m^2=0$, 0.01, 2/27, there is an extreme solution (the point A). The lines between A to B, B to C correspond to the outer inner horizon and the inner horizon, respectively. Note that below the point C, there are black hole solutions again. For $a/Q_m^2=0.1$, 1, there is not an extreme solution as we showed. In all cases, $M\to-\infty$ for $r_H\to 0$. We also show the gravitational mass M and the inverse temperature $1/T$ relation in Fig. 2. Above the mass corresponding to the point A, it is similar to the RN's qualitatively. But below the mass corresponding to the point C is quite different from RN's. The temperature is finite but nonzero at the point C. The curve from C to B means that if we regard an inner horizon as an event horizon, the ''temperature'' goes to zero when r_H approaches the point B. If we apply the turning point method in this case, the line below the point C would be unstable. So we can regard this as unphysical. But for $a/Q_m^2=0.1$, 1, this method suggest that there is no stability change if we think in the isolated system though thermodynamical properties are different in these two cases. The specific heat of the black hole never changes for $a/Q_m^2=1$, while it changes twice at the points D and E for $a/Q_m^2=0.1$. In all cases, the temperature diverges for $r_H\to 0$. It is reasonable that higher order curvature terms would change the results in this region. But even if we believe that this system describes black hole solutions correctly, it is difficult to observe the negative mass black holes since it will evaporate very quickly.

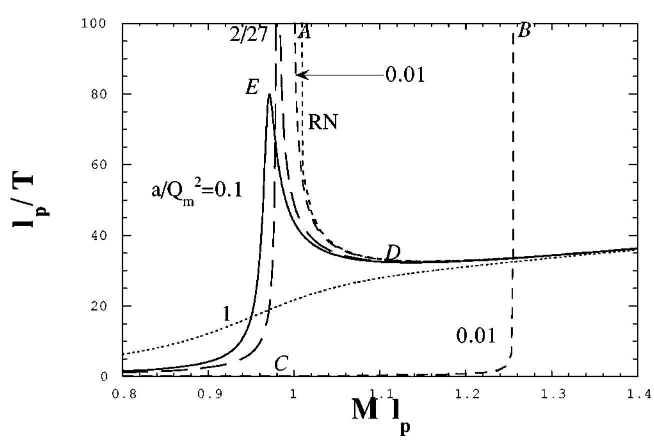

FIG. 2. M-$1/T$ relation for the same parameters in Fig. 1. For $a\leqslant a_{\mathrm{crit}}$, there is an extreme solution where the temperature is zero. For arbitrary a, the temperature diverges in the $r_H\to 0$ limit.

B. Electrically charged case

In this case, from Eq. (2.13),

$$4ar^2(A')^3+A'r^2=Q_e, \tag{3.6}$$

which has only one real solution and two imaginary solutions. The real solution is

$$A'(r)=\frac{-2\times 3^{1/3}r^2+2^{1/3}B^{2/3}}{6^{2/3}\sqrt{x}B^{1/3}}. \tag{3.7}$$

We used an abbreviation as

$$B\equiv 9\sqrt{x}Q_e+\sqrt{12r^6+81xQ_e^2}, \tag{3.8}$$

$$x(r)\equiv 4ar^2. \tag{3.9}$$

There remains only F_e part in Eq. (2.12) which shows $Gm'\geqslant 0$. This is one of the main differences from the magnetically charged case. We show that an extreme solution always exists. If we take $Gm'(r_H)=1/2$, $A'(r_H)$ is evaluated from Eq. (2.12) as

$$A'(r_H)=\frac{(y-1)^{1/2}}{2(3a)^{1/2}}. \tag{3.10}$$

We introduced a dimensionless variable y as

$$y=\sqrt{1+\frac{24a}{r_H^2}}. \tag{3.11}$$

Substituting Eq. (3.10) into Eq. (3.7) derives

$$g(y)\equiv(y-1)^{3/2}+3(y-1)^{1/2}-\frac{Q_e(y^2-1)}{4(3a)^{1/2}}=0. \tag{3.12}$$

Thus,

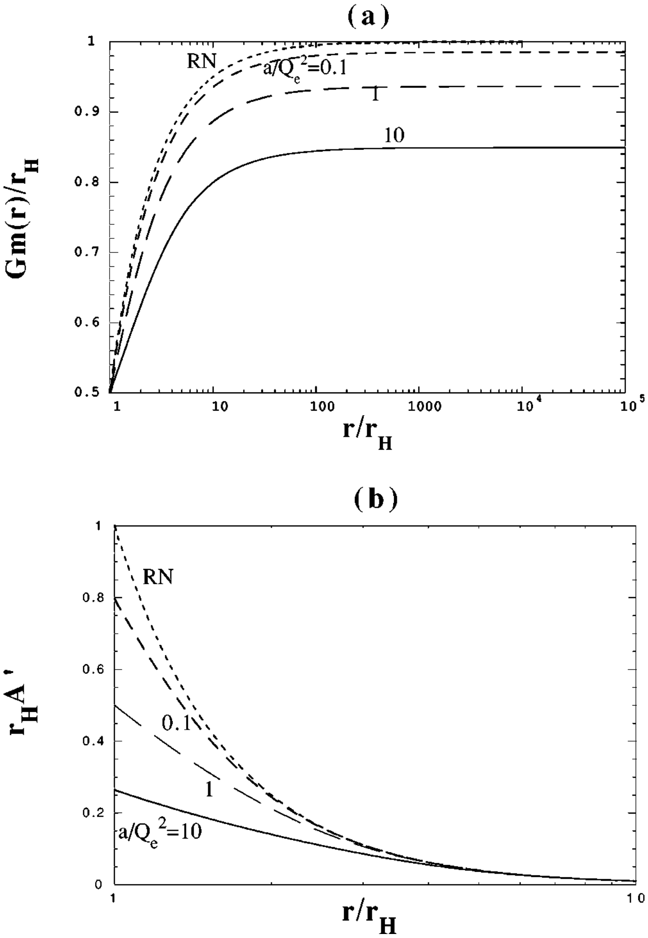

FIG. 3. Field distributions of black holes with electric charge for $r_h/l_p=1$, $Q_e/l_p=1$, and $a/Q_e^2=0,\ 0.1,\ 1,\ 10$ [(a) r-m, (b) r-A']. Because of the difference from Maxwell field at small scale, the resulting solution deviates from the RN black hole near the horizon. A' monotonically decreases as $r\to\infty$ as is easily shown.

$$g'(y)=\frac{3}{2}\frac{y}{(y-1)^{1/2}}-\frac{Q_e y}{2(3a)^{1/2}}. \tag{3.13}$$

Because of $y>1$, the solution of $g'(y)=0$ is $y=y_0=1+27a/Q_e^2(>1)$. We can see $g'(y)>0$ for $1<y<y_0$ and $g'(y)<0$ for $y_0<y$. So if we notice that $g(1)=0$ and $g\to-\infty(y\to\infty)$, we find there is only one positive solution. So we can conclude that there is one extremal black hole for $a>0$.

Electrically charged case has already been investigated previously [18]. The solution can be expressed using the hypergeometric function. But we need numerical calculation to investigate their detailed properties, particularly their thermodynamical properties. The inner horizon only appears for black hole solution $M<M_0$ as he showed. M_0 is

$$M_0=\frac{\Gamma(1/4)}{2\Gamma(3/2)}\frac{Q_e^{3/2}}{(2a)^{1/4}}. \tag{3.14}$$

We first show the field distributions of the solutions [(a)r-m, (b)r-A'] for $r_H/l_p=1$, $Q_e/l_p=1$, and $a/Q_e^2=0,\ 0.1,\ 1,\ 10$

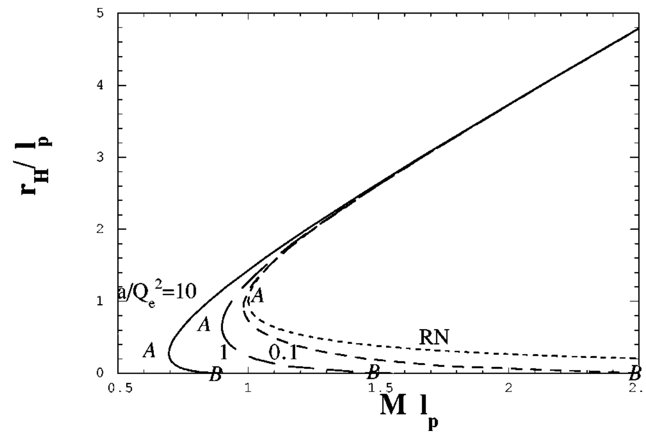

FIG. 4. M-r_H relation for $a/Q_e^2=0,\ 0.1,\ 1,\ 10$. The causality changes at the point B below which the inner horizon appears. Though the lower limit of the horizon decreases as we take a large, the extreme solution always exists.

in Fig. 3. Because its difference from the Maxwell field is particularly large near the event horizon, this makes a nontrivial change for a small black hole. We can evaluate that for $a^{1/2}Q_e\gg r^2$,

$$A'\sim\frac{Q_e^{1/3}}{(2r)^{2/3}a^{1/3}}. \tag{3.15}$$

Differentiating Eq. (3.7) shows $A''<0$, so A' monotonically decreases as $r\to\infty$. We investigate the M-r_H and M-$1/T$ relations for electrically charged black holes in Figs. 4 and 5, respectively. We take $a/Q_e^2=0,\ 0.1,\ 1,\ 10$. The point A corresponds to the extreme solution and the curve A to B shows an inner horizon. For finite a, there exists an extreme solution as we noted above and this approaches to $r_H\to0$ for $a\to\infty$. Thermodynamical properties are similar to the case for the RN solution. The point D corresponds to the point where the specific heat changes and this is not equivalent to the

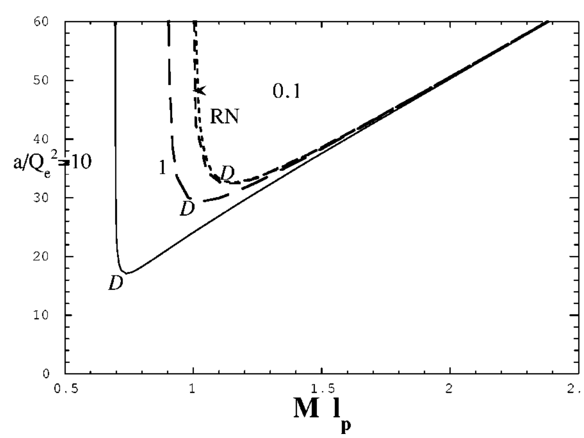

FIG. 5. M-$1/T$ relation for the same parameters in Fig. 4. The extreme solution always exists where the temperature becomes 0. So the M-$1/T$ relation is similar to the one for RN black hole. Note that the point where the sign of the specific heat changes does not necessarily correspond to the point B which suggests that the causality change will not be irrelevant to the stability change.

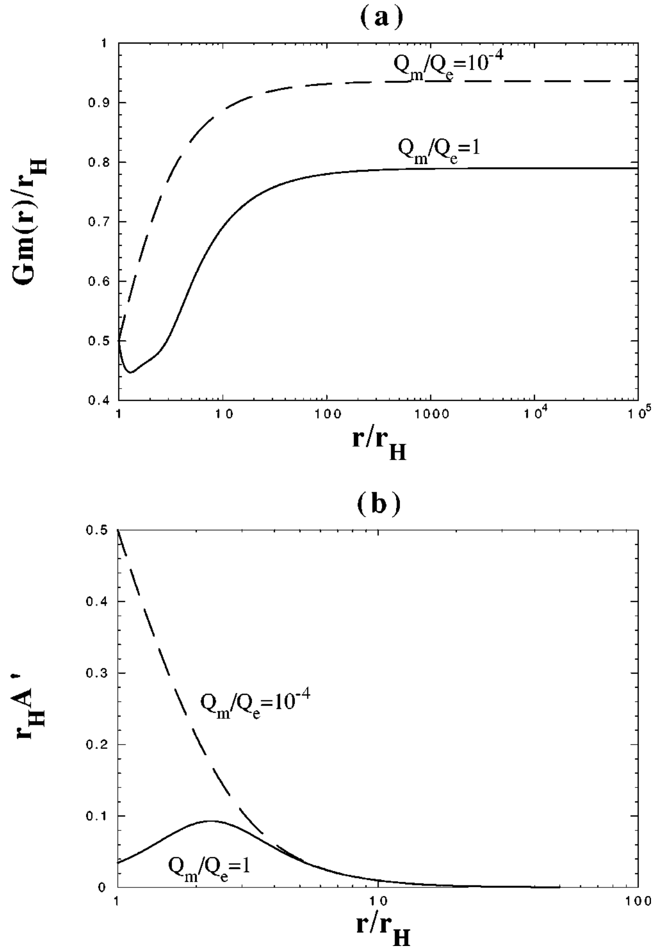

FIG. 6. Field distributions of dyon black holes for $r_H/l_p=1$, $a/Q_e^2=1$, $b/Q_e^2=-1$, $Q_e/l_p=1$, and $Q_m/l_p=10^{-4}$, 1 [(a) r-m, (b) r-A']. As we can see for $Q_m/l_p=1$ monotonically decreasing of A' is broken and $m'<0$ region appears which can be seen in the $r_H \to 0$ limit unless $Q_m \neq 0$.

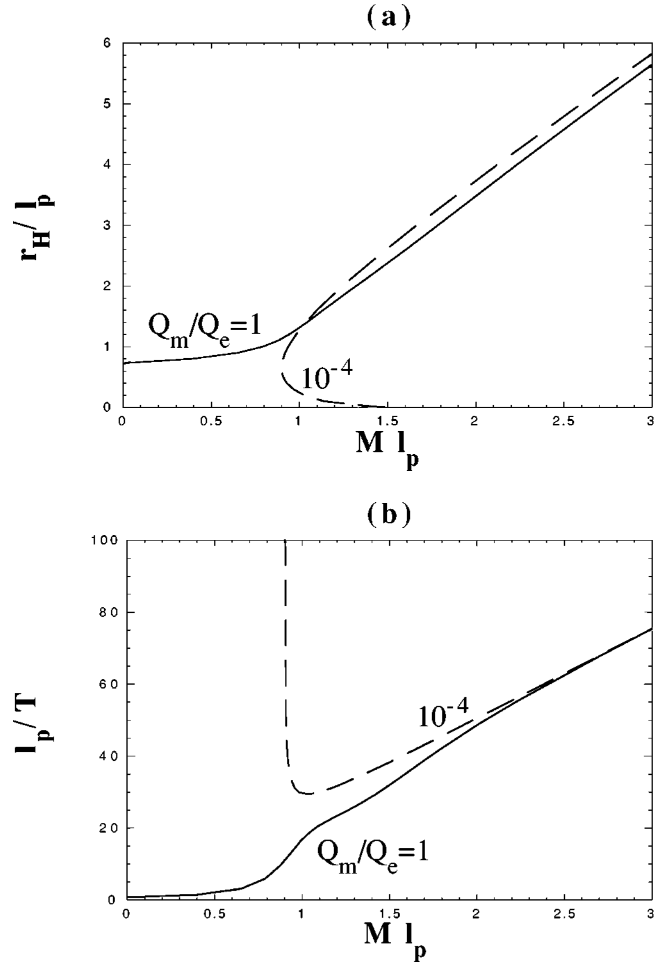

FIG. 7. (a) M-r_H, (b) M-$1/T$ relations for dyon black holes for $a/Q_e^2=1$, $b/Q_e^2=-1$, (i.e., $a+8b<0$) $Q_e/l_p=1$, and $Q_m/l_p=10^{-4}$,1. It seems that solutions in the $r_H \to 0$ limit only exist for $Q_m/l_p=1$. But it is not true.

point B. So there is no relation between the point which is relevant to the causality change and the point at which the specific heat changes.

IV. DYON BLACK HOLE

As we showed above, the properties of black holes have very different aspects, depending on whether it has electric charge or magnetic charge. In this section, one of the main purposes is to survey how thermodynamical properties change when we change the Q_m/Q_e ratio or the coupling constants a, b. In this case, from Eq. (2.13), the three solutions are expressed as

$$A'(r) = \frac{-2\times 3^{1/3} z + 2^{1/3} B^{2/3}}{6^{2/3}\sqrt{x} B^{1/3}}, \tag{4.1}$$

$$A'(r) = \frac{(1\pm i\sqrt{3})x}{2^{2/3}3^{1/3}\sqrt{x}B^{1/3}} - \frac{(1\mp i\sqrt{3})B^{1/3}}{2^{4/3}3^{2/3}\sqrt{x}}. \tag{4.2}$$

In this case,

$$B = 9\sqrt{x} Q_e + \sqrt{12z^3 + 81xQ_e^2}. \tag{4.3}$$

Note that for $12z^3+81xQ_e^2 \geqslant 0$, the only real solution is Eq. (4.1). For $a+8b>0$, because z can be negative for a small r, $12z^3+81xQ_e^2<0$ is possible only near the horizon. But even in that case, there is only one positive solution (4.1). We should take a positive solution because z eventually becomes positive for large values of r. So we take Eq. (4.1) in any case.

We can classify solutions in the $r_H \to 0$ limit three types as follows.

(I) If $a+8b=0$, $A'(r)$ approach Eq. (3.15) for $a^{1/2}Q_e \gg r^2$, which is the same as in the electrically charged case. On the contrary, Eq. (2.12) approaches

$$Gm' \sim -a\frac{Q_m^4}{r^6}, \tag{4.4}$$

which has same form as in the magnetically charged case. So the characteristic feature of small r_H is like that of the magnetically charged case.

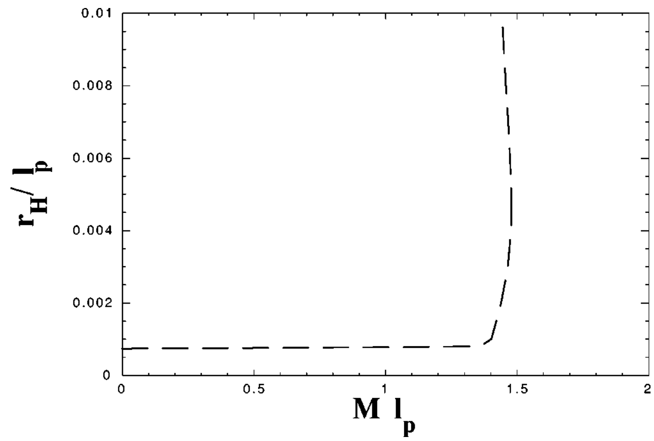

FIG. 8. Magnification of Fig. 7(a) which shows that there exist solutions in the $r_H \rightarrow 0$ limit even for $Q_m/l_p = 10^{-4}$.

For $a+8b \neq 0$, we can see its nature if we rewrite Eq. (4.1) as

$$A' = \frac{A_1^3 - A_2^3}{6^{2/3}\sqrt{x}B^{1/3}(A_1^2 + A_1 A_2 + A_2^2)}, \tag{4.5}$$

where

$$A_1 \equiv 2^{1/3} B^{2/3}, \quad A_2 \equiv 2 \times 3^{1/3} z. \tag{4.6}$$

(II) For $a+8b<0$, we can evaluate

$$A_1^3 - A_2^3 = 36 Q_e \sqrt{x} B, \tag{4.7}$$

which shows $A' \propto r^2$ in the $r \rightarrow 0$ limit. So we can conclude that if $(a+8b) \leqslant 0$, Eq. (2.12) has same asymptotically form (4.4) in the $r_H \rightarrow 0$ limit as in the magnetically charged case.

(III) For $a+8b>0$, there exists $r=r_0$ below which $12z^3 + 81 Q_e^2 x < 0$ is satisfied. For a while we consider $r < r_0$ case. Then we can evaluate

$$A_1^3 - A_2^3 = -24 z^3, \tag{4.8}$$

which shows $A' \propto r^{-2}$ in the $r \rightarrow 0$ limit. So we can not conclude whether or not solutions in the r_H limit exists. It depends on a, b as we see below.

Next, we show the field distributions in Fig. 6(a) r-m, 6(b) r-A' for $a/Q_e^2 = 1$, $b/Q_e^2 = -1$, (i.e., $a+8b<0$), $r_H/l_p = 1$, $Q_e/l_p = 1$, and $Q_m/l_p = 10^{-4}$, 1. Monotonically decrease of A' is broken and $m'<0$ region is specific for $Q_m/l_p = 1$ contrary to the case for $Q_m/l_p = 10^{-4}$. But they are universal in the $r_H \rightarrow 0$ limit unless $Q_m \neq 0$ as is shown above.

We also studied r_H and $1/T$ relations in terms of M for the above three cases. We first show those for $a+8b<0$ in Figs. 7(a) and 7(b), respectively. We fixed the parameters $a/Q_e^2 = 1$, $b/Q_e^2 = -1$, $Q_e/l_p = 1$, and $Q_m/l_p = 10^{-4}$, 1. For $Q_m/l_p = 1$, we can easily see specific properties of the magnetically charged case though for $Q_m/l_p = 10^{-4}$ we cannot. But it is not true. Even for $Q_m/l_p = 10^{-4}$, there exist solutions in the $r_H \rightarrow 0$ limit where the temperature diverges and $M \rightarrow -\infty$. They are clear from Fig. 8 which is a magnification of Fig. 7(a).

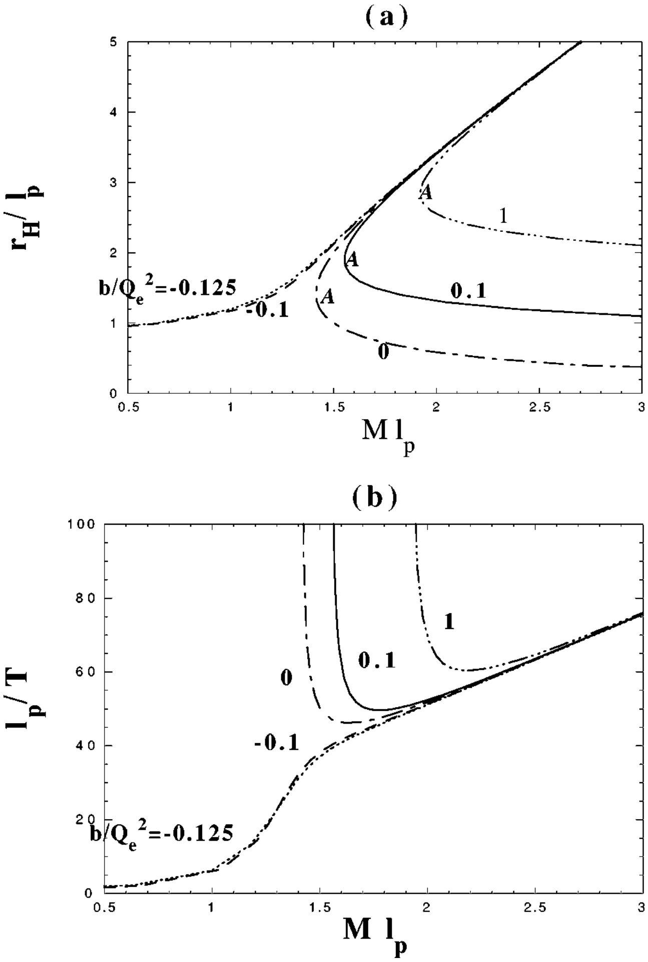

FIG. 9. (a) M-r_H, (b) M-$1/T$ relations for dyon black holes for $a/Q_e^2 = 1$, $b/Q_e^2 = -0.125$, -0.1, 0, 0.1, $Q_e/l_p = 1$, and $Q_m/l_p = 1$. For $b/Q_e^2 = -0.125$, -0.1, these figures are almot indistinguishable for these two cases and resemble those for the magnetically charged case. But for $b/Q_e^2 = 0$, 0.1 they resemble those for the electrically charged case.

We show corresponding diagrams for $a+8b \geqslant 0$ in Figs. 9(a) and 9(b), respectively. We fixed the parameters $a/Q_e^2 = 1$, $b/Q_e^2 = -0.125$ (i.e., $a+8b=0$), -0.1, 0, 0.1, $Q_e/l_p = 1$, and $Q_m/l_p = 1$. For $b/Q_e^2 = -0.125$, -0.1, it is almost indistinguishable in this diagram though the electric field has a different limit for $r_H \rightarrow 0$ in these two cases. We can see that the character is similar to the magnetically charged case. But for $b/Q_e^2 = 0$, 0.1, we can see a character similar to the electrically charged case, i.e., solutions below an extreme solution do not exist. The curve below the points A is a sequence of inner horizons. We also investigated those for various Q_e/Q_m ratio which suggest that whether or not solutions in the $r_H \rightarrow 0$ limit exist depends only on a, b. Thus if we believe that this system is realistic, the coupling constants decide the final fate of black holes.

V. CONCLUSION AND DISCUSSION

We note our conclusions and future work. We investigate black hole solutions in the EEH system for electrically

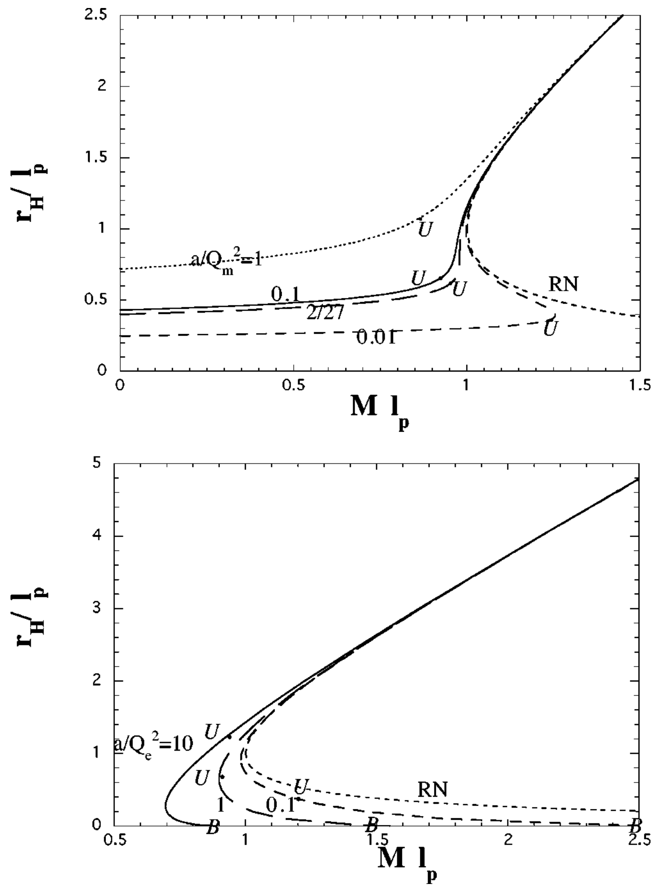

FIG. 10. M-r_H relations for (a) magnetically charged case, (b) electrically charged case which correspond to Figs. 1 and 4, respectively. The points U show that the aP^2 term becomes dominant compared with P term below U.

charged, magnetically charged and dyonic solutions. They have remarkable thermodynamical properties.

(i) For the magnetically charged case, the properties of the black holes change qualitatively for $a=a_{\text{crit}}$. There is an extreme solution only for $a\leqslant a_{\text{crit}}$. There are solutions in the $r_H\to 0$ limit for arbitrary a and the temperature diverges in this limit.

(ii) For the electrically charged case which was analyzed previously, though the lower limit of the horizon becomes small as we take a to be large, the final state of the black hole when we consider the evaporating process is similar to the RN one. The causality change has also already been pointed out, i.e., the inner horizon appears only for $M<M_0$, as we confirmed it. But the point $M=M_0$ is not relevant to the change of the stability if we apply the turning point method.

(iii) As for the dyon case, we showed that there exists solutions in the $r_H\to 0$ limit for $a+8b\leqslant 0$ and approach the magnetically charged case. For $a+8b>0$, our results suggest that whether or not solutions in the $r_H\to 0$ limit exists depends only on a and b not on the Q_m/Q_e ratio except for vanishing Q_m or Q_e.

We should discuss the validity of our solutions since the EH action is a low-energy approximation of the BI action from the point of view of the string theory. We estimated the condition where the contribution from P is larger than that from aP^2 (or bQ^2) in Eq. (2.6). For the magnetically charged case, this is evaluated as $r_H>(2aQ_m^2)^{1/4}$. We show the M-r_H relation in Fig. 10(a) and plotted the point U below which this condition is violated. This shows that our solutions are justified until they reach the extreme solutions (if they have extreme). But if there is no extreme solution, the validity of the solutions are violated before they reach negative mass solution. For the electrically charged case, this is evaluated as $r_H>(2aQ_e^2/9)^{1/4}$. We show the M-r_H relation in Fig. 10(b) which shows that though this condition is violated before solutions become extreme for large a, it is well satisfied by large in our parameter range. Dyonic black hole follows these two cases.

We now comment on what happens below the points U actually. Though we considered the Einstein-Hilbert action as a gravitational part, it is important to generalize to think higher order curvature corrections. It may be interesting to think about black hole solutions in the action which generalize the EH action to preserve supersymmetry [19]. Our solutions have pathological properties like the negative gravitational mass. There may exist mechanisms which prevent such properties as in Ref. [20]. Another concern we have is to think about black hole solutions including cosmological term, because its importance is recognized both in observational and in theoretical perspectives.

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