The QCD potential at $O(1/m^2)$: Complete spin-dependent and spin-independent result

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Within an effective field theory framework, we obtain an expression, with $O(1/m^2)$ accuracy, for the energies of the gluonic excitations between heavy quarks, which holds beyond perturbation theory. For the singlet heavy-quark–antiquark energy, in particular, we also obtain an expression in terms of Wilson loops. This provides, twenty years after the seminal work of Eichten and Feinberg, the first complete expression for the heavy quarkonium potential up to $O(1/m^2)$ for pure gluodynamics. Several errors present in the previous literature (also in the work of Eichten and Feinberg) have been corrected. We also briefly discuss the power counting of NRQCD in the nonperturbative regime.

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I. INTRODUCTION

Measured spectroscopy suggests that the charm and bottom quark masses are large enough to consider their heavy-quark–antiquark bound-state systems (generically denoted as heavy quarkonia: ψ, Υ, B_c, ...) as nonrelativistic (NR). These systems are, therefore, characterized by, at least, three widely separated scales: hard (the mass m of the heavy quarks), soft (the relative momentum of the heavy-quark–antiquark $|\mathbf{p}|\sim mv$, $v\ll 1$), and ultrasoft (the typical kinetic energy $E\sim mv^2$ of the heavy quark in the bound-state system). Inspired by this NR behavior, the investigation of heavy quarkonia has been traditionally performed by all sorts of potential models, where an ansatz potential is introduced in a Schrödinger equation (for some reviews see [1–3]). The phenomenological success of these suggests that, to some extent, a potential picture may, in fact, be appropriate and justified from QCD. This triggered attempts to derive these potentials from QCD by relating them to Wilson loops. These standard derivations used an expansion in $1/m$ (also named adiabatic or Born-Oppenheimer approximation). However, a full derivation of the potential from QCD, as well as a study of the validity of the potential picture itself, has not yet been done in the nonperturbative regime, where most of the heavy quarkonium spectrum lies. It is the aim of this paper to explicitly derive the complete nonperturbative $1/m^2$ QCD potential for pure gluodynamics within an effective field theory framework [4,5], where higher order potentials in $1/m$ and nonpotential effects could also be incorporated in a systematic way.

Since the derivation of the potential has a long story, it may be useful to summarize its main steps. The expression for the leading spin-independent potential, of $O(1/m^0)$, corresponds to the static Wilson loop and was derived and discussed in the seminal works of Wilson [6] and Susskind [7]. Expressions for the leading spin-dependent potentials in the $1/m$ expansion, of $O(1/m^2)$, were given in Refs. [8–10]. The procedure followed in these works proved to be very difficult to extend beyond these leading-order potentials. Indeed, the first attempts [11], using tools similar to those in Ref. [8], failed to obtain suitable finite expressions. In Ref. [12], a new method to calculate the potentials was proposed, where new spin-independent (some of them momentum-dependent) potentials at $O(1/m^2)$ were obtained. In these original works, the potentials obtained did not correctly reproduce the ultraviolet behavior expected from perturbative QCD (the hard logs $\sim\log m$). This was first implemented in the framework of QCD effective field theories, for both spin-dependent and spin-independent potentials, in [13–15]. At that point, the set of potentials obtained at $O(1/m^2)$ seemed to be complete and the timely study of the different Wilson loop operators describing the nonperturbative dynamics of the potentials started. For instance, a lattice study was performed in [14] and a study in the framework of QCD vacuum models was done in [16].

Nevertheless, this view has been recently challenged in Ref. [5] where (i) a systematic study of the potentials has been started within an effective field theory framework—potential NRQCD (pNRQCD) [4]—and (ii) the $O(1/m)$ potential, previously missed in the literature, has been calculated. It is the aim of this paper to explain in more detail the Hamiltonian formalism, sketched in Ref. [5], and to compute the $O(1/m^2)$ potentials. The formalism appears to be quite powerful and suitable to obtain the quarkonium potentials and the energies of any gluonic excitation at any finite order in $1/m$. A similar idea, but in the Coulomb gauge and only for the leading spin-dependent quarkonium potentials, has also been used in [17]. We will give an expression in terms of quantum-mechanical corrections to the energies of the gluonic excitations between static quarks, valid for all the gluonic excitations up to $O(1/m^2)$. For the quarkonium state (the ground state), we will express our complete $1/m^2$ result in terms of Wilson loops eventually calculable on the lattice or by means of QCD vacuum models, concluding in this way an ideal journey started over twenty years ago.

The theoretical framework of our work is NRQCD [18]

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and pNRQCD, suitable effective field theories for systems made up by two heavy quarks. NRQCD has proved to be extremely successful in studying heavy-quark–antiquark systems near threshold. It is obtained from QCD by integrating out the hard scale m. It is characterized by an ultraviolet cutoff much smaller than the mass m and much larger than any other scale, in particular much larger than $\Lambda_{\rm QCD}$. This means that the matching from QCD to NRQCD can always be done perturbatively, as well as within an expansion in $1/m$ [19,20]. The Lagrangian of NRQCD can also be organized in powers of $1/m$, thus making explicit the nonrelativistic nature of the physical systems. So far, NRQCD and pNRQCD have only been studied in detail in the perturbative situation [21,4].

By integrating out degrees of freedom with energies larger than mv^2, one is left with a new effective field theory called pNRQCD where the soft and ultrasoft scales have been disentangled and where the connection between NRQCD and a NR quantum-mechanical description of the system can be formalized in a systematic way. pNRQCD has two ultraviolet cut-offs, Λ_1 and Λ_2. The former satisfies the relation $mv^2 \ll \Lambda_1 \ll mv$ and is the cut-off of the energy of the quarks, and of the energy and the momentum of the gluons, whereas the latter satisfies $mv \ll \Lambda_2 \ll m$ and is the cutoff of the relative momentum of the quark-antiquark system, $\mathbf{p}$. In the nonperturbative situation (we understand by nonperturbative a typical situation where $mv \sim \Lambda_{\rm QCD}$, i.e., where the potential cannot be computed perturbatively), we will assume that the matching between NRQCD and pNRQCD can be performed, as in the perturbative case, order by order in the $1/m$ expansion. We will present, for the general situation $\Lambda_{\rm QCD} \lesssim mv$, the matching of NRQCD to pNRQCD at $O(1/m^2)$ for the singlet sector (to be defined later). This will prove to be equivalent to computing the heavy quarkonium potential that we can now derive from QCD by a systematic procedure. Moreover, the expression for the potential that we obtain will also be correct at any power in α_s in the perturbative regime.

A pure potential picture emerges in pure gluodynamics under the condition that all gluonic excitations have a gap larger than mv^2. Extra ultrasoft degrees of freedom such as hybrids and pions can be systematically included and may eventually affect the leading potential picture (as ultrasoft gluons in the perturbative regime [4]).

In this paper we consider the general situation of particles with different masses. Therefore, our results, in addition to the traditional Q-$\overline{Q}$ systems, may be applied to the B_c system, which, after its recent discovery by the Collider Detector at Fermilab (CDF) Collaboration [22], has received a lot of attention in theoretical investigations [23].

The paper is organized in the following way. In Sec. II we introduce NRQCD up to $O(1/m^2)$. In Sec. III, using a Hamiltonian formulation of NRQCD, we explicitly calculate up to $O(1/m^2)$ the energies of the gluonic excitations between heavy quarks. In Sec. IV we define what pNRQCD will be in the present context. In Sec. V we write the heavy quarkonium potential up to $O(1/m^2)$ in terms of Wilson loops and compare with previous results. In Sec. VI we discuss the power counting of pNRQCD in the nonperturbative regime and in Sec. VII we give our conclusions and outlook.

II. NRQCD

After integrating out the hard scale m, one obtains NRQCD [18]. Neglecting operators that involve light quark fields [24], the most general NRQCD Lagrangian (up to field redefinitions) for a quark of mass m_1 and an antiquark of mass m_2 up to $O(1/m^2)$ is given by

$$\begin{aligned}
\mathcal{L}_{\rm NRQCD} = {} & \psi^\dagger \left\{ iD_0 + \frac{\mathbf{D}^2}{2m_1} + c_F^{(1)} g \frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{2m_1} + c_D^{(1)} g \frac{[\mathbf{D}\cdot,\mathbf{E}]}{8m_1^2} + i c_S^{(1)} g \frac{\boldsymbol{\sigma}\cdot[\mathbf{D}\times,\mathbf{E}]}{8m_1^2} \right\} \psi \\
& + \chi^\dagger \left\{ iD_0 - \frac{\mathbf{D}^2}{2m_2} - c_F^{(2)} g \frac{\boldsymbol{\sigma}\cdot\mathbf{B}}{2m_2} + c_D^{(2)} g \frac{[\mathbf{D}\cdot,\mathbf{E}]}{8m_2^2} + i c_S^{(2)} g \frac{\boldsymbol{\sigma}\cdot[\mathbf{D}\times,\mathbf{E}]}{8m_2^2} \right\} \chi \\
& + \frac{d_{ss}}{m_1 m_2} \psi^\dagger \psi \chi^\dagger \chi + \frac{d_{sv}}{m_1 m_2} \psi^\dagger \boldsymbol{\sigma} \psi \chi^\dagger \boldsymbol{\sigma} \chi + \frac{d_{vs}}{m_1 m_2} \psi^\dagger \mathrm{T}^a \psi \chi^\dagger \mathrm{T}^a \chi + \frac{d_{vv}}{m_1 m_2} \psi^\dagger \mathrm{T}^a \boldsymbol{\sigma} \psi \chi^\dagger \mathrm{T}^a \boldsymbol{\sigma} \chi \\
& - \frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \left(\frac{d_2^{(1)}}{m_1^2} + \frac{d_2^{(2)}}{m_2^2} \right) G^a_{\mu\nu} D^2 G^{a\,\mu\nu} + \left(\frac{d_3^{(1)}}{m_1^2} + \frac{d_3^{(2)}}{m_2^2} \right) g f_{abc} G^a_{\mu\nu} G^b_{\mu\alpha} G^c_{\nu\alpha},
\end{aligned} \tag{1}$$

where ψ is the Pauli spinor field that annihilates the fermion and χ is the Pauli spinor field that creates the antifermion, $iD^0 = i\partial_0 - gA^0$, $i\mathbf{D} = i\boldsymbol{\nabla} + g\mathbf{A}$, $[\mathbf{D}\cdot,\mathbf{E}] = \mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D}$ and $[\mathbf{D}\times,\mathbf{E}] = \mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}$. This Lagrangian is sufficient to obtain the $O(1/m^2)$ potentials. The coefficients c_F, c_D, c_S, d_2 and d_3 can be found in Ref. [19] and d_{ij} $(i,j=s,v)$ in [20] for the modified minimal subtraction ($\overline{\rm MS}$) scheme.

Some words of caution are in order here. Even if the above matching coefficients have been computed using dimensional regularization and the $\overline{\rm MS}$ scheme, there could

still remain some ambiguity depending on the different prescriptions for the $\boldsymbol{\epsilon}^{ijk}$ tensors and the definition of the Pauli matrices $\boldsymbol{\sigma}$. For instance, the use of a scheme where the $\boldsymbol{\epsilon}^{ijk}$ only takes values for dimension equal to three ('t Hooft–Veltmann-like scheme) in the computation of Ref. [20] would change the value of d_{vv} as $d_{vv}\to[2/(D-2)]d_{vv}$, where D is the number of space-time dimensions. One should therefore be careful and make sure that the matching coefficients one is working with really are computed in the same scheme. A deep study of these ambiguities in the framework of NRQCD remains to be done. This may be especially important for higher order calculations. See also Refs. [25,26], where the authors have to deal with equivalent problems.

We are interested in the Hamiltonian of the above Lagrangian. The construction of the Hamiltonian of one effective (nonrenormalizable) Lagrangian may be complicated (for a related discussion we refer to [27]); in particular because there are higher time derivatives acting on the different fields. In order to get rid of those at $O(1/m^2)$ we have to eliminate the term $G^a_{\mu\nu}D^2G^{a\,\mu\nu}$ from the Lagrangian. This can be achieved by a field redefinition as follows. We consider the field redefinition of the gluon field ($c\sim 1/m^2$):

$$A_\mu\to A_\mu+c[D^\alpha,G_{\alpha\mu}]+O(c^2), \tag{2}$$

where c is real. This transformation preserves the gauge transformation properties and the hermiticity of the A_μ field. Equation (2) produces the following change in the gluon Lagrangian (at the order of interest):

$$-\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu}\to-\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu}-\frac{c}{2}G^a_{\mu\nu}D^2G^{a\,\mu\nu}-cgf_{abc}G^a_{\mu\nu}G^b_{\mu\alpha}G^c_{\nu\alpha}+O(c^2). \tag{3}$$

We can therefore cancel the GD^2G term by fixing

$$c=\frac{2d_2^{(1)}}{m_1^2}+\frac{2d_2^{(2)}}{m_2^2}. \tag{4}$$

This changes the value of d_3 to d_3':

$$d_3^{(1)\prime}=d_3^{(1)}-2d_2^{(1)},\quad d_3^{(2)\prime}=d_3^{(2)}-2d_2^{(2)}. \tag{5}$$

Let us now see the modifications that the above field redefinition will produce in other sectors of the theory, in particular, in the heavy fermion bilinear Lagrangian. Since we have the following change for the D_0 covariant derivative that appears at $O(1/m^0)$,

$$iD_0\to iD_0-cg[\mathbf{D}\cdot,\mathbf{E}], \tag{6}$$

the matching coefficients change, at $O(1/m^2)$, as (all the others remain unchanged)

$$c_D^{(1)\prime}=c_D^{(1)}-16d_2^{(1)}-16\frac{m_1^2}{m_2^2}d_2^{(2)},$$

$$c_D^{(2)\prime}=c_D^{(2)}-16d_2^{(2)}-16\frac{m_2^2}{m_1^2}d_2^{(1)}. \tag{7}$$

In summary, eliminating the term $G^a_{\mu\nu}D^2G^{a\,\mu\nu}$, up to order $1/m^2$, is equivalent to the redefinition of the matching coefficients $d_3\to d_3'$ and $c_D\to c_D'$ found above. We will assume this field redefinition in the following.

III. GLUONIC EXCITATIONS IN A HAMILTONIAN FORMULATION

The Hamiltonian associated to the Lagrangian (1) is, up to order $1/m^2$,

$$H=H^{(0)}+\frac{1}{m_1}H^{(1,0)}+\frac{1}{m_2}H^{(0,1)}+\frac{1}{m_1^2}H^{(2,0)}+\frac{1}{m_2^2}H^{(0,2)}+\frac{1}{m_1m_2}H^{(1,1)}, \tag{8}$$

$$H^{(0)}=\int d^3\mathbf{x}\frac{1}{2}(\boldsymbol{\Pi}^a\boldsymbol{\Pi}^a+\mathbf{B}^a\mathbf{B}^a), \tag{9}$$

$$H^{(1,0)}=-\frac{1}{2}\int d^3\mathbf{x}\psi^\dagger(\mathbf{D}^2+gc_F^{(1)}\boldsymbol{\sigma}\cdot\mathbf{B})\psi,$$

$$H^{(0,1)}=\frac{1}{2}\int d^3\mathbf{x}\chi^\dagger(\mathbf{D}^2+gc_F^{(2)}\boldsymbol{\sigma}\cdot\mathbf{B})\chi, \tag{10}$$

$$H^{(2,0)}=\int d^3\mathbf{x}\,\psi^\dagger\left\{-c_D^{(1)\prime}g\frac{[\mathbf{D}\cdot,\mathbf{E}]}{8}-ic_S^{(1)}g\frac{\boldsymbol{\sigma}\cdot[\mathbf{D}\times,\mathbf{E}]}{8}\right\}\psi-\int d^3\mathbf{x}\,d_3^{(1)\prime}gf_{abc}G^a_{\mu\nu}G^b_{\mu\alpha}G^c_{\nu\alpha}, \tag{11}$$

$$H^{(0,2)}=H^{(2,0)}(\psi\leftrightarrow\chi;1\leftrightarrow 2), \tag{12}$$

$$H^{(1,1)}=-\int d^3\mathbf{x}(d_{ss}\psi^\dagger\psi\chi^\dagger\chi+d_{sv}\psi^\dagger\boldsymbol{\sigma}\psi\chi^\dagger\boldsymbol{\sigma}\chi+d_{vs}\psi^\dagger\mathrm{T}^a\psi\chi^\dagger\mathrm{T}^a\chi+d_{vv}\psi^\dagger\mathrm{T}^a\boldsymbol{\sigma}\psi\chi^\dagger\mathrm{T}^a\boldsymbol{\sigma}\chi), \tag{13}$$

where $\boldsymbol{\Pi}^a$ is the canonical momentum conjugated to $\mathbf{A}^a$ and the physical states are constrained to satisfy the Gauss law:

$$\mathbf{D}\cdot\boldsymbol{\Pi}^a|\mathrm{phys}\rangle=g(\psi^\dagger T^a\psi+\chi^\dagger T^a\chi)|\mathrm{phys}\rangle. \tag{14}$$

Since $\boldsymbol{\Pi}^a=\mathbf{E}^a+O(1/m^2)$, in Eqs. (11)–(12) and in the rest of the paper, we will use the chromoelectric field instead of the canonical momentum where, to the order we are interested in, it does not affect our results.

A. The static limit

We are interested in the one-quark–one-antiquark sector of the Fock space. In the static limit the one-quark–one-antiquark sector of the Fock space can be spanned by

$$|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)} \equiv \psi^\dagger(\mathbf{x}_1)\chi_c^\dagger(\mathbf{x}_2)|n;\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}, \quad \forall \mathbf{x}_1,\mathbf{x}_2, \tag{15}$$

where $|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}$ is a gauge-invariant eigenstate (up to a phase) of $H^{(0)}$, as a consequence of the Gauss law, with energy $E_n^{(0)}(\mathbf{x}_1,\mathbf{x}_2)$. For convenience, we use here the field $\chi_c(\mathbf{x})=i\sigma^2\chi^*(\mathbf{x})$, instead of $\chi(\mathbf{x})$, because it is the one to which a particle interpretation can be easily given: it corresponds to a Pauli spinor that annihilates a fermion in the 3^* representation of color SU(3) with the standard, particlelike, spin structure. $|n;\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}$ encodes the gluonic content of the state; namely, it is annihilated by $\chi_c(\mathbf{x})$ and $\psi(\mathbf{x})$ $(\forall \mathbf{x})$. It transforms as a $3_{\mathbf{x}_1}\otimes 3^*_{\mathbf{x}_2}$ under color SU(3). The normalizations are taken as follows:

$$^{(0)}\langle m;\mathbf{x}_1,\mathbf{x}_2|n;\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}=\delta_{nm},$$

$$^{(0)}\langle \underline{m};\mathbf{x}_1,\mathbf{x}_2|\underline{n};\mathbf{y}_1,\mathbf{y}_2\rangle^{(0)}=\delta_{nm}\delta^{(3)}(\mathbf{x}_1-\mathbf{y}_1)\delta^{(3)}(\mathbf{x}_2-\mathbf{y}_2).$$

We have made it explicit that the positions $\mathbf{x}_1$ and $\mathbf{x}_2$ of the quark and antiquark respectively are good quantum numbers for the static solution $|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}$, whereas n generically denotes the remaining quantum numbers, which are classified by the irreducible representations of the symmetry group $D_{\infty h}$ (substituting the parity generator by CP). We also choose the basis such that $T|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}=|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}$ where T is the time-inversion operator. The ground-state energy $E_0^{(0)}(\mathbf{x}_1,\mathbf{x}_2)$ can be associated to the static potential of the heavy quarkonium under some circumstances (see Sec. IV). The remaining energies $E_n^{(0)}(\mathbf{x}_1,\mathbf{x}_2)$, $n\neq 0$, are usually associated to the potential used in order to describe heavy hybrids or heavy quarkonium (or other heavy hybrids) plus glueballs (see Sec. IV). They can be computed on the lattice (see for instance [28]). Translational invariance implies that $E_n^{(0)}(\mathbf{x}_1,\mathbf{x}_2)=E_n^{(0)}(r)$, where $\mathbf{r}=\mathbf{x}_1-\mathbf{x}_2$.

B. Beyond the static limit

Beyond the static limit, but still working order by order in $1/m$, the normalized eigenstates $|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle$ and eigenvalues $E_n(\mathbf{x}_1,\mathbf{x}_2;\mathbf{p}_1,\mathbf{p}_2)$ of the Hamiltonian H satisfy the equations

$$H|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle=\int d^3x_1' d^3x_2' |\underline{n};\mathbf{x}_1',\mathbf{x}_2'\rangle E_n(\mathbf{x}_1',\mathbf{x}_2';\mathbf{p}_1',\mathbf{p}_2') \times \delta^{(3)}(\mathbf{x}_1'-\mathbf{x}_1)\delta^{(3)}(\mathbf{x}_2'-\mathbf{x}_2), \tag{16}$$

$$\langle \underline{m};\mathbf{x}_1,\mathbf{x}_2|\underline{n};\mathbf{y}_1,\mathbf{y}_2\rangle=\delta_{nm}\delta^{(3)}(\mathbf{x}_1-\mathbf{y}_1)\delta^{(3)}(\mathbf{x}_2-\mathbf{y}_2). \tag{17}$$

Note that the positions $\mathbf{x}_1$ and $\mathbf{x}_2$ of the static solution still label the states even if the position operator does not commute with H beyond the static limit. We are interested in the eigenvalues E_n, which should be understood as operators (instead of numbers, even though we call them energies). This will match the operator interpretation within a quantum-mechanical formulation that we will give to them in pNRQCD in the next section. In particular, we will see that E_0 corresponds to the quantum-mechanical Hamiltonian of the heavy quarkonium (in some specific situation). The other energies, E_n for $n>0$, are related to the quantum-mechanical Hamiltonians of the heavy hybrids or heavy quarkonium (or other heavy hybrids) plus glueballs.

Since the derivation of the corrections to E_n may not be familiar to the reader, since they are operators, we explain it in some detail. We will work in the same way as in standard quantum mechanics, but taking into account the fact that they are operators. Analogously to standard quantum mechanics, we define a state $|\underline{\tilde{n}};\mathbf{x}_1,\mathbf{x}_2\rangle$ such that

$$H|\underline{\tilde{n}};\mathbf{x}_1,\mathbf{x}_2\rangle=\int d^3x_1' d^3x_2' |\underline{\tilde{n}};\mathbf{x}_1',\mathbf{x}_2'\rangle \tilde{E}_n(\mathbf{x}_1',\mathbf{x}_2';\mathbf{p}_1',\mathbf{p}_2')\delta^{(3)}(\mathbf{x}_1'-\mathbf{x}_1)\delta^{(3)}(\mathbf{x}_2'-\mathbf{x}_2),$$

$$^{(0)}\langle \underline{n};\mathbf{x}_1,\mathbf{x}_2|\underline{\tilde{n}};\mathbf{y}_1,\mathbf{y}_2\rangle=\delta^{(3)}(\mathbf{x}_1-\mathbf{y}_1)\delta^{(3)}(\mathbf{x}_2-\mathbf{y}_2).$$

Splitting the Hamiltonian as $H=H_0+H_I$ we have

$$|\underline{\tilde{n}};\mathbf{x}_1,\mathbf{x}_2\rangle=|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}+\frac{1}{E_n^{(0)}(x)-H^{(0)}}\sum_{m\neq n}\int d^3x_1' d^3x_2' |\underline{m};\mathbf{x}_1',\mathbf{x}_2'\rangle^{(0)\,(0)}\langle \underline{m};\mathbf{x}_1',\mathbf{x}_2'| \times\left\{H_I|\underline{\tilde{n}};\mathbf{x}_1,\mathbf{x}_2\rangle-\int d^3x_1' d^3x_2' |\underline{\tilde{n}};\mathbf{x}_1',\mathbf{x}_2'\rangle \Delta\tilde{E}_n(\mathbf{x}_1',\mathbf{x}_2';\mathbf{p}_1',\mathbf{p}_2')\delta^{(3)}(\mathbf{x}_1'-\mathbf{x}_1)\delta^{(3)}(\mathbf{x}_2'-\mathbf{x}_2)\right\},$$

and

$$\Delta\tilde{E}_n(\mathbf{x}_1,\mathbf{x}_2;\mathbf{p}_1,\mathbf{p}_2)\delta^{(3)}(\mathbf{x}_1-\mathbf{y}_1)\delta^{(3)}(\mathbf{x}_2-\mathbf{y}_2)={}^{(0)}\langle \underline{n};\mathbf{x}_1,\mathbf{x}_2|H_I|\underline{\tilde{n}};\mathbf{y}_1,\mathbf{y}_2\rangle.$$

From these formulas we can obtain $\tilde{E}_n$ order by order in the expansion parameter of H_I. Moreover $|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle$ and E_n are given by

$$|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle=\int d^3x_1' d^3x_2' |\underline{\tilde{n}};\mathbf{x}_1',\mathbf{x}_2'\rangle N_n^{-1/2}(\mathbf{x}_1',\mathbf{x}_2';\mathbf{p}_1',\mathbf{p}_2')\,\delta^{(3)}(\mathbf{x}_1'-\mathbf{x}_1)\,\delta^{(3)}(\mathbf{x}_2'-\mathbf{x}_2)$$

and

$$E_n=N_n^{1/2}\tilde{E}_n N_n^{-1/2},$$

where

$$\langle\underline{\tilde{n}};\mathbf{x}_1,\mathbf{x}_2|\underline{\tilde{n}};\mathbf{y}_1,\mathbf{y}_2\rangle=N_n(\mathbf{x}_1,\mathbf{x}_2;\mathbf{p}_1,\mathbf{p}_2)\,\delta^{(3)}(\mathbf{x}_1-\mathbf{y}_1)\,\delta^{(3)}(\mathbf{x}_2-\mathbf{y}_2).$$

By using the above results, we get for E_n up to $O(1/m^2)$,

$$\begin{aligned}
&E_n(\mathbf{x}_1,\mathbf{x}_2;\mathbf{p}_1,\mathbf{p}_2)\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_1')\,\delta^{(3)}(\mathbf{x}_2-\mathbf{x}_2')\\
&\quad=E_n^{(0)}(\mathbf{x}_1,\mathbf{x}_2)\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_1')\,\delta^{(3)}(\mathbf{x}_2-\mathbf{x}_2')+{}^{(0)}\langle\underline{n};\mathbf{x}_1,\mathbf{x}_2|\frac{H^{(1,0)}}{m_1}+\frac{H^{(0,1)}}{m_2}+\frac{H^{(2,0)}}{m_1^2}+\frac{H^{(0,2)}}{m_2^2}+\frac{H^{(1,1)}}{m_1m_2}|\underline{n};\mathbf{x}_1',\mathbf{x}_2'\rangle^{(0)}\\
&\qquad-\frac{1}{2}\sum_{k\neq n}\int d^3y_1\,d^3y_2\,{}^{(0)}\langle\underline{n};\mathbf{x}_1,\mathbf{x}_2|\frac{H^{(1,0)}}{m_1}+\frac{H^{(0,1)}}{m_2}|\underline{k};\mathbf{y}_1,\mathbf{y}_2\rangle^{(0)}\,{}^{(0)}\langle\underline{k};\mathbf{y}_1,\mathbf{y}_2|\frac{H^{(1,0)}}{m_1}+\frac{H^{(0,1)}}{m_2}|\underline{n};\mathbf{x}_1',\mathbf{x}_2'\rangle^{(0)}\\
&\qquad\times\left(\frac{1}{E_k^{(0)}(\mathbf{y}_1,\mathbf{y}_2)-E_n^{(0)}(\mathbf{x}_1',\mathbf{x}_2')}+\frac{1}{E_k^{(0)}(\mathbf{y}_1,\mathbf{y}_2)-E_n^{(0)}(\mathbf{x}_1,\mathbf{x}_2)}\right).
\end{aligned}\tag{18}$$

The expansion of E_n in inverse powers of the mass can be organized up to $O(1/m^2)$ as follows:

$$E_n=E_n^{(0)}+\frac{E_n^{(1,0)}}{m_1}+\frac{E_n^{(0,1)}}{m_2}+\frac{E_n^{(2,0)}}{m_1^2}+\frac{E_n^{(0,2)}}{m_2^2}+\frac{E_n^{(1,1)}}{m_1m_2}.\tag{19}$$

From Eq. (18) and Eqs. (10)–(13), by using the identities (here and in the rest of the paper, if not explicitly stated, the dependence on $\mathbf{x}_1$ and $\mathbf{x}_2$ is understood)

$$\text{(a)}\quad {}^{(0)}\langle n|\mathbf{D}_1|n\rangle^{(0)}=\boldsymbol{\nabla}_1,\quad {}^{(0)}\langle n|\mathbf{D}_{c\,2}|n\rangle^{(0)}=\boldsymbol{\nabla}_2,$$

$$\text{(b)}\quad {}^{(0)}\langle n|\mathbf{D}_1|j\rangle^{(0)}=\frac{{}^{(0)}\langle n|g\mathbf{E}_1|j\rangle^{(0)}}{E_n^{(0)}-E_j^{(0)}},\quad {}^{(0)}\langle n|\mathbf{D}_{c\,2}|j\rangle^{(0)}=-\frac{{}^{(0)}\langle n|g\mathbf{E}_2^T|j\rangle^{(0)}}{E_n^{(0)}-E_j^{(0)}}\quad\forall\, n\neq j,$$

$$\text{(c)}\quad {}^{(0)}\langle n|g\mathbf{E}_1|n\rangle^{(0)}=-(\boldsymbol{\nabla}_1E_n^{(0)}),\quad {}^{(0)}\langle n|g\mathbf{E}_2^T|n\rangle^{(0)}=(\boldsymbol{\nabla}_2E_n^{(0)}),$$

where $F_j\equiv F(\mathbf{x}_j)$, $\boldsymbol{\nabla}_j=\boldsymbol{\nabla}_{\mathbf{x}_j}$, $\mathbf{D}_{c\,j}=\boldsymbol{\nabla}_j+ig\mathbf{A}_j^T$, and the transpose refers to the color matrices, we obtain, at $O(1/m)$,

$$E_n^{(1,0)}=\frac{1}{2}\sum_{k\neq n}\left|\frac{{}^{(0)}\langle k|g\mathbf{E}_1|n\rangle^{(0)}}{E_n^{(0)}-E_k^{(0)}}\right|^2,\quad E_n^{(0,1)}=\frac{1}{2}\sum_{k\neq n}\left|\frac{{}^{(0)}\langle k|g\mathbf{E}_2^T|n\rangle^{(0)}}{E_n^{(0)}-E_k^{(0)}}\right|^2.\tag{20}$$

By using translational invariance one can see that $E_n^{(1,0)}$ and $E_n^{(0,1)}$ only depend on the relative distance r. Moreover, by using the symmetries of the static solutions, we can also see that $E_n^{(1,0)}=E_n^{(0,1)}$. The expressions (20) were first derived in Ref. [5].

At $O(1/m^2)$, we obtain

$$\begin{aligned}
E_n^{(2,0)}=&-\frac{c_D^{(1)\,\prime}}{8}{}^{(0)}\langle n|[\mathbf{D}_1\cdot,g\mathbf{E}_1]|n\rangle^{(0)}+\frac{c_F^{(1)\,2}}{4}\sum_{k\neq n}\frac{{}^{(0)}\langle n|g\mathbf{B}_1|k\rangle^{(0)}\cdot\,{}^{(0)}\langle k|g\mathbf{B}_1|n\rangle^{(0)}}{E_n^{(0)}-E_k^{(0)}}\\
&+\frac{1}{2}\sum_{k\neq n}\left[\left\{p_1^ip_1^j,\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_1^j|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3}\right\}+\left(\nabla_1^i\nabla_1^j\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_1^j|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3}\right)\right.\\
&+2\sum_{j,l\neq n}\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|j\rangle^{(0)}\,{}^{(0)}\langle j|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_1^j|l\rangle^{(0)}\,{}^{(0)}\langle l|g\mathbf{E}_1^j|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3(E_n^{(0)}-E_j^{(0)})(E_n^{(0)}-E_l^{(0)})}
\end{aligned}$$

$$
\begin{aligned}
&+2\left(\nabla_1^i\sum_{j\neq n}\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_1|j\rangle^{(0)}\cdot{}^{(0)}\langle j|g\mathbf{E}_1|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3(E_n^{(0)}-E_j^{(0)})}\right)-\left(\nabla_1^i\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|[\mathbf{D}_1\cdot,g\mathbf{E}_1]|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3}\right)\\
&+3\left(\nabla_1^i\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_1|n\rangle^{(0)}\cdot(\boldsymbol{\nabla}_1E_n^{(0)})}{(E_n^{(0)}-E_k^{(0)})^4}\right)-2\sum_{j\neq n}\frac{{}^{(0)}\langle n|g\mathbf{E}_1|j\rangle^{(0)}\cdot{}^{(0)}\langle j|g\mathbf{E}_1|k\rangle^{(0)}\,{}^{(0)}\langle k|[\mathbf{D}_1\cdot,g\mathbf{E}_1]|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3(E_n^{(0)}-E_j^{(0)})}\\
&+6\sum_{j\neq n}\frac{{}^{(0)}\langle n|g\mathbf{E}_1|j\rangle^{(0)}\cdot{}^{(0)}\langle j|g\mathbf{E}_1|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_1|n\rangle^{(0)}\cdot(\boldsymbol{\nabla}_1E_n^{(0)})}{(E_n^{(0)}-E_k^{(0)})^4(E_n^{(0)}-E_j^{(0)})}-3\frac{{}^{(0)}\langle n|[\mathbf{D}_1\cdot,g\mathbf{E}_1]|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_1|n\rangle^{(0)}\cdot(\boldsymbol{\nabla}_1E_n^{(0)})}{(E_n^{(0)}-E_k^{(0)})^4}\\
&+4\frac{(\boldsymbol{\nabla}_1E_n^{(0)})\cdot{}^{(0)}\langle n|g\mathbf{E}_1|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_1|n\rangle^{(0)}\cdot(\boldsymbol{\nabla}_1E_n^{(0)})}{(E_n^{(0)}-E_k^{(0)})^5}+\frac{1}{2}\frac{{}^{(0)}\langle n|[\mathbf{D}_1\cdot,g\mathbf{E}_1]|k\rangle^{(0)}\,{}^{(0)}\langle k|[\mathbf{D}_1\cdot,g\mathbf{E}_1]|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3}\Bigg]\\
&-d_3^{(1)\prime}f_{abc}\int d^3\mathbf{x}\,g\,{}^{(0)}\langle n|G^a_{\mu\nu}(\mathbf{x})G^b_{\mu\alpha}(\mathbf{x})G^c_{\nu\alpha}(\mathbf{x})|n\rangle^{(0)}+\frac{c_F^{(1)}}{2}\sum_{k\neq n}\left\{\nabla_1^i,\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|\boldsymbol{\sigma}_1\cdot g\mathbf{B}_1|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^2}\right\}\\
&-i\frac{c_S^{(1)}}{4}\frac{1}{r}\frac{d\,E_n^{(0)}}{dr}\boldsymbol{\sigma}_1\cdot(\mathbf{r}\times\boldsymbol{\nabla}_1),
\end{aligned}
\tag{21}
$$

$$
E_n^{(0,2)}=E_n^{(2,0)}(g\mathbf{E}_1\to-g\mathbf{E}_2^T,g\mathbf{B}_1\to-g\mathbf{B}_2^T,\boldsymbol{\sigma}_1\to\boldsymbol{\sigma}_2,\boldsymbol{\nabla}_1\to\boldsymbol{\nabla}_2,\mathbf{D}_1\to\mathbf{D}_{c\,2},m_1\leftrightarrow m_2),
\tag{22}
$$

and

$$
\begin{aligned}
E_n^{(1,1)}=\sum_{k\neq n}\Bigg[&-\left\{p_1^ip_2^j,\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^{j\,T}|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3}\right\}-\left(\nabla_1^i\nabla_2^j\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^{j\,T}|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3}\right)\\
&+2\sum_{j,l\neq n}\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|j\rangle^{(0)}\,{}^{(0)}\langle j|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^{j\,T}|l\rangle^{(0)}\,{}^{(0)}\langle l|g\mathbf{E}_2^{j\,T}|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3(E_n^{(0)}-E_j^{(0)})(E_n^{(0)}-E_l^{(0)})}\\
&+\left(\nabla_1^i\sum_{j\neq n}\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^T|j\rangle^{(0)}\cdot{}^{(0)}\langle j|g\mathbf{E}_2^T|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3(E_n^{(0)}-E_j^{(0)})}\right)\\
&-\left(\nabla_2^i\sum_{j\neq n}\frac{{}^{(0)}\langle n|g\mathbf{E}_1|j\rangle^{(0)}\cdot{}^{(0)}\langle j|g\mathbf{E}_1|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^{i\,T}|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3(E_n^{(0)}-E_j^{(0)})}\right)+\frac{1}{2}\left(\nabla_1^i\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|[\mathbf{D}_{c\,2}\cdot,g\mathbf{E}_2^T]|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3}\right)\\
&+\frac{1}{2}\left(\nabla_2^i\frac{{}^{(0)}\langle 0|[\mathbf{D}_1\cdot,g\mathbf{E}_1]|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^{i\,T}|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3}\right)-\frac{3}{2}\left(\nabla_1^i\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^T|n\rangle^{(0)}\cdot(\boldsymbol{\nabla}_2E_n^{(0)})}{(E_n^{(0)}-E_k^{(0)})^4}\right)\\
&-\frac{3}{2}\left(\nabla_2^i\frac{(\boldsymbol{\nabla}_1E_n^{(0)})\cdot{}^{(0)}\langle n|g\mathbf{E}_1|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^{i\,T}|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^4}\right)+\sum_{j\neq n}\frac{{}^{(0)}\langle n|g\mathbf{E}_1|j\rangle^{(0)}\cdot{}^{(0)}\langle j|g\mathbf{E}_1|k\rangle^{(0)}\,{}^{(0)}\langle k|[\mathbf{D}_{c\,2}\cdot,g\mathbf{E}_2^T]|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3(E_n^{(0)}-E_j^{(0)})}\\
&-\sum_{j\neq n}\frac{{}^{(0)}\langle n|[\mathbf{D}_1\cdot,g\mathbf{E}_1]|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^T|j\rangle^{(0)}\cdot{}^{(0)}\langle j|g\mathbf{E}_2^T|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3(E_n^{(0)}-E_j^{(0)})}\\
&-3\sum_{j\neq n}\frac{{}^{(0)}\langle n|g\mathbf{E}_1|j\rangle^{(0)}\cdot{}^{(0)}\langle j|g\mathbf{E}_1|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^T|n\rangle^{(0)}\cdot(\boldsymbol{\nabla}_2E_n^{(0)})}{(E_n^{(0)}-E_k^{(0)})^4(E_n^{(0)}-E_j^{(0)})}\\
&+3\sum_{j\neq n}\frac{(\boldsymbol{\nabla}_1E_n^{(0)})\cdot{}^{(0)}\langle n|g\mathbf{E}_1|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^T|j\rangle^{(0)}\cdot{}^{(0)}\langle j|g\mathbf{E}_2^T|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^4(E_n^{(0)}-E_j^{(0)})}\\
&+\frac{3}{2}\frac{{}^{(0)}\langle n|[\mathbf{D}_1\cdot,g\mathbf{E}_1]|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^T|n\rangle^{(0)}\cdot(\boldsymbol{\nabla}_2E_n^{(0)})}{(E_n^{(0)}-E_k^{(0)})^4}+\frac{3}{2}\frac{(\boldsymbol{\nabla}_1E_n^{(0)})\cdot{}^{(0)}\langle n|g\mathbf{E}_1|k\rangle^{(0)}\,{}^{(0)}\langle j|[\mathbf{D}_{c\,2}\cdot,g\mathbf{E}_2^T]|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^4}
\end{aligned}
$$

$$-4\frac{(\boldsymbol{\nabla}_1 E_n^{(0)})\cdot {}^{(0)}\langle n|g\mathbf{E}_1|k\rangle^{(0)}\,{}^{(0)}\langle k|g\mathbf{E}_2^T|n\rangle^{(0)}\cdot(\boldsymbol{\nabla}_2 E_n^{(0)})}{(E_n^{(0)}-E_k^{(0)})^5}-\frac{1}{2}\frac{{}^{(0)}\langle n|[\mathbf{D}_1\cdot,g\mathbf{E}_1]|k\rangle^{(0)}\,{}^{(0)}\langle k|[\mathbf{D}_{c\,2}\cdot,g\mathbf{E}_2^T]|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^3}\Bigg]$$

$$+(d_{ss}+d_{vs}\,{}^{(0)}\langle n|T_1^a T_2^{a\,T}|n\rangle^{(0)})\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2)-\frac{c_F^{(1)}}{2}\sum_{k\neq n}\left\{\nabla_2^i,\frac{{}^{(0)}\langle n|g\mathbf{E}_2^{i\,T}|k\rangle^{(0)}\,{}^{(0)}\langle k|\boldsymbol{\sigma}_1\cdot g\mathbf{B}_1|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^2}\right\}$$

$$-\frac{c_F^{(2)}}{2}\sum_{k\neq n}\left\{\nabla_1^i,\frac{{}^{(0)}\langle n|g\mathbf{E}_1^i|k\rangle^{(0)}\,{}^{(0)}\langle k|\boldsymbol{\sigma}_2\cdot g\mathbf{B}_2^T|n\rangle^{(0)}}{(E_n^{(0)}-E_k^{(0)})^2}\right\}-\frac{c_F^{(1)}c_F^{(2)}}{2}\sum_{k\neq n}\frac{{}^{(0)}\langle n|\boldsymbol{\sigma}_1\cdot g\mathbf{B}_1|k\rangle^{(0)}\,{}^{(0)}\langle k|\boldsymbol{\sigma}_2\cdot g\mathbf{B}_2^T|n\rangle^{(0)}}{E_n^{(0)}-E_k^{(0)}}$$

$$-(d_{sv}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+d_{vv}\,{}^{(0)}\langle n|T_1^a\boldsymbol{\sigma}_1\cdot T_2^{a\,T}\boldsymbol{\sigma}_2|n\rangle^{(0)})\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2). \tag{23}$$

The above equations (20)–(23) give the energies of the gluonic excitations between heavy quarks within an expansion in $1/m$ up to $O(1/m^2)$. From these expressions, in the case of the ground state ($n=0$), we will derive, in Sec. V, the equivalent Wilson loop expressions.

A similar approach has been used in Ref. [17] in order to derive, from the QCD Hamiltonian in the Coulomb gauge, the spin-dependent part of the potential up to $O(1/m^2)$. However, the behavior at scales of $O(m)$ was not correctly incorporated there. If we take our NRQCD matching coefficients at the tree level and neglect the tree-level annihilation contributions in the equal-mass case, we find agreement for the spin-dependent potentials (up to some transpose color matrices). Nevertheless, our general expression (18) differs from the one used in [17], which, in general, will not give the correct spin-independent potentials. This has to do, in our opinion, with the fact that in order to derive Eq. (18) one has to deal with operators rather than with numbers.

IV. PNRQCD

In the previous section we have studied the static limit of NRQCD and its corrections within a $1/m$ expansion. Let us now connect those results with pNRQCD.

In the static limit, the gap between different states at fixed $\mathbf{r}$ will depend on the dimensionless parameter $\Lambda_{\rm QCD}r$. In a general situation, there will be a set of states $\{n_{\rm us}\}$ such that $E_{n_{\rm us}}^{(0)}(r)\sim mv^2$ for the typical r of the actual physical system. We denote these states as ultrasoft. The aim of pNRQCD is to describe the behavior of the ultrasoft states. Therefore, all the physical degrees of freedom with energies larger than mv^2 will be integrated out from NRQCD in order to obtain pNRQCD. It is in this context that one may work order by order in $1/m$ (in particular for the kinetic energy), and the calculation of the previous section becomes the matching calculation between NRQCD and pNRQCD and provides a rigorous connection with the adiabatic approximation (this approximation is implicit in all the attempts at deriving the nonperturbative potentials from QCD we are aware of). Whereas this can be justified within a perturbative framework, in the nonperturbative case, we cannot, in general, guarantee the validity of the $1/m$ expansion and one may think of examples where certain degrees of freedom cannot be integrated out in the $1/m$ expansion (see [29]). We believe that this possibility, which, to our knowledge, has never been mentioned before, except in Ref. [5], deserves further study. Note that this does not have to do with the consideration of ultrasoft effects, which, unlike in earlier approaches, can be readily incorporated within our formalism.

In the perturbative situation $\Lambda_{\rm QCD}r\ll 1$, which has been studied in detail in [4], $\{n_{\rm us}\}$ corresponds to a heavy-quark–antiquark state, in either a singlet or an octet configuration, plus gluons and light fermions, all of them with energies of $O(mv^2)$. In a nonperturbative situation, which we will generically denote by $\Lambda_{\rm QCD}r\sim 1$, it is not so clear what $\{n_{\rm us}\}$ is. One can think of different possibilities. Each of them will give, in principle, different predictions and, therefore, it should be possible to experimentally discriminate among them. In particular, one could consider the situation where, because of a mass gap in QCD, the energy splitting between the ground state and the first gluonic excitation is larger than mv^2 and, because of chiral symmetry breaking of QCD, Goldstone bosons (pions and kaons) appear. Hence, in this situation, $\{n_{\rm us}\}$ would be the ultrasoft excitations about the static ground state (i.e., the solutions of the corresponding Schrödinger equation), which will be named the singlet, plus the Goldstone bosons. If one switches off the light fermions (pure gluodynamics), only the singlet survives and pNRQCD reduces to a pure two-particle NR quantum-mechanical system, usually referred as a pure potential model.

In this paper, we will study the pure singlet sector, with no reference to further ultrasoft degrees of freedom. In this situation, pNRQCD only describes the ultrasoft excitations about the static ground state of NRQCD. In terms of static NRQCD eigenstates, this means that only $|\underline{0};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}$ is kept as an explicit degree of freedom whereas $|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}$ with $n\neq 0$ are integrated out.[1] This provides the only dy-

[1]In fact, we are only integrating out states with energies larger than mv^2 and all the states with $n\neq 0$ will be understood in this way throughout the paper. Since, in practice, we are integrating over all the states, if we are in the situation where some states, different from the singlet, are ultrasoft, these have to be subtracted later on. This is analogous to what happens in the perturbative situation, where the subtraction is done order by order in the multipole expansion. In this situation our calculation should be understood as the leading term in the multipole expansion.

namical degree of freedom of the theory. It is described by means of a bilinear color singlet field $S(\mathbf{x}_1,\mathbf{x}_2,t)$, which has the same quantum numbers and transformation properties under symmetries as the static ground state of NRQCD in the one-quark–one-antiquark sector. In the above situation, the Lagrangian of pNRQCD reads

$$\mathcal{L}_{\rm pNRQCD}=S^\dagger[i\partial_0-h_s(\mathbf{x}_1,\mathbf{x}_2,\mathbf{p}_1,\mathbf{p}_2)]S, \tag{24}$$

where h_s is the Hamiltonian of the singlet (actually h_s is only a function of $\mathbf{r}$, $\mathbf{p}_1$, $\mathbf{p}_2$, which is analytic in the two last operators but typically contains nonanalyticities in $\mathbf{r}$), $\mathbf{p}_1=-i\boldsymbol{\nabla}_{\mathbf{x}_1}$, and $\mathbf{p}_2=-i\boldsymbol{\nabla}_{\mathbf{x}_2}$. It has the following expansion up to order $1/m^2$:

$$h_s(\mathbf{x}_1,\mathbf{x}_2,\mathbf{p}_1,\mathbf{p}_2)=\frac{\mathbf{p}_1^2}{2m_1}+\frac{\mathbf{p}_2^2}{2m_2}+V^{(0)}+\frac{V^{(1,0)}}{m_1}+\frac{V^{(0,1)}}{m_2}+\frac{V^{(2,0)}}{m_1^2}+\frac{V^{(0,2)}}{m_2^2}+\frac{V^{(1,1)}}{m_1m_2}. \tag{25}$$

The integration of higher excitations is trivial using the basis $|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle$ since, in this case, they are decoupled from $|\underline{0};\mathbf{x}_1,\mathbf{x}_2\rangle$. Then, the matching of NRQCD to pNRQCD consists in renaming things in a way such that pNRQCD reproduces the matrix elements of NRQCD for the ground state and, in particular, the energy. This fixes the matching condition

$$E_0(\mathbf{x}_1,\mathbf{x}_2,\mathbf{p}_1,\mathbf{p}_2)=h_s(\mathbf{x}_1,\mathbf{x}_2,\mathbf{p}_1,\mathbf{p}_2). \tag{26}$$

Although our main concern in this paper is to provide a well-controlled derivation of the potential for the heavy quarkonium, we would like to say a few words about the expressions E_n $(n\neq 0)$ we have found in the previous section. In the static limit, the different $E_n^{(0)}$ $(n\neq 0)$ are identified with the static potentials to be used in a Schrödinger equation to obtain the spectra of the bound systems composed of a heavy quark and an antiquark (plus glueballs) different from the heavy quarkonium such as, for instance, heavy hybrids. This assignment is argued within the adiabatic approximation and corresponds to what is actually done in lattice simulations [28]. In this respect, since we have given a systematic method to obtain the corrections to the energy within a $1/m$ expansion, the energies E_n correspond to the quantum-mechanical Hamiltonians of the different bound systems made by a heavy quark and an antiquark (up to glueballs) and the $1/m$ and $1/m^2$ terms should be understood as the relativistic corrections to the static potentials. It is still an open problem if this procedure is the sensible thing to do for heavy hybrids, if (and whichever) other possibilities may occur, and if these potentials, like the heavy quarkonium potential, may eventually be written in terms of Wilson loops. We will not deal with these problems here, which, however, deserve further investigations. We refer to [3] for related discussions.

V. HEAVY QUARKONIUM POTENTIAL AND WILSON LOOPS

In this section we express the heavy quarkonium potential in terms of Wilson loop operators. These kinds of expressions are quite convenient for lattice simulations or for QCD-vacuum-model studies (see, for instance, [14,16]). We shall use the following definitions. The angular brackets $\langle\cdots\rangle$ will stand for the average value over the Yang-Mills action, $W_\square$ for the rectangular static Wilson loop of dimensions $r\times T_W$,

$$W_\square\equiv \mathrm{P}\,\exp\left\{-ig\oint_{r\times T_W}dz^\mu A_\mu(z)\right\},$$

and $\langle\!\langle\cdots\rangle\!\rangle\equiv\langle\cdots W_\square\rangle/\langle W_\square\rangle$; P is the path-ordering operator. Moreover, we define the *connected* Wilson loop with $O_1(t_1)$, $O_2(t_2)$, ..., $O_n(t_n)$ operator insertions for $T_W/2\geqslant t_1\geqslant t_2\geqslant\cdots\geqslant t_n\geqslant -T_W/2$ by

$$\begin{aligned}\langle\!\langle O_1(t_1)O_2(t_2)\cdots O_n(t_n)\rangle\!\rangle_c&=\langle\!\langle O_1(t_1)O_2(t_2)\cdots O_n(t_n)\rangle\!\rangle\\&\quad-\sum_{j=1}^{n-1}\langle\!\langle O_1(t_1)O_2(t_2)\cdots O_j(t_j)\rangle\!\rangle_c\\&\quad\times\langle\!\langle O_{j+1}(t_{j+1})\cdots O_n(t_n)\rangle\!\rangle_c,\end{aligned} \tag{27}$$

$$\langle\!\langle O_j(t_j)\rangle\!\rangle_c=\langle\!\langle O_j(t_j)\rangle\!\rangle. \tag{28}$$

We also define in a shorthand notation

$$\lim_{T\to\infty}\equiv\lim_{T\to\infty}\lim_{T_W\to\infty}, \tag{29}$$

where T_W is the time length of the Wilson loop and T the time length appearing in the time integrals. By performing first the $T_W\to\infty$, the averages $\langle\!\langle\cdots\rangle\!\rangle$ become independent of T_W and thus invariant under global time translations.

By using the matching condition (26) and the quantum-mechanical expressions (20), it has already been proved in [5] that the quarkonium singlet static potential and the $O(1/m)$ potential can be expressed in terms of Wilson loops with field strength insertions in it as

$$V^{(0)}(r)=\lim_{T\to\infty}\frac{i}{T}\ln\langle W_\square\rangle, \tag{30}$$

$$V^{(1,0)}(r)=-\frac{1}{2}\lim_{T\to\infty}\int_0^T dt\,t\,\langle\!\langle g\mathbf{E}_1(t)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c. \tag{31}$$

Owing to invariance under charge conjugation plus $m_1\leftrightarrow m_2$ transformation we have

$$V^{(1,0)}(r)=V^{(0,1)}(r).$$

The way to prove the equivalence of Eq. (31) and Eq. (20) has been discussed in Ref. [5], where more details can be found. Here we only mention that this equivalence proof as

well as the following ones can be done straightforwardly by inserting complete sets of intermediate states in the Wilson loop operators and by explicitly computing the time integrals.

Let us now consider the terms of $O(1/m^2)$. It is convenient to split them into a spin-dependent and a spin-independent part. For the $V^{(2,0)}$ and $V^{(0,2)}$ potentials we define

$$V^{(2,0)}=V_{SD}^{(2,0)}+V_{SI}^{(2,0)}, \quad V^{(0,2)}=V_{SD}^{(0,2)}+V_{SI}^{(0,2)}. \tag{32}$$

The spin-independent terms can be written as

$$V_{SI}^{(2,0)}=\frac{1}{2}\{\mathbf{p}_1^2,V_{\mathbf{p}^2}^{(2,0)}(r)\}+\frac{V_{\mathbf{L}^2}^{(2,0)}(r)}{r^2}\mathbf{L}_1^2+V_r^{(2,0)}(r) \tag{33}$$

and

$$V_{SI}^{(0,2)}=\frac{1}{2}\{\mathbf{p}_2^2,V_{\mathbf{p}^2}^{(0,2)}(r)\}+\frac{V_{\mathbf{L}^2}^{(0,2)}(r)}{r^2}\mathbf{L}_2^2+V_r^{(0,2)}(r), \tag{34}$$

where $\mathbf{L}_1\equiv\mathbf{r}\times\mathbf{p}_1$ and $\mathbf{L}_2\equiv\mathbf{r}\times\mathbf{p}_2$. Note that neither $\mathbf{L}_1$ nor $\mathbf{L}_2$ corresponds to the orbital angular momentum of the particle and antiparticle. By using invariance under charge conjugation plus $m_1\leftrightarrow m_2$ transformation, we obtain

$$V_{\mathbf{p}^2}^{(2,0)}(r)=V_{\mathbf{p}^2}^{(0,2)}(r), \quad V_{\mathbf{L}^2}^{(2,0)}(r)=V_{\mathbf{L}^2}^{(0,2)}(r),$$
$$V_r^{(2,0)}(r)=V_r^{(0,2)}(r;m_2\leftrightarrow m_1). \tag{35}$$

The spin-dependent part of $V^{(2,0)}$ is of the type

$$V_{SD}^{(2,0)}=V_{LS}^{(2,0)}(r)\mathbf{L}_1\cdot\mathbf{S}_1. \tag{36}$$

Analogously, for the $V^{(0,2)}$ potential we can write

$$V_{SD}^{(0,2)}=-V_{LS}^{(0,2)}(r)\mathbf{L}_2\cdot\mathbf{S}_2. \tag{37}$$

From invariance under charge conjugation plus $m_1\leftrightarrow m_2$ transformation, we obtain

$$V_{LS}^{(2,0)}(r)=V_{LS}^{(0,2)}(r;m_2\leftrightarrow m_1).$$

By using Eqs. (26) and (21) we get, in terms of Wilson loop operators,

$$V_{\mathbf{p}^2}^{(2,0)}(r)=\frac{i}{2}\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\lim_{T\to\infty}\int_0^T dt\,t^2\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_1^j(0)\rangle\rangle_c, \tag{38}$$

$$V_{\mathbf{L}^2}^{(2,0)}(r)=\frac{i}{4}(\delta^{ij}-3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j)\lim_{T\to\infty}\int_0^T dt\,t^2\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_1^j(0)\rangle\rangle_c, \tag{39}$$

$$\begin{aligned}
V_r^{(2,0)}(r)=&-\frac{c_D^{(1)\prime}}{8}\lim_{T_W\to\infty}\langle\langle[\mathbf{D}_1,g\mathbf{E}_1](t)\rangle\rangle_c-\frac{ic_F^{(1)\,2}}{4}\lim_{T\to\infty}\int_0^T dt\langle\langle g\mathbf{B}_1(t)\cdot g\mathbf{B}_1(0)\rangle\rangle_c+\frac{1}{2}(\boldsymbol{\nabla}_r^2V_{\mathbf{p}^2}^{(2,0)})\\
&-\frac{i}{2}\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3\,(t_2-t_3)^2\langle\langle g\mathbf{E}_1(t_1)\cdot g\mathbf{E}_1(t_2)g\mathbf{E}_1(t_3)\cdot g\mathbf{E}_1(0)\rangle\rangle_c\\
&+\frac{1}{2}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1}dt_2\,(t_1-t_2)^2\langle\langle g\mathbf{E}_1^i(t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c\right)\\
&-\frac{i}{2}(\boldsymbol{\nabla}_r^iV^{(0)})\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1}dt_2\,(t_1-t_2)^3\langle\langle g\mathbf{E}_1^i(t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c-\frac{1}{2}\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1}dt_2\,(t_1-t_2)^2\\
&\times\langle\langle[\mathbf{D}_1.,g\mathbf{E}_1](t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c+\frac{i}{8}\lim_{T\to\infty}\int_0^T dt\,t^2\langle\langle[\mathbf{D}_1.,g\mathbf{E}_1](t)[\mathbf{D}_1.,g\mathbf{E}_1](0)\rangle\rangle_c\\
&-\frac{i}{4}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt\,t^2\langle\langle g\mathbf{E}_1^i(t)[\mathbf{D}_1.,g\mathbf{E}_1](0)\rangle\rangle_c\right)-\frac{1}{4}\lim_{T\to\infty}\int_0^T dt\,t^3\langle\langle[\mathbf{D}_1.,g\mathbf{E}_1](t)g\mathbf{E}_1^j(0)\rangle\rangle_c(\boldsymbol{\nabla}_r^jV^{(0)})\\
&+\frac{1}{4}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt\,t^3\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_1^j(0)\rangle\rangle_c(\boldsymbol{\nabla}_r^jV^{(0)})\right)-\frac{i}{12}\lim_{T\to\infty}\int_0^T dt\,t^4\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_1^j(0)\rangle\rangle_c(\boldsymbol{\nabla}_r^iV^{(0)})(\boldsymbol{\nabla}_r^jV^{(0)})\\
&-d_3^{(1)\prime}f_{abc}\int d^3\mathbf{x}\lim_{T_W\to\infty}g\langle\langle G_{\mu\nu}^a(x)G_{\mu\alpha}^b(x)G_{\nu\alpha}^c(x)\rangle\rangle
\end{aligned} \tag{40}$$

(note that, although formally the first and last terms depend on the time where the operator insertion is made, this is not so after doing the $T_W \to \infty$ limit[2]),

$$V_{LS}^{(2,0)}(r) = -\frac{c_F^{(1)}}{r^2} i\mathbf{r}\cdot \lim_{T\to\infty}\int_0^T dt\, t \langle\!\langle g\mathbf{B}_1(t)\times g\mathbf{E}_1(0)\rangle\!\rangle + \frac{c_S^{(1)}}{2r^2}\mathbf{r}\cdot(\boldsymbol{\nabla}_r V^{(0)}). \tag{41}$$

For the $V^{(1,1)}$ potential we define

$$V^{(1,1)} = V_{SD}^{(1,1)} + V_{SI}^{(1,1)}. \tag{42}$$

The spin-independent part can be written as

$$V_{SI}^{(1,1)} = -\frac{1}{2}\{\mathbf{p}_1\cdot\mathbf{p}_2, V_{\mathbf{p}^2}^{(1,1)}(r)\} - \frac{V_{\mathbf{L}^2}^{(1,1)}(r)}{2r^2}(\mathbf{L}_1\cdot\mathbf{L}_2 + \mathbf{L}_2\cdot\mathbf{L}_1) + V_r^{(1,1)}(r), \tag{43}$$

while the spin-dependent part contains the following operators:

$$V_{SD}^{(1,1)} = V_{L_1S_2}^{(1,1)}(r)\mathbf{L}_1\cdot\mathbf{S}_2 - V_{L_2S_1}^{(1,1)}(r)\mathbf{L}_2\cdot\mathbf{S}_1 + V_{S^2}^{(1,1)}(r)\mathbf{S}_1\cdot\mathbf{S}_2 + V_{\mathbf{S}_{12}}^{(1,1)}(r)\mathbf{S}_{12}(\hat{\mathbf{r}}), \tag{44}$$

where $\mathbf{S}_{12}(\hat{\mathbf{r}}) \equiv 3\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\,\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$. Because of the invariance under charge conjugation plus $m_1 \leftrightarrow m_2$ transformation, we have

$$V_{L_1S_2}^{(1,1)}(r) = V_{L_2S_1}^{(1,1)}(r; m_1 \leftrightarrow m_2).$$

By using Eqs. (26) and (23) we get, in terms of Wilson loop operators,

$$V_{\mathbf{p}^2}^{(1,1)}(r) = i\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j \lim_{T\to\infty}\int_0^T dt\, t^2 \langle\!\langle g\mathbf{E}_1^i(t) g\mathbf{E}_2^j(0)\rangle\!\rangle_c, \tag{45}$$

$$V_{\mathbf{L}^2}^{(1,1)}(r) = i\frac{\delta^{ij} - 3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j}{2} \lim_{T\to\infty}\int_0^T dt\, t^2 \langle\!\langle g\mathbf{E}_1^i(t) g\mathbf{E}_2^j(0)\rangle\!\rangle_c, \tag{46}$$

$$\begin{aligned}
V_r^{(1,1)}(r) = &-\frac{1}{2}(\boldsymbol{\nabla}_r^2 V_{\mathbf{p}^2}^{(1,1)}) - i\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3\,(t_2-t_3)^2 \langle\!\langle g\mathbf{E}_1(t_1)\cdot g\mathbf{E}_1(t_2) g\mathbf{E}_2(t_3)\cdot g\mathbf{E}_2(0)\rangle\!\rangle_c \\
&+\frac{1}{2}\left(\boldsymbol{\nabla}_r^i \lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2 (t_1-t_2)^2 \langle\!\langle g\mathbf{E}_1^i(t_1) g\mathbf{E}_2(t_2)\cdot g\mathbf{E}_2(0)\rangle\!\rangle_c\right) + \frac{1}{2}\left(\boldsymbol{\nabla}_r^i \lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2 (t_1-t_2)^2\right. \\
&\left.\times\langle\!\langle g\mathbf{E}_2^i(t_1) g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c\right) - \frac{i}{2}(\boldsymbol{\nabla}_r^i V^{(0)}) \lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2 (t_1-t_2)^3 \langle\!\langle g\mathbf{E}_1^i(t_1) g\mathbf{E}_2(t_2)\cdot g\mathbf{E}_2(0)\rangle\!\rangle_c \\
&-\frac{i}{2}(\boldsymbol{\nabla}_r^i V^{(0)}) \lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2 (t_1-t_2)^3 \langle\!\langle g\mathbf{E}_2^i(t_1) g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c - \frac{1}{2}\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2 (t_1-t_2)^2 \\
&\times\langle\!\langle [\mathbf{D}_1\cdot, g\mathbf{E}_1](t_1) g\mathbf{E}_2(t_2)\cdot g\mathbf{E}_2(0)\rangle\!\rangle_c + \frac{1}{2}\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2 (t_1-t_2)^2 \langle\!\langle [\mathbf{D}_2\cdot, g\mathbf{E}_2](t_1) g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c \\
&-\frac{i}{4}\lim_{T\to\infty}\int_0^T dt\, t^2 \langle\!\langle [\mathbf{D}_1\cdot, g\mathbf{E}_1](t)[\mathbf{D}_2\cdot, g\mathbf{E}_2](0)\rangle\!\rangle_c + \frac{i}{4}\left(\boldsymbol{\nabla}_r^i \lim_{T\to\infty}\int_0^T dt\, t^2 \{\langle\!\langle g\mathbf{E}_1^i(t)[\mathbf{D}_2\cdot, g\mathbf{E}_2](0)\rangle\!\rangle_c\right.
\end{aligned}$$

[2] $V^{(0)}$ could also be written in a similar way:

$$V^{(0)} = \frac{1}{2}\int d^3\mathbf{x} \lim_{T_W\to\infty} \langle\!\langle (\boldsymbol{\Pi}^a\boldsymbol{\Pi}^a + \mathbf{B}^a\mathbf{B}^a)(x)\rangle\!\rangle.$$

$$-\langle\!\langle g\mathbf{E}_2^i(t)[\mathbf{D}_1\cdot,g\mathbf{E}_1](0)\rangle\!\rangle_c\}\Bigg)-\frac{1}{4}\lim_{T\to\infty}\int_0^T dt\,t^3\{\langle\!\langle[\mathbf{D}_1\cdot,g\mathbf{E}_1](t)g\mathbf{E}_2^j(0)\rangle\!\rangle_c-\langle\!\langle[\mathbf{D}_2\cdot,g\mathbf{E}_2](t)g\mathbf{E}_1^j(0)\rangle\!\rangle_c\}(\boldsymbol{\nabla}_r^j V^{(0)})$$
$$+\frac{1}{4}\Bigg(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt\,t^3\{\langle\!\langle g\mathbf{E}_1^i(t)g\mathbf{E}_2^j(0)\rangle\!\rangle_c+\langle\!\langle g\mathbf{E}_2^i(t)g\mathbf{E}_1^j(0)\rangle\!\rangle_c\}(\boldsymbol{\nabla}_r^j V^{(0)})\Bigg)$$
$$-\frac{i}{6}\lim_{T\to\infty}\int_0^T dt\,t^4\langle\!\langle g\mathbf{E}_1^i(t)g\mathbf{E}_2^j(0)\rangle\!\rangle_c(\boldsymbol{\nabla}_r^i V^{(0)})(\boldsymbol{\nabla}_r^j V^{(0)})+(d_{ss}+d_{vs}\lim_{T_W\to\infty}\langle\!\langle T_1^aT_2^a\rangle\!\rangle)\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2) \tag{47}$$

(here and in the following formulas the two color matrices in $\langle\!\langle T_1^aT_2^a\rangle\!\rangle$ are inserted in the Wilson loop at the same time: $-T_W/2\leqslant t\leqslant T_W/2$; the t dependence disappears in the $T_W\to\infty$ limit),

$$V_{L_2S_1}^{(1,1)}(r)=-\frac{c_F^{(1)}}{r^2}i\mathbf{r}\cdot\lim_{T\to\infty}\int_0^T dt\,t\,\langle\!\langle g\mathbf{B}_1(t)\times g\mathbf{E}_2(0)\rangle\!\rangle, \tag{48}$$

$$V_{S^2}^{(1,1)}(r)=\frac{2c_F^{(1)}c_F^{(2)}}{3}i\lim_{T\to\infty}\int_0^T dt\,\langle\!\langle g\mathbf{B}_1(t)\cdot g\mathbf{B}_2(0)\rangle\!\rangle$$
$$-4(d_{sv}+d_{vv}\lim_{T_W\to\infty}\langle\!\langle T_1^aT_2^a\rangle\!\rangle)\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2), \tag{49}$$

$$V_{\mathbf{S}_{12}}^{(1,1)}(r)=\frac{c_F^{(1)}c_F^{(2)}}{4}i\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\lim_{T\to\infty}\int_0^T dt\Bigg[\langle\!\langle g\mathbf{B}_1^i(t)g\mathbf{B}_2^j(0)\rangle\!\rangle$$
$$-\frac{\delta^{ij}}{3}\langle\!\langle g\mathbf{B}_1(t)\cdot g\mathbf{B}_2(0)\rangle\!\rangle\Bigg]. \tag{50}$$

We now compare our results with previous ones. For the spin-dependent potentials we find agreement with the Eichten-Feinberg results [8] (once the NRQCD matching coefficients have been taken into account) *except* for the $1/m_1m_2$ spin-orbit potential $V_{L_2S_1}^{(1,1)}$. Since the Eichten-Feinberg results have been checked by, at least, three independent groups [10,12,13], we perform a more detailed comparison in Appendix B. We show that our expression in terms of Wilson loops and theirs give different results in terms of intermediate states and, more important, we show that they give different perturbative results at leading order in α_s. Ours coincides with the well-known tree-level calculation, whereas the Eichten-Feinberg expression gives 1/2, the expected result. Moreover, our perturbative result satisfies the Gromes relation [10]. The fact that the same mistake has been done by several groups can only be explained by a systematic error. We believe that their systematic error has to do with the common assumption in the literature that one may neglect, *in general*, the dependence of the Wilson loops on the gluonic strings or on any other gluonic operator at $t=\pm T_W/2$. An analysis of the calculation done by Eichten and Feinberg in [8] supports this belief. Finally, we would like to mention that several different expressions for the spin-dependent potentials, in particular the correct one, can be found in the literature dealing with the lattice evaluation of them [30–32,3,14]. All these refer to the work of Eichten and Feinberg [8] for the derivation. We believe that our result makes mandatory a clarification of all previous lattice evaluations of the spin-dependent potentials.

The spin-independent potentials have only been computed before by Barchielli *et al.* [12] (the analysis done in [11], which appears to be inconclusive, has never been published). We agree (once the NRQCD matching coefficients have been taken into account) with their results for the momentum-dependent terms, but not for the momentum-independent terms, where we find new contributions. Moreover, since the potential we get here is *complete* up to order $1/m^2$, it is not affected by the ordering ambiguity, which affects the derivation in [12]. In this context, we would like to mention that our result may be of particular relevance for the study of the properties of the QCD vacuum in the presence of heavy sources. So far the lattice data for the spin-dependent and spin-independent potentials are consistent with a flux-tube picture, whereas it is only for the spin-dependent terms that the so-called ''scalar confinement'' is consistent with lattice data [1,33] (however, the lattice data are still not conclusive). It will be interesting to see how these pictures compare with the new momentum- and spin-independent potentials, once lattice data will be available for them. We note that some of them are not simply expressed by two field insertions on a static Wilson loop, such as the spin- or momentum-dependent terms. In particular, an extended object coming from the Yang-Mills sector is required (similar extended objects would also show up by taking into account operators with light quarks).

Gauss law and further identities

The above results may be simplified and rewritten in several ways. For instance, by using the quantum-mechanical identities (a)–(c) given in Sec. III, we obtain

$$\lim_{T_W\to\infty}\langle\!\langle[\mathbf{D}_1,g\mathbf{E}_1](t)\rangle\!\rangle_c$$
$$=-\left(\boldsymbol{\nabla}_r^2V^{(0)}+2i\lim_{T\to\infty}\int_0^T dt\,\langle\!\langle g\mathbf{E}_1(t)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c\right), \tag{51}$$

changing the expression of the Darwin term (which now

looks similar to the analogous expression given in Ref. [12]). In fact, by using the quantum-mechanical identities (a)–(c) of Sec. III, we could systematically transform $[\mathbf{D},g\mathbf{E}]$ in terms of normal derivatives acting on matrix elements or on static energies.

Another possibility, which turns out to be more powerful, is the use of the Gauss law (14). It allows us to write all the terms of the type $[\mathbf{D},g\mathbf{E}]$ in terms of $\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2)$ times some color matrices [up to some terms proportional to $\delta^{(3)}(0)$ that vanish in dimensional regularization]. More information can be obtained by using the behavior of the Wilson loops (or of the states) at short distances for the terms proportional to the deltas (assuming they are regular enough). It follows that all the original terms with $[\mathbf{D},g\mathbf{E}]$ disappear except the Darwin term. Moreover, we have $[C_f=(N_c^2-1)/(2N_c)]$

$$\lim_{T_W\to\infty}\langle\langle T_1^aT_2^a\rangle\rangle\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2)=C_f\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2).$$

Therefore, some potentials get simplified into the following expressions:

$$\begin{aligned}
V_r^{(2,0)}(r)=&\frac{\pi C_f\alpha_s c_D^{(1)'}}{2}\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2)-\frac{ic_F^{(1)\,2}}{4}\lim_{T\to\infty}\int_0^T dt\langle\langle g\mathbf{B}_1(t)\cdot g\mathbf{B}_1(0)\rangle\rangle_c+\frac{1}{2}(\boldsymbol{\nabla}_r^2V_{\mathbf{p}^2}^{(2,0)})\\
&-\frac{i}{2}\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3\,(t_2-t_3)^2\langle\langle g\mathbf{E}_1(t_1)\cdot g\mathbf{E}_1(t_2)g\mathbf{E}_1(t_3)\cdot g\mathbf{E}_1(0)\rangle\rangle_c\\
&+\frac{1}{2}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1}dt_2\,(t_1-t_2)^2\langle\langle g\mathbf{E}_1^i(t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c\right)\\
&-\frac{i}{2}(\boldsymbol{\nabla}_r^iV^{(0)})\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1}dt_2\,(t_1-t_2)^3\langle\langle g\mathbf{E}_1^i(t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c\\
&+\frac{1}{4}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt\,t^3\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_1^j(0)\rangle\rangle_c(\boldsymbol{\nabla}_r^jV^{(0)})\right)-\frac{i}{12}\lim_{T\to\infty}\int_0^T dt\,t^4\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_1^j(0)\rangle\rangle_c(\boldsymbol{\nabla}_r^iV^{(0)})(\boldsymbol{\nabla}_r^jV^{(0)})\\
&-d_3^{(1)'}f_{abc}\int d^3\mathbf{x}\lim_{T_W\to\infty}g\langle\langle G_{\mu\nu}^a(x)G_{\mu\alpha}^b(x)G_{\nu\alpha}^c(x)\rangle\rangle,
\end{aligned}\tag{52}$$

$$\begin{aligned}
V_r^{(1,1)}(r)=&-\frac{1}{2}(\boldsymbol{\nabla}_r^2V_{\mathbf{p}^2}^{(1,1)})-i\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3\,(t_2-t_3)^2\langle\langle g\mathbf{E}_1(t_1)\cdot g\mathbf{E}_1(t_2)g\mathbf{E}_2(t_3)\cdot g\mathbf{E}_2(0)\rangle\rangle_c\\
&+\frac{1}{2}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1}dt_2(t_1-t_2)^2\langle\langle g\mathbf{E}_1^i(t_1)g\mathbf{E}_2(t_2)\cdot g\mathbf{E}_2(0)\rangle\rangle_c\right)\\
&+\frac{1}{2}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1}dt_2(t_1-t_2)^2\langle\langle g\mathbf{E}_2^i(t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c\right)\\
&-\frac{i}{2}(\boldsymbol{\nabla}_r^iV^{(0)})\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1}dt_2(t_1-t_2)^3\langle\langle g\mathbf{E}_1^i(t_1)g\mathbf{E}_2(t_2)\cdot g\mathbf{E}_2(0)\rangle\rangle_c\\
&-\frac{i}{2}(\boldsymbol{\nabla}_r^iV^{(0)})\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1}dt_2(t_1-t_2)^3\langle\langle g\mathbf{E}_2^i(t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c\\
&+\frac{1}{4}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt\,t^3\{\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_2^j(0)\rangle\rangle_c+\langle\langle g\mathbf{E}_2^i(t)g\mathbf{E}_1^j(0)\rangle\rangle_c\}(\boldsymbol{\nabla}_r^jV^{(0)})\right)\\
&-\frac{i}{6}\lim_{T\to\infty}\int_0^T dt\,t^4\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_2^j(0)\rangle\rangle_c(\boldsymbol{\nabla}_r^iV^{(0)})(\boldsymbol{\nabla}_r^jV^{(0)})+(d_{ss}+d_{vs}C_f)\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2)
\end{aligned}\tag{53}$$

$$V_{S^2}^{(1,1)}(r)=\frac{2c_F^{(1)}c_F^{(2)}}{3}i\lim_{T\to\infty}\int_0^T dt\,\langle\langle g\mathbf{B}_1(t)\cdot g\mathbf{B}_2(0)\rangle\rangle-4(d_{sv}+d_{vv}C_f)\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2). \tag{54}$$

Similar considerations also apply to the results in terms of states of Sec. III.

VI. POWER COUNTING

The standard power counting of NRQCD (organized in powers of v and α_s) used to assess the relative importance of the different matrix elements, as discussed, for instance, in [34], can only be proved in the perturbative regime. Even in this regime, owing to the different dynamical scales still involved, the matrix elements of NRQCD do not have a unique power counting in v. In the nonperturbative regime the problem of the power counting of NRQCD is still open. In principle, it is possible that a different power counting may be appropriate in this situation and this would influence, for instance, the studies of the charmonium system or of higher bottomonium states.[3] We believe that our result, through the connection between NRQCD and the quantum-mechanical picture, will eventually help to better understand the hierarchy of the different matrix elements in NRQCD, as well as to get a much deeper understanding of the underlying dynamics. This is due to the fact that, by going to a NR quantum-mechanical formulation, we have made the dynamics of the heavy quarks explicit, transferring the problem of the power counting of NRQCD into the problem of obtaining the power counting of the different potentials in pNRQCD. These may be expressed in terms of Wilson loops where only gluons and light quarks appear as dynamical entities and for which there are or there will be direct lattice measurements. Moreover, it is in this formulation that statements such as the virial theorem have a more rigorous, gauge-independent meaning.

Here, we only say a few words about the expected behavior of the potentials using arguments of naturalness on the scale mv, i.e., assuming that the potentials scale with mv. We first consider $V^{(0)}$. In principle, $V^{(0)}$ counts as mv, but, by definition, the kinetic energy counts as mv^2. Therefore, the virial theorem constrains $V^{(0)}$ also to count as mv^2. The extra $O(v)$ suppression has to come on dynamical grounds. In the perturbative case, it originates from the factor $\alpha_s\sim v$ in the potential. In the nonperturbative case little can be said and other mechanisms must be responsible. Using naturalness, $V^{(1,0)}/m$ scales like mv^2. Therefore, it could in principle be as large as $V^{(0)}$. This makes a lattice calculation or a model evaluation of this potential urgent. Perturbatively, owing to the factor α_s^2, it is of $O(mv^4)$. For what concerns the $1/m^2$ potentials, the naturalness argument suggests that they are of $O(mv^3)$. However, also here several constraints apply. Terms involving $\nabla V^{(0)}\sim m^2v^3$ are suppressed by an extra factor v, due to the virial theorem. The Gromes relation [10,36]

$$\frac{1}{2r}\frac{dV^{(0)}}{dr}+V_{LS}^{(2,0)}-V_{L_2S_1}^{(1,1)}=0 \tag{55}$$

suppresses by an extra factor v the combination $V_{LS}^{(2,0)}-V_{L_2S_1}^{(1,1)}$. Similar constraints also exist for the spin-independent potentials [12]. Perturbatively the $1/m^2$ potentials count at most as $O(mv^4)$, because of the extra α_s suppression. Finally, it is important to consider that some of the potentials are $O(\alpha_s)$ suppressed because of the matching coefficients inherited from NRQCD. This is, for instance, the case of the terms coming from the $1/m^2$ corrections to the purely gluonic sector of the NRQCD Lagrangian or of the terms coming from the four-fermion sector.

Terms involving two field-strength insertions in the static Wilson loop are known from lattice measurements [14] and have been studied in some QCD vacuum models [16]. For them a parametrization is possible and some supplementary information can be extracted. However, terms involving more than two field insertions in the static Wilson loop have not been studied so far, to our knowledge, by lattice simulations or within models. Consistency with the experimental data will further constrain any possible power-counting rule. In any case, a detailed study of the potentials using the above information (as well as new lattice or model-dependent results) should be performed in order to obtain the size (and thus the power-counting rules) of the different potentials for the charmonium and bottomonium systems.

VII. CONCLUSIONS AND OUTLOOK

A new formalism with which to obtain the QCD potential at arbitrary orders in $1/m$ has been explained in detail. We have obtained expressions for the energies of the gluonic excitations between heavy quarks valid beyond perturbation theory at $O(1/m^2)$. In particular, for the heavy quarkonium, we have also obtained the complete spin-dependent and spin-independent potentials at $O(1/m^2)$ for pure gluodynamics in terms of Wilson loops. For the spin-dependent piece our results correct the expressions given in [8,10,12]. For the spin-independent potentials, we agree with the momentum-dependent potentials obtained in [12], but not for the momentum-independent terms, where new contributions are found. We have also briefly discussed the power counting in the nonperturbative regime.

We conclude, commenting on two possible developments

[3] A different, nonstandard, power counting of the matrix elements of NRQCD may explain the apparent difficulties that NRQCD is facing to explain the polarization of prompt J/ψ data and to accurately determine the different matrix elements (see [35]).

of the present work. First, it is worthwhile to explore the possibility of expressing the potentials associated with higher gluonic excitations in terms of Wilson loop operators as done here for the heavy quarkonium ground state. The corresponding quantum-mechanical expressions are given in Eqs. (20)–(23). Second, our results are complete at $O(1/m^2)$ in the case of pure gluodynamics. If we want to incorporate light fermions, the procedure to be followed is analogous and our results still remain valid (considering now matrix elements and Wilson loops with dynamical light fermions incorporated), except for new terms appearing in the energies at $O(1/m^2)$ due to operators involving light fermions that appear in the NRQCD Lagrangian at $O(1/m^2)$ [24]. They may be incorporated along the same lines as the terms discussed here and will be explicitly worked out elsewhere.

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APPENDIX A: THE PARAMETRIZATION OF [12]

In this appendix, for ease of comparison, we write the spin-independent potentials in the parameterization given in [12]. They read

$$V_{SI}^{(2,0)}=\frac{1}{2}\left\{\mathbf{p}_1^i\mathbf{p}_1^j,\delta^{ij}V_d(r)+\left(\frac{\delta^{ij}}{3}-\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\right)V_e(r)\right\}+\frac{1}{8}(\boldsymbol{\nabla}_r^2[V^{(0)}+V_a]), \tag{A1}$$

$$V_{SI}^{(1,1)}=\frac{1}{2}\left\{\mathbf{p}_1^i\mathbf{p}_2^j,\delta^{ij}V_b(r)+\left(\frac{\delta^{ij}}{3}-\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\right)V_c(r)\right\}+V_f(r). \tag{A2}$$

Let us note that in [12] the $1/m_1m_2$ potential contained only momentum-dependent pieces. Therefore, the momentum-independent potential, which we name V_f, was missing. Our calculation also substantially modifies the result for V_a given in [12]. The above potentials read

$$V_d=\frac{i}{6}\lim_{T\to\infty}\int_0^T dt\,t^2\langle\langle g\mathbf{E}_1(t)\cdot g\mathbf{E}_1(0)\rangle\rangle_c, \tag{A3}$$

$$\left(\frac{\delta^{ij}}{3}-\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\right)V_e=\frac{i}{2}\lim_{T\to\infty}\int_0^T dt\,t^2\left\{\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_1^j(0)\rangle\rangle_c-\frac{\delta^{ij}}{3}\langle\langle g\mathbf{E}_1(t)\cdot g\mathbf{E}_1(0)\rangle\rangle_c\right\}, \tag{A4}$$

$$\frac{1}{8}(\boldsymbol{\nabla}_r^2[V^{(0)}+V_a])=V_r^{(2,0)}(r)-\frac{1}{2}(\boldsymbol{\nabla}_r^2V_{\mathbf{p}^2}^{(2,0)})+\frac{i}{4}\left(\nabla_r^i\nabla_r^j\lim_{T\to\infty}\int_0^T dt\,t^2\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_1^j(0)\rangle\rangle_c\right), \tag{A5}$$

$$V_b=-\frac{i}{3}\lim_{T\to\infty}\int_0^T dt\,t^2\langle\langle g\mathbf{E}_1(t)\cdot g\mathbf{E}_2(0)\rangle\rangle_c, \tag{A6}$$

$$\left(\frac{\delta^{ij}}{3}-\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\right)V_c=-i\lim_{T\to\infty}\int_0^T dt\,t^2\left\{\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_2^j(0)\rangle\rangle_c-\frac{\delta^{ij}}{3}\langle\langle g\mathbf{E}_1(t)\cdot g\mathbf{E}_2(0)\rangle\rangle_c\right\}, \tag{A7}$$

$$V_f(r)=V_r^{(1,1)}(r)+\frac{1}{2}(\boldsymbol{\nabla}_r^2V_{\mathbf{p}^2}^{(1,1)})+\frac{i}{2}\left(\nabla_r^i\nabla_r^j\lim_{T\to\infty}\int_0^T dt\,t^2\langle\langle g\mathbf{E}_1^i(t)g\mathbf{E}_2^j(0)\rangle\rangle_c\right). \tag{A8}$$

APPENDIX B: COMPARISON WITH THE EICHTEN-FEINBERG SPIN-ORBIT POTENTIAL

In order to compare our results with the Eichten-Feinberg ones properly, we set $c_F^{(1)}=c_F^{(2)}=1$. Then, our Eq. (48) reads

$$V_{L_2S_1}^{(1,1)}(r)=-\frac{i}{r^2}\mathbf{r}\cdot\lim_{T\to\infty}\int_0^T dt\,t\,\langle\langle g\mathbf{B}_1(t)\times g\mathbf{E}_2(0)\rangle\rangle$$
$$=\frac{i}{r^2}\mathbf{r}\cdot\sum_{k\neq0}\frac{{}^{(0)}\langle 0|g\mathbf{B}_1|k\rangle^{(0)}\times{}^{(0)}\langle k|g\mathbf{E}_2^T|0\rangle^{(0)}}{(E_0^{(0)}-E_k^{(0)})^2}$$
$$\underset{\text{pert}}{=}\frac{C_f\alpha_s}{r^3}+O(\alpha_s^2). \tag{B1}$$

On the other hand, Eichten and Feinberg obtain (we actually use the expression in Minkowski space given in Ref. [12])

$$
\begin{aligned}
V^{(1,1)}_{L_2S_1}(r) &= \frac{i}{2r^2}\lim_{T_W\to\infty}\frac{1}{T_W}\int_{-T_W/2}^{T_W/2}dt\int_{-T_W/2}^{T_W/2}dt'\,t'\mathbf{r}\cdot\langle\!\langle g\mathbf{B}_1(t)\times g\mathbf{E}_2(t')\rangle\!\rangle \\
&= \frac{i}{2r^2}\mathbf{r}\cdot\sum_{m\neq 0}\sum_{k\neq 0,m}\frac{a_0a_m}{a_0^2}\frac{{}^{(0)}\langle 0|g\mathbf{B}_1|k\rangle^{(0)}\times{}^{(0)}\langle k|g\mathbf{E}_2^T|m\rangle^{(0)}}{(E_0^{(0)}-E_k^{(0)})(E_0^{(0)}-E_m^{(0)})} \\
&\quad -\frac{i}{2r^2}\mathbf{r}\cdot\sum_{m\neq 0}\sum_{k\neq 0,m}\frac{a_0a_m}{a_0^2}\frac{{}^{(0)}\langle m|g\mathbf{B}_1|k\rangle^{(0)}\times{}^{(0)}\langle k|g\mathbf{E}_2^T|0\rangle^{(0)}}{(E_0^{(0)}-E_k^{(0)})(E_0^{(0)}-E_m^{(0)})}+\frac{i}{2r^2}\mathbf{r}\cdot\sum_{k\neq 0}\frac{{}^{(0)}\langle 0|g\mathbf{B}_1|k\rangle^{(0)}\times{}^{(0)}\langle k|g\mathbf{E}_2^T|0\rangle^{(0)}}{(E_0^{(0)}-E_k^{(0)})^2} \\
&\underset{\text{pert}}{=}\frac{C_f\alpha_s}{2r^3}+O(\alpha_s^2),
\end{aligned}
\tag{B2}
$$

where the $a_n(\mathbf{x}_1,\mathbf{x}_2)$ are defined by

$$
\psi^\dagger(\mathbf{x}_1)\phi(\mathbf{x}_1,\mathbf{x}_2)\chi(\mathbf{x}_2)|\text{vac}\rangle=\sum_n a_n(\mathbf{x}_1,\mathbf{x}_2)|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)},
$$

here

$$
\phi(\mathbf{y},\mathbf{x})\equiv \mathrm{P}\,\exp\left\{ig\int_0^1 ds\,(\mathbf{y}-\mathbf{x})\cdot\mathbf{A}[\mathbf{x}-s(\mathbf{x}-\mathbf{y})]\right\}
\tag{B3}
$$

being the end-point string used in the Wilson loop operators. Note that we have fixed $T_W=T$ in Eq. (B2), as corresponds to the procedure followed by Eichten and Feinberg.

The above calculation makes manifest the disagreement of Eq. (B1) with Eq. (B2) both at the perturbative level as well as in the representation in terms of intermediate states. A possible source of disagreement may be traced back in the original paper of Eichten and Feinberg [8] to their Eq. (4.9b), which seems to be incorrect. Finally, the reason for this last error seems to be the improper treatment of the Wilson loops in the large-time limit.

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