Calibrations and Fayyazuddin-Smith spacetimes

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We show that a class of spacetimes introduced by Fayyazuddin and Smith to describe intersecting M5-branes admits a generalized Kähler calibration. Equipped with this understanding, we are able to construct spacetimes corresponding to further classes of calibrated *p*-brane world-volume solitons. We note that these classes of spacetimes also describe the fields of *p*-branes wrapping certain supersymmetric cycles of Calabi-Yau manifolds.

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I. INTRODUCTION

The mathematical theory of calibrations [1] and certain extensions thereof have proven to be quite useful in classifying the supersymmetric world-volume solitons [2,3] of branes embedded in fixed supersymmetric spacetime backgrounds. Applications include smoothed intersections of branes in flat spacetime [4–9], branes wrapping supersymmetric cycles of Calabi-Yau manifolds [10,11], world-volume solitons in AdS compactifications [12], solitons involving world-volume gauge fields that describe branes ending on other branes [13–15], and branes in *p*-brane spacetime backgrounds [16].

The calibrated branes in each of these applications have so far been treated as test objects. In this paper we demonstrate that the calibration technology is also useful in understanding the spacetime fields that result from treating these world-volume solitons as charged, gravitating sources. The reason for this is quite simple. Consider the spacetime geometry generated by a supersymmetric world-volume soliton. Based on the Bogomol'nyi-Prasad-Sommerfield (BPS) ''no force'' properties of branes, it should be possible for a suitably configured test brane embedded in this spacetime to be in equilibrium. The spacetime should therefore carry a calibrating form. Moreover, near infinity, this calibrating form should approach the fixed background form that calibrated the original world-volume soliton. This situation holds, for example, for the spacetime of a single planar M-brane [16].[1]

Our starting point will be a class of supersymmetric spacetimes constructed by Fayyazuddin and Smith to describe the spacetime fields of M5-branes intersecting on 3-branes [18]. A central ingredient in these spacetimes is a warped Kähler metric residing on the four relative tranverse directions of the intersecting brane configuration. The Kähler metric depends, as well, on the overall transverse coordinates. The exact form of the Kähler metric and the warp factor are related to the M5-brane sources by a nonlinear field equation. We will see that these spacetimes indeed have calibrating forms of the appropriate type and that, equipped with this understanding, one can construct similar spacetimes corresponding to other calibrated world-volume solitons. We will refer to these different spacetimes collectively as Fayyazuddin-Smith (FS) spacetimes. We will focus here on Kähler calibrations and correspondingly on FS spacetimes built around Kähler metrics. More generally, FS spacetimes will involve other metrics of reduced holonomy. We conjecture that the spacetime fields of all calibrated world-volume solitons will be of FS type.

Of course, it has proved to be quite difficult to construct spacetimes corresponding to particular configurations of localized intersections of branes [18–24], as opposed to smeared intersections (see [25] for a complete review). It was argued in [26] that this situation may reflect interesting underlying physics. The world-volume effective field theory description of the delocalization of certain brane intersections is related to the Coleman-Mermin-Wagner theorem. In these cases, the dimensionality of the intersection is the determining factor as to whether the localization of the classical world-volume soliton persists in the supergravity solution. FS spacetimes should provide the appropriate supergravity setting to study these effects.

We note that FS spacetimes also provide the spacetime fields of branes wrapping supersymmetric cycles of Calabi-Yau manifolds. To describe intersecting branes in otherwise empty spacetime, the 4 real dimensional Kähler metric in the original FS ansatz [18] is taken to be asymptotic to 4-dimensional flat space. However, if instead it is taken to be asymptotic to a Calabi-Yau metric, e.g. to a Ricci flat metric on K3, then the FS spacetimes [18] describe M5-branes wrapping supersymmetric (1,1) cycles of K3.[2] The spacetime geometry of branes wrapping all of K3 has been shown to reflect very interesting underlying physics [28]. It seems likely that FS spacetimes will provide a rich ground for further study in this context. If we take, for example, M2-branes wrapping 2-cycles of compact Calabi-Yau 3-folds, then from the 5-dimensional viewpoint these will be black holes. Dimensionally reducing and keeping only the massless Kaluza-Klein modes should give the black holes of [29–33]. The FS

[1]It was also demonstrated in [17] that the BPS spike soliton of a test D3-brane, describing a fundamental string ending on the brane, can also be found when the test D3-brane is placed in a D3-brane spacetime background.

[2]See [27] for a recent related discussion of branes wrapping cycles of K3.

spacetimes should provide the 11-dimensional lifts of these spacetimes, including the nontrivial massive Kaluza-Klein modes as well.

Finally, we note that another extension of the FS class of spacetimes to include branes ending on branes has recently been given in [34]. We expect that these spacetimes may also be usefully organized using calibration technology.

II. CALIBRATIONS

We start with a brief and basic introduction to calibrations. Consider the action for a p-brane moving in a $(D+1)$-dimensional spacetime with metric $G_{\mu\nu}$ and $(p+1)$-form gauge potential $A_{\mu_1 \ldots \mu_{p+1}}$:

$$S_{p+1}=\int d^{p+1}\sigma\left\{\sqrt{-\det g_{ab}}-\frac{1}{(p+1)!}\varepsilon^{a_1\cdots a_{p+1}}\partial_{a_1}X^{\mu_1}\cdots\partial_{a_{p+1}}X^{\mu_{p+1}}A_{\mu_1\ldots\mu_{p+1}}\right\}, \tag{1}$$

where σ^a with $a=0,1,\ldots,p$ are world-volume coordinates, $X^\mu(\sigma)$ gives the embedding of the brane in the background spacetime and $g_{ab}=\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$ is the induced metric on the world volume. We will not consider here possible world-volume gauge fields or couplings to additional spacetime fields. To start, let us assume a flat background, $G_{\mu\nu}=\eta_{\mu\nu}$, $A_{\mu_1\ldots\mu_{p+1}}=0$, and consider static brane configurations. These will minimize the spatial volume of the brane. Calibrations are a mathematical technique for finding classes of such minimal submanifolds. A calibration for a p-dimensional submanifold is a p-form ϕ on the embedding space that satisfies two properties:

(1) The calibration ϕ is a closed form

$$d\phi=0. \tag{2}$$

(2) The pullback of ϕ onto any p-dimensional submanifold Σ is always less than or equal to the induced volume form on the submanifold:

$$^*\phi\leqslant\varepsilon_\Sigma . \tag{3}$$

It then follows via a simple argument that, if the inequality (3) is saturated at every point on a p-dimensional submanifold Σ, then Σ minimizes volume within its homology class. Assume Σ saturates the inequality (3) at every point. Pick a closed $(p-1)$-dimensional surface S in Σ, and within S continuously deform Σ into a new submanifold Σ'. The following chain of equalities and inequalities then shows that $\mathrm{Vol}(\Sigma)\leqslant\mathrm{Vol}(\Sigma')$:

$$\begin{aligned}\mathrm{Vol}(\Sigma)&=\int_\Sigma\varepsilon_\Sigma=\int_\Sigma {}^*\phi=\int_{\Sigma'}{}^*\phi+\int_B d\phi\\&=\int_{\Sigma'}{}^*\phi\leqslant\int_{\Sigma'}\varepsilon_{\Sigma'}=\mathrm{Vol}(\Sigma'),\end{aligned} \tag{4}$$

where B is the p-dimensional region bounded by Σ and Σ'.

A. Kähler calibrations

The simplest examples of calibrating forms and the ones that will concern us below are the Kähler calibrations. Start with even dimensional flat space, $D=2n$, with real Cartesian coordinates $x^1,\ldots,x^{2n}$. Choose a complex structure, i.e. a pairing of real coordinates into complex coordinates, for example,

$$z^1=x^1+ix^2,\ldots,z^n=x^{2n-1}+ix^{2n}; \tag{5}$$

then, the Kähler form is given by

$$\begin{aligned}\omega&=dx_1\wedge dx_2+\cdots+dx_{2n-1}\wedge dx_{2n}\\&=\frac{i}{2}(dz_1\wedge d\overline{z_1}+\cdots dz_n\wedge d\overline{z_n}).\end{aligned} \tag{6}$$

The forms $\phi_{2k}=\omega^k/k$ can then be shown to be calibrations [1]. The corresponding calibrated submanifolds are simply the complex submanifolds of real dimension $2k$.

We recall some examples of calibrated surfaces [4,5] that will be useful to keep in mind below. Our focus below will be with M2-branes and M5-branes and we frame the examples in this context. First consider a static M2-brane configuration whose world volume lies entirely in the (1,2,3,4) subspace of 10 dimensional flat space. We can then take $D=4$ above and the calibrating 2-form $\phi=\omega=dx_1\wedge dx_2+dx_3\wedge dx_4$. Clearly, if the M2-brane lies either in the (1,2) plane, or in the (3,4) plane, then the inequality (3) is saturated and these are calibrated surface. A more nontrivial example is the family of complex curves

$$z_1z_2=\alpha^2, \tag{7}$$

with α an arbitrary constant. These curves interpolate smoothly between the (1,2) and (3,4) planes and represent a smoothed version of two static M2-branes intersecting at a point. The singular limit $\alpha=0$ gives the pure orthogonal intersection of the two planes. If we added on 3 additional flat spatial directions to the brane, then the curve (7) gives two M5-branes intersecting on a 3-brane.

Now take $D=6$ and consider the 4-form calibration $\phi=\frac{1}{2}\omega\wedge\omega$. In terms of the real coordinates this is

$$\begin{aligned}\phi=&dx_1\wedge dx_2\wedge dx_3\wedge dx_4+dx_1\wedge dx_2\wedge dx_5\wedge dx_6\\&+dx_3\wedge dx_4\wedge dx_5\wedge dx_6 .\end{aligned} \tag{8}$$

Clearly the (1,2,3,4), (1,2,5,6) and (3,4,5,6) planes are examples of submanifolds calibrated by ϕ, and complex surfaces exist that interpolate smoothly between these planes. Adding another spatial direction x^7 to get M5-branes, there will then be calibrated surfaces describing the smoothed intersection of three 5-branes in the directions

$$(t,1,2,3,4,x,x,7)$$
$$(t,1,2,x,x,5,6,7)$$
$$(t,x,x,3,4,5,6,7) \tag{9}$$

where the x's are placeholders. Note that each pair of M5-branes intersects on a 3-brane and that altogether they intersect on a string.

B. Calibrations and spinors

The calibration technology applies in curved spaces as well. For example, if ω is now the Kähler form for an arbitrary Kähler space, then the forms $\phi_{2k}=\omega^k/k$ are again calibrations and the calibrated submanifolds are again the set of complex submanifolds. In general the existence of calibrating forms is tied to the property of reduced holonomy (see e.g. [35]). Reduced holonomy in turn is tied to the existence of spinor fields having special properties.

For example, for an N complex dimensional Kähler manifold with metric $g_{m\bar{n}}$, the holonomy group is $U(N)\subset SO(2N)$. Covariantly constant spinors exist only in the Calabi-Yau case of vanishing Ricci tensor, for which the holonomy group is further reduced to $SU(N)$. For a general Kähler metric, though, there exists a pair of spinors ϵ_+ and ϵ_- transforming as singlets of the holonomy group. These satisfy the relations

$$\Gamma_m\epsilon_+=\Gamma_{\bar{m}}\epsilon_-=0, \tag{10}$$

from which follow the projection conditions

$$\Gamma_{m\bar{n}}\epsilon_\pm=\pm g_{m\bar{n}}\epsilon_\pm\,. \tag{11}$$

If we normalize $\epsilon_\pm^\dagger\epsilon_\pm=1$, then the Kähler form can be written as

$$\omega_{ab}=\pm i\epsilon_\pm^\dagger\Gamma_{ab}\epsilon_\pm\,. \tag{12}$$

The vanishing of the components ω_{mn} and $\omega_{\bar{m}\bar{n}}$ follows from the relations (10), which also imply that only even dimensional forms with equal numbers of holomorphic and antiholomorphic indices can be built in this way. The covariant derivatives of $\epsilon_\pm$ are given by

$$\nabla_p\epsilon_\pm=\partial_p\epsilon_\pm\pm\frac{1}{2}(\bar{E}^{-1}\partial_p\bar{E})\epsilon_\pm\,,$$
$$\nabla_{\bar{p}}\epsilon_\pm=\partial_{\bar{p}}\epsilon_\pm\mp\frac{1}{2}(E^{-1}\partial_{\bar{p}}E)\epsilon_\pm\,, \tag{13}$$

where E and $\bar{E}$ are determinants of the complex frame fields $E_n^{\hat{m}}$ and $E_{\bar{n}}^{\hat{\bar{m}}}$ respectively, with the caret denoting flat space frame indices. For the Ricci flat case, these second terms vanish identically giving covariantly constant spinors.

C. Generalized calibrations

A certain amount of care is neccesary in applying the calibration technology to find static solutions for p-branes in curved spacetimes [16], because even for a static p-brane, the time-time component G_{00} of the spacetime metric enters the p-brane effective action (1). Assume that the embedding spacetime is static with timelike Killing vector $\xi^a=(\partial/\partial x^0)^a$. If we fix the static gauge $\sigma^0=x^0$ for the coordinates on the brane world volume, then the $\sqrt{-\det g_{ab}}$ term in the brane action (1) includes a contribution from $g_{00}=G_{00}$, called the redshift factor in [16]. This factor can be absorbed by defining a new effective spatial metric $\hat{G}_{\alpha\beta}=(-G_{00})^{1/p}G_{\alpha\beta}$ where $\alpha,\beta=1,\ldots,D$ now run over only spatial directions in the embedding space. We then have

$$\sqrt{-\det g_{ab}}=\sqrt{\det\hat{g}_{kl}} \tag{14}$$

where $k,l=1,\ldots,p$ are purely spatial world-volume indices and $\hat{g}_{kl}$ is the spatial metric induced on the brane via embedding in the rescaled metric $\hat{G}_{\alpha\beta}$ defined above. If there are additional spatial symmetry directions of the embedding space that are shared by the p-brane configuration, then these can be handled in a similar manner [16] by appropriately modifying the definitions of $\hat{G}_{\alpha\beta}$, $\hat{A}$ and $\hat{F}$.

Finally, if the spacetime has a nonzero $(p+1)$-form gauge potential $A_{\mu_1\ldots\mu_{p+1}}$, then a static brane configuration will satisfy equations of motion involving the corresponding field strength. An appropriately generalized definition of calibrating forms taking this additional force into account was given in [12,16]. The modification required is quite simple. Condition (2) becomes

$$d\phi=\hat{F} \tag{15}$$

where $\hat{F}=d\hat{A}$ and $\hat{A}_{\alpha_1\ldots\alpha_p}=A_{0\alpha_1\ldots\alpha_p}$. Therefore the calibrating form ϕ is equal to the reduced gauge potential $\hat{A}$ up to a gauge transformation. This new condition then yields a chain of equalities and inequalities similar to Eq. (4), showing that if a static surface saturates the calibration bound then it minimizes the action (1).

D. M2-brane spacetime

The planar M2-brane itself provides a good example of a spacetime with a generalized calibrating form [16]:

$$ds^2=H^{-2/3}(-dt^2+dx_1^2+dx_2^2)+H^{+1/3}(dx_3^2+\cdots+dx_{10}^2),$$
$$A_{t12}=cH^{-1}, \tag{16}$$

where $c=\pm1$. For a static test M2-brane in this background the effective spatial metric and gauge potential defined above are given by

$$d\hat{s}^2 = H^{-1}(dx_1^2 + dx_2^2) + dx_3^2 + \cdots + dx_{10}^2, \quad \hat{A}_{12} = cH^{-1}. \tag{17}$$

As discussed in [16], test M2-branes will then be calibrated by the form

$$\phi = cH^{-1} dx_1 \wedge dx_2 + \omega_\perp, \tag{18}$$

where $\omega_\perp$ is an arbitrary Kähler form on the transverse space, equivalent to a choice of complex structure in the transverse space. The calibrating form ϕ is then gauge equivalent to the gauge potential $\hat{A}$ and Eq. (15) is satisfied. The calibrated surfaces are complex surfaces with respect to the associated almost complex structure obtained by raising one index on ϕ_{kl} using the rescaled metric (17). Note that the warp factor H then drops out.

III. FAYYAZUDDIN-SMITH SPACETIMES

The original FS spacetimes [18] described M5-branes intersecting on 3-branes. Here we start with the related M2-brane FS spacetimes studied in [23]. The metric and gauge potential for these are given by

$$ds^2 = H^{-2/3}(-dt^2 + 2g_{m\bar{n}} dz^m dz^{\bar{n}}) + H^{+1/3}(\delta_{\alpha\beta} dx^\alpha dx^\beta)$$

$$A_{tm\bar{n}} = icH^{-1} g_{m\bar{n}}, \quad c = \pm 1. \tag{19}$$

Here z^m with $m = 1,2$ are complex coordinates on a 4 real dimensional Kähler manifold $\mathcal{M}$ and $\alpha, \beta = 5, \ldots, 10$ are indices for the 6-dimensional transverse space. The Kähler metric $g_{m\bar{n}}$ on $\mathcal{M}$ is allowed to depend on the transverse coordinates x^α as well as on position in $\mathcal{M}$; i.e., it can be written as $g_{m\bar{n}} = \partial_m \partial_{\bar{n}} K(z^p, \overline{z^q}, x^\alpha)$ with K a Kähler potential depending on the transverse coordinates. The warp factor H is also allowed to depend on both position in $\mathcal{M}$ and position in the transverse space. Note that the Kähler metric $g_{m\bar{n}}$ at a fixed transverse position is not required to be Ricci flat.

We present a detailed review of the supersymmetry conditions in order to correct a mistake in the form of the results stated in [18,23] that has obstructed a better understanding of this class of spacetimes. The supersymmetry condition for $D = 11$ supergravity takes the form $\hat{\nabla}_A \epsilon = 0$, where

$$\hat{\nabla}_A \epsilon = \nabla_A \epsilon + \frac{1}{288}(\Gamma_A^{BCDE} - 8\delta_A^B \Gamma^{CDE}) F_{BCDE} \epsilon, \tag{20}$$

and $(A, B, \ldots)$ are $D = 11$ indices. The supercovariantly constant spinors of the FS spacetimes (19) satisfy the projection conditions

$$\Gamma^{\hat{m}\hat{\bar{n}}} \epsilon = a \delta^{\hat{m}\hat{\bar{n}}} \epsilon, \quad a = \pm 1,$$

$$\Gamma^{\hat{t}} \epsilon = ib\epsilon, \quad b = \pm 1 \tag{21}$$

where careted indices are frame indices and the signs of the two projections are correlated with the sign of the gauge potential by the relation $abc = -1$. The first projection condition is just the standard projection (11) onto singlets of the $U(2)$ holonomy group of the Kähler metric $g_{m\bar{n}}$ at fixed transverse position. The combination of the two projections reduces the fraction of supersymmetry preserved to 1/4.

Given the projections (21), the supersymmetry conditions then impose a relation between the warp factor H and the complex determinant of the Kähler metric $g = g_{1\bar{1}} g_{2\bar{2}} - g_{1\bar{2}} g_{2\bar{1}}$. In [23], and originally in [18] for the M5-brane case, this relation is given as $H = g$. However, this is not precisely correct, as the following argument shows. The form of the FS ansatz (19) is preserved by holomorphic coordinate transformations on the Kähler manifold that do not depend on the transverse coordinates:

$$z'^m = z'^m(z^p). \tag{22}$$

Under these transformations the warp factor H is invariant, but the determinant g is transformed to $g' = g f \bar{f}$ where f is holomorphic. Hence the condition $H = g$ is not covariant under the transformations (22).

In order to determine the correct conditions, we write out the requirements that each of the components of $\hat{\nabla}_A \epsilon = 0$ reduce to after having applied the projection conditions (21):

$$A = t: \partial_\alpha \log H - \partial_\alpha \log g = 0$$

$$A = p: \partial_p \epsilon + \left\{ \left(\frac{1}{6} + \frac{a}{4} \right) \partial_p \log H - \frac{a}{2} \partial_p \log \bar{E} \right\} \epsilon = 0$$

$$A = \bar{p}: \partial_{\bar{p}} \epsilon + \left\{ \left(\frac{1}{6} - \frac{a}{4} \right) \partial_{\bar{p}} \log H + \frac{a}{2} \partial_{\bar{p}} \log E \right\} \epsilon = 0$$

$$A = \alpha: \partial_\alpha \epsilon + \left\{ \frac{1}{6} \partial_\alpha \log H + \frac{a}{4} \partial_\alpha \log E - \frac{a}{4} \partial_\alpha \log \bar{E} \right\} \epsilon = 0. \tag{23}$$

One can check that the whole set of equations can be solved provided that

$$\partial_\alpha \log H = \partial_\alpha \log g$$

$$\partial_m \partial_{\bar{n}} \log H = \partial_m \partial_{\bar{n}} \log g. \tag{24}$$

It is worth noting that the second condition involves the Kähler metric at fixed transverse position through its Ricci tensor $\mathcal{R}_{m\bar{n}} = -\partial_m \partial_{\bar{n}} g$, which transforms as a tensor under the coordinate transformations (22). These new relations are then covariant under the holomorphic coordinate transformations (22). These conditions imply that the general form of the relation between H and g is given by

$$g = H f \bar{f}, \tag{25}$$

where $f(z^m)$ is a holomorphic function of the complex coordinates and is independent of the transverse coordinates x^α. We then find using Eq. (25) that the supercovariantly constant spinors are then given by

$$\epsilon = (E/f)^{-(1/6 + a/4)} (\bar{E}/\bar{f})^{-(1/6 - a/4)} \epsilon_0, \tag{26}$$

where ϵ_0 is a constant spinor satisfying the projections (21). The supercovariantly constant spinors are invariant under the coordinate transformations (22).

All spacetimes of the form (19) satisfying the relations (24) are supersymmetric. However, we have not yet imposed the gauge field equations of motion. For the FS spacetimes (19), these reduce to the equations

$$2\partial_m\partial_{\bar{n}}H+\delta^{\alpha\beta}\partial_\alpha\partial_\beta g_{m\bar{n}}=0, \tag{27}$$

which are covariant with respect to the holomorphic coordinate transformations (22).[3] Combining the gauge field equation of motion (27) with the condition (25) gives a set of coupled nonlinear equations that has proved difficult to solve. Solutions have been given in the M5-brane case in the near horizon limit [18,36,37] and to first order in the far field limit [23].

The gauge field equation of motion (27) can be rewritten as an equation for the Ricci tensor of the Kähler metric $\mathcal{R}_{m\bar{n}}$ at fixed transverse position, giving

$$\mathcal{R}_{m\bar{n}}=\frac{(\partial_m H)(\partial_{\bar{n}}H)}{H^2}+\frac{1}{2H}\delta^{\alpha\beta}\partial_\alpha\partial_\beta g_{m\bar{n}}. \tag{28}$$

It is worth noting that given the correct relation (25) between g and H, the standard supersymmetric supergravity vacua now solve the field equations. If $g_{m\bar{n}}$ is Ricci flat and independent of the transverse coordinates, then one can choose complex coordinates so that $g=1$ everywhere. Taking $H=g=1$ then clearly gives a solution of Eq. (28). Performing a holomorphic coordinate transformation as in Eq. (22) on this spacetime yields $g=f\bar{f}$, with f holomorphic, and $H=1$, which is still obviously a solution to Eq. (28). However, if we instead also change H, so that as in [18] $H=g=f\bar{f}$, then the spacetime no longer solves Eq. (28). In this case, by referring to Eq. (19), we see that the gauge potential has nonzero field strength. These spacetimes correspond to nontrivial configurations of M2-brane sources. This is consistent because these spacetimes are not related by coordinate transformations to the the original Ricci flat vacuum spacetime.

IV. CALIBRATIONS AND NEW M-BRANE SPACETIMES

We now want to look at the FS spacetimes from the point of view of calibrations. The perspective we gain will prove useful in finding FS spacetimes for other types of M-brane world-volume solitons. It is straightforward to check that the FS spacetimes (19) discussed above have generalized calibrating forms in the sense defined in [12,16]. The effective spatial metric and gauge potential seen by a static M2-brane probe are

$$d\hat{s}^2=2H^{-1}g_{m\bar{n}}dz^m dz^{\bar{n}}+\delta_{\alpha\beta}dx^\alpha dx^\beta$$

$$\hat{A}_{m\bar{n}}=icH^{-1}g_{m\bar{n}},\quad c=\pm 1. \tag{29}$$

The corresponding generalized calibrating 2-form is given by

$$\phi=cH^{-1}\omega_{\mathcal{M}}+\omega_\perp, \tag{30}$$

where $\omega_{\mathcal{M}}=ig_{m\bar{n}}dz^m\wedge dz^{\bar{n}}$ is the Kähler form associated with the metric $g_{m\bar{n}}$, and $\omega_\perp$ is an arbitrary Kähler form on the transverse space. The calibrated surfaces are complex surfaces with respect to the almost complex structure obtained by raising one index on ϕ. Note that the warp factor H again drops out from the almost complex structure.

What can we learn from this structure that will be useful in constructing FS spacetimes for other types of world-volume solitons? The FS spacetimes (19) arise in two diferent physical contexts. In [23], the FS spacetimes were considered to be generated by static M2-brane sources lying on a nontrivial holomorphic curves in a 4 dimensional subspace of $D=10$ flat space. The Kähler metric $g_{m\bar{n}}$ was taken to be flat near infinity in the transverse space. A second application is to take the Kähler manifold $\mathcal{M}$ to be K3 and letting the Kähler metric $g_{m\bar{n}}$ approach a Ricci flat K3 metric near infinity. The FS spacetimes then describe M2-branes wrapping (1,1) cycles of K3. In each of these cases the original source branes were calibrated by the corresponding Kähler forms of these supersymmetric vacua. The warped Kähler form ϕ in Eq. (30) approaches the corresponding vacuum Kähler form near infinity, since as we have argued above, H must approach unity near infinity.

We conjecture that a similar structure will hold for spacetimes corresponding to other calibrated world-volume solitons. For Kähler calibrated solitons, we expect to find an FS spacetime built around a Kähler metric, with a gauge potential simply related to the original calibrating form. For another type of calibrated world-volume soliton, we would expect to find an FS spacetime built around a general curved space that admits this type of calibration. For special Lagrangian solitons, for example, we would expect an FS spacetime built around a Ricci flat Kähler metric. For a soliton calibrated by an exceptional calibration, we expect to find an FS spacetime built by a warped construction around a space with the corresponding reduced holonomy. Below, we give results for Kähler calibrated solitons. We will return to the other cases in future work.

A. New M2-brane spacetimes

The most straightforward generalization of the FS construction is to increase the number of dimensions of the Kähler manifold, maintaining the same basic form of the FS spacetimes (19). For e.g. a 3 complex dimensional space this would correspond to 1/8 supersymmetric, smoothed intersections of 3 M2-branes or to M2-branes wrapping (1,1) cycles of Calabi-Yau 3-folds. Setting the complex dimension to be N, we make the ansatz

$$ds^2=-H^{-2A}dt^2+2H^{-2B}g_{m\bar{n}}dz^m dz^{\bar{n}}+H^{2C}(\delta_{\alpha\beta}dx^\alpha dx^\beta)$$

$$A_{tm\bar{n}}=icH^{-1}g_{m\bar{n}},\quad c=\pm 1 \tag{31}$$

[3] $D=11$ supergravity can be coupled to M-brane sources by combining the bulk supergravity action with the M2-brane and M5-brane Born-Infeld actions. For M2-brane sources, the resulting current contribution to the right-hand side of Eq. (27) is given in [23].

where now the complex coordinates $m,n=1,\ldots,N$ and the transverse coordinates $\alpha,\beta=1,\ldots,10-2N$. The nontrivial possibilities are $N=2,3,4,5$. These spacetimes describe either $1/2^N$ supersymmetric smoothed intersections of M2-branes in otherwise empty spacetime or to M2-branes wrapping (1,1) cycles of Calabi-Yau N-folds.

We find that these spacetimes preserve $1/2^N$ supersymmetry, if the exponents are given by

$$A=\frac{1}{3}(N-1),\quad B=\frac{1}{6}(4-N),\quad C=\frac{1}{6}(N-1) \tag{32}$$

and H, $g_{m\bar{n}}$ related in general as in Eq. (25).[4] The supercovariantly constant spinors are given by

$$\epsilon=(E/f)^{-[(N-1)/6+a/4]}(\overline{E}/\overline{f})^{-[(N-1)/6-a/4]}\epsilon_0, \tag{33}$$

where ϵ_0 is a constant spinor satisfying the projection conditions (21). The source free equations of motion again reduce to Eq. (27).

The effective spatial metric and gauge potential for test M2-branes embedded in these spacetimes again have the form given in Eq. (29). This implies that the warped Kähler forms ϕ in Eq. (30) are again generalized calibrating forms for test M2-branes.

B. New M5-brane spacetimes

A more nontrivial application of our strategy is to start with world-volume solitons calibrated by the square of the Kähler form $\phi=\frac{1}{2}\omega\wedge\omega$. Since this requires that the spatial dimension of the brane be at least 4, in the context of M theory we will be looking at M5-branes. These spacetimes again will have two physical settings. One could start with smoothed intersections of M5-branes that share a common string [4,5] in otherwise empty space as in the discussion above Eq. (9). Alternatively, one can start with M5-branes wrapping a (2,2) cycle of a Calabi-Yau manifold, leaving a string in the remaining noncompact directions.

We build an FS ansatz similar to Eq. (31) that reflects these new physical settings. In particular, the calibrating form $\phi=\frac{1}{2}\omega\wedge\omega$ of the world-volume soliton is built into the 6-form gauge potential. Consider the 1/8 supersymmetric case, corresponding to a 3 complex dimensional space. Accordingly, let

$$ds^2=H^{-2A}(-dt^2+dy^2)+2H^{-2B}g_{m\bar{n}}dz^m dz^{\bar{n}} +H^{2C}(\delta_{\alpha\beta}dx^\alpha dx^\beta)$$

$$A_{tym\bar{n}r\bar{s}}=cH^{-1}(g_{m\bar{n}}g_{r\bar{s}}-g_{m\bar{s}}g_{r\bar{n}}),\quad c=\pm 1 \tag{34}$$

where $m,n=1,\ldots 3$, and $\alpha,\beta=1,2,3$. We find that supersymmetry (a) requires the projection conditions

$$\Gamma^{m\bar{n}}\epsilon=aH^{2B}g^{m\bar{n}}\epsilon,\quad a=\pm 1$$

$$\Gamma^{ty}\epsilon=bH^{2A}\epsilon,\quad b=\pm 1 \tag{35}$$

with $bc=-1$; (b) fixes the values of the exponents to be $A=B=1/6$ and $C=1/3$; and (c) imposes the general relation (25) between H and $g_{m\bar{n}}$. The source free gauge field equations of motion again reduce to Eq. (27). We find that the supercovariantly constant spinors are given by

$$\epsilon=(E/f)^{-(1/12+a/4)}(\overline{E}/\overline{f})^{-(1/12-a/4)}\epsilon_0, \tag{36}$$

where ϵ_0 is a constant spinor satisfying the projection conditions (35).

Following [16], we introduce a rescaled effective spatial metric $d\hat{s}^2$ for test M5-branes that are both static and translationally invariant in the y direction. The appropriate rescaling is $\hat{G}_{kl}=(-G_{tt}G_{yy})^{1/4}G_{kl}$, where k,l run over all directions except t,y. The 1/4 power arises because these factors are now shared by the remaining 4 spatial dimensions of the brane. This yields

$$d\hat{s}^2=2H^{-1/2}g_{m\bar{n}}dz^m dz^{\bar{n}}+H^{1/2}(dx_8^2+dx_9^2+dx_{10}^2). \tag{37}$$

The calibrating form is then given by the expression $\phi=\frac{1}{2}\omega\wedge\omega$ in terms of the Hermitian form

$$\omega=cH^{-1/2}\omega_{\mathcal{M}}+H^{+1/2}dx_8\wedge dx_9. \tag{38}$$

The resulting form

$$\phi=\frac{1}{2}H^{-1}\omega_{\mathcal{M}}\wedge\omega_{\mathcal{M}}+c\,\omega_{\mathcal{M}}\wedge dx_8\wedge dx_9 \tag{39}$$

is gauge equivalent to the effective spatial gauge potential $\hat{A}_{ijkl}=A_{tyijkl}$. This can be seen by using the closure property of the Kähler metric $g_{m\bar{n}}$.

V. CONCLUSION

We have conjectured that the spacetime fields of p-brane world-volume solitons are spacetimes of the FS type. We have seen that thinking of FS spacetimes in terms of calibrations is useful both in understanding their structure and in generating new examples. In this paper we have focused on Kähler calibrations. As discussed above, we plan to investigate further examples in future work.

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[4]Note that for $N=1$ the exponents in Eq. (32) yield flat Minkowski spacetime. This seems puzzling because the ansatz (31) should cover the original M2-brane spacetime (16). It turns out that the supersymmetry condition (20) can also be satisfied by taking the Kähler metric $g_{m\bar{n}}$ in Eq. (31) to be flat, so that the relations (24) between g and H no longer hold. The original M2-brane spacetimes are recovered in this way for $N=1$. For $N>1$ one recovers the intersecting M2-brane spacetimes of [38] in this way.

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