General relativistic hydrodynamics in multiple coordinate systems

Chongming Xu and Xuejun Wu
Lohrmann Observatory, TU Dresden, D-01062 Dresden, Germany
and Department of Physics, Nanjing Normal University, Nanjing 210097, China
(Received 9 October 2000; published 1 February 2001)

In this paper, the general relativistic hydrodynamic equations of a thermally conducting, viscous and compressible fluid in multiple coordinate systems are deduced in terms of the scheme developed by Damour, Sofel, and Xu (DSX scheme). Our paper is the first one to describe the hydrodynamic equations of a nonperfect fluid in every local coordinate system at the first post-Newtonian approximation of Einstein's theory of gravity. The hydrodynamic equations in local coordinate systems are useful for calculating multipole moments of post-Newtonian N-body problems in the DSX scheme. Therefore, this paper is a supplement to the DSX scheme in some meaning. The corresponding PN thermodynamic equations in local coordinate systems are also represented. Lastly, some remarks on the possible applications are mentioned.

DOI: 10.1103/PhysRevD.63.064001 PACS number(s): 04.25.−g, 04.40.−b, 47.75.+f

I. INTRODUCTION

General relativistic hydrodynamics is an important subject in classical general relativity [1–4]. The importance results not only from a theoretical point of view, but also from the practical application, such as compact stars, precise measurements in geophysics and cosmological problems. First order post-Newtonian (1PN) hydrodynamics has been investigated by many authors (e.g., Chandrasekhar [5]; Greenberg [6]; Taub [2]; Blanchet, Damour, and Schäfer [7]; Will [8] *et al.*). But we should point out that most of the calculations are in one-global coordinate system, or we will call them one coordinate hydrodynamics. Only in very few cases [9] did someone consider the hydrodynamic equations of a perfect fluid in a local coordinate system by means of matching technique [10]. As we know, the calculation of matching technique is quite long and complex. Until now, no one has considered complete hydrodynamic equations of a nonperfect fluid and corresponding thermodynamic equations in multiple coordinate systems. Multiple coordinate systems mean N-local coordinate systems for N astronomical bodies (or N different regions of fluid) and one-global coordinate system. In some cases, for example in binary or N-body systems, stars in globular clusters *et al.*, it is necessary to introduce a local coordinate system for each object to calculate local multipole moments [11]. That is the reason for this paper.

Before 1991 no complete theory of astronomical reference systems was available. Damour, Soffel, and Xu (DSX) presented a new formalism for treating the general-relativistic celestial mechanics of systems of N arbitrarily composed and shaped, weakly self-gravitating, rotating, deformable bodies [12–15]. This formalism is aimed at yielding a complete description, at the first post-Newtonian level, of the global dynamics of such N-body systems, the local gravitational structure of each body, and the way the external and internal problems fit together (''a complete theory of reference systems''). In the DSX scheme, there are $N+1$ coordinate systems: one global coordinate system $x^{\mu}=(ct,x^i)$ and N-local coordinate systems $X^{\alpha}=(cT,X^a)$. The description of the metric tensor $g_{\mu\nu}$ in the global system with four metric potentials (w,w_i) as functions of x^{μ}; similarly, a description of $G^{A}_{\mu\nu}$ in each A-frame moving of body A with four potentials W^A,W^A_a also. Now we will briefly review the content in the DSX scheme that we shall use later.

Since in the DSX scheme the equations of the gravitational metric potentials are linear, the metric can be split as

$$W^A_{\alpha}=\overset{+}{W}{}^A_{\alpha}+\overline{W}{}^A_{\alpha}, \tag{1.1}$$

where $\overset{+}{W}{}^A_{\alpha}$ is the self-part, $\overline{W}{}^A_{\alpha}$ is the external part. The self potentials $\overset{+}{W}{}^A_{\alpha}$ are linearly related to the corresponding Ath global contribution, i.e.,

$$w^A_{\mu}=\mathcal{A}^A_{\mu\alpha}\overset{+}{W}{}^A_{\alpha}+\mathcal{O}(4,2), \tag{1.2}$$

where we use the notation $\mathcal{O}(n)\equiv\mathcal{O}(c^{-n})$, and $\mathcal{O}(4,2)$ means $\mathcal{O}(4)$ corresponding to $\mu=0$ and $\mathcal{O}(2)$ to $\mu=a$. The external one is linearly related to the part of the global potentials generated by all the external bodies

$$\overline{w}{}^A_{\mu}=\sum_{B\neq A} w^B_{\mu}=\mathcal{A}^A_{\mu\alpha}\overline{W}{}^A_{\alpha}+\mathcal{B}^A_{\mu}+\mathcal{O}(4,2), \tag{1.3}$$

where $\mathcal{A}^A_{\mu\alpha}$ are the components of the Jacobian matrix, $\mathcal{B}^A_{\mu}$ are the inhomogeneous terms. By solving Einstein field equations we can get $\overset{+}{W}{}^A{}_{\alpha}$, then

$$\overset{+}{W}{}^B_{\alpha}\rightarrow w^B_{\mu}\rightarrow\overline{w}{}^A_{\mu}\rightarrow\overline{W}{}^A_{\mu}=\mathcal{A}^{-1}_{\alpha\mu}(\overline{w}{}^A_{\mu}-\mathcal{B}_{\mu}). \tag{1.4}$$

Finally, we get external potential $\overline{W}{}^A_{\mu}$ in local coordinate system (of body) A from the self-potentials $\overset{+}{W}{}^B_{\alpha}$ of every local coordinate system B. Then from Eq. (1.1) we get the W^A_{α} in local coordinate systems A. Therefore, to obtain W^A_{α} we have to derive the self-potentials in each local coordinate system first. The self part $\overset{+}{W}{}^B_{\alpha}$ is determined by a set of mass- and spin-multipole moments (we call them B-D

moments), (M_L^B, S_L^B), of body B, whereas the external influence can be described by means of suitably defined tidal moments (G_L, H_L).

The metric in the local coordinate system A is written in the form

$$G_{00}^A = -\exp\left(-\frac{2W^A}{c^2}\right) + \mathcal{O}(6), \tag{1.5}$$

$$G_{0a}^A = -\frac{4W_a^A}{c^3} + \mathcal{O}(5), \tag{1.6}$$

$$G_{ab}^A = \delta_{ab}\exp\left(\frac{2W^A}{c^2}\right) + \mathcal{O}(4). \tag{1.7}$$

Later we omit the label A (or B) on all quantities pertaining to the local frame if it is unnecessary to point out the special coordinate system. In the DSX scheme we still do not know the time evolution of multipole moments, because it is dependent on the hydrodynamic equations of each body. In some problems, like the coalescence of compact binary systems [11], the time evolution of multipole moments is of considerable importance. From the definition of mass-multipole moments [12] and spin-multipole moments [14] we need to know the time evolution of Σ and Σ^a in each local frame, which are defined by

$$\Sigma \equiv \frac{T^{00}+T^{ss}}{c^2}, \tag{1.8}$$

$$\Sigma^a \equiv \frac{T^{0a}}{c}, \tag{1.9}$$

where $T^{\alpha\beta}$ is stress-energy tensor. By means of the sources Σ and Σ^a, the Einstein field equations in the harmonic gauge can be written as

$$\nabla^2 \overset{+}{W} - \frac{1}{c^2}\frac{\partial^2 \overset{+}{W}}{\partial T^2} = -4\pi G\Sigma + \mathcal{O}(4), \tag{1.10}$$

$$\nabla^2 \overset{+}{W}_a = -4\pi G\Sigma^a + \mathcal{O}(2). \tag{1.11}$$

The hydrodynamic equations (energy equation and Euler equation) read

$$\frac{\partial}{\partial T}\Sigma + \frac{\partial}{\partial X^a}\Sigma^a = \frac{1}{c^2}\frac{\partial}{\partial T}T^{bb} - \frac{1}{c^2}\Sigma\frac{\partial}{\partial T}W + \mathcal{O}(4), \tag{1.12}$$

$$\frac{\partial}{\partial T}\left[\left(1+\frac{4W}{c^2}\right)\Sigma^a\right] + \frac{\partial}{\partial X^b}\left[\left(1+\frac{4W}{c^2}\right)T^{ab}\right] = F^a(T,X^a) + \mathcal{O}(4), \tag{1.13}$$

where

$$F^a = \Sigma E_a + \frac{1}{c^2}B_{ab}\Sigma^b, \tag{1.14}$$

$$E_a = \partial_a W + \frac{4}{c^2}\partial_T W_a, \tag{1.15}$$

$$B_{ab} = -4(\partial_a W_b - \partial_b W_a). \tag{1.16}$$

These equations are valid in every local coordinate system. Even in global coordinate systems the form of equations is the same after replacing Σ, Σ^a, W, and W_a by σ, σ^i, w, and w_i.

In the original DSX papers, the sources are described by Σ and Σ^a, but the stress-energy tensor $T^{\alpha\beta}$ has not been explicitly expressed by Σ, Σ^a and their derivatives, since a 4-velocity has not been introduced. Therefore Eq. (1.12) and Eq. (1.13) cannot be used directly to calculate the time evolution of multipole moments, in this sense the DSX scheme is incomplete for the description of hydrodynamic problems. Why shall we discuss hydrodynamics in the DSX scheme? There are two reasons for that: first, in this scheme the transformation properties of gravitational field variable (W and W^a) are known. Second, all equations in every local coordinate system as well as in global coordinate systems have the same form. That means if we write down any equation in one local coordinate system, then we have it in every local coordinate system and in the global coordinate system as well. Therefore it might be easier for us to use the DSX scheme in the discussion of PN hydrodynamics in multiple coordinate systems.

In this paper, our symbols and signature follow the DSX scheme [12]. All the notations and convention are taken from the DSX scheme. Here we summarize the notations in their scheme which we shall use in this paper. The signature is $-+++$, spacetime indices go from 0 to 3 and are denoted by greek indices, while spatial indices (1 to 3) are denoted by latin indices. We use Einstein's summation convention for both types of indices, whatever the position of repeated indices. The ''global'' (or ''common view'') coordinate system used for describing the overall dynamics of the multiple coordinate systems will be denoted by $(x^\mu) \equiv (ct, x^i)$. By contrast, each of the ''local'' coordinate systems, used for describing the internal dynamics of each body, will be denoted by capital letters $(X^\alpha) \equiv (cT, X^\alpha)$. We shall distinguish the second part of the latin alphabet ($i,j,k,\ldots$) for global spatial coordinate system from the first part of the latin alphabet ($a,b,c,\ldots$) for local spatial coordinate system as done in the DSX scheme. The symmetrization of indices will be denoted by round brackets [e.g., $V^{(a}{}_{,b)} = \frac{1}{2}(V^a{}_{,b} + V^b{}_{,a})$]. In the following discussion on hydrodynamics, we consider a ''simple fluid'' in each body as usual [16], certainly in a different body (or a different region) a different simple fluid could be taken. By a so-called simple fluid we mean the chemical composition of the fluid is fixed uniquely by two thermodynamic variables: the total number density of baryons n and the entropy per baryon s. We can express the stress-energy tensor T^{ab} in terms of Σ,Σ^a (4 functions) and their derivatives at the 1-PN level for a nonperfect fluid. Then we write down the hydrodynamic equations in multiple coordinate systems. As a special example we express the formulas for a perfect fluid explicitly. With these equations and some thermodynamic ones (the equation of state, the equation of heat transfer and so on), we can obtain the time evolution of the multipole moments in principle. Therefore in one sense, our work is a supplement of the DSX scheme. In Sec. II, we derived the PN hydrodynamic equations of a nonperfect fluid in multiple systems. The corresponding thermodynamic equations and the completeness of the whole picture of hydrodynamics are discussed in Sec. III. Finally, we give a brief discussion in Sec. IV. In the Appendix, the main physical quantities are represented by means of Σ,Σ^a without 4-velocity.

II. GENERAL RELATIVISTIC HYDRODYNAMIC EQUATION FOR NONPERFECT FLUID

Many years ago, we already knew the general expression of $\overline{T}^{\alpha\beta}$ for a thermally conducting, viscous and compressible fluid in a global coordinate system [16,6,4]

$$\overline{T}_{\mu\nu}=\epsilon u_\mu u_\nu+p\overline{h}_{\mu\nu}-\frac{1}{3}\beta\theta\overline{h}_{\mu\nu}-\lambda\overline{\sigma}_{\mu\nu}+\frac{2}{c^2}q_{(\mu}u_{\nu)}, \tag{2.1}$$

where p is the isotropic pressure, u_μ is the 4-velocity ($u^\mu\equiv dx^\mu/d\tau$, $u^\mu u_\mu=-c^2$), ϵ is the density of total mass-energy (including rest mass, thermal energy, compressional energy, and so on), here the density means the total mass energy contained in a unit three-dimensional volume of the rest frame, $\overline{\sigma}_{\mu\nu}$ is the shear tensor

$$\overline{\sigma}_{\mu\nu}=u_{(\mu;\nu)}+\frac{1}{c^2}a_{(\mu}u_{\nu)}-\frac{1}{3}\theta\overline{h}_{\mu\nu}, \tag{2.2}$$

where

$$\overline{h}_{\mu\nu}\equiv g_{\mu\nu}+\frac{1}{c^2}u_\mu u_\nu \tag{2.3}$$

is the project operator, $\lambda\geqslant0$ is the coefficient of shear viscosity, $\beta\geqslant0$ is the coefficient of bulk viscosity,

$$\theta=u^\mu{}_{;\mu} \tag{2.4}$$

is the expansion scalar, q_μ is the heat flux vector

$$q_\mu=-k\overline{h}_\mu{}^\nu\left(\mathcal{T}_{,\nu}+\frac{1}{c^2}\mathcal{T}\dot{u}_\nu\right), \tag{2.5}$$

where $\dot{u}_\mu\equiv\partial_T u_\mu$, $k\geqslant0$ is the coefficient of thermal conductivity, $\mathcal{T}$ is the temperature and

$$a_\mu\equiv u_{\mu;\nu}u^\nu \tag{2.6}$$

is the 4-acceleration. It is easy to prove that

$$h_{\mu\nu}u^\nu=\overline{\sigma}_{\mu\nu}u^\nu=a_\nu u^\nu=q_\nu u^\nu=0. \tag{2.7}$$

From Eqs. (2.1)–(2.6), it is obvious that the stress-energy tensor $\overline{T}_{\mu\nu}$ includes only u_μ and their derivatives plus several scalar functions: $\lambda,\beta,k,\epsilon,p,\mathcal{T},\ldots$ which are determined by the physical property of material. Normally, they are defined in the rest frame of an observer comoving with an element of the fluid. As usual treatment, λ, β, and k can be considered as constant in a certain region. In the extreme condition (e.g., neutron star), they are functions of temperature and pressure, also of space-time.

Since we hope to express the hydrodynamic equations in each local coordinate system, we first express the stress-energy tensor $T^{\alpha\beta}$ in a local coordinate system. As we know $\lambda,\beta,k,\epsilon,p,\mathcal{T},\theta$ are scalars. $u_\mu,a_\mu,\overline{h}_{\mu\nu},\overline{\sigma}_{\mu\nu},q_\mu$ are four-dimension vectors or tensors in the global coordinate system, in local coordinate system they are substituted by $U_\alpha,A_\alpha,h_{\alpha\beta},\sigma_{\alpha\beta},Q_\alpha$ without changing the physical meaning. Then the stress-energy tensor in local coordinate systems reads

$$T^{\alpha\beta}=\epsilon U^\alpha U^\beta+ph^{\alpha\beta}-\frac{1}{3}\beta\theta h^{\alpha\beta}-\lambda\sigma^{\alpha\beta}+\frac{2}{c^2}Q^{(\alpha}U^{\beta)}, \tag{2.8}$$

where we raise the indices by means of project operator, since we have similar equations as Eq. (2.7) in local coordinate systems.

In the following part, we will express θ, A_α, $\sigma^{\alpha\beta}$, Q^α by means of ϵ, p, $\mathcal{T}$, λ, β, k, W, W^a and three-dimensional velocity V^a. Then we express ϵ and V^a in terms of Σ, Σ^a, p, $\mathcal{T}$, λ, β, and k. Finally, the stress-energy tensor $T^{\alpha\beta}$ are presented by means of Σ, Σ^a, W, W^a, p, $\mathcal{T}$, λ, β, and k. At first glance we might doubt why $T^{\alpha\beta}$ can be expressed by Σ, Σ^a and their derivatives, since $T^{\alpha\beta}$ have 10 components, but Σ and Σ^a only 4. In fact $T^{\alpha\beta}$ are not only expressed by Σ and Σ^a, but also by their derivatives. As we know, the stress-energy tensor of nonperfect fluid can be expressed by 4-velocity and its derivatives [6]. Therefore if we can establish the relation between U^α and Σ, Σ^a in 1-PN level, it is not difficult to express $T^{\alpha\beta}$ by means of Σ, Σ^a and their derivatives.

The 4-velocity $U^\alpha=dX^\alpha/d\tau$ can be expressed to require order as

$$U^0=c\left[1+\frac{1}{c^2}\left(W+\frac{1}{2}V^2\right)\right]+\mathcal{O}(3), \tag{2.9}$$

$$U^a=\frac{V^aU^0}{c}=V^a+\frac{V^a}{c^2}\left(W+\frac{V^2}{2}\right)+\mathcal{O}(4), \tag{2.10}$$

where V^a and V are the three-dimension velocity and its value. First we calculate the expansion scalar and get

$$\theta=U^\mu_{;\mu}=\frac{\partial V^a}{\partial X^a}+\frac{1}{c^2}\left[\frac{\partial V^a}{\partial X^a}\left(W+\frac{V^2}{2}\right)+3\frac{dW}{dT}+\frac{1}{2}\frac{d}{dT}V^2\right]+\mathcal{O}(4). \tag{2.11}$$

The components of 4-acceleration A_α ($A_\alpha \equiv U_{\alpha;\beta}U^\beta$) take the form

$$A_0 = U_{0;\beta}U^\beta = -\frac{1}{2c}\frac{d}{dT}V^2 + \frac{1}{c}W_{,b}V^b + \mathcal{O}(3), \tag{2.12}$$

$$A_a = U_{a,\beta}U^\beta = \frac{dV^a}{dT} - W_{,a} + \mathcal{O}(2). \tag{2.13}$$

The shear tensor turns out

$$\sigma^{00} = \frac{1}{c^2}\left(\frac{1}{2}V^2{}_{,c}V^c - \frac{1}{3}V^2V^c{}_{,c}\right) + \mathcal{O}(4), \tag{2.14}$$

$$\sigma^{0a} = \frac{1}{c}\left(V^{(a}{}_{,b)} - \frac{1}{3}\delta_{ab}V^c{}_{,c}\right)V^b + \mathcal{O}(3), \tag{2.15}$$

$$\sigma^{ab} = V^{(a}{}_{,b)} - \frac{1}{3}\delta_{ab}V^c{}_{,c} + \frac{1}{c^2}\Bigg[\left(\frac{1}{2}V^2 - W\right) \times \left(V^{(a}{}_{,b)} - \frac{1}{3}\delta_{ab}V^c{}_{,c}\right) + \frac{dV^{(a}}{dT}V^{b)} - \frac{1}{6}\delta_{ab}\frac{dV^2}{dT} + \frac{1}{2}V^2{}_{,(b}V^{a)} - \frac{1}{3}V^aV^bV^c{}_{,c}\Bigg] + \mathcal{O}(4). \tag{2.16}$$

The components of the project operator $h^{\alpha\beta}$ are given by

$$h^{00} = \frac{V^2}{c^2} + \mathcal{O}(4), \tag{2.17}$$

$$h^{0a} = \frac{V^a}{c} + \frac{1}{c^3}V^a(2W + V^2) - \frac{4}{c^3}W^a + \mathcal{O}(5), \tag{2.18}$$

$$h^{ab} = \delta_{ab}\left(1 - \frac{2W}{c^2}\right) + \frac{V^aV^b}{c^2} + \mathcal{O}(4). \tag{2.19}$$

Then the heat flux Q^α can be written as

$$Q^0 = -kh^{0\beta}\left(\mathcal{T}_{,\beta} + \frac{1}{c^2}\mathcal{T}\dot{U}_\beta\right) = -\frac{k}{c}V^b\mathcal{T}_{,b} + \mathcal{O}(3), \tag{2.20}$$

$$Q^a = -kh^{\alpha\beta}\left(\mathcal{T}_{,\beta} - \frac{1}{c^2}\mathcal{T}\dot{U}_\beta\right) = -k\mathcal{T}_{,a} - \frac{k}{c^2}\left(\frac{d}{dT}(\mathcal{T}V^a) - W_{,a}\mathcal{T} - 2W\mathcal{T}_{,a}\right) + \mathcal{O}(4). \tag{2.21}$$

The stress-energy tensor then can be deduced as

$$T^{00} = c^2\epsilon + (2W + V^2)\epsilon + \mathcal{O}(2), \tag{2.22}$$

$$T^{0a} = c\epsilon V^a + \frac{1}{c}\Bigg[\epsilon V^a(2W + V^2) + pV^a - \lambda V^{(a}{}_{,b)}V^b - \frac{1}{3}(\beta - \lambda)V^aV^c{}_{,c} - k\mathcal{T}_{,a}\Bigg] + \mathcal{O}(3), \tag{2.23}$$

$$T^{ab} = \epsilon V^aV^b + p\delta_{ab} - \frac{1}{3}(\beta - \lambda)\delta_{ab}V^c{}_{,c} - \lambda V^{(a}{}_{,b)} + \frac{1}{c^2}\Bigg\{\epsilon(2W + V^2)V^aV^b - 2pW\delta_{ab} + pV^aV^b + \frac{1}{3}(\beta - \lambda)\delta_{ab} \times \left[V^c{}_{,c}\left(W - \frac{1}{2}V^2\right) - \frac{1}{2}\frac{dV^2}{dT}\right] - \frac{1}{3}(\beta - \lambda)V^aV^bV^c{}_{,c} - \lambda\left[\left(\frac{1}{2}V^2 - W\right) \times V^{(a}{}_{,b)} + \frac{1}{2}V^2{}_{,(a}V^{b)} + \frac{dV^{(a}}{dT}V^{b)}\right] - \beta\delta_{ab}\frac{dW}{dT} - 2k\mathcal{T}_{,(a}V^{b)}\Bigg\} + \mathcal{O}(4). \tag{2.24}$$

As we mentioned already, in the DSX scheme all equations in different coordinate systems have the same form, therefore Eqs. (2.22)–(2.24) are valid also in the global coordinate system (only need to change capital letters to small letters). According to the definition of source Σ and Σ^i [Eqs. (1.8) and (1.9)], we get

$$\Sigma = \epsilon + \frac{1}{c^2}[2\epsilon(W + V^2) + 3p - \beta V^d{}_{,d}] + \mathcal{O}(4), \tag{2.25}$$

$$\Sigma^a = V^a\left[\epsilon + \frac{1}{c^2}\left(\epsilon(2W + V^2) + p - \frac{1}{3}(\beta - \lambda)V^d{}_{,d}\right)\right] - \frac{\lambda}{c^2}V^{(a}_{,b)}V^b - \frac{k}{c^2}\mathcal{T}_{,a} + \mathcal{O}(4). \tag{2.26}$$

From Eqs. (2.25), (2.26) and considering PN approximation, we can get the expressions of three-dimension velocity V^a and ϵ by means of Σ, Σ^a, p, λ, β, k, and $\mathcal{T}$ as

$$V^a = \frac{\Sigma^a}{\Sigma}\left[1 + \frac{1}{c^2}\left(\frac{\Sigma^d\Sigma^d}{\Sigma^2} + \frac{2p}{\Sigma} - \frac{2\beta + \lambda}{3\Sigma}\left(\frac{\Sigma^d}{\Sigma}\right)_{,d}\right)\right] + \frac{\lambda}{c^2}\frac{\Sigma^d}{\Sigma^2}\left(\frac{\Sigma^{(a}}{\Sigma}\right)_{,d)} + \frac{k}{c^2\Sigma}\mathcal{T}_{,a} + \mathcal{O}(4), \tag{2.27}$$

$$\epsilon=\Sigma-\frac{1}{c^2}\left[2\Sigma\left(W+\frac{\Sigma^d\Sigma^d}{\Sigma^2}\right)+3p-\beta\left(\frac{\Sigma^d}{\Sigma}\right)_{,d}\right]+\mathcal{O}(4). \tag{2.28}$$

We also can rewrite the expressions of U^α, θ, A^α, $\sigma^{\alpha\beta}$, Q^α, and $T^{\alpha\beta}$ by means of Σ, Σ^a instead of ϵ and V^a (to see in the Appendix).

Substituting $T^{\alpha\beta}$ into evolution equations, then the hydrodynamic equations for nonperfect fluid in local coordinate system turns out

$$\frac{\partial\Sigma}{\partial T}+\frac{\partial\Sigma^a}{\partial X^a}=\frac{1}{c^2}\left[\frac{2\Sigma^d\dot{\Sigma}^d}{\Sigma}-\frac{\dot{\Sigma}\Sigma^d\Sigma^d}{\Sigma^2}+3\dot{p}-\beta\dot{\theta}-\Sigma\dot{W}\right]+\mathcal{O}(4), \tag{2.29}$$

$$\frac{\partial}{\partial T}\left[\left(1+\frac{4W}{c^2}\right)\Sigma^a\right]+\frac{\partial}{\partial X^b}\left[\left(1+\frac{4W}{c^2}\right)T^{ab}\right]=\Sigma\partial_a W+\frac{4}{c^2}[\Sigma\partial_T W_a-\Sigma^b(\partial_a W_b-\partial_b W_a)]+\mathcal{O}(4), \tag{2.30}$$

where $\dot{\Sigma}\equiv\partial\Sigma/\partial T$ *et al.*, T^{ab} is shown in Eq. (2.24) [a complete representation with substituting ϵ and V^a by Eqs. (2.27) and (2.28) is given in Appendix Eq. (A13)]. Therefore we now have hydrodynamic equations of nonperfect fluid in local coordinate systems. In principle we can write down hydrodynamic equations in any local coordinate system and global coordinate systems with the same form in the DSX scheme. Then the time evolution of B-D moments can be calculated out.

When $\lambda=\beta=k=0$, Eqs. (2.29) and (2.30) turn to the equations of perfect fluid, they read

$$\dot{\Sigma}+\Sigma^a_{,a}=\frac{1}{c^2}\left(\frac{2\dot{\Sigma}^d\Sigma^d}{\Sigma}-\frac{\dot{\Sigma}\Sigma^d\Sigma^d}{\Sigma^2}+3\dot{p}-\Sigma\dot{W}\right)+\mathcal{O}(4), \tag{2.31}$$

$$\frac{\partial}{\partial T}\left[\left(1+\frac{4W}{c^2}\right)\Sigma^a\right]+\frac{\partial}{\partial X^b}\left[\left(1+\frac{4W}{c^2}\right)\times\left(\frac{\Sigma^a\Sigma^b}{\Sigma}\left(1+\frac{\Sigma^c\Sigma^c}{\Sigma^2c^2}+\frac{2p}{\Sigma c^2}\right)+p\delta^{ab}\left(1-\frac{2W}{c^2}\right)\right)\right]=\Sigma\partial_a W+\frac{4}{c^2}[\Sigma\partial_T W_a-\Sigma^b(\partial_a W_b-\partial_b W_a)]+\mathcal{O}(4). \tag{2.32}$$

These are the PN hydrodynamic equations of perfect fluid in a local coordinate system.

III. PN THERMODYNAMIC EQUATIONS IN THE DSX SCHEME

The hydrodynamic equations are incomplete without discussion of the corresponding thermodynamic equations. As we mentioned before, for a ''simple fluid'' only the number density n and the entropy per baryon s determine the whole composition of the fluid. Therefore equations of state are

$$\mathcal{T}=\mathcal{T}(n,s), \tag{3.1}$$

$$p=p(n,s). \tag{3.2}$$

The number density n satisfies the law of baryon conservation $(nU^\alpha)_{;\alpha}=0$, i.e.,

$$\frac{dn}{d\tau}=\frac{dn}{dT}\left[1+\frac{1}{c^2}\left(W+\frac{1}{2}\left(\frac{\Sigma^d\Sigma^d}{\Sigma^2}\right)\right)\right]+\mathcal{O}(4)=-nU^\alpha_{;\alpha}=-n\theta. \tag{3.3}$$

By means of s, we may define the entropy 4-vector as

$$S^\alpha=nsU^\alpha+Q^\alpha/\mathcal{T}, \tag{3.4}$$

S^α satisfies PN equation of heat transfer

$$\mathcal{T}S^\alpha{}_{;\alpha}=\frac{1}{3}\beta\theta^2+\lambda\sigma_{\alpha\beta}\sigma^{\alpha\beta}+\frac{k}{\mathcal{T}}h^{\alpha\beta}\left(\mathcal{T}_{,\alpha}+\frac{1}{c^2}\mathcal{T}A_\alpha\right)\left(\mathcal{T}_{,\beta}+\frac{1}{c^2}\mathcal{T}A_\beta\right). \tag{3.5}$$

The PN equation of heat transfer in local coordinate systems of the DSX scheme has an explicit form:

$$\begin{aligned}n\mathcal{T}\frac{ds}{d\tau}=n\mathcal{T}\frac{ds}{dT}\left[1+\frac{1}{c^2}\left(W+\frac{1}{2}V^2\right)\right]&=\frac{1}{3}(\beta-\lambda)\theta^2+\lambda V^a{}_{,b}V^{(a}{}_{,b)}+k\mathcal{T}_{,aa}\\&+\frac{1}{c^2}\Bigg\{\lambda\Bigg[2V^a{}_{,b}V^{(a}{}_{,b)}\left(W+\frac{1}{2}V^2\right)\\&+2V^a{}_{,a}\frac{dW}{dT}-\frac{1}{2}V^a{}_{,b}V^bV^a{}_{,c}V^c+2V^{(a}{}_{,b)}\frac{dV^{(a}}{dT}V^{b)}\\&+\frac{1}{8}\lambda V^2{}_{,a}V^2{}_{,a}\Bigg]+k\Bigg[2\mathcal{T}_{,a}\left(\frac{dV^a}{dT}-W_{,a}\right)+\mathcal{T}\left(\frac{dV^a}{dT}\right)_{,a}\\&+(V^a\mathcal{T}_{,a})_{,T}+\left(V^a\frac{d\mathcal{T}}{dT}\right)_{,a}-2W\mathcal{T}_{,aa}-\mathcal{T}W_{,aa}\Bigg]\Bigg\}\\&+\mathcal{O}(4),\end{aligned} \tag{3.6}$$

where V^a is expressed in Eq. (2.27).

Sometimes Navier-Stokes equation $(h^\alpha_\mu T^{\mu\upsilon}{}_{;\nu}=0)$ is also considered as a thermohydrodynamic equation. It is equivalent to Eqs. (2.29) and (2.30).

At the end of this section we will point out equations as a whole to be close. For 1-PN nonperfect fluid in every local coordinate system A, we have 44 functions: Σ, Σ^a, $\overset{+}{W}{}^A$, $\overset{+}{W}{}^A_a$, $h^{\alpha\beta}$, θ, A_α, $\sigma_{\alpha\beta}$, Q^α, V^a, p, $\mathcal{T}$, n, s, and 3 coefficients: β, λ, k. Normally β, λ, and k are determined by solid state physics which is beyond our discussion. We just have 10 differential equations, 2 algebraic equations, and 32 definitions and relations of function to fit 44 functions: $\overset{+}{W}^A$, $\overset{+}{W}_a^A$ satisfy 1-PN Einstein field equations with harmonic condition Eqs. (1.10), (1.11); Σ, Σ^a satisfy 1 PN hydrodynamic equations Eqs. (2.29), (2.30); n and s satisfy the law of baryon conservation [Eq. (3.3)] and equation of heat transfer [Eq. (3.6)]; p and $\mathcal{T}$ satisfy two algebraic equations of state [Eq. (3.1) and Eq. (3.2)]; $h^{\alpha\beta}$, θ, A_α, $\sigma_{\alpha\beta}$, Q^α, and V^a are defined by Eqs. (2.17)–(2.19), Eq. (2.11), Eqs. (2.12), (2.13), Eqs. (2.14)–(2.16), Eqs. (2.20), (2.21), and Eq. (2.27) which are 32 definitions or relations corresponding to 32 functions.

It seems close for every local coordinate system, but that is not true, because in the equations of Secs. II and III there are W^A and W_a^A, but not $\overset{+}{W}^A$ and $\overset{+}{W}_a^A$. As we mentioned in Sec. I, to get W and W_a we have to know $\overset{+}{W}^B$ and $\overset{+}{W}_a^B$ in all other local coordinate systems [see Eq. (1.4)]. If there are N local coordinate systems, we have $44\times N$ functions and $44\times N$ equations and definitions. Therefore the equations as a whole are close. Although equations are complex, if we know the boundary conditions and initial conditions, we could calculate these equations numerically. In principle, the 1 PN hydrodynamic problem of a simple fluid in N-multiple coordinate systems could be solved.

IV. CONCLUSION

(1) In this paper it is the first time to present hydrodynamic equations of a thermally conducting, viscous, and compressible fluid in every local coordinate system. We are not only writing the hydrodynamic equations in every local coordinate system formally, but we also clearly know the relations of W and W_a between different local coordinate systems by means of the theory of reference system in the DSX scheme [12].

(2) $T^{\alpha\beta}$ can be expressed by four functions (Σ and Σ^a) and their derivatives. Then the evolution equations of Σ, Σ^a in the DSX scheme is a complete set. Therefore, the time dependent mass-multiple moments $M_L(T)$ and spin-multipole moments $S_L(T)$ in the DSX scheme can be numerically calculated out from the evolution equation, equation of state, physical properties of material, and the above-mentioned relations and definitions. In that meaning our work is a supplement of the DSX scheme.

(3) We would like to say some words about the possible applications of our work. In the problem of coalescence of compact binaries [11], we can calculate PN spin and quadrupole moments and their time derivatives by means of the hydrodynamic equations in each local coordinate system in the coalescing process. Besides this, we might consider the PN influence of a globular cluster on binary systems. As we know, the Newtonian influence of globular cluster on binary systems is considered already as a tidal force. Certainly the PN influence must be very small, but it should exist.

ACKNOWLEDGMENTS

This work is supported by the Deutsche Forschungs Gemeinschaft (DFG). We thank also the Chinese National Science Foundation (19873008 and 19835040). The authors thank Professor T. Damour, Professor M. Soffel, and Professor R. Maartens for helpful discussions.

APPENDIX

In the DSX scheme, 4-velocity is not introduced. Therefore if we want to express our discussion on hydrodynamic equations with the symbols in the DSX scheme only, we have to reexpress U^α, θ, A_α, $\sigma^{\alpha\beta}$, Q^α, $T^{\alpha\beta}$ by means of Σ and Σ^a without velocity, which we are giving in this appendix.

According to Eqs. (2.9), (2.10), and (2.27), the 4-velocity can be represented as

$$U^0/c=1+\frac{1}{c^2}\left(W+\frac{1}{2}\frac{\Sigma^b\Sigma^b}{\Sigma^2}\right)+\mathcal{O}(4), \tag{A1}$$

$$U^a=\frac{V^aU^0}{c}=\frac{\Sigma^a}{\Sigma}\left[1+\frac{1}{c^2}\left(\frac{3}{2}\frac{\Sigma^d\Sigma^d}{\Sigma^2}+W+\frac{2p}{\Sigma}-\frac{2\beta+\lambda}{3\Sigma}\left(\frac{\Sigma^d}{\Sigma}\right)_{,d}\right)\right]+\frac{\lambda}{c^2}\frac{\Sigma^d}{\Sigma^2}\left(\frac{\Sigma^{(d}}{\Sigma}\right)_{,a)}+\frac{k}{c^2}\frac{\mathcal{T}_{,a}}{\Sigma}+\mathcal{O}(4). \tag{A2}$$

Then the expansion scalar θ reads

$$\theta=\left(\frac{\Sigma^a}{\Sigma}\right)_{,a}+\frac{1}{c^2}\Bigg\{\left(\frac{\Sigma^a}{\Sigma}\right)_{,a}\left(W+\frac{3}{2}\frac{\Sigma^d\Sigma^d}{\Sigma^2}\right)+\frac{2\Sigma^a\Sigma^d}{\Sigma^2}\left(\frac{\Sigma^d}{\Sigma}\right)_{,a}+3\frac{dW}{dT}+\frac{\Sigma^d}{\Sigma}\left(\frac{d}{dT}\frac{\Sigma^d}{\Sigma}\right)+\left[\frac{\Sigma^a}{\Sigma^2}\left(2p-\frac{2\beta+\lambda}{3}\left(\frac{\Sigma^d}{\Sigma}\right)_{,d}\right)\right]_{,a}+\lambda\left[\frac{\Sigma^d}{\Sigma^2}\left(\frac{\Sigma^{(d}}{\Sigma}\right)_{,a)}\right]_{,a}+\left(\frac{k}{\Sigma}\mathcal{T}_{,a}\right)_{,a}\Bigg\}+\mathcal{O}(4). \tag{A3}$$

The components of acceleration are

$$A_0=\frac{\Sigma^d}{c\Sigma}\left(W_{,d}-\frac{d}{dT}\left(\frac{\Sigma^d}{\Sigma}\right)\right)+\mathcal{O}(3), \tag{A4}$$

$$A_a=\frac{d}{dT}\left(\frac{\Sigma^a}{\Sigma}\right)-W_{,a}+\mathcal{O}(2). \tag{A5}$$

The shear tensor $\sigma_{\alpha\beta}$ can be expressed as

$$\sigma^{00}=\frac{1}{c^2}\frac{\Sigma^d}{\Sigma}\left[\left(\frac{\Sigma^d}{\Sigma}\right)_{,a}\frac{\Sigma^a}{\Sigma}-\frac{1}{3}\frac{\Sigma^d}{\Sigma}\theta\right]+\mathcal{O}(4), \tag{A6}$$

$$\sigma^{0a}=\frac{1}{c}\left[\left(\frac{\Sigma^{(a}}{\Sigma}\right)_{,d)}\frac{\Sigma^d}{\Sigma}-\frac{1}{3}\frac{\Sigma^a}{\Sigma}\theta\right]+\mathcal{O}(3), \tag{A7}$$

$$\sigma^{ab}=\left(\frac{\Sigma^{(a}}{\Sigma}\right)_{,b)}-\frac{1}{3}\delta_{ab}\theta+\frac{1}{c^2}\left\{\frac{\Sigma^{(a}}{\Sigma}\frac{d}{dT}\left(\frac{\Sigma^{b)}}{\Sigma}\right)+\frac{3}{2}\left(\frac{\Sigma^d\Sigma^d\Sigma^{(a}}{\Sigma^3}\right)_{,b)}-\frac{\theta}{3}\frac{\Sigma^a\Sigma^b}{\Sigma^2}-W\left(\frac{\Sigma^{(a}}{\Sigma}\right)_{,b)}+\left[\left(2p-\frac{2\beta+\lambda}{3}\theta\right)\frac{\Sigma^{(a}}{\Sigma^2}\right]_{,b)}\right.$$
$$\left.+\frac{2}{3}\delta_{ab}W\theta+\frac{\lambda}{2}\left[\frac{\Sigma^d}{\Sigma}\left(\left(\frac{\Sigma^{(a}}{\Sigma}\right)_{,d}+\left(\frac{\Sigma^d}{\Sigma}\right)_{,(a}\right)\right]_{,b)}+k\left(\frac{\mathcal{T}_{,(a}}{\Sigma}\right)_{,b)}+\delta_{ab}\frac{dW}{dT}\right\}+\mathcal{O}(4). \tag{A8}$$

The heat flux Q^α can be written as

$$Q^0=-\frac{k}{c}\mathcal{T}_{,b}\frac{\Sigma^b}{\Sigma}+\mathcal{O}(3), \tag{A9}$$

$$Q^a=-k\mathcal{T}_{,a}-\frac{k}{c^2}\left(\frac{d}{dT}\left(\mathcal{T}\frac{\Sigma^a}{\Sigma}\right)-\mathcal{T}W_{,a}-2W\mathcal{T}_{,a}\right)+\mathcal{O}(4). \tag{A10}$$

Substituting above quantities into $T^{\alpha\beta}$ and considering λ, β, k, as constants, finally the stress-energy tensor can be represented by Σ, Σ^a, W, p, λ, β, k, and $\mathcal{T}$ as

$$T^{00}=c^2\Sigma-\frac{\Sigma^d\Sigma^d}{\Sigma}-3p+\beta\left(\frac{\Sigma^a}{\Sigma}\right)_{,d}+\mathcal{O}(2), \tag{A11}$$

$$T^{0a}=c\Sigma^a, \tag{A12}$$

$$T^{ab}=\frac{\Sigma^a\Sigma^b}{\Sigma}+p\delta_{ab}-\frac{1}{3}(\beta-\lambda)\delta_{ab}\theta-\lambda\left(\frac{\Sigma^{(a}}{\Sigma}\right)_{,b)}+\frac{1}{c^2}\left\{2p\left(\frac{\Sigma^a\Sigma^b}{\Sigma^2}-\delta_{ab}W\right)+\frac{\Sigma^a\Sigma^b\Sigma^d\Sigma^d}{\Sigma^3}+\frac{2}{3}\beta\theta\left(\delta_{ab}W-\frac{\Sigma^a\Sigma^b}{\Sigma^2}\right)\right.$$
$$-\lambda\left[\left(\frac{\Sigma^{(a}}{\Sigma}\left(\frac{\Sigma^d\Sigma^d}{\Sigma^2}+\frac{2p}{\Sigma}-\frac{2\beta+\lambda}{3\Sigma}\theta\right)\right)_{,b)}+\frac{\lambda}{2}\left(\frac{\Sigma^d}{\Sigma^2}\left[\left(\frac{\Sigma^d}{\Sigma}\right)_{,(a}+\left(\frac{\Sigma^{(a}}{\Sigma}\right)_{,d}\right]\right)_{,b)}+\left(\frac{\Sigma^{(a}}{\Sigma}\right)_{,b)}\left(\frac{\Sigma^d\Sigma^d}{2\Sigma^2}-W\right)+\frac{\Sigma^{(a}}{\Sigma}\left(\frac{d}{dT}\frac{\Sigma^{b)}}{\Sigma}\right)\right.$$
$$\left.\left.-\frac{\Sigma^{(a}}{\Sigma^2}\left(\frac{\Sigma^{b)}}{\Sigma}\right)_{,d}\Sigma^d+\delta_{ab}\frac{dW}{dT}+\frac{2}{3}\delta_{ab}W\theta+\frac{\theta}{3}\frac{\Sigma^a\Sigma^b}{\Sigma^2}+k\left(\frac{\mathcal{T}_{,(a}}{\Sigma}\right)_{,b)}\right]\right\}+\mathcal{O}(4). \tag{A13}$$

When $\lambda=\beta=k=0$ Eqs. (A11)–(A13) return to the stress-energy tensor of a perfect fluid. Substituting $T^{\alpha\beta}$ into Eqs. (2.29) and (2.30), we have the equations for a perfect fluid [Eqs. (2.31) and (2.32)].

[1] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics*, Lectures on the Existence of Solutions (Benjamin, New York, 1967).
[2] A. H. Taub, Annu. Rev. Fluid Mech. **10**, 301 (1978).
[3] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1980).
[4] J. Ehlers, Gen. Relativ. Gravit. **25**, 1225 (1993).
[5] S. Chandrasekhar, Astrophys. J. **142**, 1488 (1965).
[6] P. J. Greenberg, Astrophys. J. **164**, 569 (1971).
[7] L. Blanchet, T. Damour, and G. Schäfer, Mon. Not. R. Astron. Soc. **242**, 289 (1989).
[8] C. M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, England, 1993).
[9] S. M. Kopeikin, Astron. Zh. **62**, 889 (1985) [Sov. Astron. **29**, 516 (1985)].
[10] P. D. D'Eath, Phys. Rev. D **12**, 2183 (1975).
[11] C. Xu, X. Wu, and G. Schäfer, Phys. Rev. D **55**, 528 (1997).
[12] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **43**, 3273 (1991).
[13] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **45**, 1017 (1992).
[14] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **47**, 3124 (1993).
[15] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **49**, 618 (1994).
[16] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).