Born-Infeld theory and stringy causality

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Fluctuations around a nontrivial solution of Born-Infeld theory have a limiting speed given not by the Einstein metric but the Boillat metric. The Boillat metric is *S*-duality invariant and conformal to the open string metric. It also governs the propagation of scalars and spinors in Born-Infeld theory. We discuss the potential clash between causality determined by the closed string and open string light cones and find that the latter never lie outside the former. Both cones touch along the principal null directions of the background Born-Infeld field. We consider black hole solutions in situations in which the distinction between bulk and brane is not sharp such as space-filling branes and find that the location of the event horizon and the thermodynamic properties do not depend on whether one uses the closed or open string metric. Analogous statements hold in the more general context of nonlinear electrodynamics or effective quantum-corrected metrics. We show how Born-Infeld action to second order might be obtained from higher-curvature gravity in Kaluza-Klein theory. Finally we point out some intriguing analogies with Einstein-Schrödinger theory.

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I. INTRODUCTION

A striking feature of much recent work on open string states in string or M theory is the considerable insights afforded by the Born-Infeld [1] approximation. Reciprocally, string or M theory has provided a rationale for some of the hitherto mysterious and only partially understood properties of this remarkable theory.

The existence of a limiting electric field strength, which was originally the *raison d'etre* of Born-Infeld theory, now finds a dynamical justification in the increasingly copious production of electrically charged open string states as one approaches the critical value [2]. More subtly, the electric-magnetic duality symmetry of Born-Infeld theory, a nonlinear generalization of Hodge duality first recognized by Schrödinger [3], may be viewed as a special case of *S* duality. In fact electric-magnetic duality is a special case of Born reciprocity [4], a transformation which acts as a rotation in phase space $(p,q)\rightarrow(p\cos\theta+q\sin\theta, q\cos\theta-p\sin\theta)$. In nonlinear electrodynamics, the phase space variables might be considered to be $\mathbf{B}$ and $\mathbf{D}=\partial L/\partial\mathbf{E}$, which are canonically conjugate variables in the sense of the Poisson brackets[1]

$$\{B_i(\mathbf{x}),D_j(\mathbf{y})\}_{P.B.}=-\epsilon_{ijk}\partial_k\delta(\mathbf{x}-\mathbf{y}). \tag{1.1}$$

Born reciprocity applied to string theory gives rise to *T* duality [5]. According to Hull and Townsend [6] T and S duality are included in the more general *U*-duality symmetry. This leads naturally to the question of whether Born-Infeld theory is the only nonlinear electrodynamic theory admitting electric-magnetic duality. It is not [7].

Another striking feature of Born-Infeld theory is that it admits BIon solutions. These are exact solutions of the full nonlinear theory with distributional sources with finite total energy. They can now be understood in terms of strings ending on D-branes [8,9].

Schrödinger recognized yet another remarkable property of Born-Infeld theory: viewed as a nonlinear optical theory, Born-Infeld theory exhibits uncommon properties with respect to the scattering of light by light. He constructed exact wave like solutions of the full nonlinear equations representing light pulses with solitonic properties. They pass through one another without scattering [10,11]. This can be also understood from a string theory point of view [8].

Some time after Shrödinger's work it was realized that the propagation of Born-Infeld fluctuations around a background solution has exceptional causal properties [12,13] and that the theory also admits exact solutions exhibiting these exceptional properties [14]. From the string theory perspective, this relates to the recent interest in open string theory in a constant background Kalb-Ramond potential $B_{\mu\nu}$ and thus with gauge theory in a flat non-commutative spacetime [15]. The Kalb-Ramond potential $B_{\mu\nu}$ appears in the Born-Infeld action in the combination $F_{\mu\nu}+B_{\mu\nu}$ and so from the Born-Infeld point of view, a constant $B_{\mu\nu}$ field may be regarded as a background solution of Born-Infeld theory. Some of the open string states propagating around the constant background $B_{\mu\nu}$ field may be identified (at least in the Abelian case) with fluctuations of the Born-Infeld theory. Thus we need to understand the causal structure of their propagation. It turns out that this is governed by a metric, $G_{\mu\nu}=g_{\mu\nu}-B_{\mu\alpha}g^{\alpha\beta}B_{\beta\nu}$,[2] which *differs* from the usual spacetime metric $g_{\mu\nu}$ [15]. In fact we have two light-cones: the usual light-cone given by $g_{\mu\nu}$ which governs the propagation of closed string states such as the graviton and that given by $G_{\mu\nu}$

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[1]We use the convention $\{x^i,p_j\}_{P.B.}=\delta^i_j$.

[2]$B_{\mu\nu}$ may be taken to stand for the background field or for the constant Kalb-Ramond 2-form. In string language we are using units in which $2\pi\alpha'=1$.

which governs the propagation of open string states such as the Born-Infeld photon. In general, the former lies outside the latter except in two privileged directions corresponding to the two principal null directions or eigenvectors of the background two-form field. To put things provocatively, gravitons almost always travel faster than light.

One immediate consequence is that if the closed string metric admits no closed timelike curves then neither will the open string metric. Another obvious consequence is that if the closed string metric contains an event horizon then the open string metric will also contain an event horizon which lies inside or on the closed string event horizon.

The comments in the last two paragraphs encode the global theme of this paper. Within many physical theories, two non-conformal metric structures arise:

$$g_{\mu\nu} \quad \text{and} \quad g_{\mu\nu}+S_{\mu\nu}, \tag{1.2}$$

where $S_{\mu\nu}$ is some symmetric two-tensor. Geometrically they represent two sets of light-cones. Questions regarding causality or the existence of event horizons might then become particularly subtle. It is our purpose to discuss such issues in several contexts. For the aforementioned open versus closed string theory causal structures, there is some advantage in placing these properties in the general context of nonlinear electrodynamic theories, just as in the case of electric-magnetic duality. Particularly so because a quite separate strand of recent research has been concerned with analogues of black holes, closed timelike curves and circular null-geodesics in nonlinear electrodynamics [16]. There is also considerable interest in black holes in theories of nonlinear electrodynamics coupled to Einstein gravity (some recent references are [17]).

In Sec. II we look at general nonlinear electrodynamics in a four-dimensional flat background. Hence $g_{\mu\nu}$ is the Minkowski metric. The study of the propagation of fluctuations of the electromagnetic field in a given electromagnetic background introduces naturally a second metric structure: the Boillat metric $A^{Boillat}_{\mu\nu}$. We emphasize the special properties of Born-Infeld theory in at least four senses: the existence of both electric-magnetic and Legendre duality and the absence of both birefringence and shocks.

In Sec. III, we specialize to the case when the electrodynamics theory is Born-Infeld. One can deal with a general curved background. So the natural background geometry is described by $g_{\mu\nu}$, the closed string metric. But open strings propagating in a nontrivial electromagnetic field or Kalb-Ramond potential see a different metric: the open string metric $G_{\mu\nu}$, conformal to $A^{Boillat}_{\mu\nu}$. We shall then see that if the closed string metric is static and the Born-Infeld field is pure electric or pure magnetic then the open string metric cannot have a non-singular event horizon distinct from the one given by the closed string metric, because on it the electric field **E** must either equal its limiting value or the magnetic field **B** must diverge. Note that the metric $G_{\mu\nu}$ is not invariant, even up to a conformal factor, under Hodge duality $\delta B_{\mu\nu}=\star B_{\mu\nu}$ but, as we shall see, it *is* invariant up to a conformal factor under electric-magnetic duality rotations.

In Sec. IV we show that scalar and spinor fluctuations around a Born-Infeld background are governed by the open string metric.

In Sec. V the bi-metric theme takes another perspective. Recently [18] examples have been given of how dimensional reduction can alter the causal structure of stringy black holes. Considering a trivial dilaton field, the relation between the lower dimensional metric $g_{\mu\nu}$ and the higher dimensional one $\hat{g}_{\mu\nu}$ is of the form (1.2), with $\hat{g}_{\mu\nu}=g_{\mu\nu}+A_\mu A_\nu$. With this motivation, we look to higher-order Kaluza-Klein theory. We notice that it is possible to obtain Born-Infeld theory to second order and still avoid ghosts, as long as the higher dimensional graviton is only excited along the compact dimensions. We similarly show that the effective theory for QED, the Euler-Heisenberg theory, may be obtained in this fashion, and discuss some properties of the theory obtained by starting with an Einstein-Hilbert plus Gauss-Bonnet action in higher dimensions [19]. This gives an application of the general concepts discussed in Sec. II.

In Sec. VI we start by reviewing the results of Sec. II considering the gravitational effects of the electromagnetic background field. One is then led to consider besides the usual Einstein metric $g_{\mu\nu}$ an effective co-metric of the form $g^{\mu\nu}+AR^{\mu\nu}$. Another possible origin for such effective metric is quantum renormalization of the propagator of test fields in a fixed background. We then discuss the universality of black holes event horizons and thermodynamic properties, by applying a result derived in the context of quantum renormalized metrics to the case of the Boillat metric for nonlinear electrodynamics coupled to gravity.

In Sec. VII, we review an old attempt of Einstein and Schrödinger to construct a unified theory of gravity and electromagnetism (see [20] for original references). One then introduces a metric which has an antisymmetric part. The symmetric part $g_{\mu\nu}$ and the inverse of the symmetric part of the inverse of the full metric, $A^{\text{Eins-Schro}}_{\mu\nu}$ have remarkable similarities with the closed and open string metric, as first noticed by Boillat [21]. We discuss some exact solutions found by Papapetrou [22].

We close with a discussion.

II. CAUSALITY IN NONLINEAR ELECTRODYNAMICS

A. Characteristics and effective geometry

We consider a general Lagrangian $L=L(x,y)$ depending on the Lorentz invariants $x=\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $y=\frac{1}{4}F_{\mu\nu}\star F^{\mu\nu}$.[3] These are the only independent Lorentz invariants in four spacetime dimensions. The energy momentum tensor is given by

$$T_{\mu\nu}=-L_x T^{\text{Maxwell}}_{\mu\nu}+\frac{1}{4}Tg_{\mu\nu}, \tag{2.1}$$

[3]We use a mainly minus metric signature and, contrary to Boillat and some other references who use the opposite sign we choose L to have the standard sign such that for Maxwell theory $L=-x$. Subscripts indicate partial differentiation.

where the trace and the Maxwell energy-momentum tensor are given by

$$T \equiv T^{\mu}_{\mu} = -4(L - xL_x - yL_y),$$

$$T^{\text{Maxwell}}_{\mu\nu} = -F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} + xg_{\mu\nu}. \tag{2.2}$$

Since both $T^{\text{Maxwell}}_{\mu\nu}$ and $g_{\mu\nu}$ (with mainly minus signature) satisfy the dominant energy condition, and the set of energy momentum tensors satisfying the dominant energy condition is a convex cone, a sufficient requirement for $T_{\mu\nu}$ to satisfy the dominant energy condition is that $L_x < 0$ and $T \geqslant 0$. An argument of Hawking and Ellis [23] then shows that propagation in the full non-linear theory is causal in the sense that if at time zero all fields vanish outside some compact set, then they will vanish outside the future of that set. In general one expects the fields to advance into empty space with no background field at the speed of light and this expectation is supported by the observation (originally due to Schrödinger [10]) that any solution of Maxwell's linear electrodynamics with vanishing invariants, $x = y = 0$, will also be an exact solution of nonlinear electrodynamics. Among such so-called self-conjugate solutions are the usual plane wave solutions which have unit speed.

If a background field is present however these arguments require re-examination. One approach might be to look at the energy momentum tensor of the fluctuations. We shall not do this here but begin by considering the *characteristics*, which by definition are hypersurfaces along which *weak* discontinuities propagate. Assuming $F_{\mu\nu}$ to be discontinuous across the surface $S(x^{\mu}) = \text{const}$, the characteristics are given by [24,12,13,25]

$$(T^{\mu\nu}_{\text{Maxwell}} + \mu g^{\mu\nu})\partial_{\mu}S\partial_{\nu}S = 0. \tag{2.3}$$

This has the form of a relativistic Hamilton-Jacobi equation for massless particles with effective co-metric $T^{\mu\nu}_{\text{Maxwell}} + \mu g^{\mu\nu}$, and where S would be the action function. This effective metric also governs the propagation of weak, but not necessarily discontinuous fluctuations around a background. Later we will turn to the propagation of shocks and the behavior of fully non-linear fluctuations. The function $\mu = \mu(x,y)$ satisfies

$$\varpi\mu^2 + \mu + \omega - \varpi(x^2 + y^2) = 0, \tag{2.4}$$

where

$$\varpi = \frac{L_{xx}L_{yy} - L_{xy}^2}{L_x(L_{xx} + L_{yy})}, \tag{2.5}$$

and

$$\omega = \frac{L_x + x(L_{xx} - L_{yy}) + 2yL_{xy}}{L_{xx} + L_{yy}}. \tag{2.6}$$

In the general case the characteristics exhibits bi-refrigence: $\mu(x,y)$, which for convenience we parametrize as $\mu(x,y) = x + \zeta_{\pm}(x,y)$, can take *two values*, depending upon the polarization state and the background field. Thus there are, in general, two metrics. The interpretation of the quantities $\zeta_{\pm}$ is that they correspond to critical electric field strengths above which the theory breaks down. For exceptional theories the two values of $\zeta_{\pm}$ coincide and there is a single light-cone and no bi-refringence. Exceptional theories fall into two classes. The first has $\varpi = 0$. This happens for instance if the Lagrangian L is independent of y, which includes Maxwell's theory as a special case. But not all theories with $\varpi = 0$ are exceptional in this sense. In fact, although Eq. (2.4) still encodes relevant information for this case, it does not contain *all* the information. One example will be given in Sec. V.

For $\varpi \neq 0$, the only exceptional theory is Born-Infeld [12]. The latter is also very special in that $\zeta_{\pm}$ is a constant independent of x and y. It is the only theory for which this is true. We shall use units in which this constant is taken to be one.

The condition that the theory admit electric-magnetic duality rotations is rather weaker. It suffices that $\mathbf{B}\cdot\mathbf{E} = \mathbf{D}\cdot\mathbf{H}$ [7], which implies that the Lagrangian satisfy the first order Hamilton-Jacobi type equation

$$y(L_x^2 - L_y^2) - 2xL_xL_y = y. \tag{2.7}$$

The characteristics or wave surfaces may be thought of as null hypersurfaces of a metric whose null geodesics correspond to the *rays*. Note that the characteristics and the rays depend only a conformal equivalence class of metrics, defined by Eq. (2.3). A particular choice of conformal representative used by Boillat, which we shall refer to as the Boillat metric and co-metric, is given by

$$A^{\text{Boillat}}_{\mu\nu} = \frac{1}{\sqrt{\mu^2 - x^2 - y^2}}(\mu g_{\mu\nu} - T^{\text{Maxwell}}_{\mu\nu}) \tag{2.8}$$

$$C^{\mu\nu}_{\text{Boillat}} = \frac{1}{\sqrt{\mu^2 - x^2 - y^2}}(\mu g^{\mu\nu} + T^{\mu\nu}_{\text{Maxwell}}), \tag{2.9}$$

so that[4] $A^{\text{Boillat}}_{\mu\alpha}C^{\alpha\nu}_{\text{Boillat}} = \delta^{\nu}_{\mu}$. As we shall see in detail later, in the case of Born-Infeld theory, the open-string metric $G_{\mu\nu}$ and the Boillat metric $A^{\text{Boillat}}_{\mu\nu}$ are conformal.

Because

$$A^{\text{Boillat}}_{\mu\nu} = \frac{\mu - x}{\sqrt{\mu^2 - x^2 - y^2}}\left(g_{\mu\nu} - \frac{1}{\mu - x}F_{\mu\alpha}g^{\alpha\beta}F_{\beta\nu}\right), \tag{2.10}$$

the Boillat metric has a remarkable expression as a sort of square root:

[4]Throughout this paper all indices will be raised or lowered using the usual Einstein or closed string metric $g_{\mu\nu}$ *with the exception* of the open string metric $G_{\mu\nu}$ whose inverse is denoted by $G^{\mu\nu}$ in accordance with string theory conventions.

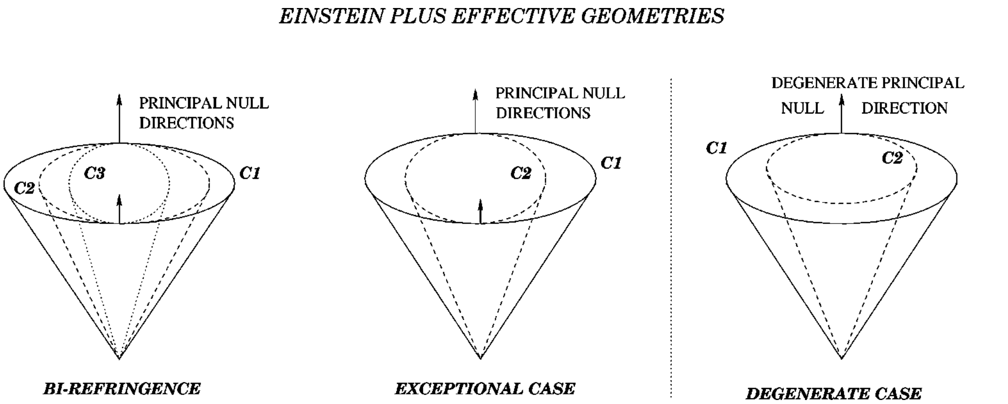

FIG. 1. Cones for the Einstein geometry and effective geometry describing the propagation of fluctuations in a nontrivial background $F_{\mu\nu}$ field. If condition (2.16) is obeyed, $C1$ is the Einstein cone, $C2$ (and $C3$ for the non-exceptional case) the effective geometry cones. The cone on the right represents the exceptional degenerate case.

$$A^{\text{Boillat}}=\frac{\mu-x}{\sqrt{\mu^2-x^2-y^2}}\times\left(g+\frac{1}{\sqrt{\mu-x}}F\right)g^{-1}\left(g-\frac{1}{\sqrt{\mu-x}}F\right). \tag{2.11}$$

It follows easily that

$$\sqrt{-\det A^{\text{Boillat}}_{\mu\nu}}=\sqrt{-\det g_{\mu\nu}}, \tag{2.12}$$

in other words, the Boillat metric and the spacetime metric induce the same volume element. The two principal null vectors common to both cones are annihilated by $g+F/\sqrt{\mu-x}$ or by $g-F/\sqrt{\mu-x}$.

We record for later use that if the background Einstein metric g is flat, then up to the conformal factor $1/\sqrt{\mu^2-x^2-y^2}$ the Boillat metric is

$$(\mu-x)(dt^2-d\mathbf{x}^2)-\mathbf{E}^2dt^2+(\mathbf{E}\cdot d\mathbf{x})^2+2\mathbf{E}\times\mathbf{B}\cdot d\mathbf{x}dt-\mathbf{B}^2d\mathbf{x}^2+(\mathbf{B}\cdot d\mathbf{x})^2. \tag{2.13}$$

In the generic case, one may diagonalize the Boillat metric with respect to the usual spacetime metric $g_{\mu\nu}$. This gives the speeds of propagation of the fluctuations in the associated inertial frame. In this frame the Poynting vector $\mathbf{E}\times\mathbf{H}=-L_x\mathbf{E}\times\mathbf{B}$ vanishes. The velocities, i.e. the ratio of spacelike to timelike eigenvalues, turn out to be

$$\left(1,\frac{\mu-\sqrt{x^2+y^2}}{\mu+\sqrt{x^2+y^2}},\frac{\mu-\sqrt{x^2+y^2}}{\mu+\sqrt{x^2+y^2}}\right). \tag{2.14}$$

Thus in general there are two directions in which the Boillat-cone touches the usual Einstein light-cone, corresponding to the first component of Eq. (2.14). These are the principal null directions of $F_{\mu\nu}$. Note that $F_{\mu\nu}$ and any duality rotation of it have the same principal null directions. In Fig. 1, we represent the light cones for the effective plus Einstein geometry. The left cones illustrate the case with bi-refringence; we then have the Einstein plus two effective geometry cones. For the causal case, the Einstein cone will be $C1$. All the cones touch in two points, along the principal null directions of $F_{\mu\nu}$. The cones in the center of Fig. 1 illustrate the exceptional case (like the Born-Infeld or open string theory case) where the effective geometry only possesses one light cone.

It may happen that the two principal null directions coincide. This occurs if and only if $x=0=y$. In this case the metric takes the form

$$A^{\text{Boillat}}_{\mu\nu}=g_{\mu\nu}-l_\mu l_\nu, \tag{2.15}$$

where l_μ is parallel to the principal null direction. The characteristic cone touches the Einstein cone along a single generator. This degenerate case is illustrated for exceptional theories in Fig. 1 right. For a generic electromagnetic field the principal null directions will coincide on a submanifold of dimension (and also co-dimension) two, $\mathcal{N}$. The complement $\mathcal{M}\backslash\mathcal{N}$ in the spacetime manifold $\mathcal{M}$, may not be simply connected. This gives rise to ambiguities in defining the ‘‘complexion’’ $\frac{1}{2}\arctan y/x$ of the electromagnetic field. In many ways, particularly if it is timelike, $\mathcal{N}$ behaves rather like a cosmic string [26].

In string theory, if a dilaton Φ is present, one distinguishes between the Einstein metric $g_{\mu\nu}$ and the (closed)-string metric $e^{-2\Phi}g_{\mu\nu}$. However both have the same (Einstein) light-cone, i.e., they are conformal. This is because the dilaton is a state of the closed string. It seems therefore, at least at the level of approximation we are considering, that there are just two causal structures and two sets of cones: the open and the closed. Of course from the strict string theory point of view one refers brane and the other to bulk propagation but we have in mind situations where the distinction is not sharp, such as for example in the case of space-filling branes, or when considering gravitons confined to, or at least moving parallel to, the surface of a brane. A sufficient condition that the Boillat-cone does not lie outside the usual Einstein light-cone, i.e. that the speeds never exceed unity is that both μ's must satisfy

$$\mu>\sqrt{x^2+y^2}\equiv r. \tag{2.16}$$

In terms of the coefficients in Eq. (2.4) this requirement reads $\omega<-r$, $-1/(2r)<\varpi<0$. Specialized to the Born-Infeld case (2.16) yields positive the quantity under the square root in the Born-Infeld action.

B. Wave surfaces and ether drift

Boillat [12] has calculated the wave-front produced by waves moving outwards from a point source with respect to an inertial frame in which the Poynting vector $\mathbf{E}\times\mathbf{H}=-L_x\mathbf{E}\times\mathbf{B}$ does not vanish. If

$$w=\frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2),\quad a=\sqrt{\frac{\mu+\sqrt{x^2+y^2}}{\mu+w}},$$

$$b=\sqrt{\frac{\mu-\sqrt{x^2+y^2}}{\mu+w}}, \tag{2.17}$$

and $c=ab$, so that $1>a\geqslant b>c$, he finds that it is given by the family of ellipsoids [in Cartesian coordinates (x,y,z)]

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{(z-tv_{\rm drift})^2}{c^2}=t^2, \tag{2.18}$$

where the drift velocity is given by

$$\mathbf{v}_{\rm drift}=\frac{\mathbf{E}\times\mathbf{B}}{\mu+w}. \tag{2.19}$$

Since $v_{\rm drift}^2=(1-a^2)(1-b^2)$, the drift velocity is always less than one. Therefore, the presence of the background electromagnetic field causes the drift of the origin of disturbances and establishes preferred directions in spacetime; in a sense plays the role of ''ether.''

C. Convexity of the Hamiltonian function

The energy density or Hamiltonian density T_{00} should be considered as a function $H(\mathbf{D},\mathbf{B})$ of the canonically conjugate variables $(\mathbf{D},\mathbf{B})$ [in the sense of Eq. (1.1)]. Their time evolution is obtained by taking the curl of $(\mathbf{H},-\mathbf{E})$ where the constitutive relation $(\mathbf{E},\mathbf{H})=(\partial H/\partial\mathbf{D},\partial H/\partial\mathbf{B})$ holds. In other words $(\mathbf{E},\mathbf{H})$ and $(\mathbf{D},\mathbf{B})$ are related by a Legendre transformation and in this sense one may regard the variables $(\mathbf{E},\mathbf{H})$ as canonically conjugate to the variables $(\mathbf{D},\mathbf{B})$. Of course this is a different sense of canonically conjugate than that in which $\mathbf{B}$ and $\mathbf{D}$ are canonically conjugate. It is a covariant sense in which one thinks of the space of Faraday tensors $F_{\mu\nu}$ (possibly subject to the closure constraint $\partial_{[\mu}F_{\nu\tau]}=0$) as the covariant configuration space rather than the non-covariant configuration space of magnetic induction fields $\mathbf{B}$ subject to the constraint div$\mathbf{B}=0$.

The Legendre transformation will be well defined and invertible if and only if the Hamiltonian density $H(\mathbf{B},\mathbf{D})$ is a convex function of its arguments. In other words the matrix of second derivatives or Hessian is positive definite. Note that in general H may be defined only in a portion of the six-dimensional space of possible $\mathbf{D}$ and $\mathbf{B}$'s and the Legendre transform may only map into part of six-dimensional space of possible $\mathbf{E}$ and $\mathbf{H}$'s. Thus for example, in the case of Born-Infeld theory

$$H=\sqrt{(1+\mathbf{B}^2)(1+\mathbf{D}^2)-(\mathbf{B}\cdot\mathbf{D})^2}-1, \tag{2.20}$$

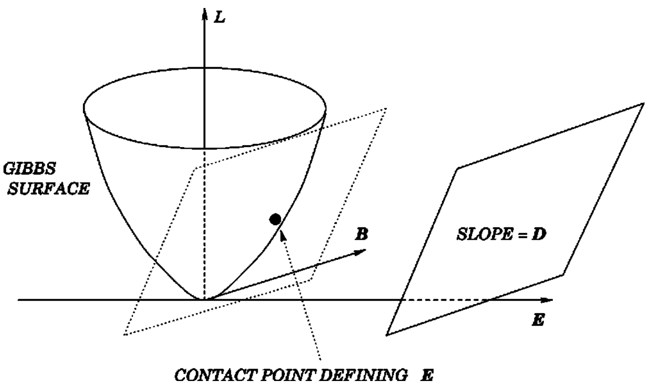

FIG. 2. The Gibbs surface is the Lagrangian function. Inverting the constitutive relations, to find $\mathbf{E}=\mathbf{E}(\mathbf{B},\mathbf{D})$, corresponds to finding the $\mathbf{E}$ coordinate of the contact point of the Gibbs surface with a plane with slope $\mathbf{D}$ along a line of constant $\mathbf{B}$. Convexity is necessary for the inverse constitutive relations to be well defined.

which is, in fact, defined for all $\mathbf{B}$ and $\mathbf{D}$. However the inverse Legendre transformation is effected by means of the function

$$1-\sqrt{(1-\mathbf{H}^2)(1-\mathbf{E}^2)-(\mathbf{E}\cdot\mathbf{H})^2}, \tag{2.21}$$

which is defined only over the domain of $(\mathbf{E},\mathbf{H})$ given by

$$\mathbf{E}^2+\mathbf{H}^2<1+(\mathbf{E}\times\mathbf{H})^2. \tag{2.22}$$

Born-Infeld theory is one with the same constant upper bound for both the electric (at zero magnetic field) and magnetic field strengths (at zero electric field). Of course it should be borne in mind that singling out a particular pair of variables is rather artificial. The underlying invariant geometric structure is the 12-dimensional symplectic vector space with symplectic form $d\mathbf{B}\cdot\wedge d\mathbf{H}+d\mathbf{D}\cdot\wedge d\mathbf{E}$ and a Lagrangian submanifold which defines the constitutive relations. If one wishes one may pass to a 13 dimensional contact manifold with contact form $dL-\mathbf{D}\cdot d\mathbf{E}+\mathbf{H}\cdot d\mathbf{E}$. Then the constitutive relation provides a Legendre submanifold, which of course on projection onto the L coordinate gives back the Lagrangian submanifold. One may instead perform a projection onto any pair of the 12 vector coordinates to obtain a ''Gibbs surface'' in a seven-dimensional space. Picking for example the pair $(\mathbf{E},\mathbf{B})$ the Gibbs surface is given by

$$L=1-\sqrt{1-\mathbf{E}^2+\mathbf{B}^2-(\mathbf{B}\cdot\mathbf{E})^2}. \tag{2.23}$$

This is defined only in the domain $D\subset\mathbb{R}^6$ connected to the origin for which $\det(g+F)<0$, that is $\mathbf{E}^2-\mathbf{B}^2+(\mathbf{B}\cdot\mathbf{E})^2<1$. Geometrically for example, to find $\mathbf{E}$ as a function of $\mathbf{D}$ and $\mathbf{B}$ one brings up a 6-plane parallel to the $\mathbf{B}$ axis whose slope is given by $\mathbf{D}$ until it touches the Gibbs surface. The point of contact defines $\mathbf{E}$. If the Gibbs surface is convex there will be only one such contact point. This is illustrated in Fig. 2.

Convexity will guarantee that all these projections are well defined over the relevant domain and that the surface has no folds for example as it would if the system exhibited some sort of hysteresis phenomenon. For a general nonlinear electrodynamic theory the Hessian will only be positive definite over some domain in $(\mathbf{B},\mathbf{D})$ space. Outside that domain the constitutive relation is just that: a relation rather than a function.

The components of the Hessian are just the electric permitivities and magnetic permeabilities. They govern the behavior of small disturbances around a background. Thus the background will be stable as long as the Hessian is positive definite. The equations for small fluctuations will also be hyperbolic as long as the Hessian is positive definite [27].

D. Shock waves and exceptionality

In Maxwell theory, in flat space $\mathbb{E}^{3,1}$, there exist traveling wave solutions of the form

$$F_{\mu\nu}=f(S)F^0_{\mu\nu}, \tag{2.24}$$

where f is an arbitrary function of its argument, $S=\mathbf{n}\cdot\mathbf{x}-vt$, and $\mathbf{n}$ is a constant unit 3-vector. For fixed $\mathbf{n}$ these represent a train of parallel waves moving with unit speed in a fixed direction. The arbitrary function f allows us to pick the profile of the wave train arbitrarily. One may even choose it to be discontinuous. The amplitude of the wave is constant on a family of wave surfaces $S=\text{const}$ which correspond to a family of spacetime parallel null hyperplanes whose intersection with any surface of constant time gives a family of parallel 2-planes in $\mathbb{E}^3$. Because they move at the speed of light, wave trains cannot be brought to rest by means of a Lorentz transformation.

In nonlinear theories in flat space one may, by analogy, adopt the ansatz

$$F_{\mu\nu}=F^0_{\mu\nu}(f(S)), \tag{2.25}$$

where $F^0_{\mu\nu}$ will now in general depend on the arbitrary function f and where

$$S=\mathbf{n}\cdot\mathbf{x}-v(\mathbf{n},S)t. \tag{2.26}$$

Now we get a family of hyperplanes $S=\text{const}$ in $\mathbb{E}^{3,1}$ but they are no-longer parallel, although their intersections with any surface of constant time still gives a family of parallel 2-planes in $\mathbb{E}^3$. The wave train therefore moves in a constant direction but not with constant speed. They may slow down or speed up in the sense that a hyperplane which passes a given point in space at a later time may have a smaller or greater speed $v(\mathbf{n},S)$. The hyperplanes will thus in general intersect (see Fig. 3). At these locations the ansatz breaks down. Neighboring hyperplanes will envelop a caustic hypersurface obtained by eliminating S from the equations

$$S=\mathbf{n}\cdot\mathbf{x}-v(S)t, \quad 1=-v'(S)t, \tag{2.27}$$

where $'$ indicates differentiation with respect to S. Exceptional waves are those for which

$$v'(S)=0. \tag{2.28}$$

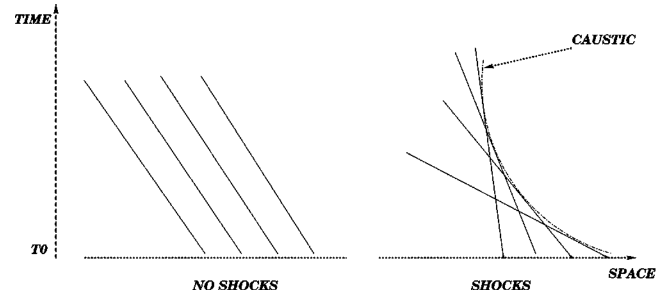

FIG. 3. Family of hyperplanes describing the propagation of wave fronts. If the hyperplanes intersect (right figure) the theory will be singular. Regularity (left figure) arises for exceptional theories only, like Born-Infeld, which have no shock formation.

If all waves are exceptional, i.e. if $v'=0\ \forall S$, then parallel hyperplanes are possible. If $v\neq 1$ these can be brought to rest by means of a Lorentz transformation. One then has stationary solutions[5] depending upon two arbitrary functions $f_1(z)$ and $f_2(z)$ of a single spatial coordinate, z say. We shall give concrete examples in the next section for the Born-Infeld case.

To understand the physical significance of exceptionality, in the sense of the absence of shock waves, one should consider non-exceptional theories which do admit shock waves. As theories they are essentially incomplete. One needs extra physical assumptions to render the evolution beyond the shock. This typically may come from some underlying more fundamental theory. Thus the predictions of classical theory admitting shocks, or indeed other singularities, cannot be trusted in situations where they arise or are about to arise. In this sense such theories ''predict their own demise,'' something that is often said of classical general relativity. By contrast a classical theory, such as Yang-Mills theory, for which the evolution of regular finite energy initial data remains non-singular for all times [28] is certainly complete as a theory, even though, because of quantum mechanics one does not trust every classical prediction. To check the reliability of a classical prediction we must check to see how it might be effected by quantum effects. Generally speaking, we expect classical Yang-Mills theory to be useful in the weak coupling limit and when dealing with very massive excitations such as magnetic monopoles.

Classical general relativity is known to admit singularities as a consequence of gravitational collapse. Only for weak data do we expect non-singular evolution for all time [29]. There exist fully nonlinear non-singular solutions of general relativity depending upon two arbitrary functions propagating at unit speed. These are the pp-waves. They may be generalized to propagate in an anti–de Sitter background [30]. In some ways AdS is analogous to a background B field. But pp-waves wave-fronts in AdS are null hypersurfaces of the AdS metric. This is in contrast with Born-Infeld theory and other nonlinear electrodynamical theories, where there are plane wave solutions traveling in some non-flat

[5]We say stationary rather than static because the Poynting vector may not vanish.

background spacetime at a slower speed. Again, by contrast with Born-Infeld theory, the collision of two pp-waves gives rise to a spacetime singularity [31]. Such waves definitely cannot pass through one another. In this respect Born-Infeld theory resembles Yang-Mills theory more than it does general relativity [28]. One is tempted to speculate that it may be a complete classical theory. Even if this is so, any of its classical predictions is subject to quantum correction unless there is some reason, such as supersymmetry, for believing that the quantum corrections vanish.

E. Covariant Legendre transformation

We introduce here a dual notation via the fields $P_{\mu\nu}$ and $N_{\mu\nu}\equiv\star P_{\mu\nu}$. This notation has the advantage of making Legendre self-duality of Born-Infeld theory manifest.

In the dual notation, the field equations of *any* nonlinear theory of electrodynamics are

$$\nabla_\mu P^{\mu\nu}=0, \tag{2.29}$$

or, in form language, $d\star P=0$, where the field $P^{\mu\nu}$ is defined by

$$dL=-\tfrac{1}{2}P^{\mu\nu}dF_{\mu\nu}. \tag{2.30}$$

$P_{\mu\nu}$ coincides with $F_{\mu\nu}$ for Maxwell's theory. In general it reads

$$P_{\mu\nu}=-(L_xF_{\mu\nu}+L_y\star F_{\mu\nu}). \tag{2.31}$$

The components of $P_{\mu\nu}$ are just **D** and **H**. Using this two-form, the energy-momentum tensor can be cast in a form identical to $T^{\rm Maxwell}_{\mu\nu}$

$$T_{\mu\nu}=-P_{\mu\alpha}g^{\alpha\beta}F_{\nu\beta}-g_{\mu\nu}L. \tag{2.32}$$

The formulation of the theory in terms of $P_{\mu\nu}$ is dual to the $F_{\mu\nu}$ formulation in the sense of a Legendre transformation. In fact if one takes the Legendre transform with respect to $\hat{L}$ by

$$\hat{L}=-\tfrac{1}{2}P^{\mu\nu}F_{\mu\nu}-L, \tag{2.33}$$

one has

$$d\hat{L}=-\tfrac{1}{2}F_{\mu\nu}dP^{\mu\nu}, \tag{2.34}$$

in analogy to Eq. (2.30). For the special case of a purely electric configuration in flat space, $\hat{L}$ is the ordinary Hamiltonian. Introducing the Hodge dual field $N_{\mu\nu}\equiv\star P_{\mu\nu}$, and defining $s\equiv\frac{1}{4}N^{\mu\nu}N_{\mu\nu}=-\frac{1}{4}P^{\mu\nu}P_{\mu\nu}$ and $t\equiv\frac{1}{4}\star N^{\mu\nu}N_{\mu\nu}=-\frac{1}{4}P^{\mu\nu}\star P_{\mu\nu}$ then the theory is specified by giving $\hat{L}$ as a function of s and t. Then we have

$$F_{\mu\nu}=\hat{L}_sP_{\mu\nu}+\hat{L}_t\star P_{\mu\nu}. \tag{2.35}$$

The energy momentum in tensor in terms of the dual variables follows from Eqs. (2.33) and (2.35):

$$T_{\mu\nu}=\hat{L}_sT^{\rm Maxwell}_{\mu\nu}[P]-g_{\mu\nu}(s\hat{L}_s+t\hat{L}_t-\hat{L}),$$
$$T^{\rm Maxwell}_{\mu\nu}[P]=-P_{\mu\alpha}g^{\alpha\beta}P_{\nu\beta}-sg_{\mu\nu}. \tag{2.36}$$

Sufficient conditions for the dominant energy condition to hold are

$$\hat{L}_s>0,\quad s\hat{L}_s+t\hat{L}_t-\hat{L}\geqslant 0. \tag{2.37}$$

In the case of Born-Infeld theory one has $t=-y$ by electric-magnetic duality invariance and expressing also x in terms of (s,t) one gets

$$-\hat{L}=1-\sqrt{1+2s-t^2}\quad\Leftrightarrow\quad L(F_{\mu\nu})=-\hat{L}(N_{\mu\nu}). \tag{2.38}$$

For Legendre self-dual theories like Born-Infeld, the equations describing propagation of perturbations (2.3), (2.4), will have exactly the same form in terms of the variables (x,y) as they do in terms of the variables (s,t).

III. BORN-INFELD OR STRING THEORY

A. Open and closed string metrics

The open string metric $G_{\mu\nu}$ is usually obtained as follows [15].[6] One starts with the matrix $g+F$ whose components are $g_{\mu\nu}+F_{\mu\nu}$. Then one inverts to obtain a matrix with components

$$\left(\frac{1}{g+F}\right)^{\mu\nu}=G^{\mu\nu}+\theta^{\mu\nu}, \tag{3.1}$$

where $G^{\mu\nu}$ is symmetric and $\theta^{\mu\nu}$ is antisymmetric. Let $G_{\mu\nu}$ be the inverse of $G^{\mu\nu}$, i.e. $G_{\alpha\mu}G^{\mu\beta}=\delta^\beta_\alpha$. Calculation reveals that

$$G^{\mu\nu}=(G^{-1})^{\mu\nu}=((g-F)^{-1}g(g+F)^{-1})^{\mu\nu}, \tag{3.2}$$

which is conformal to the inverse of Eq. (2.11) specialized to the Born-Infeld case. Then one checks that

$$G_{\mu\nu}=g_{\mu\nu}-F_{\mu\alpha}g^{\alpha\beta}F_{\beta\nu}. \tag{3.3}$$

A slightly more involved calculation shows that

$$\theta^{\mu\nu}=-\frac{1}{1+2x-y^2}(F^{\mu\nu}-y\star F^{\mu\nu})=-\frac{1}{\sqrt{1+2x-y^2}}P^{\mu\nu}, \tag{3.4}$$

where $P^{\mu\nu}$ is the dual Maxwell field in the sense of Eq. (2.30). In verifying Eq. (3.4) the following four dimensional matrix identities are useful (where **1** stands for the identity matrix):

$$g^{-1}Fg^{-1}\star F=-y\mathbf{1}, \tag{3.5}$$

[6] We will always use $F_{\mu\nu}$ for the gauge field, but it might represent the Kalb-Ramond potential $B_{\mu\nu}$.

$$g^{-1}Fg^{-1}F-g^{-1}\star Fg^{-1}\star F=-2x\mathbf{1}, \tag{3.6}$$

and hence

$$(g^{-1}F-yg^{-1}\star F)\left(g^{-1}F+\frac{1}{y}g^{-1}\star F\right)=-(1+2x-y^2)\mathbf{1}. \tag{3.7}$$

Comparing Eq. (3.3) with Eq. (2.8) one sees that the open string metric is equal, up to a conformal factor, to the Boillat metric governing the propagation of fluctuations around a Born-Infeld background:

$$G_{\mu\nu}=\sqrt{1+2x-y^2}A^{\text{Boillat}}_{\mu\nu}. \tag{3.8}$$

Relations (3.8) and (3.4) translate the stringy quantities $G_{\mu\nu}$ and $\theta_{\mu\nu}$ into pure nonlinear electrodynamics language, i.e. the metric describing fluctuations around a fixed background and the dual Maxwell field.

An essential requirement on the causal structure defined by the open string metric is to be invariant under electric-magnetic duality rotations. To examine this we recall that the stress tensor of Born-Infeld theory, which is known to be invariant [7], is given by

$$g^{\mu\nu}-T^{\mu\nu}_{\text{Born-Infeld}}=\frac{2}{\sqrt{-\det g}}\frac{\partial\sqrt{-\det(g+F)}}{\partial g_{\mu\nu}}. \tag{3.9}$$

But

$$\delta\sqrt{-\det(g+F)}=\frac{1}{2}\sqrt{-\det(g+F)}\left(\frac{1}{g+F}\right)^{\mu\nu}\delta g_{\mu\nu}. \tag{3.10}$$

Thus

$$g^{\mu\nu}-T^{\mu\nu}_{\text{Born-Infeld}}=\frac{\sqrt{-\det(g+F)}}{\sqrt{-\det g}}G^{\mu\nu}. \tag{3.11}$$

Since the left hand side of Eq. (3.11) is invariant so is the right hand side. Notice that the scalar $\sqrt{-\det(g+F)}/\sqrt{-\det g}$ is not invariant but its change merely induces a conformal transformation in $G^{\mu\nu}$ and hence in $G_{\mu\nu}$, preserving the causal structure. It is worthwhile noticing that the right hand side of Eq. (3.11) coincides with the Boillat co-metric

$$g^{\mu\nu}-T^{\mu\nu}_{\text{Born-Infeld}}=C^{\mu\nu}_{\text{Boillat}}, \tag{3.12}$$

which is therefore completely invariant under electric-magnetic duality rotations. It is easily seen from Eq. (2.12) that the determinant of both sides of Eq. (3.12) equals $\det g^{\mu\nu}$. Thus we get the remarkable result that

$$\det(\delta^{\mu}_{\nu}-T_{\text{Born-Infeld}}{}^{\mu}{}_{\nu})=1. \tag{3.13}$$

In the next subsections we illustrate the results above by analyzing the geometries seen by open strings in several special cases. We use the simplest exact solutions to Born-Infeld theory: plane wave solutions and spherically symmetric solutions.

B. Exact plane wave solutions

Boillat [32,33] found an exact stationary solution to Born-Infeld theory given in terms of two arbitrary functions $f_{1,2}$ of only one of the Cartesian coordinates, say z, with an electric field and a magnetic induction given by

$$\begin{aligned}\mathbf{E}&=\cosh\alpha\mathbf{i}+(\cosh\alpha\sinh\beta f_1(z)-\sinh\alpha f_2(z))\mathbf{k},\\ \mathbf{B}&=(\cosh\alpha f_2(z)-\sinh\alpha\sinh\beta f_1(z))\mathbf{i}\\ &\quad-\cosh\beta f_1(z)\mathbf{j}+\sinh\alpha\mathbf{k},\end{aligned} \tag{3.14}$$

where α,β are arbitrary constants. The magnetic field and electric induction are easily obtained via the constitutive relations. The two Lorentz-invariants are

$$-2x=1-f_1^2-f_2^2,\quad y=f_2, \tag{3.15}$$

so that the Born-Infeld Lagrangian equals $1-|f_1|$. The Poynting vector $\mathbf{P}=\mathbf{E}\times\mathbf{H}$ is given by

$$\begin{aligned}2|f_1|\mathbf{P}=&(f_1^2\cosh\alpha\sinh2\beta-2f_1f_2\sinh\alpha\cosh\beta)\mathbf{i}\\ &+(2f_1f_2\sinh\beta\cosh2\alpha\\ &-\sinh2\alpha(f_1^2\sinh^2\beta+f_2^2+1))\mathbf{j}\\ &-2\cosh\alpha\cosh\beta f_1\mathbf{k}.\end{aligned} \tag{3.16}$$

One might wonder if these stationary solutions may be interpreted as domain wall solutions. That is can one choose the asymptotic values of the arbitrary functions f_1 and f_2 so as to interpolate between two stable ‘‘ground states’’? One would then expect to have a static family of domain walls, that is, a nontrivial solution for which the Poynting vector would be zero. However, this is not allowed by P_z in Eq. (3.16): f_1 would need to be zero for which case $\mathbf{D},\mathbf{H}$ blow up. Therefore one finds no domain walls, just as in Maxwell's theory.

By performing a Lorentz transformation on Eq. (3.14), one gets the general fully nonlinear sub-luminal plane wave solution. It presents no shocks, in accordance to Sec. II D, since it propagates with constant speed. In general we do not expect superposition to hold in nonlinear electrodynamics and therefore such plane waves propagating on top of some background solution should not solve the equations of motion anymore. However, the plane waves obtained by boosting Eq. (3.14) may be superimposed to a background field and still yield a solution to Born-Infeld, as shown in [14]. Therein a background magnetic field along the x-axis and electric field along the y-axis are considered: $\mathbf{B}=B\mathbf{i}$, $\mathbf{E}=E\mathbf{j}$ so that $\mathbf{E}\times\mathbf{B}=-EB\mathbf{k}$. If we set

$$v_{\pm}=\frac{-EB\pm\sqrt{1-E^2+B^2}}{1+B^2}=\frac{1-E^2}{EB\pm\sqrt{1-E^2+B^2}}, \tag{3.17}$$

one checks that plane waves traveling in the z-direction can be superimposed to the background field. The waves can do so in two polarization states, with the electric and magnetic field given by

$(\mathbf{e},\mathbf{b})=(v_{\pm}\mathbf{j},-\mathbf{i})f_{\parallel}(z-v_{\pm}t)$ for the parallel polarization state,

$(\mathbf{e},\mathbf{b})=(v_{\pm}\mathbf{i},\mathbf{j})f_{\perp}(z-v_{\pm}t)$ for the perpendicular polarization state,

where $f_{\parallel}$ and $f_{\perp}$ are arbitrary functions of their argument. Note that there is a net drift in the z direction

$$v_{\rm drift}=\frac{1}{2}(v_{+}+v_{-})=-\frac{EB}{1+B^2}, \tag{3.18}$$

in agreement with Eq. (2.19). This drift effect may be understood as a consequence of Lorentz-invariance. If $B^2>E^2$ and one performs a Lorentz boost with velocity $u=E/B$ one may pass to a frame in which the electric field vanishes and the magnetic field becomes equal to $B_0=\sqrt{B^2-E^2}$. Now the velocity v_0 in this frame is symmetric with respect to reversing the z-direction and is given by

$$v_0=\frac{1}{\sqrt{1+B_0^2}}. \tag{3.19}$$

One may check that $v_{\pm}$, v_0 and u satisfy the usual relative velocity addition formula

$$v_{\pm}=\frac{u\pm v_0}{1\pm uv_0}. \tag{3.20}$$

If $E^2>B^2$ one may reduce the magnetic field to zero. The electric field in the de-boosted frame will be $E_0=\sqrt{E^2-B^2}$ and $v_0=\sqrt{1-E_0^2}$. In these two cases the open string metrics are [using Eq. (2.13)],

$$ds^2_{\rm open}=dt^2-dx^2-(1+B_0^2)(dy^2+dz^2) \tag{3.21}$$

and

$$ds^2_{\rm open}=(1-E_0^2)(dt^2-dy^2)-dx^2-dz^2. \tag{3.22}$$

In terms of the electric induction D_0 the latter is

$$ds^2_{\rm open}=\frac{1}{1+D_0^2}(dt^2-dy^2)-dx^2-dz^2, \tag{3.23}$$

which illustrates invariance of the open string metric up to a conformal factor under the discrete electric-magnetic duality transformation $(\mathbf{B},\mathbf{D})\rightarrow(-\mathbf{D},\mathbf{B})$. The general metric may be obtained using a Lorentz transformation.

C. BIons and other static solutions

Non-linear electrodynamic theories typically admit static finite energy solutions with distributional sources. Because they have sources and also have (albeit mild) singularities, these solutions are not solitons in the usual sense of the word. In [8] they were called BIons. For the electrically charged BIon of Born-Infeld theory we find the open string metric to be

$$ds^2_{\rm open}=\frac{r^4}{1+r^4}(dt^2-dr^2)-r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{3.24}$$

The scattering of null geodesics is most conveniently represented as geodesics of the optical metric

$$ds^2_{\rm optical}=dr^2+\frac{1+r^4}{r^2}(d\theta^2+\sin^2\theta d\phi^2), \tag{3.25}$$

which is easily seen to admit a 2-sphere of circular geodesics at $r=1$ surrounding an infinite redshift infinite area naked singularity at finite proper distance situated at $r=0$. Such geodesics correspond to null geodesics of Eq. (3.24).

The open string metric for the magnetically charged bion is different:

$$ds^2_{\rm open}=dt^2-dr^2-\frac{1+r^4}{r^2}(d\theta^2+\sin^2\theta d\phi^2). \tag{3.26}$$

However the optical metrics are identical and in fact the two metrics are conformally related.

This example may be generalized to any static configuration in Minkowski spacetime. By static one means that the Poynting vector vanishes, so that $\mathbf{E}\times\mathbf{B}=0$. The open string metric is then also static and given by

$$ds^2_{\rm open}=(1-\mathbf{E}^2)dt^2-d\mathbf{x}^2+(\mathbf{E}\cdot d\mathbf{x})^2-\mathbf{B}^2d\mathbf{x}^2+(\mathbf{B}\cdot d\mathbf{x})^2. \tag{3.27}$$

Since $|\mathbf{E}|\leqslant 1$ we have $G_{00}\geqslant 0$, moreover $G_{00}=0$ implies that $|\mathbf{E}|=1$. Thus any static event horizon of the Boillat metric which is not an event horizon of the Einstein metric must be singular, just as in the case of a single bion solution.

Now consider what happens if the closed string metric g ceases to be flat but remains static. One has

$$G_{00}=g_{00}+F_{i0}F_{j0}g^{ij}=g_{00}(1+2x). \tag{3.28}$$

Clearly, as long as $x>-1/2$ the sign of G_{00} is determined entirely by the sign of g_{00}. Thus unless the electric field reaches the critical value, there can be no open-string static event horizon which is not also a closed string event horizon.

This result is really obvious from the viewpoint of electric-magnetic duality because we could instead have considered a purely magnetic field. In this case

$$G_{00}=g_{00}, \tag{3.29}$$

and the magnetic field has no effect on that part of the metric which governs the location of event horizons. Actually, these results do not depend upon the detailed form of the open string metric obtained from Born-Infeld theory, nor upon electric-magnetic duality. They hold quite generally, as may be seen directly from the general expression (2.13) for the metric. At an event horizon we need

$$\mathbf{E}^2 = \zeta_{\pm}. \tag{3.30}$$

That is the electric field attains its limiting value at which point the theory breaks down.

D. The Boillat metric and BIonic scattering

In this subsection we apply the fore-going theory to the problem of scattering off the supersymmetric BIon or spike solution of the Dirac-Born-Infeld equations of motion [8,9]. This has been the subject of a number of detailed studies (see [34,35] and references therein). Physically the solution represents a fundamental string attached to a D-brane. It is static and the transverse displacement of the brane is given by a scalar field $\phi(\mathbf{x})$. The metric g induced on the brane is thus

$$ds^2 = dt^2 - (d\mathbf{x})^2 - (\nabla\phi\cdot d\mathbf{x})^2. \tag{3.31}$$

Using the Bogomol'nyi conditions

$$\mathbf{E} = \pm\nabla\phi, \tag{3.32}$$

where

$$\nabla^2\phi = 0, \tag{3.33}$$

the induced metric becomes

$$ds^2 = dt^2 - (d\mathbf{x})^2 - (\mathbf{E}\cdot d\mathbf{x})^2. \tag{3.34}$$

The open string metric then becomes

$$ds^2_{\text{open}} = \frac{dt^2}{1+\mathbf{E}^2} - (d\mathbf{x})^2. \tag{3.35}$$

This metric generates the same classical scattering as the Lagrangian (77) of [35]. In the case of a single SUSY BIonic spike we get

$$ds^2_{\text{open}} = \left(\frac{r^4}{1+r^4}\right) dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.36}$$

All the information about the classical scattering is now contained in $G_{\mu\nu}$.

IV. OTHER SPINS

Born-Infeld actions may be extended to include scalars and spinors. In this section we shall investigate the characteristics of these fields around a background constant B field.

A. Scalars

The Born-Infeld action with a single scalar field reads

$$S = \int d^4x\left(\sqrt{-\det(g_{\mu\nu})} - \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu} - \partial_\mu\phi\partial_\nu\phi)}\right). \tag{4.1}$$

Expanding around a background F-field and retaining only quadratic terms in ϕ we get the S-duality invariant expression

$$S \simeq S_{BI} + \frac{1}{2}\int d^4x\sqrt{-g}\,C^{\mu\nu}_{\text{Boillat}}\partial_\mu\phi\partial_\nu\phi. \tag{4.2}$$

S_{BI} is the usual Born-Infeld action. Therefore, as expected, scalar perturbations propagate according to the characteristics of the open string metric.

B. Spinors

Consider a general Dirac action of the form

$$S_D = \frac{i}{2}\int d^4x\,\mu(\bar{\Psi}\gamma_\alpha a^{\alpha\beta}\nabla_\beta\Psi + \cdots), \tag{4.3}$$

where μ is a scalar density, $a^{\mu\nu} = (a)^{\mu\nu}$ are the components of a contravariant second rank tensor which need be neither symmetric or antisymmetric and the ellipsis denotes other possible terms in fermions but with no derivatives.

The gamma matrices γ generate the Clifford algebra associated with the closed string metric g

$$\{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha\beta}. \tag{4.4}$$

The characteristics of this system are easily seen to be given by the co-metric

$$a^{\alpha\mu}g_{\alpha\beta}a^{\beta\nu}, \tag{4.5}$$

which are the components of $a^t g a$. Note that we could rewrite the action as

$$S_D = \frac{i}{2}\int d^4x\,\mu(\bar{\Psi}\Gamma^\alpha\nabla_\alpha\Psi + \cdots), \tag{4.6}$$

where

$$\Gamma^\alpha = \gamma_\beta a^{\beta\alpha}. \tag{4.7}$$

In the case of Born-Infeld theory, it is natural to take $a = (g+B)^{-1}$, in which case use of Eq. (3.2) shows that the characteristics as determined by Eq. (4.5) are given by the open string metric. Moreover, the gamma matrices introduced in Eq. (4.7) generate the Clifford algebra associated with the metric $a^t g a$

$$\{\Gamma^\alpha, \Gamma^\beta\} = 2a^{\alpha\mu}g_{\alpha\beta}a^{\beta\nu}. \tag{4.8}$$

For the Born-Infeld case this is the open string Clifford algebra.

Consider the Born-Infeld-Volkov-Akulov action which arises when one supersymmetrizes the Born-Infeld action;

$$S_{DBIVA}=\int d^4x\left(\sqrt{-\det(g_{\mu\nu})}-\sqrt{-\det(g_{\mu\nu}+F_{\mu\nu}+B_{\mu\nu}-\partial_\mu\phi\partial_\nu\phi+i\bar{\Psi}\gamma_\mu\nabla_\nu\Psi)}\right). \tag{4.9}$$

In the absence of B,ϕ and A fields this reduces to the Volkov-Akulov action [36]. Expanding to quadratic order in fermions gives a spinor action of the form S_D, with $\mu=\sqrt{-\det(g+B)}$ and $a=(g+B)^{-1}$. Thus, as one might have anticipated, the fermions characteristic cone is also given by the open string metric.

C. Gravitons

It is not clear whether gravitons propagating in an external B field would have their characteristics modified, since these are closed string modes and propagate on the bulk. However, in the light of the fact that we now believe that gravity can be localized on the brane [37], one might be tempted to speculate that under some circumstances that should happen. If that is the case the obvious guess for the characteristics would be the open string metric. Indeed precisely this happens in Einstein-Schrödinger theory. This is a unified theory in which the usual symmetric Einstein metric is replaced by an arbitrary 4×4 (or more generally in n spacetime dimensions an $n\times n$) tensor field a which we write suggestively as

$$(a^{-1})_{\alpha\beta}=a_{\alpha\beta}=g_{\alpha\beta}+B_{\alpha\beta}. \tag{4.10}$$

Lichnerowicz and Maurer-Tison [38,39] showed that some of the small fluctuations have characteristics given by the symmetric part of the co-metric, i.e. $a^{(\mu\nu)}\equiv G^{\mu\nu}_{Eins\text{-}Schro}$, in striking analogy to the open string or Born-Infeld case. Therefore the properties of these characteristics are the same as the ones presented in Secs. II and III. However, the theory exhibits a kind of bi-refringence, due to the existence of a second set of (co-)cones for small fluctuations, given by [39][7]

$$2\frac{\det g_{\alpha\beta}}{\det(g_{\alpha\beta}+B_{\alpha\beta})}g^{\mu\nu}-G^{\mu\nu}_{\text{Eins-Schro}}. \tag{4.11}$$

If we define $T^{\mu\nu}_{\text{Eins-Schro}}$ by an expression similar to Eq. (3.11) replacing $G^{\mu\nu}$ by $G^{\mu\nu}_{\text{Eins-Schro}}$ and F by B, our two co-metrics are conformal to, respectively

$$g^{\mu\nu}-T^{\mu\nu}_{\text{Eins-Schro}},$$
$$\left(2\sqrt{\frac{\det g_{\alpha\beta}}{\det(g_{\alpha\beta}+B_{\alpha\beta})}}-1\right)g^{\mu\nu}+T^{\mu\nu}_{\text{Eins-Schro}}. \tag{4.12}$$

Just as in the BI case the first set of light cones will lie inside the Einstein cones. But because of the opposite sign in $T^{\mu\nu}_{\text{Eins-Schro}}$, the second set of light cones will be outside both the first set of cones and the Einstein light cones. The former result was pointed out in [39] while the latter appears to confirm the pathological properties of this theory, in that some fluctuations are tachyonic with respect to the Einstein metric. Presumably these fluctuations can carry negative energy.

We shall return to this theory in Sec. VII. Before doing so we should recall that Einstein-Schrödinger theory appears to break invariance under the gauge transformation $B\to B+dA$ and for this reason it has been claimed to admit negative energy states [20].

D. Gravitinos?

This is a short subsection because as yet we have no consistent supergravity brane solution and thus as far as we know no consistent theory of a gravitino propagating on a brane. However if such a theory exists and the gravitino propagation is affected by a background B field then there is an obvious suggestion for the characteristics.

V. NONLINEAR ELECTRODYNAMICS FROM $U(1)$ KALUZA-KLEIN THEORY

Kaluza-Klein theory stems from the fact that the Ricci scalar for the $(D+1)$ dimensional ansatz

$$d\hat{s}^2=ds^2+(dy+A_\mu dx^\mu)^2, \tag{5.1}$$

is $\hat{R}=-x$, i.e., the Maxwell Lagrangian. Here ds^2 is the D-dimensional Minkowski metric and y the coordinate along the extra dimension. It is known, however, that the truncation of Kaluza-Klein theory to pure electromagnetism is not consistent. In fact, considering a trivial scalar field implies $x=$const via the scalar equation of motion. We will not be concerned about this point in what follows, but rather study some properties of the electrodynamical theory that arises from considering the lowest order in α' tree level string theory corrections to the Einstein-Hilbert action in dimensions higher than four. Full study of such Kaluza-Klein theory must be performed by considering also gravitational and scalar excitations.

With the ansatz (5.1) the curvature invariants of second order in $D+1$ dimensions are (excluding a possible Chern-Simons term)

$$\hat{R}^2=x^2,$$
$$\hat{R}_{MN}\hat{R}^{MN}=x^2+\tfrac{1}{2}\partial_\beta F_\alpha{}^\beta\partial_\mu F^{\alpha\mu}+\tfrac{1}{4}F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}, \tag{5.2}$$
$$\hat{R}_{MNPQ}\hat{R}^{MNPQ}=6x^2+\tfrac{5}{8}F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}+\partial_\alpha F_{\mu\sigma}\partial^\alpha F^{\mu\sigma}.$$

[7] We would like to thank M. Clayton for poining out to us the existence of this second light cone.

The most general parity conserving term quadratic in the curvature is then

$$\hat{R}_{MNPQ}\hat{R}^{MNPQ}+a\hat{R}_{MN}\hat{R}^{MN}+b\hat{R}^2 = (6+a+b)x^2+\frac{5+2a}{8}F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu} + \frac{4+a}{4}\partial_\alpha F_{\mu\nu}\partial^\alpha F^{\mu\nu}+\cdots, \tag{5.3}$$

where the dots stand for total derivatives. Hence, the terms with derivatives of the field strength in Eq. (5.2) cancel (up to total derivatives) in the combination $\hat{R}_{MNPQ}\hat{R}^{MNPQ}-4\hat{R}_{MN}\hat{R}^{MN}$, thus avoiding ghosts in the propagation of the electromagnetic field. Actions with such derivative terms have nevertheless been considered in the past, as in the Bopp-Podolsky action [40]. We will require the cancellation of ghosts and therefore consider the dimensional reduction of an action of the type

$$S=\frac{1}{16\pi G_{D+1}}\int d^{D+1}\hat{x}\sqrt{\hat{g}} \times(\hat{R}+\Upsilon(\hat{R}_{MNPQ}\hat{R}^{MNPQ}-4\hat{R}_{MN}\hat{R}^{MN}+b\hat{R}^2)). \tag{5.4}$$

Specializing to $D=4$, where one can use the identity $F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}=8x^2+4y^2$, we get the Lagrangian

$$\mathcal{L}_{KK}=-x+\Upsilon((b-1)x^2-\tfrac{3}{2}y^2). \tag{5.5}$$

One notices the absence of an xy parity breaking term to this order. In principle one could bring such term into the theory by including a Chern-Simons term. In $D+1=5$, two such possible terms are

$$S_{CS}=\int \mathrm{tr}(\hat{R}_{AB}\wedge\hat{R}_C{}^B\wedge X), \quad \text{or}$$

$$S_{CS}=\int \mathrm{tr}(\hat{R}_{AB}\wedge\hat{R}_C{}^B\wedge\hat{w}_D{}^C), \tag{5.6}$$

for some one form field X, or using the one form connection $\hat{w}$. The second and most natural possibility gives, however, terms of order higher than the ones considered in $\mathcal{L}_{KK}$. For the first possibility, the most natural choice of X is as being dual to the fiber direction $\partial/\partial y$; then the first possibility contributes only to the ghosts. Hence we will not consider them anymore.

By arranging the constants b and Υ in Eq. (5.5), one can recover several interesting cases which analyze in the following subsections.

A. Gauss-Bonnet electromagnetism

This theory is obtained for $b=1$ [19] (see also earlier work in [41]). As pointed out in [42], an analysis for linear perturbations of the gravitational field shows that ghost cancellation requires $\hat{R}^2$ to enter the combination (5.3) as

$$\Omega_2=\hat{R}_{MNPQ}\hat{R}^{MNPQ}-4\hat{R}_{MN}\hat{R}^{MN}+\hat{R}^2 \overset{4D}{=}\frac{1}{4}\hat{R}_{MN}{}^{AB}\hat{R}_{PQ}{}^{CD}\eta^{MNPQ}\eta_{ABCD}. \tag{5.7}$$

η is the Levi-Civita tensor, not density. This is the Gauss-Bonnet combination. The last equality, which holds in four dimensions where the four-form η is the volume form, shows it is the second Euler density, a topological term in four dimensions but dynamical in higher dimensions. We recall that the first Euler density, topological in two dimensions but dynamical in higher is just the Ricci scalar:

$$\Omega_1=\hat{R}\overset{2D}{=}\frac{1}{2}\hat{R}_{MN}{}^{AB}\eta^{MN}\eta_{AB}. \tag{5.8}$$

The last equality holds in two dimensions, where the two-form η is the volume form. The Gauss-Bonnet combination is usually referred to as describing the first order string theory corrections to general relativity [42]. The first order (in α') stringy gravitational action can then be written exclusively in terms of Euler densities (that does not seem to be the case already at third order):

$$S^{(1)}=\frac{1}{16\pi G_{D+1}}\int d^{D+1}\hat{x}\sqrt{-\hat{g}}(\Omega_1+\Upsilon\Omega_2), \tag{5.9}$$

with $\Upsilon\propto\alpha'$. That $S^{(1)}$ is the correct effective action relies on two arguments. Matching the amplitude for the scattering of three on-shell gravitons in bosonic closed string theory only fixes the $(\hat{R}_{MNPQ})^2$ term; the $(\hat{R}_{MN})^2$ and $\hat{R}^2$ do not contribute to the on-shell amplitude. These are fixed by the no-ghost requirement, since one does not see any ghosts in the string spectrum. But for purely electromagnetic excitations within a Kaluza-Klein context, the no-ghost requirement is more relaxed and makes sense to consider an arbitrary $\hat{R}^2$ coefficient.

In non-covariant language, the Gauss-Bonnet Lagrangian is described by

$$\mathcal{L}_{GB}=\tfrac{1}{2}(\mathbf{E}^2-\mathbf{B}^2-3\Upsilon(\mathbf{E}\cdot\mathbf{B})^2). \tag{5.10}$$

Properties of Gauss-Bonnet electromagnetism

The constitutive relations are very simple and easily invertible. **E** and **H** may be expressed in terms of **B** and **D** as

$$\mathbf{E}=\mathbf{D}+\frac{3\Upsilon(\mathbf{B}\cdot\mathbf{D})}{1-3\Upsilon\mathbf{B}^2}\mathbf{B},$$

$$\mathbf{H}=\mathbf{B}+\frac{3\Upsilon(\mathbf{B}\cdot\mathbf{D})}{1-3\Upsilon\mathbf{B}^2}\mathbf{D}+\left(\frac{3\Upsilon(\mathbf{B}\cdot\mathbf{D})}{1-3\Upsilon\mathbf{B}^2}\right)^2\mathbf{B}. \tag{5.11}$$

It follows from the constitutive relations that $\mathbf{B}\cdot\mathbf{E}\neq\mathbf{D}\cdot\mathbf{H}$. Therefore this theory does not admit electric-magnetic duality.

The Hamiltonian becomes

$$\mathcal{H}_{GB}=\frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2-3\Upsilon(\mathbf{E}\cdot\mathbf{B})^2)$$
$$=\frac{1}{2}\left(\mathbf{D}^2+\mathbf{B}^2+\frac{3\Upsilon(\mathbf{B}\cdot\mathbf{D})^2}{1-3\Upsilon\mathbf{B}^2}\right). \tag{5.12}$$

The dominant energy condition for the theory is obeyed if $\Upsilon<0$. Hence the first expression for the Hamiltonian shows the energy is positive, whereas the one in terms of the canonically conjugate variables $\mathbf{B}$ and $\mathbf{D}$ imposes no upper-bound on the magnitude of the magnetic induction. However, the expressions for $\mathbf{B}$ and $\mathbf{D}$ in terms of $\mathbf{E}$ and $\mathbf{H}$,

$$\mathbf{B}=\mathbf{H}-\frac{3\Upsilon(\mathbf{E}\cdot\mathbf{H})}{1+3\Upsilon\mathbf{E}^2}\mathbf{E},$$

$$\mathbf{D}=\mathbf{E}-\frac{3\Upsilon(\mathbf{E}\cdot\mathbf{H})}{1+3\Upsilon\mathbf{E}^2}\mathbf{H}+\left(\frac{3\Upsilon(\mathbf{E}\cdot\mathbf{H})}{1+3\Upsilon\mathbf{E}^2}\right)^2\mathbf{E}, \tag{5.13}$$

do constrain the value of the electric field to be bounded by

$$\mathbf{E}^2=\frac{1}{3|\Upsilon|}. \tag{5.14}$$

Another way to see this is by using our analysis of Sec. II. For the Gauss-Bonnet electromagnetic theory (2.4) becomes

$$\mu=x-\frac{1}{3\Upsilon}, \tag{5.15}$$

from where we can read immediately the limiting field value (5.14), in agreement with the discussion following (2.4). What happens to the light cones in this limit? Considering $\mathbf{B}=0$, we see from Eq. (2.14) that the Boillat light cone collapses in the two non-principal directions, manifesting the breakdown of the theory. Moreover, beyond such limit, the causality inequality (2.16) is violated.

Yet another manifestation of the limiting electric field can be seen by studying the convexity of the Hamiltonian function, as discussed in Sec. II C. The latter property is equivalent to the positive-definiteness of the following six dimensional quadratic form (of the variables $\mathbf{b}$, $\mathbf{d}$):

$$\mathbf{b}^2+\mathbf{d}^2+\frac{3\Upsilon}{1-3\Upsilon\mathbf{B}^2}[(\mathbf{D}\cdot\mathbf{b})^2+(\mathbf{B}\cdot\mathbf{d})^2+2(\mathbf{b}\cdot\mathbf{d})(\mathbf{D}\cdot\mathbf{B})$$
$$+2(\mathbf{D}\cdot\mathbf{b})(\mathbf{B}\cdot\mathbf{d})]+\frac{27\Upsilon^3}{(1-3\Upsilon\mathbf{B}^2)^3}$$
$$\times(\mathbf{b}\cdot\mathbf{B})^2(\mathbf{D}\cdot\mathbf{B})^2+\frac{9\Upsilon^2}{2(1-3\Upsilon\mathbf{B}^2)^2}(\mathbf{D}\cdot\mathbf{B})$$
$$\times[4(\mathbf{D}\cdot\mathbf{b})(\mathbf{B}\cdot\mathbf{b})+4(\mathbf{B}\cdot\mathbf{d})(\mathbf{B}\cdot\mathbf{b})+(\mathbf{D}\cdot\mathbf{B})\mathbf{b}^2]. \tag{5.16}$$

Again, for vanishing magnetic induction both eigenvalues will be positive if and only if the electric field is smaller than Eq. (5.14).

Since Eq. (2.4) reduces to Eq. (5.15), which has a unique solution for μ one might think that Gauss-Bonnet electromagnetism admits no bi-refringence. However, as discussed in Sec. II, when ϖ in Eq. (2.4) vanishes, the information contained in Eq. (2.4) might be incomplete. As shown in [19] this theory exhibits bi-refringence, with one cone given by the Boillat cone with Eq. (5.15) and the second coinciding with the Minkowski light cone. So, the middle illustration in Fig. 1 is the one to bear in mind, but now, the Minkowski light cone is degenerate; it represents both the background geometry and one of the effective geometries describing the propagation of fluctuations.

B. The Born-Infeld theory to second order

The *Born-Infeld* case, $\mathcal{L}_{KK}=\mathcal{L}_{BI}{}^{(2)}$, $b=-1/2$. The action matches the Born-Infeld action to second order. To match the constants Υ with β one must remember that in the Kaluza-Klein ansatz one should replace $A_\mu\rightarrow\zeta A_\mu$ where ζ is a constant with dimension length (we are using quantum units, i.e., $c=\hbar=1$). If we write down $\mathcal{L}_{BI}$ as

$$\mathcal{L}_{BI}=\frac{1}{\beta^2}(\sqrt{-g}-\sqrt{-\det(g_{\mu\nu}+\beta F_{\mu\nu})}), \tag{5.17}$$

the constants match as

$$\beta^2=-\frac{3\Upsilon\zeta^4L}{16\pi G_5}, \tag{5.18}$$

where L is the perimeter of the compact dimension (constant since we considered a trivial dilaton).

C. The Euler-Heisenberg action

The *Euler-Heisenberg* case, $\mathcal{L}_{KK}=\mathcal{L}_{EH}$, $b=1/7$. This is the effective action to QED due to one-loop corrections [43]. The constant Υ should then be

$$\frac{\Upsilon\zeta^4L}{16\pi G_5}=-\frac{28\alpha^2}{135m_e^4}, \tag{5.19}$$

where α is the fine structure constant.

VI. METRIC INDEPENDENCE OF BLACK HOLE PROPERTIES

In this section we consider the propagation of fluctuations of fields (including the electromagnetic) in curved backgrounds. Our main theme will be that even though there may be more than one metric present in theory, many properties of black holes and their thermodynamic behavior are metric invariant. In this sense we find that the event horizon and its properties have a universality which goes beyond the universality implied by the equivalence principle.

A. Causality and the strong energy condition

The presence of gravity as a background field is expected to induce changes in the propagation of electromagnetic fluctuations, in the same way a background electromagnetic field does. In fact the former is a consequence of the latter via the Einstein equations. Let us start by using the Boillat metric presented in Sec. II to ask when such propagation is causal. The Boillat co-metric (2.9) is conformal to

$$((\mu-x)L_x-yL_y+L)g^{\mu\nu}-\left(T^{\mu\nu}-\frac{T}{2}g^{\mu\nu}\right). \tag{6.1}$$

From now on in this section we assume that g is the Einstein metric, rather than some conformal multiple, such as the closed string metric. This is because we wish to assume that the Einstein equations hold. Then the Boillat co-metric is conformal to

$$g^{\mu\nu}_{\text{effective}}=g^{\mu\nu}+AR^{\mu\nu}, \tag{6.2}$$

where $R^{\mu\nu}$ is the Ricci tensor and

$$\frac{1}{8\pi G_N A}=(x-\mu)L_x+yL_y-L. \tag{6.3}$$

If $T_{\mu\nu}$ satisfies the strong energy condition and the Einstein equations hold, then $R^{\alpha\beta}p_\alpha p_\beta\geqslant 0$ for all co-vectors lying inside or on the Einstein co-cone. Thus if $A\geqslant 0$, the Boillat co-cone lies outside or on the Einstein co-cone. Passing back to the original Einstein and Boillat cones, remembering that duality reverses inclusions we see that the strong energy condition together with the requirement that $A\geqslant 0$ is a sufficient condition that the Boillat cone lies inside the Einstein cone. In these circumstances small disturbances travel no faster than gravitons.

B. Stationary event horizons and the touching theorem

Before discussing the even horizon given by the co-metric (6.2) it seems worth recalling that quantum mechanical effects renormalize the propagation equations in a background gravitational field. For scalars ϕ and spinors ψ, additional terms appear in the effective action of the form

$$\frac{A}{2}R^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi, \quad \frac{iA}{2}R^{\alpha\beta}\overline{\psi}\gamma_\alpha\nabla_\beta\psi, \tag{6.4}$$

for some coefficients A. In the case of scalars these give an effective metric of the form (6.2). In the case of spinors the discussion given in Sec. IV B applies. In the notation used there one has

$$a^{\alpha\beta}=g^{\alpha\beta}+\frac{A}{2}R^{\alpha\beta} \tag{6.5}$$

and from Eq. (4.5) it follows that

$$g^{\mu\nu}_{\text{effective}}=g^{\mu\nu}+AR^{\mu\nu}+\frac{A^2}{4}R_\alpha{}^\mu R^{\alpha\nu}. \tag{6.6}$$

In perturbative calculations one neglects the last third term. The second was computed by Ohkuwa within the Weinberg-Salam model [44] yielding

$$A=-\frac{11}{192\pi^2}\frac{e^2\hbar}{M_W^2\sin^2\theta_W c^3}, \tag{6.7}$$

where θ_W is the Weinberg angle and M_W is the W-boson mass.[8] Since A is negative the effective cones lie outside the Einstein cone. Physically however it is not clear that this implies the neutrino speeds faster than light, because the approximation of retaining only first derivatives in the effective action may break down.

The case of photons is more complex and it involves the Riemann tensor [45].[9]

Work on the causality properties of such effective metrics ([46] and references therein) uncovered a striking result which is also relevant in the context of non-linear electrodynamics.

If the Einstein metric g contains a stationary event horizon $\mathcal{H}$ with null generators l_α and the weak energy condition holds, $T^{\alpha\beta}l_\alpha l_\beta\geqslant 0$, then Hawking has shown that restricted to $\mathcal{H}$

$$R^{\alpha\beta}l_\alpha l_\beta=0. \tag{6.8}$$

It follows that

$$g^{\alpha\beta}_{\text{effective}}l_\alpha l_\beta=0. \tag{6.9}$$

Thus the null generator of the horizon lies on the effective co-cone. Passing to the dual space we see that the Einstein cone and the effective cone actually touch along the null generator of the horizon. In the case that $A\geqslant 0$ the effective cone will touch from the inside. This makes the existence of another effective event horizon outside the Einstein event horizon unlikely.

This ''touching theorem'' shows that the concept of an absolute event horizon is more ''absolute'' than one might

[8]Notice that the different sign in [44] is due to the opposite convention for the Riemann tensor.

[9]It maybe of interest to note that if one has as many scalar as spinor degrees of freedom with the same mass going around the loop then the Drummond-Hathrell correction vanishes.

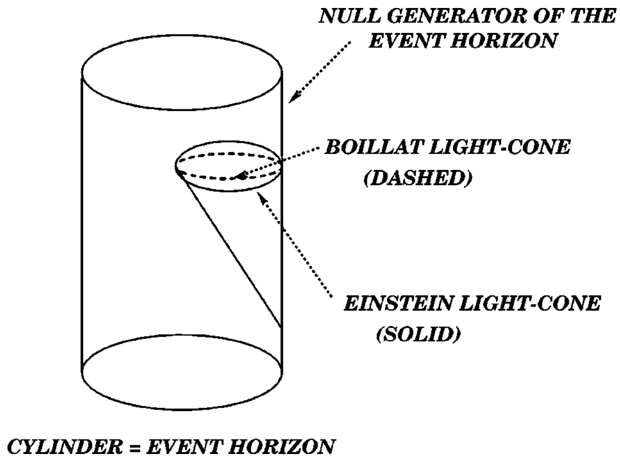

FIG. 4. The touching theorem.

have thought. After all because quantum fluctuations will in general affect different particles differently and because the effective metric they see clearly depends upon their couplings one might have imagined that in quantum theory different particles would have different effective event horizons, in contradiction with the classical equivalence principle. However we have seen that to the order we have been working this is not so. All particles see the same event horizon (see Fig. 4). In other words, the concept of a black hole remains universal in the quantum theory.

C. The surface gravity and the universality of the Hawking temperature

As well as the location of the event horizon one might ask whether the thermodynamic properties, such as the temperature, are universal. Because more than one metric is involved, this is not immediately obvious. In the case of static solutions, the simplest way of obtaining the surface gravity κ and hence the Hawking temperature $T_H = \kappa/(2\pi)$ is by setting $t=\sqrt{-1}\tau, \tau$ real and calculating the period $\beta=(T_H)^{-1} = 2\pi/\kappa$ required to remove the potential conical singularity at the horizon. It is clear that there will be no conical singularity in one metric if and only if there is no conical singularity in the other metric. Thus we get the same period β for both metrics.

If the timelike Killing vector, which is of course a Killing vector of both metrics, is normalized to have unit magnitude at infinity with respect the Einstein metric, then this calculation yields the temperature in Einstein units as judged by closed observers at infinity. If a background dilaton Φ is non-zero then this must be rescaled to get the temperature in closed string units. Similarly if the background Kalb-Ramond field is non-vanishing we must rescale to get the temperature in open string units. For previous work on the universality of the thermodynamic properties of black holes in generally covariant theories including arbitrary higher derivative interactions see [47].

D. Black hole in a magnetic field in Einstein-Maxwell theory

One stimulus for this work is the current activity on physics in an external B field. It is worth recalling therefore the properties of a black hole immersed in an external magnetic field according to Einstein-Maxwell theory. The main point we wish to make is that the thermodynamic properties of the black hole are unaffected by the magnetic field passing through it. This is perhaps not unexpected if one believes that the thermodynamics has its origin in microscopic degrees of freedom whose number and nature are essentially unchanged by external fields. To be concrete the metric is [48]

$$ds^2=\Lambda(r,\theta)^2\left(-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-\frac{2M}{r}}+r^2d\theta^2\right)+\Lambda(r,\theta)^{-2}r^2\sin^2\theta d\phi^2, \tag{6.10}$$

where M is the analogue of the ADM mass for asymptotically Melvin boundary conditions and

$$\Lambda(r,\theta)=1+\tfrac{1}{4}B_0^2r^2\sin^2\theta. \tag{6.11}$$

The Hawking temperature T_H and the area of the event horizon A_H are easily seen to be the same as for the Schwarzchild solution. For some other comments on ‘‘noncommutative black holes’’ see [49].

VII. EINSTEIN-SCHRÖDINGER THEORY

It is well known that there are many similarities between Born-Infeld theory and the Einstein-Schrödinger theory of gravity. In Sec. IV C we discussed that the characteristics and therefore the causal structure relevant for fluctuations is analogous to the one for the open string. We now specialize the discussion to some black hole solutions found by Papapetrou.

The connection in this theory is not the usual Levi-Civita connection, but rather computed from the relation

$$a_{\alpha\beta,\mu}-a_{\nu\beta}\Gamma^{\nu}_{\alpha\mu}-a_{\alpha\nu}\Gamma^{\nu}_{\mu\beta}=0. \tag{7.1}$$

The notation is the one of Sec. IV C. The Ricci tensor is computed by an expression formally identical to the one in general relativity, but has both a symmetric piece $R_{(\alpha\beta)}$ and an antisymmetric one $R_{[\alpha\beta]}$. In analogy to the dual Maxwell tensor introduced in Sec. II E we define $P^{\mu\nu}=(a)^{[\mu\nu]}$. The vacuum field equations read

$$R_{(\alpha\beta)}=0, \quad \partial_\beta(\mathfrak{P}^{\alpha\beta})=0, \quad R_{[[\alpha\beta],\mu]}=0. \tag{7.2}$$

The contravariant second rank tensor density $\mathfrak{P}^{\mu\nu} = \sqrt{-\det a^{-1}}P^{\mu\nu}$. These are similar to the usual Einstein equations, equation of motion and Bianchi identities in nonlinear electrodynamics, provided one thinks of $F_{\mu\nu}$ as being analogous to $R_{[\mu\nu]}$. We will avoid the issue of positivity of energy in this theory.

It is perhaps worth remarking that every Ricci flat Kähler metric including Calabi-Yau spaces, provides a Euclidean solution to this theory. In fact, if g is Kähler, choosing for B a multiple of the Kähler form, which is covariantly constant, the Levi-Civita connection of g will solve Eq. (7.1). Hence all equations of motion are obeyed. For these solutions $G^{\mu\nu}_{\text{Eins-Schro}} \propto g^{\mu\nu}$ and so there is no ambiguity as to which metric to use. For the analogous phenomenon in Born-Infeld theory see [49].

The last comment is not true in general. Spherically symmetric solutions were found by Papapetrou [22], corresponding to ''electrically'' and ''magnetically'' charged spherically symmetric objects. The most general electrical solution reads

$$(a^{-1})_{\mu\nu}dx^{\mu}dx^{\nu}=\left(1+\frac{Q^2}{r^4}\right)\left(1-\frac{2MG_N}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2MG_N}{r}\right)} - r^2(d\theta^2+\sin^2\theta d\phi^2)+\frac{Q}{r^2}dt\wedge dr. \tag{7.3}$$

A short calculation reveals that

$$ds^2_{\text{Eins-Schro}}=\left(1-\frac{2G_NM}{r}\right)dt^2 - \frac{dr^2}{\left(1+\frac{Q^2}{r^4}\right)\left(1-\frac{2G_NM}{r}\right)} - r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{7.4}$$

It is striking that some of our previous findings concerning the invariance under the change of metric of the black hole properties still hold in this theory. For example, in general the causal structure of g and $G^{\text{Eins-}Schro}_{\mu\nu}$ differ but both agree about the location of the event horizon $r=2G_NM$ and its surface gravity which is

$$\kappa=\frac{1}{4G_NM}\left(1+\frac{Q^2}{16G_NM^4}\right). \tag{7.5}$$

Note, while the area of the event horizon is given by the same formula in terms of the mass as it is in the Schwarzchild solution, a black hole with $Q\neq 0$ is hotter than the Schwarzchild hole with the same mass. The hotter temperature is ascribable to the fact that the factor $(1+Q^2/r^4)$ in g_{00} is blue-shifting rather than redshifting.

A magnetic solution found by Papapetrou reads

$$(a^{-1})_{\mu\nu}dx^{\mu}dx^{\nu}=\left(1-\frac{2MG_N}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2MG_N}{r}\right)} - r^2(d\theta^2+\sin^2\theta d\phi^2) + B_0 r^2 \sin\theta d\theta\wedge d\phi. \tag{7.6}$$

The Einstein-Schrödinger metric then becomes

$$ds^2_{\text{Eins-Schro}}=\left(1-\frac{2MG_N}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2MG_N}{r}\right)} - (1+B_0^2)r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{7.7}$$

The physical meaning of the two form in Eq. (7.6) is not clear. It is spherically symmetric and of constant magnitude. It is striking that the metric has a similar form to that of Eq. (3.21). The location of the horizon and the surface gravity are independent of B_0 and indeed the $r-t$ metric is identical to that of a Schwarzchild black hole.

VIII. CONCLUSIONS

The background geometry determined by a gravitational theory might not be the relevant one seen by fluctuations of some test field. This is true even at the classical level, but quantum effects can also renormalize the geometry describing the propagation of fluctuations. One quite interesting example of such distinction was uncovered in work on string propagation in a background B field [15]: in this setup open and closed string fluctuations move, in general, at different velocities. Gravitons and Born-Infeld photons see different light cones. The discussion in Secs. II and III shows that the latter causal structure is the Boillat causal structure, studied long ago in the context of non-linear electrodynamics. Moreover, the $\theta^{\mu\nu}$ parameter describing the non-commutativity of spacetime in the duality established in [15] is just the dual Maxwell tensor of Born-Infeld theory.

The open string metric is intrinsically connected to the Born-Infeld action, as we showed in Sec. IV by including scalars and fermions in a Born-Infeld type action, and showing the characteristics are determined by the open string metric. At this point a question requires more thorough understanding. In the context of string theory, the Born-Infeld action describes brane dynamics. The brane world scenario motivated by [37] tries to bind gravitons to the brane. The difficulty is, of course, that gravitons are closed rather than open string modes. But if this program is successful, either the brane graviton sees different light cones from the other spin brane fields or, if it is governed by the open string metric, the question arises to what effective field theory describes such gravitons. The Einstein-Schrödinger theory seems to have exactly the characteristics we would then be looking for. But it seems to suffer from instabilities [20].

In Sec. V we looked at higher order gravity and Kaluza-Klein theory. To lowest order in α', the abelian truncation of the effective open string theory (Maxwell's theory) is obtained by Kaluza-Klein compactification of the effective closed string theory (Einstein's gravity). But this does not seem to hold to the next order in α': the Gauss-Bonnet contribution to the effective closed string theory gives upon Kaluza-Klein reduction what we named as ''Gauss-Bonnet electromagnetism,'' distinct from both the Euler-Heisenberg theory and the Born-Infeld theory to this order.[10] We remarked however that for purely electromagnetic excitations, the no-ghost requirement is weaker and by an appropriate choice of couplings one can obtain the latter two theories to this order. It would be interesting to consider also the full Kaluza-Klein theory, with all excitations present. This would give non-minimal gravitational-electromagnetic couplings, violating the equivalence principle. Gravitational birefringence and dispersion effects might also be present, although the latter seem only to occur at even higher order in α' [50].

In Sec. VI we made use of a result know in the literature as the ''touching theorem'' to show that the propagation of fluctuations in non-linear electrodynamics coupled to gravity will see a universal event horizon and black hole temperature. Such comment also holds for one-loop corrected propagators in curved spacetime. In this way black holes do not seem to ''leak.'' We also noticed that similar invariance is seen for a black hole immersed in a magnetic field in Einstein-Maxwell theory. It would perhaps be interesting to look explicitly at such black holes in a ''Melvin Universe'' for the case of non-linear electrodynamics.

[10]We remark that the Born-Infeld theory to second order does not coincide with the Euler Heisenberg theory, which might be puzzling. But at least in the supersymmetric case they do coincide to this order [7].

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