Radiative corrections of $O(\alpha)$ for pion beta decay in the light-front quark model

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If the CKM matrix element V_{ud} that can be derived from superallowed nuclear decays, neutron decay and pion beta decay is used for a precision test of the unitarity of the CKM matrix, the combination of the present world data seems to indicate a small violation of the unitarity condition for the first row. While an accurate calculation of the radiative corrections (RC) of $O(\alpha)$ is crucial in order to determine the value of V_{ud} as precisely as possible, the theoretical analysis has been limited in the past by the rather crude estimate of the effect of the hadronic structure. Only the contribution due to the axial vector current depends on the hadronic environment. We develop a strategy to deal with the influence of the hadronic structure on the decay properties of the simplest hadron, the pion, and calculate the contribution of the axial vector current to the RC, using a light-front model for the pion. Its $q\bar{q}$ bound state structure is well described by two parameters, the constituent quark mass and confinement scale, that have been fixed by a comparison with the data. We take into consideration three different groups of two-loop diagrams, and derive their light-front representations. We discuss the associated zero-mode problem and show that the respective light-front amplitudes are free of spurious contributions. There is only a small model dependent uncertainty of the final result for the RC for pion beta decay.

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I. INTRODUCTION

For three fermion generations, unitarity of the Cabibbo-Kobayashi-Maskawa (CKM) matrix requires the sum of the squared moduli of the first three elements to be equal to one:

$$V^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1. \tag{1.1}$$

A test of this property is of crucial importance since a violation of unitarity would be evidence for new physics, and the use of such a result to constrain possible extensions of the standard model would require a precise value of V^2 and its uncertainty.

The unitarity sum V^2 critically depends upon the precise value of the matrix element V_{ud} for the decays $u \to d\bar{e}\nu$ and $d \to ue\bar{\nu}$. These quark level transitions give rise to superallowed Fermi beta decays, the decay of the free neutron $n \to pe\bar{\nu}$ and pion beta decay $\pi^+ \to \pi^0 \bar{e}\nu$. In each case the measured rate can be used to determine the value of V_{ud}, after radiative corrections (RC) and the effect of the hadronic environment have been separated out.

A general formula for the RC of order α to the transition rates has been given by Sirlin [1]. The total decay rate $1/\tau$ can be separated into the uncorrected expression, denoted by $1/\tau_0$, and an overall factor as

$$1/\tau = 1/\tau_0(1+\delta)$$

$$\delta = \frac{\alpha}{2\pi}\left[g(E_0) + 3\ln\frac{M_Z}{M_p} + A_g\right] + \frac{\alpha}{2\pi}\left[3(Q_u+Q_d)\ln\frac{M_Z}{M_A} + 2C\right], \tag{1.2}$$

where Q_u and Q_d are the quark charges of u and d quarks. The Sirlin function $g(E,E_0)$ has been defined in [2] as a function of the electron or positron energy E and represents the RC to the electron or positron spectrum in allowed beta decay. In the total decay rate $1/\tau$ it is replaced by the averaged value $g(E_0)$; E_0 is the end-point energy of the spectrum.

The correction terms of $O(\alpha)$ consist of three distinct parts. The first two terms $g(E_0) + 3\ln(M_Z/M_p)$ represent the contribution of the vector current and are independent of hadron dynamics. The Z boson mass M_Z is a consequence of short-distance effects while the proton mass M_p cancels in the sum of the two terms. The third term A_g is a small asymptotic QCD correction term: $A_g = -0.34$ [1,3]. Finally there is a contribution $\ln(M_Z/M_A) + 2C$ induced by the axial vector current, where the logarithm is again the result of short-distance effects, with M_A acting as an effective low-energy cutoff (presumably roughly equal to the a_1 meson mass), and $2C$ stands for the remaining low-energy part.

The value of M_A is uncertain; Marciano and Sirlin [3] suggested a range 400 MeV$\leqslant M_A \leqslant$1600 MeV, while Sirlin [4] proposed an even wider range

$$M_{a1}/2 \leqslant M_A \leqslant 2M_{a1}, \tag{1.3}$$

with the central value at the a_1 meson mass $M_{a1} = 1.26$ GeV. The quantity $2C$ is model dependent and has been calculated in the Born approximation in Refs. [3,5] using nucleon electromagnetic and axial form factors. For pion beta decay $C = 0$ in the Born approximation since the axial vector current does not couple to a pseudoscalar meson. The resulting values for C are

$$C = C_{Born} = \begin{cases} 0.885 & \text{(superallowed and neutron decays)}, \\ 0 & \text{(pion beta decay)}. \end{cases} \tag{1.4}$$

For superallowed beta decays there are additional nuclear structure dependent contributions to C which have been proposed and discussed in Ref. [6].

The uncorrected decay rate $1/\tau_0$, defined by Eq. (1.2), still incorporates Coulomb corrections and Z-dependent radiative corrections of $O(Z\alpha^2)$ and $O(Z^2\alpha^3)$ for superallowed nuclear decays, and depends on hadronic form factors, which encode the effect of the quark structure of the decaying hadron.

Recently the current status of V_{ud} has been reviewed by Towner and Hardy [7], based upon the current world data for the three decay modes indicated above. To date, nine superallowed $0^+ \to 0^+$ transitions have been measured to $\pm 0.1\%$ precision or better, and the result for V_{ud} obtained from the average ft value is

$$|V_{ud}| = 0.9740 \pm 0.0005. \tag{1.5}$$

From this value of V_{ud} the unitarity sum, Eq. (1.1), becomes

$$V^2 = 0.9968 \pm 0.0014, \tag{1.6}$$

where the 1998 Particle Data Group (PDG 98) [8] recommendations for V_{us} and V_{ub} have been used in Ref. [7]. The value for the CKM matrix element V_{us} determined from an analysis of kaon and hyperon decays is $|V_{us}| = 0.2196 \pm 0.0023$, while the value for V_{ub} is $|V_{ub}| = 0.0032 \pm 0.0008$ and does not affect the unitarity sum at its present level of accuracy.

According to the analysis of Towner and Hardy, the error bar associated with the value of V_{ud} is caused mainly by the uncertainty in the RC (± 0.0004) due to the prescription (1.3) for the effective low-energy cutoff and the uncertainty in the nuclear isospin symmetry-breaking correction (± 0.0003), while the average experimental uncertainty is quite small (± 0.0001).

The problems associated with a precise treatment of nuclear structure effects can be avoided if the beta decay of free hadrons is considered instead. A survey of world data on neutron decay observables has been presented in Ref. [7] and it has been noted that the derivation of the value of V_{ud} from n decay is limited largely by the uncertainty in the overall average value of $\lambda = g_A/g_V$. However, there is a new result for the beta asymmetry obtained by the PERKEO II Collaboration [9] which leads to the value $|\lambda| = 1.2735 \pm 0.0021$. This single value, combined with the world average for the neutron lifetime, leads to the following value for V_{ud} [7]:

$$|V_{ud}| = 0.9714 \pm 0.0015. \tag{1.7}$$

The unitarity sum is then

$$V^2 = 0.9919 \pm 0.0030. \tag{1.8}$$

The error given in Eq. (1.7) is three times larger than the error in Eq. (1.5) and is dominated by the uncertainty in the measurement of the beta asymmetry but, as in the analysis of the superallowed decays, still contains the contribution of the uncertainty in the RC.

The results for V_{ud} and the unitarity sum V^2 given in Eqs. (1.5)–(1.8) are consistent with each other and seem to indicate a substantial violation of the unitarity condition (1.1) for three generations. Moreover, they support the conclusion reached in Ref. [7], that the treatment of the effect of the nuclear environment in superallowed nuclear decays is reliable, with only a small error, and that there is no evidence that the unitarity problem can be solved by improvements in the calculation of nuclear structure effects.

In order to obtain more information on the unitarity problem accurate measurements of the pion beta decay observables would be of great importance. Like the superallowed nuclear decays pion beta decay is a pure vector transition and the matrix element of the axial vector current, which complicates the analysis of neutron decay, does not contribute to the lowest order amplitude. In higher orders both the vector and the axial vector parts of the weak current contribute. The expression for the radiative corrections of $O(\alpha)$ is given in Eq. (1.2). Moreover, since the decaying pion is free, the nuclear structure dependent corrections that complicate nuclear beta decay are absent. Based on the lifetime [8]

$$\tau_{exp} = (2.6033 \pm 0.0005) \times 10^{-8}\ \text{s} \tag{1.9}$$

and the branching ratio [10]

$$BR = (1.025 \pm 0.034) \times 10^{-8}, \tag{1.10}$$

the value of V_{ud} was determined in Ref. [7] to be

$$|V_{ud}| = 0.9670 \pm 0.0161 \tag{1.11}$$

and the unitarity sum

$$V^2 = 0.9833 \pm 0.0311. \tag{1.12}$$

The price to pay for the advantage of a simple theoretical analysis of pion beta decay is a large error in V_{ud} due to the considerable experimental difficulty in measuring the π branching ratio with a precision comparable to the one obtained in superallowed beta decays. However, there is a proposal for an experiment at PSI [11] with the aim of making a precise determination of the pion beta decay rate. In the first phase of the experiment it is intended to measure the branching ratio with an accuracy of 0.5%. The proposed experimental method was designed to finally achieve an overall level of uncertainty in the range of $0.2 - 0.3\,\%$.

The decay rate for pion beta decay including the RC of order α is given by Eq. (1.2), where an approximate expression for the uncorrected decay rate has been derived long ago by Källén [12]:

$$1/\tau_0 = \frac{G_F^2 |V_{ud}|^2}{30\pi^3}\left(1 - \frac{\Delta}{2M_+}\right)\Delta^5 f(\epsilon,\Delta), \tag{1.13}$$

$$f(\epsilon,\Delta) = \sqrt{1-\epsilon}\left[1 - \frac{9\epsilon}{2} - 4\epsilon^2 + \frac{15}{2}\epsilon^2 \ln\left(\frac{1+\sqrt{1-\epsilon}}{\sqrt{\epsilon}}\right) - \frac{3}{7}\frac{\Delta^2}{(M_+ + M_0)^2}\right], \tag{1.14}$$

with $\epsilon = m_e^2/\Delta^2$ and $\Delta = M_+ - M_0$, where M_+ and M_0 are the masses of π^+ and π^0; G_F is the Fermi coupling constant. Equation (1.14) includes the leading correction in an expansion in powers of $\Delta^2/(M_+ + M_0)^2$ [1]. The effect of the quark structure has been neglected entirely, and in Sec. II we shall study the error made by this approximation. In particular, we shall investigate the effect of isospin violation due to the quark mass difference $m_d - m_u$, in order to make sure that isospin breaking effects do not produce unexpectedly large contributions.

For a precision test of the unitarity of the CKM matrix, i.e. of the standard model, an accurate calculation of the RC, in particular a reliable determination of the effect of the hadronic structure, is crucial. The terms in the electromagnetic radiative corrections of $O(\alpha)$ that are generated by the vector current [the first two terms in Eq. (1.2)] are firmly founded on a current algebra formulation and the details of the underlying quark structure are of only minor importance. We shall not further consider that part of the RC of $O(\alpha)$ which is induced by the vector current. While the short-distance contribution of the axial vector current is well established too, its role at low energies strongly depends upon the detailed quark structure of the decaying hadron and its influence on the decay properties has been estimated only very roughly in terms of an effective low-energy cutoff M_A and the quantity C. We do not know of any published work that attempts to obtain the contribution of the axial vector current using a model of hadronic structure. However, it is evident that a reliable interpretation of the experimental data and a conclusive analysis of the unitarity problem necessarily requires a more refined treatment of the effect of the quark structure in order to substantially reduce the theoretical uncertainties and to firmly establish the size of the hadronic corrections.

In this paper we shall calculate the axial vector contribution to the RC in the case of pion beta decay in the framework of the light-front quark model (LFQM), which is a relativistic constituent quark model based on the light-front formalism [13]. The LFQM provides a conceptually simple, phenomenological method for the determination of hadronic form factors and coupling constants, and has become a much used tool for investigating various electroweak properties of light and heavy mesons (see e.g. [14,15] and references therein). In Ref. [16] we have presented a covariant extension of the LFQM which permits the calculation of all the form factors that are necessary to represent the Lorentz structure of a hadronic matrix element. In this approach a meson is composed of valence quarks with constituent quark masses and the structure of the bound $q\bar{q}$ meson state is approximated by a covariant model vertex function, which depends on a parameter $1/\beta$ which essentially determines the confinement scale, i.e. The size of the composite meson. Form factors are given in the one-loop approximation as light-front momentum integrals. As an example, it was shown in Ref. [16] that a prediction of the electromagnetic form factor of the pion for small values of the momentum transfer can be made that is in good agreement with the data.

The simple structure of the $q\bar{q}$ bound state should allow definite conclusions about the relative importance of the hadronic environment in a calculation of the RC. Radiative corrections of order α to the form factors that describe pion beta decay arise from the virtual exchange of Z, γ or W and are represented by two-loop diagrams. We shall extend the approach of Ref. [16] and derive unique LFQM expressions (that are free of spurious contributions) for the two-loop diagrams associated with the axial vector current, and derive in this way the effect of the hadronic structure on the $O(\alpha)$ corrections for pion beta decay. This determination of the effect of the hadronic environment by means of a two-loop calculation should be just as reliable as the one-loop calculation of the electromagnetic form factor of the pion.

We shall show in this work that the uncertainty of the hadronic corrections due to the particular quark structure of the pion is small for pion beta decay. This result is in contrast to the situation for superallowed nuclear decays and neutron decay where the large value of C, Eq. (1.4), signals a much greater importance of the detailed quark structure with all its model dependent uncertainty. We shall analyze superallowed nuclear decays and neutron decay in a future work in a similar manner as for pion beta decay. But even without knowing the result of such an investigation it is clear that pion beta decay, once precise data are available, will always have a unique position due to the simple quark structure of the pion which generates hadronic corrections with very small uncertainties.

In Sec. II we present the general formalism for pion beta decay without radiative corrections, which is analyzed in terms of two form factors that describe the quark structure of the pion. We investigate the effect on the decay rate of both the isospin violation due to the quark mass difference and the momentum transfer dependence of the form factors. In Sec. III the detailed calculation of the RC due to the axial vector current is presented. We consider three different groups of two-loop diagrams, and derive their light-front representations. We discuss the associated zero-mode problem in the Appendix and show that the respective light-front amplitudes are unique, i.e. free of spurious contributions. The Appendix contains also a general discussion of the covariance properties of one- and two-loop light-front integrals. We approximate higher order gluon exchange effects by means of ρ exchange diagrams, which are shown to be of only minor importance if appropriate off-shell form factors are used. Section IV contains our result for the RC for pion beta decay.

II. GENERAL FORMALISM FOR PION BETA DECAY WITHOUT RADIATIVE CORRECTIONS

The amplitude without radiative corrections for the decay $\pi^+ \to \pi^0 \bar{e} \nu$ is given by

$$T_1 = \frac{G_F}{\sqrt{2}} V_{ud} \langle \pi^0(P'') | \bar{d} \gamma_\mu (1-\gamma_5) u | \pi^+(P') \rangle L^\mu \tag{2.1}$$

where the matrix element of the leptonic current is

$$L_\mu=\overline{u}_\nu(k_\nu)\gamma_\mu(1-\gamma_5)v_e(l) \tag{2.2}$$

and k_ν, l are the 4-momenta of the neutrino and the positron respectively. We represent the hadronic matrix element for pion beta decay in terms of appropriate form factors

$$\langle \pi^0(P'')|\overline{d}\gamma_\mu u|\pi^+(P')\rangle=\sqrt{2}\{(P'+P'')_\mu F_1(q^2)+q_\mu F_2(q^2)\} \tag{2.3}$$

$$\langle \pi^0(P'')|\overline{d}\gamma_\mu\gamma_5 u|\pi^+(P')\rangle=0 \tag{2.4}$$

where $q=P'-P''$ is the 4-momentum transfer which varies within the range $m_e^2\leqslant q^2\leqslant(M_+-M_0)^2$.

It is convenient to analyze semileptonic decays of pseudoscalar mesons in terms of the form factors $F_1(q^2)$ and $F_0(q^2)$, where the scalar form factor $F_0(q^2)$ is defined by

$$F_0(q^2)=F_1(q^2)+\frac{q^2}{M_+^2-M_0^2}F_2(q^2). \tag{2.5}$$

The differential partial width in terms of these form factors is then

$$\frac{d\Gamma_0(\pi^+\to\pi^0\overline{e}\nu)}{dq^2}=\frac{G_F^2|V_{ud}|^2M_+^3}{32\pi^3}\rho(q^2) \tag{2.6}$$

and the density $\rho(q^2)$ consists of spin 0 and spin 1 contributions as follows:

$$\rho(q^2)=\rho_0(q^2)+\rho_1(q^2) \tag{2.7}$$

$$\rho_0(q^2)=\frac{m_e^2}{q^2}(F_0(q^2))^2\left(1-\frac{m_e^2}{q^2}\right)^2\left(1-\frac{M_0^2}{M_+^2}\right)^2\frac{p_\pi}{M_+} \tag{2.8}$$

$$\rho_1(q^2)=\frac{8}{3}(F_1(q^2))^2\left(1-\frac{m_e^2}{q^2}\right)^2\left(1+\frac{m_e^2}{2q^2}\right)\left(\frac{p_\pi}{M_+}\right)^3 \tag{2.9}$$

where p_π is the recoil momentum of the π^0 in the π^+ rest frame:

$$p_\pi^2=\frac{1}{4M_+^2}\{(M_+^2-M_0^2)^2+q^4-2q^2(M_+^2+M_0^2)\}. \tag{2.10}$$

If the quark structure of the pion is neglected, i.e. in the limit $F_1(q^2)=F_0(q^2)=1$ and for the approximation $p_\pi^2\simeq(M_++M_0)^2(\Delta^2-q^2)/4M_+^2$, the integrated partial width leads to the approximate expression for the total decay rate $1/\tau_0$, Eq. (1.13).

In this section we shall briefly discuss the exact integrated partial width (2.6) based upon the formulas for the form factors $F_1(q^2)$ and $F_2(q^2)$, which we have derived in the framework of the quark model in Ref. [16].

The hadronic matrix element (2.3) is given in the one-loop approximation, corresponding to the diagrams of Fig. 1, as a light-front momentum integral, denoted by A_μ. The 4-momentum of a meson of mass M' in terms of light-front components is $P'=(P'^-,P'^+,P'_\perp)$, where the transverse vector is $P'_\perp=(P'^1,P'^2)$. Its constituent quarks have masses m'_1, m_2 and 4-momenta p'_1, p_2, respectively, and the total 4-momentum of the meson state is given by $p'_1+p_2=P'$, i.e. The quarks are in general off the mass-shell. The appropriate variables for the internal motion of the constituents, $(x,p'_\perp)$, are defined by

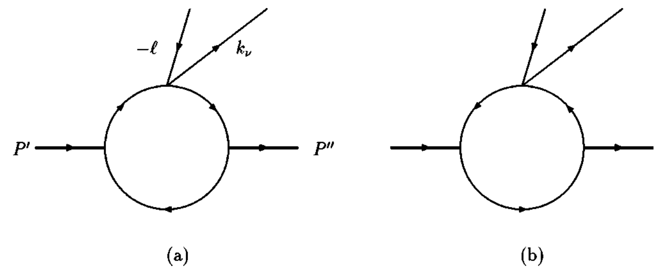

FIG. 1. The one-loop contributions to pion beta decay.

$$p_1'^+=xP'^+,\quad p_2^+=(1-x)P'^+$$

$$p'_{1\perp}=xP'_\perp+p'_\perp,\quad p_{2\perp}=(1-x)P'_\perp-p'_\perp$$

and the kinematic invariant mass is

$$M_0'^2=\frac{p_\perp'^2+m_1'^2}{x}+\frac{p_\perp'^2+m_2^2}{1-x}. \tag{2.11}$$

For the transition between an initial $\pi^+=u\overline{d}$ with 4-momentum P', mass M', and internal variables and masses of its constituent quarks $(x,p'_\perp,m'_1,m_2)$ and a final $\pi^0=(d\overline{d}-u\overline{u})/\sqrt{2}$ with 4-momentum P'', mass M'', and the corresponding internal quantities $(x,p''_\perp,m''_1,m_2)$, the momentum integral A_μ, in a Lorentz frame with $q^+=0$, consists of two parts that describe the $u\to d$ transition of Fig. 1(a) and the $\overline{d}\to\overline{u}$ transition of Fig. 1(b), and is given by

$$A_\mu=\frac{1}{\sqrt{2}}(H_\mu(m_u,m_d,m_d)+H_\mu(m_d,m_u,m_u)) \tag{2.12}$$

where

$$H_\mu(m'_1,m''_1,m_2)=\frac{N_c}{16\pi^3}\int_0^1 dx\int d^2p'_\perp\frac{h'_0h''_0}{(1-x)N'_1N''_1}S_\mu \tag{2.13}$$

with

$$S_\mu=\mathrm{tr}[\gamma_5(\not p''_1+m''_1)\gamma_\mu(\not p'_1+m'_1)\gamma_5(-\not p_2+m_2)], \tag{2.14}$$

where N_c is the number of colors, i.e. $N_c=3$. The light-front momentum integral H_μ, Eq. (2.13), is computed at the pole of the spectator quark:

$$N_2\equiv p_2^2-m_2^2=0. \tag{2.15}$$

In our formalism [16] four-momentum is conserved and the 4-vectors appearing in the trace (2.14) are then given by

$$p_2=\left(\frac{m_{2\perp}^2}{p_2^+},p_2^+,p_{2\perp}\right)$$

$$p_1'=P'-p_2$$

$$p_1''=p_1'-q, \tag{2.16}$$

where $m_{2\perp}^2=m_2^2+p_\perp'^2$. It follows from Eq. (2.16) that

$$N_1'\equiv p_1'^2-m_1'^2=x(M'^2-M_0'^2)$$

$$N_1''\equiv p_1''^2-m_1'^2=x(M''^2-M_1''^2)$$

$$M_0''^2=\frac{p_\perp''^2+(1-x)m_1''^2+xm_2^2}{x(1-x)}, \tag{2.17}$$

and $p_\perp''=p_\perp'-(1-x)q_\perp$. In our phenomenological approach we have chosen a pseudoscalar vertex operator for the $q\bar{q}$ pair bound in a S-state state, with the matrix structure of γ_5 and vertex functions h_0' and h_0'', where [16]

$$h_0'=h_0'(M_0')=\left[\frac{M_0'^4-(m_1'^2-m_2^2)^2}{4M_0'^3}\right]^{1/2}$$

$$\times\frac{M'^2-M_0'^2}{[M_0'^2-(m_1'-m_2)^2]^{1/2}}\phi(M_0'^2) \tag{2.18}$$

for the $q\bar{q}$ bound state of mass M', and a similar equation for h_0''. The orbital wave function is assumed to be a simple function of the kinematic invariant mass as

$$\phi(M_0'^2)=N'\exp(-M_0'^2/8\beta'^2), \tag{2.19}$$

where N' is the normalization constant and the parameter $1/\beta'$ determines the confinement scale. The normalization condition is obtained for $M'=M''$, $\beta'=\beta''=\beta$, $m_1'=m_1''=m_2=m$ and $q^2=0$, either as a relation for $H_\mu(m,m,m)$:

$$H_\mu(m,m,m)=(P'+P'')_\mu, \tag{2.20}$$

or as a relation for the orbital wave function:

$$\frac{N_c}{16\pi^3}\int_0^1 dx\int d^2p_\perp'\frac{M_0'}{2x(1-x)}|\phi(M_0'^2)|^2=1, \tag{2.21}$$

which for the equal mass case is given explicitly by

$$\phi(M_0'^2)=\pi^{-3/4}\beta^{-3/2}\left(\frac{8\pi^3}{3}\right)^{1/2}\exp(-(M_0'^2-4m^2)/8\beta^2). \tag{2.22}$$

While the form factor $F_1(q^2)$ in the one-loop approximation can be derived directly from the plus component of the momentum integral A_μ (2.12), the calculation of the form factor $F_2(q^2)$ requires an appropriate account of the effect of zero-modes, as we have shown in [16]. We shall not write down the formulas for the form factors, they can be found in Ref. [16], but quote the results of the numerical calculation. In the limit of exact isospin symmetry the quark masses and the pion masses are equal, i.e. $m_u=m_d=m$ and $M_+=M_0=M_\pi$, and the form factors can be predicted: $F_1(q^2)=F_\pi(q^2)$, where F_π is the charge form factor of the pion with $F_\pi(0)=1$, and $F_2(q^2)=0$. In our model the effect of isospin symmetry breaking is generated by a finite quark mass difference $\Delta m=m_d-m_u$, while the parameters for the wave functions of π^+ and π^0 are kept equal: $\beta_+=\beta_0=\beta_\pi$. We use the parameters which we have found to reproduce the properties of pions in very good agreement with the data in Ref. [16]:

$$m=(m_u+m_d)/2=260\ \text{MeV}$$

$$\beta_\pi=308.8\ \text{MeV}. \tag{2.23}$$

For the calculations of this section we assumed a mass difference $m_d-m_u=4$ MeV.

The momentum transfer in pion beta decay is small and the form factors can be approximated by monopole forms

$$F_i(q^2)=\frac{F_1(0)}{1-q^2/\Lambda_i^2},\quad i=0,1. \tag{2.24}$$

The explicit calculation gives $\Lambda_1=719$ MeV (the corresponding quantity for the charge form factor of the pion is $\Lambda_\pi=720$ MeV), while

$$F_1(0)\approx 1-\frac{\Delta m^2}{(902\ \text{MeV})^2}=1-2.0\times10^{-5}, \tag{2.25}$$

from which it is seen that the effect of symmetry breaking on $F_1(0)$ is of second order, in accordance with the Ademollo-Gatto theorem [17]. In contrast, $F_2(0)$ is of first order in the pion mass difference M_+-M_0 and takes the value

$$F_2(0)=-1.44\times10^{-3}, \tag{2.26}$$

which leads to the monopole approximation (2.24) for $F_0(q^2)$ with $\Lambda_0=1.123$ GeV.

The width Γ_0 can be obtained from Eq. (2.6) by a numerical integration over q^2 with the result

$$\Gamma_0(\pi^+\to\pi^0\bar{e}\nu)=1/\tau_0(1-1.2\times10^{-5}), \tag{2.27}$$

where $1/\tau_0$ is the approximate expression given by Eqs. (1.13) and (1.14). The correction is essentially due to the quark structure of the pion. Obviously, the effect of the sym-

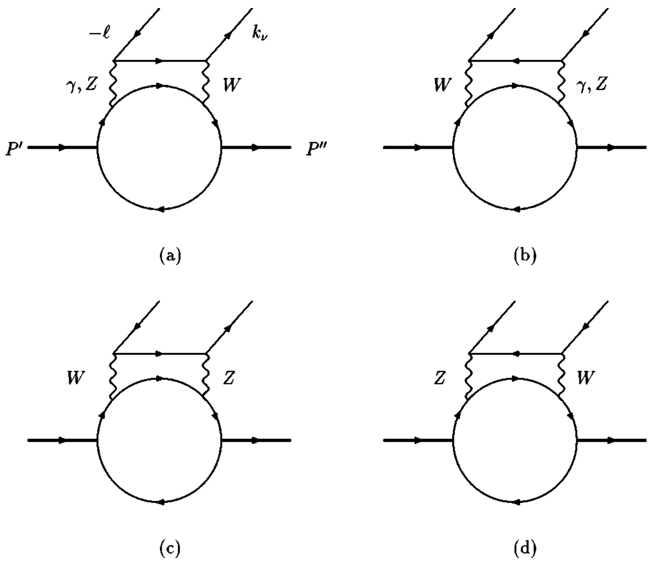

FIG. 2. Vertex corrections for pion beta decay.

metry breaking, Eq. (2.25), is largely compensated by the effect of the q^2-dependence of the form factors, and the sum of all structure dependent contributions to the transition probability, Eq. (2.27), is indeed very small, and can be safely neglected. We shall continue to analyze pion beta decay in the isospin symmetry limit $m_u = m_d = m$, with m given by Eq. (2.23).

III. THE RADIATIVE CORRECTIONS OF $O(\alpha)$ FROM THE AXIAL VECTOR CURRENT

The axial vector current essentially contributes to pion beta decay in $O(\alpha)$ only in the two-loop processes which are represented by the vertex correction diagrams of Fig. 2 and the exchange diagrams of Fig. 3. The amplitude corresponding to the photon-exchange diagrams of Fig. 2, involving the axial vector current, is given by

$$T_2^{(\gamma)} = \frac{G_F}{\sqrt{2}} V_{ud} \frac{\alpha}{4\pi^3} \times \int d^4k \, \frac{A_{\mu\lambda} L^{\mu\lambda}}{(k^2 + i\varepsilon)(k^2 - 2lk + i\varepsilon)} \, \frac{M_W^2}{M_W^2 - (q-k)^2 + i\varepsilon}. \tag{3.1}$$

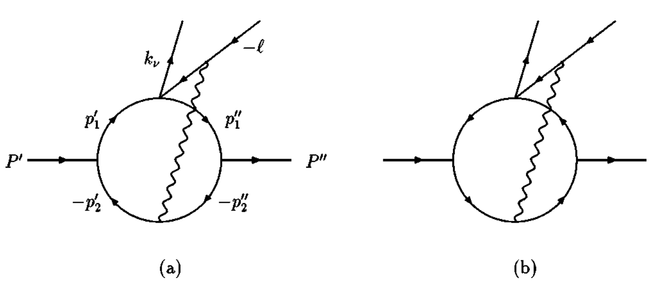

FIG. 3. Exchange corrections for pion beta decay.

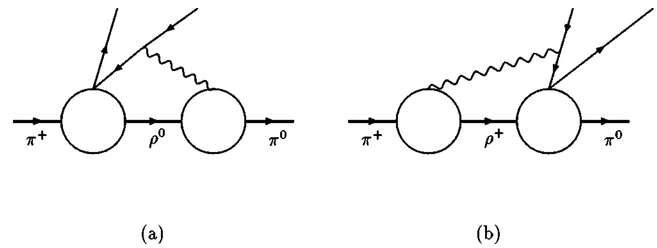

FIG. 4. ρ exchange corrections for pion beta decay.

The leptonic tensor is

$$\begin{aligned} L_{\mu\lambda} &= \bar{u}_\nu(k_\nu)\gamma_\mu(1-\gamma_5)(-\not{l} + \not{k} + m_e)\gamma_\lambda v_e(l) \\ &= -2 l_\lambda L_\mu + k_\lambda L_\mu + k_\mu L_\lambda - g_{\mu\lambda} kL + i\varepsilon_{\mu\lambda\alpha\beta} k^\alpha L^\beta \end{aligned} \tag{3.2}$$

and the leptonic current L_μ has been defined in Eq. (2.2). The hadronic tensor $A_{\mu\lambda}$ contains only the axial vector part of the weak current. Current algebra methods have been used in Ref. [18] to derive its asymptotic behavior, which leads to the following expression for $A_{\mu\lambda}$:

$$\begin{aligned} A_{\mu\lambda} = &-(Q_u + Q_d)\, i\varepsilon_{\mu\lambda\alpha\beta} k^\alpha \langle \pi^0(P'') | \bar{d}\gamma^\beta u | \pi^+(P') \rangle \\ &\times \frac{i}{k^2 - M_A^2} + O\left(\frac{1}{k^2}\right), \end{aligned} \tag{3.3}$$

where an arbitrary hadronic mass M_A is introduced to avoid a spurious infrared divergence in Eq. (3.1). The low-energy part of $A_{\mu\lambda}$ depends on the quark structure of the pion and is unknown.

If the result (3.3) is inserted into Eq. (3.1) for $T_2^{(\gamma)}$ and added to the corresponding Z-exchange contribution $T_2^{(Z)}$ of Fig. 2, one obtains the correction terms of $O(\alpha)$ in Eq. (1.2) that are induced by the axial vector current, where the unknown low-energy contribution is parametrized in terms of the constant M_A.

It is the main purpose of this work to evaluate those contributions of the vertex correction and exchange diagrams that come from the axial vector current, in the light-front quark model of Ref. [16]. The model calculation coincides with the result of Eq. (3.3) in the asymptotic limit and completes the current algebra approach by filling in the details that depend upon the quark structure of the pion.

In addition we shall estimate the contribution of higher order gluon exchange by means of the ρ exchange diagrams of Fig. 4.

A. The vertex corrections of Fig. 2

We shall calculate the contribution of the vertex correction diagrams of Fig. 2 by treating separately the vertex correction for an off-shell quark. In the limit $l = k_\nu = q = 0$ the matrix element for the exchange of a photon, that consists of a part that describes the $u \to d$ transitions of Figs. 2(a),(b) and an analogous part for the $\bar{d} \to \bar{u}$ transitions, is given by

$$T_2^{(\gamma)}=G_F V_{ud}\frac{N_c}{16\pi^3}\int_0^1 dx\int d^2p'_\perp \frac{h_0'^2}{(1-x)N_1'^2}\frac{\alpha}{4\pi}[Q_u\Pi^{(a)}+Q_d\Pi^{(b)}] \tag{3.4}$$

and the contribution of the vertex correction is

$$\Pi^{(a)}=\frac{i}{\pi^2}\int d^4k\frac{S^{(a)}_{\mu\lambda}L^{\mu\lambda}}{(k^2+i\varepsilon)^2(k^2-2p'_1k+N'_1+i\varepsilon)}\frac{M_W^2}{M_W^2-k^2+i\varepsilon}, \tag{3.5}$$

$$\Pi^{(b)}=\frac{i}{\pi^2}\int d^4k\frac{S^{(b)}_{\mu\lambda}L^{\mu\lambda}}{(k^2+i\varepsilon)^2(k^2+2p''_1k+N''_1+i\varepsilon)}\frac{M_W^2}{M_W^2-k^2+i\varepsilon}, \tag{3.6}$$

with

$$S^{(a)}_{\mu\lambda}=\mathrm{tr}[\gamma_5(\not p''_1+m''_1)(-\gamma_\mu\gamma_5)(\not p'_1-\not k+m)\gamma_\lambda(\not p'_1+m'_1)\gamma_5(-\not p_2+m_2)],$$

$$S^{(b)}_{\mu\lambda}=\mathrm{tr}[\gamma_5(\not p''_1+m''_1)\gamma_\lambda(\not p''_1+\not k+m)(-\gamma_\mu\gamma_5)(\not p'_1+m'_1)\gamma_5(-\not p_2+m_2)],$$

where $p''_1=p'_1$ and $N''_1=N'_1$, since $q=0$. The evaluation of the traces gives the result

$$S^{(a)}_{\mu\lambda}=-i\varepsilon_{\mu\lambda\alpha\beta}(4N'_1p_1'^{\alpha}p_2^{\beta}+k^{\alpha}S^{\beta}),$$

$$S^{(b)}_{\mu\lambda}=-i\varepsilon_{\mu\lambda\alpha\beta}(-4N'_1p_1'^{\alpha}p_2^{\beta}+k^{\alpha}S^{\beta}),$$

where S^β has been defined in Eq. (2.14). Only the term $i\varepsilon_{\mu\lambda\alpha\beta}k^\alpha L^\beta$ of the leptonic tensor $L_{\mu\lambda}$, Eq. (3.2), contributes to the momentum integrals $\Pi^{(a)}$ and $\Pi^{(b)}$, which can be written as

$$\Pi^{(a)}=\frac{2}{i\pi^2}\int d^4k\frac{k^2\cdot LS-kS\cdot kL+4N'_1(p'_1k\cdot P'L-p'_1L\cdot kP')}{(k^2+i\varepsilon)^2(k^2-2p'_1k+N'_1+i\varepsilon)}\frac{M_W^2}{M_W^2-k^2+i\varepsilon}, \tag{3.7}$$

$$\Pi^{(b)}=\frac{2}{i\pi^2}\int d^4k\frac{k^2\cdot LS-kS\cdot kL-4N'_1(p'_1k\cdot P'L-p'_1L\cdot kP')}{(k^2+i\varepsilon)^2(k^2+2p''_1k+N''_1+i\varepsilon)}\frac{M_W^2}{M_W^2-k^2+i\varepsilon}, \tag{3.8}$$

where we have used that $p_2=P'-p'_1$. The momentum integrals of Eqs. (3.7) and (3.8) can be calculated in terms of the usual space-time components by the standard Feynman parameter method. Using the detailed results that have been collected in Appendix A we find

$$\begin{aligned}\Pi^{(a)}=\Pi^{(b)}&\equiv\Pi\\ &=2\{(3b_1+b_2p_1'^2)LS-b_2p'_1S\cdot p'_1L\\ &\quad-4a_1N'_1(p_1'^2\cdot P'L-p'_1L\cdot p'_1P')\},\end{aligned} \tag{3.9}$$

where a_1, b_1 and b_2 are functions of $p_1'^2$ and are given by Eqs. (A3)–(A5).

The matrix element $T_2^{(\gamma)}$ for the exchange of a photon has been expressed in terms of the light-front momentum integral (3.4) which is to be computed at the pole of the spectator quark. However, it is well known (see e.g. Ref. [16] and references therein) that this straightforward light-front representation of a hadronic matrix element is in general incomplete and contains spurious contributions that violate Lorentz covariance. These difficulties are a consequence of the fact that the effect of the associated zero-modes is not included. (Examples of zero-mode contributions can be found in Appendix B, Eq. (B5), and in Ref. [16]. A more general discussion in the context of light-front quantization is given in Ref. [19].) This is the zero-mode problem which in the present case can be circumvented by the decomposition of the matrix element $T_2^{(\gamma)}$ into a covariant (physical) part, that is not associated with a zero-mode, and a spurious part that is canceled by the appropriate zero-mode contribution. We are only interested in the physical part of $T_2^{(\gamma)}$ that can be identified by choosing a special representation of the 4-vector L:

$$L=(L^-,0,0_\perp). \tag{3.10}$$

The method which we have developed in Ref. [16] can be used to show that the resulting expression for $T_2^{(\gamma)}$ is unique, i.e. the contribution of the associated zero-mode vanishes exactly.

We thus conclude that the condition (3.10) guarantees that all spurious contributions are eliminated and the momentum integral (3.4), calculated at the pole of the spectator quark, uniquely defines the complete light-front representation of the matrix element $T_2^{(\gamma)}$. In order to express the quantity Π, Eq. (3.9), in terms of light-front variables, we compute the following scalar products, using Eqs. (2.15)–(2.17),

$$p_1'L = x\,P'L,$$

$$LS = 4xM_0'^2\,P'L,$$

$$p_1'P' = \frac{1}{2}(M_\pi^2 + N_1'),$$

$$p_1'S = 2(p_1'^2\,M_\pi^2 + m^2M_\pi^2 - N_1'^2). \tag{3.11}$$

In this manner one finds Π as a function of $p_1'^2 = m^2 + N_1' = m^2 + x(M_\pi^2 - M_0'^2)$:

$$\Pi = 8P'L\left\{3b_1xM_0'^2 - (a_1+b_2)N_1'\left[p_1'^2 - \frac{x}{2}(N_1' + M_\pi^2)\right]\right\}. \tag{3.12}$$

Inserting Eq. (3.12) into Eq. (3.4) we can express the matrix element $T_2^{(\gamma)}$ in terms of T_1, Eq. (2.1), as

$$T_2^{(\gamma)} = \frac{1}{2}T_1\,\delta_{axial}^{(2\gamma)},$$

$$\delta_{axial}^{(2\gamma)} = \frac{\alpha}{2\pi}(Q_u+Q_d)\frac{N_c}{16\pi^3}\int_0^1 dx \times \int d^2p_\perp'\,\frac{h_0'^2}{(1-x)N_1'^2} \times \left\{6b_1xM_0'^2 - 2(a_1+b_2)N_1' \times \left(p_1'^2 - \frac{x}{2}(N_1'+M_\pi^2)\right)\right\}. \tag{3.13}$$

In arriving at Eq. (3.13) we have finally established the invariant LFQM expression for the matrix element $T_2^{(\gamma)}$ (see also Appendix C).

In order to complement the above remarks regarding the zero-mode problem, we note that the 3-dimensional light-front momentum integral (3.13) with pointlike $\pi q\bar{q}$ vertices (i.e. for $h_0 = \text{const}$), and the covariant 4-dimensional momentum integral that represents the photon-exchange Feynman diagrams of Fig. 2 (with the same pointlike $\pi q\bar{q}$ vertices) are equal, which is another proof that there are no zero-mode contributions.

We can reverse this argument and conclude that the transition from the covariant Feynman perturbation theory to the LFQM proceeds in two steps: In the first step the manifestly covariant 4-dimensional momentum integral, that corresponds to a given Feynman diagram, is represented exactly in terms of a 3-dimensional light-front momentum integral. In the second step appropriate phenomenological $\pi q\bar{q}$ vertex functions are introduced into the light-front representation.

For the amplitude of the Z-exchange diagrams of Fig. 2, that can be derived by an analogous analysis, one finds, that the quark structure contributes to the amplitude only to order m^2/M_W^2 and m^2/M_Z^2; this is a small effect which can be safely neglected. Therefore, the published result, that is due originally to Sirlin [1], remains essentially unchanged and is

$$T_2^{(Z)} = \tfrac{1}{2}\,T_1\,\delta_{axial}^{(2Z)}$$

$$\delta_{axial}^{(2Z)} = \frac{\alpha}{2\pi}(Q_u+Q_d)\left(3\ln\frac{M_Z}{M_W} + O\left(\frac{m^2}{M_W^2}, \frac{m^2}{M_Z^2}\right)\right), \tag{3.14}$$

where the minimal standard model relation $M_W = M_Z\cos\Theta_W$ (Θ_W is the weak angle) has been used (see e.g. Ref. [8]).

In the calculation of the vertex-loop integrals the value of the mass of the internal quark line is important only in the low photon momentum range, where the quark mass is essentially equal to the constituent mass. Therefore, for the numerical computation of Eq. (3.13), we take the values (2.23) for the parameters m and β_π, and obtain the result

$$\delta_{axial}^{(2\gamma)} = \frac{\alpha}{2\pi}(Q_u+Q_d)\left(3\ln\frac{M_W}{m} + \frac{9}{4} + \Delta + O\left(\frac{m^2}{M_W^2}\right)\right),$$

$$\Delta = -4.498, \tag{3.15}$$

where we have used the normalization condition (2.21).

The combined correction is

$$\delta_{axial}^{(2)} = \delta_{axial}^{(2\gamma)} + \delta_{axial}^{(2Z)} = \frac{\alpha}{2\pi}\left(3(Q_u+Q_d)\ln\frac{M_Z}{m} + \frac{3}{4} + \frac{\Delta}{3}\right). \tag{3.16}$$

B. The exchange corrections of Fig. 3

The exchange diagrams of Fig. 3 are different from the diagrams of Figs. 1 and 2 in that both quark lines are involved in the decay process, and there is no well defined spectator quark line. In order to establish the LFQM expression for the amplitude T_3 that corresponds to the exchange diagrams of Fig. 3 we shall use the analysis of the covariant two-loop diagram with pointlike $\pi q\bar{q}$ vertices (a Feynman diagram) presented in Appendix B, as a guide.

Just as in the case of the one-loop momentum integrals we can calculate the two-loop 4-momentum integrals, expressed in terms of light-front variables, by performing the integrals over the minus components of the loop momenta by contour methods, whereby the momentum integrals are given as residues of the respective quark poles. We have emphasized in the previous subsection that this straightforward procedure leads in general to an incomplete result since it misses the contribution of the zero-modes, and without including this effect the contour method not only violates Lorentz covariance, but is an uncertain approximation of the 4-momentum integrals. The zero-mode problem can be circumvented only if the special representation (3.10) is used. In Appendix B we prove that the contribution of the zero modes vanishes exactly for the resulting amplitude T_3. Therefore, the contour method already leads to the complete result for the momentum integrals, which are given as residues of the respective quark poles, and uniquely defines their light-front representation.

We shall calculate the amplitude T_3 in the limit $l=k_\nu=0$, $q=l+k_\nu=0$ and $P'=P''$, with $P'^2=M_\pi^2$. Only the contribution of the axial vector current will be considered. It consists of two parts, which depend upon the sign of the plus component of the photon momentum $k=p_1''-p_1'=p_2'-p_2''$. For $k^+>0$ the residue is determined by the poles of the quarks with momenta p_1' and p_2''. The remaining momenta are determined by 4-momentum conservation, i.e. $p_2'=P'-p_1'$ and $p_1''=P'-p_2''$. These conditions lead to the relations

$$k^+>0:\quad p_1'^2-m^2=0,\quad p_2''^2-m^2=0,$$
$$p_2'^2-m^2=(1-x')(M_\pi^2-M_0'^2),\ \ p_1''^2-m^2=x''(M_\pi^2-M_0''^2),$$
$$k^2=k_>^2=\kappa\{M_\pi^2-(1-x')M_0'^2-x''M_0''^2\}-k_\perp^2, \tag{3.17}$$

where $\kappa=k^+/P'^+=x''-x'$ and $0\leqslant\kappa\leqslant 1-x'$.

For $k^+<0$ the residue is determined by the poles of the quarks with momenta p_2' and p_1''. These conditions lead to the relations

$$k^+<0:\quad p_2'^2-m^2=0,\quad p_1''^2-m^2=0,$$
$$p_1'^2-m^2=x'(M_\pi^2-M_0'^2),\quad p_1''^2-m^2=(1-x'')(M_\pi^2-M_0''^2),$$
$$k^2=k_<^2=\kappa\{-M_\pi^2+x'M_0'^2+(1-x'')M_0''^2\}-k_\perp^2, \tag{3.18}$$

and κ varies within the range $-x'\leqslant\kappa\leqslant 0$. The resulting amplitude T_3 is then given by

$$T_3=G\,V_{ud}\frac{\alpha N_c}{128\pi^5}\int_0^1 dx'\int d^2p_\perp'\int_0^1 dx''$$
$$\times\int d^2p_\perp''\frac{h_0'\,h_0''}{x'(1-x')x''(1-x'')(M_\pi^2-M_0'^2)(M_\pi^2-M_0''^2)}$$
$$\times\left(\frac{\Theta(x''-x')}{k_>^4}+\frac{\Theta(x'-x'')}{k_<^4}\right)(Q_dS_{\mu\lambda}^{(a)}-Q_uS_{\mu\lambda}^{(b)})L^{\mu\lambda}. \tag{3.19}$$

The hadronic tensors associated with the diagrams of Figs. 3(a),(b) are

$$S_{\mu\lambda}^{(a)}=\mathrm{tr}[\gamma_5(\not p_1''+m)(-\gamma_\mu\gamma_5)(\not p_1'+m)\gamma_5(-\not p_2'+m)\gamma_\lambda(-\not p_2''+m)],$$
$$S_{\mu\lambda}^{(b)}=\mathrm{tr}[\gamma_5(\not p_2''+m)\gamma_\lambda(\not p_2'+m)\gamma_5(-\not p_1'+m)(-\gamma_\mu\gamma_5)(-\not p_1''+m)],$$
$$=-S_{\mu\lambda}^{(a)}. \tag{3.20}$$

The leptonic tensor $L_{\mu\lambda}$ has been defined in Eq. (3.2). The product of the hadronic and leptonic tensors consists of a scalar and a pseudoscalar part:

$$S_{\mu\lambda}^{(a)}L^{\mu\lambda}=i\varepsilon^{\mu\lambda\alpha\beta}k_\alpha L_\beta S_{\mu\lambda}^{(a)}+\text{pseudoscalar}, \tag{3.21}$$

where the pseudoscalar terms do not contribute to the amplitude T_3 and will be ignored in the following presentation. The scalar terms are given by

$$S_{\mu\lambda}^{(a)}L^{\mu\lambda}=8\,(p_1''p_2''+m^2)(p_1'k\cdot P'L-p_1'L\cdot P'k)-8\,(p_1'p_2'+m^2)(p_1''k\cdot P'L-p_1''L\cdot P'k). \tag{3.22}$$

Using the special representation (3.10) we find that

$$S_{\mu\lambda}^{(a)}L^{\mu\lambda}=8\,P'L\,(p_1''p_2''+m^2)(p_1'k-x'P'k)-8\,P'L\,(p_1'p_2'+m^2)(p_1''k-x''P'k). \tag{3.23}$$

This equation can be written such that its value at the various quark poles becomes obvious:

$$S^{(a)}_{\mu\lambda}L^{\mu\lambda}=2\,P'L\,(M_\pi^2-(p_1''^2-m^2)-(p_2''^2-m^2))((1-x')(p_1''^2-m^2)$$
$$-(1-x')(p_1'^2-m^2)+x'(p_2''^2-m^2)-x'(p_2'^2-m^2)-k^2)$$
$$+2\,P'L\,(M_\pi^2-(p_1'^2-m^2)-(p_2'^2-m^2))((1-x'')(p_1'^2-m^2)$$
$$-(1-x'')(p_1''^2-m^2)+x''(p_2'^2-m^2)-x''(p_2''^2-m^2)-k^2). \tag{3.24}$$

The contribution of $S^{(a)}_{\mu\lambda}L^{\mu\lambda}$ to the integrand of Eq. (3.19) for $k^+>0$ is, according to the conditions (3.17), given by

$$k^+>0:\quad S^{(a)}_{\mu\lambda}L^{\mu\lambda}=2\,P'L\cdot R(x',x''), \tag{3.25}$$

where

$$R(x',x'')=((1-x'')M_\pi^2+x''M_0''^2)(x''(1-x')(M_\pi^2-M_0''^2)-x'(1-x')(M_\pi^2-M_0'^2)-k_>^2)$$
$$+(x'M_\pi^2+(1-x')M_0'^2)(x''(1-x')(M_\pi^2-M_0'^2)-x''(1-x'')(M_\pi^2-M_0''^2)-k_>^2). \tag{3.26}$$

For $k^+<0$ we use the conditions (3.18) to find

$$k^+<0:\quad S^{(a)}_{\mu\lambda}L^{\mu\lambda}=2\,P'L\cdot R(1-x',1-x''). \tag{3.27}$$

The computation of the amplitude T_3 can be simplified by the observation that the integrals for $k^+>0$ and $k^+<0$ are equal, which can be shown by substituting x' for $1-x'$ and x'' for $1-x''$.

Inserting Eq. (3.25) into Eq. (3.19) we find the invariant LFQM expression for the matrix element T_3 (see also Appendix C):

$$T_3=\tfrac{1}{2}\,T_1\,\delta^{(3)}_{axial} \tag{3.28}$$

where

$$\delta^{(3)}_{axial}=(Q_u+Q_d)\,\frac{\alpha N_c}{16\pi^5}\int_0^1 dx'\int d^2p'_\perp\int_{x'}^1 dx''\int d^2p''_\perp\,\frac{h_0'\,h_0''\,R(x',x'')}{x'(1-x')x''(1-x'')(M_\pi^2-M_0'^2)(M_\pi^2-M_0''^2)k_>^4}. \tag{3.29}$$

For the numerical calculation of $\delta^{(3)}_{axial}$ we take again the values (2.23) for the parameters m and β_π and find the correction

$$\delta^{(3)}_{axial}=\frac{\alpha}{2\pi}0.256. \tag{3.30}$$

C. The ρ exchange corrections of Fig. 4

Besides the diagrams of Figs. 2 and 3 there is also the sum of all irreducible higher order gluon exchange diagrams. The effect of this contribution can be approximated by means of diagrams of the type drawn in Fig. 4, where appropriate meson states are exchanged between the weak axial vector and the electromagnetic vertices. The Born approximation (exchange of a pion), which in general is expected to give the dominant contribution, vanishes, since the pion does not couple to the axial vector current. We shall consider only the lowest mass exchange process, i.e. the ρ exchange diagrams of Fig. 4, and calculate the corresponding amplitude first for on-shell vertex structures, which are defined by the appropriate matrix elements.

The matrix element of the electromagnetic current for the transition $\rho^0\to\pi^0$ is

$$j_\lambda=Q_u\bar{u}\gamma_\lambda u+Q_d\bar{d}\gamma_\lambda d$$
$$\langle P'|j_\lambda|P'';1J_3\rangle=\epsilon^\nu\Gamma^{(\gamma)}_{\lambda\nu}=\epsilon^\nu\,(Q_u+Q_d)g(k^2)i\varepsilon_{\lambda\nu\alpha\beta}P^\alpha k^\beta. \tag{3.31}$$

The matrix element of the axial vector current for the transition $\pi^+\to\rho^0$ is

$$\langle P'';1J_3|-\bar{d}\gamma_\mu\gamma_5 u|P'\rangle=\sqrt{2}\,\epsilon^{*\nu}\Gamma^{(A)}_{\lambda\nu}$$
$$=\sqrt{2}\,\epsilon^{*\nu}\{-f(k^2)g_{\mu\nu}-a_+(k^2)P_\mu P_\nu+a_-(k^2)k_\mu P_\nu\}, \tag{3.32}$$

where $\epsilon=\epsilon(J_3)$ is the polarization vector of the ρ, $P=P'+P''$, and $k=P''-P'$.

The amplitude that corresponds to the diagram of Fig. 4(a) is then given in the limit $l=k_\nu=0$ by

$$T_{4a}=-G\,V_{ud}(Q_u+Q_d)\frac{ie^2}{(2\pi)^4}\int d^4k\frac{\Gamma^{(\gamma)}_{\lambda\nu}g^{\nu\rho}\Gamma^{(A)}_{\mu\rho}L^{\mu\lambda}}{(k^2+i\varepsilon)^2((P'+k)^2-M_\rho^2+i\varepsilon)}, \tag{3.33}$$

where we have used the bare ρ propagator

$$D_{\mu\nu}(P)=\left(g_{\mu\nu}-\frac{P_\mu P_\nu}{P^2}\right)\Delta_0(P),$$

$$\Delta_0^{-1}(P)=P^2-M_\rho^2+i\varepsilon, \tag{3.34}$$

and the leptonic tensor $L_{\mu\lambda}$ is given by Eq. (3.2) and $P'^2=M_\pi^2$. Using the hadronic tensors as defined by Eqs. (3.31) and (3.32) gives the result

$$T_{4a}=-G\,V_{ud}(Q_u+Q_d)\frac{ie^2}{(2\pi)^4}\int d^4k\,g(k^2)f(k^2)\frac{4(P'k\cdot kL-k^2\cdot P'L)}{(k^2+i\varepsilon)^2((P'+k)^2-M_\rho^2+i\varepsilon)}. \tag{3.35}$$

In the isospin symmetry limit the amplitudes corresponding to the diagrams of Fig. 4(a) and 4(b) are equal, and the total contribution is given by

$$T_4=T_{4a}+T_{4b}=2\,T_{4a}.$$

The amplitude T_4 depends only on the form factors $g(k^2)$ and $f(k^2)$, which we have determined in the framework of the light-front formalism in Ref. [16]. The results of [16] can be written as

$$g(k^2;P'^2,P''^2)=-\frac{N_c}{8\pi^3}\int_0^1dx\int d^2p'_\perp\frac{h_\pi(M'_0)h_\rho(M''_0)}{(1-x)x^2(P'^2-M_0'^2)(P''^2-M_0''^2)}\times\left\{m+\frac{2}{M''_0+2m}\left[p_\perp'^2+\frac{(p'_\perp k_\perp)^2}{k^2}\right]\right\}, \tag{3.36}$$

$$f(k^2;P'^2,P''^2)=\frac{N_c}{8\pi^3}\int_0^1dx\int d^2p'_\perp\frac{h_\pi(M'_0)h_\rho(M''_0)}{(1-x)x^2(P'^2-M_0'^2)(P''^2-M_0''^2)}\times\left\{-2xmM_0'^2-2xmP'^2-mkP+mk^2+2m(k^2-kP)\frac{p'_\perp k_\perp}{k^2}-2\frac{p_\perp'^2+\frac{(p'_\perp k_\perp)^2}{k^2}}{M''_0+2m}\left[2xP'^2+2xM_0'^2-k^2+kP-2(k^2-kP)\frac{p'_\perp k_\perp}{k^2}\right]\right\}, \tag{3.37}$$

where $p''_\perp=p'_\perp-(1-x)k_\perp$, $k_\perp^2=-k^2$ and $kP=P''^2-P'^2$. We have designated the $\pi q\bar{q}$ vertex function h'_0 by $h_\pi(M'_0)$ and the $\rho q\bar{q}$ vertex function h''_0 by $h_\rho(M''_0)$; they are given by Eq. (2.18) in terms of the π mass M_π and ρ mass M_ρ, respectively. The on-shell form factors are given by

$$g(k^2)=g(k^2;M_\pi^2,M_\rho^2),\quad f(k^2)=f(k^2;M_\pi^2,M_\rho^2). \tag{3.38}$$

For the evaluation of the 4-momentum integral (3.35) it is convenient to approximate the form factors by monopole forms:

$$g(k^2)=\frac{g(0)}{1-k^2/\Lambda_g^2},\quad f(k^2)=\frac{f(0)}{1-k^2/\Lambda_f^2}, \tag{3.39}$$

where the pole masses Λ_g and Λ_f are determined by the derivatives of the form factors at $k^2=0$. For the numerical calculation we take the values (2.23) for the parameters m and β_π, and $\beta_\rho=0.26$ GeV [16], and obtain the results

$$g(0)=-1.21\ \text{GeV}^{-1},\quad \Lambda_g=0.664\ \text{GeV},$$

$$f(0)=-0.85\ \text{GeV},\quad \Lambda_f=1.72\ \text{GeV}. \tag{3.40}$$

The 4-momentum integral (3.35) can now be calculated by the standard Feynman parameter method, with the result

$$T_4 = \tfrac{1}{2} T_1 \, \delta^{(4)}_{axial},$$

$$\delta^{(4)}_{axial} = \frac{\alpha}{2\pi}(-0.69) = -8.0\times 10^{-4} \quad \text{(on-shell)}. \tag{3.41}$$

Sirlin has estimated the contribution of the diagrams of Fig. 4 on the basis of vector dominance and Weinberg sum rule arguments in Ref. [1] and found the correction to the decay rate ''to be a few times 10^{-4},'' in accordance with Eq. (3.41).

However, the intermediate ρ in the diagrams of Fig. 4 is off-shell, and for a rigorous evaluation of the corresponding amplitude the off-shell structure of the hadronic vertices must be accounted for. Moreover, a consistent treatment requires the calculation of the ρ self-energy operator in the same $q\bar{q}$-loop approximation that has been used for the calculation of the hadronic form factors. The corresponding renormalized ρ propagator that arises from the $q\bar{q}$ structure of the vector meson is obtained from Eq. (3.34) by the modification

$$\Delta_0^{-1}(P) \rightarrow \Delta^{-1}(P) = (P^2 - M_\rho^2 + i\varepsilon)\, F_\rho(0;M_\rho^2,P^2),$$

where $F_\rho(k^2;M_\rho^2,P^2)$ is the half-off-shell charge form factor of the ρ, with the normalization condition $F_\rho(0;M_\rho^2,M_\rho^2) = 1$. An analogous result has been derived for the renormalized pion propagator in Ref. [20]. For $k^2=0$ the charge form factors of π and ρ are given in the one-loop approximation by the same analytical expression (with appropriate π and ρ parameters) which has been derived in Ref. [16] to be

$$F_\rho(0;M_\rho^2,P^2) = \frac{N_c}{8\pi^3}\int_0^1 dx \int d^2p'_\perp \times \frac{(h_\rho(M'_0))^2}{(1-x)x^2(M_\rho^2 - M_0'^2)(P^2 - M_0'^2)}\, x M_0'^2 . \tag{3.42}$$

In order to estimate the importance of this structure effect we have used the light-front formulas (3.36) and (3.37) to continue the form factors $g(k^2)$ and $f(k^2)$ to their off-shell forms $g[k^2;M_\pi^2,(P'+k)^2]$ and $f[k^2;M_\pi^2,(P'+k)^2]$, with $P'^2 = M_\pi^2$. For an order of magnitude estimate it is sufficient to compare the values of the different form factors at $k=0$. We find

$$g(0;M_\pi^2,M_\pi^2) = 0.125 \ \text{GeV}^{-1},$$

$$f(0;M_\pi^2,M_\pi^2) = -0.028 \ \text{GeV},$$

$$F_\rho(0;M_\rho^2,M_\pi^2) = -0.070,$$

and consequently

$$g(0;M_\pi^2,M_\pi^2) f(0;M_\pi^2,M_\pi^2) / (g(0) f(0) F_\rho(0;M_\rho^2,M_\pi^2)) = 0.048. \tag{3.43}$$

Therefore, without going into any further computational details one can conclude that the resulting correction is reduced by at least the factor (3.43), i.e.

$$\delta^{(4)}_{axial} = O(10^{-5}) \quad \text{(off-shell)}. \tag{3.44}$$

Evidently the correction due the ρ exchange diagrams of Fig. 4 are very small, and can be safely neglected.

IV. CONCLUDING REMARKS

We have developed in this work a strategy to deal with the effect of the hadronic structure on the corrections to pion beta decay in $O(\alpha)$ due to the axial vector current. It is based on the light-front quark model for the pion, whose $q\bar{q}$ bound state structure is well described by two adjustable parameters, constituent quark mass and confinement scale, that have been shown in Refs. [14,16] to describe a large body of data. In most applications of the light-front quark model the effect of the hadronic structure could be well approximated by one-loop diagrams. However, the axial vector $O(\alpha)$ corrections to the beta decay of a $q\bar{q}$ bound state requires the calculation of two-loop diagrams, and we have considered three different types which are represented by the graphs of Figs. 2, 3 and 4. The amplitudes corresponding to the diagrams of Figs. 2 and 3 can be expressed in terms of light-front momentum integrals, which we have shown to be unique, i.e. there are no associated zero-mode contributions. Only the form factors F_2 and *f*, which are defined by Eqs. (2.3) and (3.32), respectively, are affected by zero modes; a method to account for the effect of zero-modes has been proposed in Ref. [16]. An alternative, explicitly covariant approach for a consistent evaluation of the form factors has been presented in Appendix C. Both methods lead to the same numerical results. In particular, we have shown that the correction of the decay rate due to the contributions of F_2, Eq. (2.8), and of *f*, Eq. (3.35) (corresponding to the diagrams of Fig. 4), is negligible.

The results of the quark model calculation for the corrections of $O(\alpha)$ due to the axial vector current are given by Eqs. (3.16), (3.30) and (3.44), which can be rewritten in the form of the standard representation (1.2) in terms of the quantities M_A and *C*:

$$\delta^{(2)}_{axial} + \delta^{(3)}_{axial} = \frac{\alpha}{2\pi} 3(Q_u + Q_d) \ln\frac{M_Z}{M_A},$$

$$M_A = 425.68 \pm 8 \ \text{MeV}, \tag{4.1}$$

and

$$\delta^{(4)}_{axial} = \frac{\alpha}{2\pi} 3(Q_u + Q_d)\, 2C,$$

$$C = O(10^{-2}). \tag{4.2}$$

Comparison with Eq. (1.3) shows that the value (4.1) for the effective mass M_A is clearly different from the a_1 meson mass, but close to the confinement scale of the $q\bar{q}$ pion, and is even below the guessed range (1.3). The error bar associated with M_A in Eq. (4.1) is due to the small uncertainties of the constituent quark mass $m=260\pm5$ MeV [14] and the Z mass $M_Z=91.188\pm0.007$ GeV [8]. The value of C, i.e. the correction due to the ρ exchange diagrams of Fig. 4 is so small that it can be neglected. The same is true of the effect of the form factors of the pion which we have discussed in Sec. II.

Thus our model calculation of the axial vector contribution to the radiative corrections of $O(\alpha)$ not only gives a definite value for M_A, Eq. (4.1), which leads to larger corrections, but also removes the large uncertainty in the RC due to the assumed range (1.3). Using the average value of Ref. [8] $\Delta=M_+-M_0=4.5936\pm0.0005$ MeV, which gives the end-point energy $E_0=(M_++M_0)\Delta/2M_+=4.5180\pm0.0005$ MeV, we find $(\alpha/2\pi)g(E_0)=1.0515\times10^{-2}$ and obtain from Eqs. (1.2) and (4.1) the value of the RC to pion beta decay

$$\delta=(3.230\pm0.002)\times10^{-2}, \tag{4.3}$$

where the error comes from the uncertainty of M_A, Eq. (4.1). We emphasize that the error on the RC is of the same order as the neglected corrections (2.27), due to the weak form factors of the pion, and $\delta^{(4)}_{axial}$, Eq. (3.44).

We shall investigate the effect of the detailed structure of the three quark bound state on the properties of superallowed nuclear decays and neutron decay in a future work, in a similar manner as for the pion. If we tentatively assume that the effective mass M_A associated with a qqq nucleon will be found approximately equal to the result given by Eq. (4.1), it is easy to show that the unitarity sum derived from nuclear decays becomes $V^2=0.9956\pm0.0011$, i.e. the violation of unitarity seems to be much more pronounced than indicated by Eq. (1.6).

However, a definite judgment of the unitarity problem will be possible only if the complete results of both the detailed structure calculation for qqq nucleons and the precision measurement of pion beta decay will be available.

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APPENDIX A: THE MOMENTUM INTEGRALS FOR SEC. III A

The momentum integrals encountered in the calculation of the vertex correction diagrams of Fig. 2 can be evaluated by the standard Feynman parameter method. Convenient formulas for the on-shell case can be found, e.g., in Ref. [21]. In Sec. III A the vertex correction is calculated for off-shell quarks in terms of the function Π, and the relevant integrals are given below:

$$\frac{1}{i\pi^2}\int d^4k\,\frac{k_\mu}{(k^2+i\varepsilon)^2(k^2\pm2p_1'k+N_1'+i\varepsilon)}\,\frac{M_W^2}{M_W^2-k^2+i\varepsilon}=\pm p'_{1\mu}a_1, \tag{A1}$$

$$\frac{1}{i\pi^2}\int d^4k\,\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2(k^2\pm2p_1'k+N_1'+i\varepsilon)}\,\frac{M_W^2}{M_W^2-k^2+i\varepsilon}=g_{\mu\nu}b_1+p'_{1\mu}p'_{1\nu}b_2, \tag{A2}$$

where

$$a_1=-\frac{1}{p_1'^2}-\frac{m^2}{p_1'^4}\ln\frac{m^2-p_1'^2}{m^2}+O\left(\frac{1}{M_W^2}\right), \tag{A3}$$

$$b_1=\frac{1}{4}\ln\frac{M_W^2}{m^2}+\frac{3}{8}+\frac{m^2-p_1'^2}{4p_1'^2}+\frac{m^4-p_1'^4}{4p_1'^4}\ln\frac{m^2-p_1'^2}{m^2}+O\left(\frac{m^2}{M_W^2}\right), \tag{A4}$$

$$b_2=-\frac{1}{2p_1'^2}-\frac{m^2-p_1'^2}{p_1'^4}-m^2\frac{m^2-p_1'^2}{p_1'^6}\ln\frac{m^2-p_1'^2}{m^2}+O\left(\frac{1}{M_W^2}\right). \tag{A5}$$

APPENDIX B: THE TWO-LOOP DIAGRAM AND ITS ZERO-MODE CONTRIBUTION

We shall analyze in this appendix the two-loop diagrams of Fig. 3 for the special case of pointlike $\pi q\bar{q}$ vertex functions. From the corresponding covariant amplitude, which is given by the Feynman rules, we shall derive the light-front amplitude by integrating over the minus components of the momentum variables. For the purpose of this calculation we put the $\pi q\bar{q}$

coupling constant equal to 1. In the conventional space-time formalism the covariant amplitude in the limit $l=k_\nu=0$ is given by

$$T_3^{Feynman}=\frac{1}{2}G\,V_{ud}\frac{i^2e^2N_c}{(2\pi)^8}\int d^4p_1'\int d^4p_1''\frac{(Q_dS_{\mu\lambda}^{(a)}-Q_uS_{\mu\lambda}^{(b)})L^{\mu\lambda}}{D_1'D_2'D_1''D_2''(k^2+i\varepsilon)^2}\,\frac{M_W^2}{M_W^2-k^2+i\varepsilon}, \tag{B1}$$

where $D_n'=p_n'^2-m^2+i\varepsilon$, $D_n''=p_n''^2-m^2+i\varepsilon$ for $n=1,2$, and $p_1'+p_2'=p_1''+p_2''=P'$ with $P'^2=M_\pi^2$. The photon has momentum $k=p_1''-p_1'=p_2'-p_2''$. The leptonic tensor $L_{\mu\lambda}$ has been defined in Eq. (3.2), and the hadronic tensors $S_{\mu\lambda}^{(a)}$ and $S_{\mu\lambda}^{(b)}$ in Eq. (3.20).

If the momenta are decomposed into their light-front components we have

$$d^4p_1'=\tfrac{1}{2}\,P'^{+}dp_1'^{-}dx'\,d^2p_\perp', \quad d^4p_1''=\tfrac{1}{2}\,P'^{+}dp_1''^{-}dx''\,d^2p_\perp''.$$

We are only interested in the integration with respect to $p_1'^{-}$ and $p_1''^{-}$, and use the same technique as in Ref. [16], which is based upon the integral representation

$$\frac{i}{p^2-m^2+i\epsilon}=\int_0^\infty d\alpha\, e^{i\alpha(p^2-m^2+i\epsilon)}. \tag{B2}$$

A similar procedure has been used in Refs. [22] to investigate the relation between the standard covariant quantum field theory and light-front field theory.

There are three basic integrals that contribute to the amplitude (B1); these are

$$\left(\frac{i}{2\pi}\right)^2\int dp_1'^{-}\int dp_1''^{-}\frac{1}{D_1'D_2'D_1''D_2''(k^2+i\varepsilon)^2}\,\frac{M_W^2}{M_W^2-k^2+i\varepsilon}$$
$$=\frac{1}{M_\pi^2x'(1-x')x''(1-x'')(M_\pi^2-M_0'^2)(M_\pi^2-M_0''^2)}$$
$$\times\left\{\frac{\Theta(x''-x')}{k_>^4}\,\frac{M_W^2}{M_W^2-k_>^2}+\frac{\Theta(x'-x'')}{k_<^4}\,\frac{M_W^2}{M_W^2-k_<^2}\right\}, \tag{B3}$$

$$\left(\frac{i}{2\pi}\right)^2\int dp_1'^{-}\int dp_1''^{-}\frac{p_1'^{-}}{D_1'D_2'D_1''D_2''(k^2+i\varepsilon)^2}\,\frac{M_W^2}{M_W^2-k^2+i\varepsilon}$$
$$=\frac{1}{M_\pi^3x'(1-x')x''(1-x'')(M_\pi^2-M_0'^2)(M_\pi^2-M_0''^2)}$$
$$\times\left\{\frac{(M_\pi^2-x'M_0'^2)\Theta(x''-x')}{k_>^4}\,\frac{M_W^2}{M_W^2-k_>^2}+\frac{(1-x')M_0'^2\Theta(x'-x'')}{k_<^4}\,\frac{M_W^2}{M_W^2-k_<^2}\right\}, \tag{B4}$$

where $k_>^2$ and $k_<^2$ have been defined in Eqs. (3.17) and (3.18), and we have used that $P'^{+}=M_\pi$. The result for the first two integrals (B3) and (B4) coincides with the result obtained by the contour method, i.e. in both cases one finds the residues of the respective quark poles, and zero modes do not contribute.

The third integral we represent as

$$\left(\frac{i}{2\pi}\right)^2\int dp_1'^{-}\int dp''{}^{-}\frac{p_1'^{-}\,p_1''^{-}}{D_1'D_2'D_1''D_2''(k^2+i\varepsilon)^2}\,\frac{M_W^2}{M_W^2-k^2+i\varepsilon}$$
$$=\frac{1}{M_\pi^4}\{P_1\Theta(x''-x')+P_2\Theta(x'-x'')+R_1\delta(x')\,\delta(x'')+R_2\delta(1-x')\,\delta(1-x'')\}. \tag{B5}$$

The form of the residue terms is obvious

$$P_1=\frac{(M_\pi^2-x'M_0'^2)(1-x'')M_0''^2}{x'(1-x')x''(1-x'')(M_\pi^2-M_0'^2)(M_\pi^2-M_0''^2)\,k_>^4},$$

$$P_2=\frac{(1-x')M_0'^2(M_\pi^2-x''M_0''^2)}{x'(1-x')x''(1-x'')(M_\pi^2-M_0'^2)(M_\pi^2-M_0''^2)\,k_<^4}. \tag{B6}$$

The functions $R_n=R_n(p'_\perp,p''_\perp)$ for $n=1,2$, which determine the zero-mode contribution, are independent of M_π, but their detailed form cannot be derived by the method used above. For example, R_1 is found to be the ratio of two functions, each of which becomes zero for $x'=x''=0$. However, there is an alternative way to determine R_1 and R_2. It uses the limiting behavior of the integral

$$\lim_{M_\pi\to 0}\int d^4p'_1\int d^4p''_1\frac{p'_1P'\cdot p''_1P'}{D'_1D'_2D''_1D''_2\,(k^2+i\varepsilon)^2}\,\frac{M_W^2}{M_W^2-k^2+i\varepsilon}=O(M_\pi^2), \tag{B7}$$

which is implied by Lorentz covariance.

If we define

$$P_1^{(0)}=\lim_{M_\pi\to 0}P_1=\frac{1}{(1-x')x''k_>^4},$$

$$P_2^{(0)}=\lim_{M_\pi\to 0}P_2=\frac{1}{x'(1-x'')k_<^4}, \tag{B8}$$

and integrate Eq. (B5) with respect to x' and x'', then, according to Eq. (B8) the contribution of $O(M_\pi^{-4})$ must vanish exactly, which gives the conditions

$$\int_0^1 dx'\int_{x'}^1 dx''\,P_1^{(0)}+R_1=0,$$

$$\int_0^1 dx'\int_0^{x'} dx''\,P_2^{(0)}+R_2=0. \tag{B9}$$

From Eqs. (B9) for R_1 and R_2 it can be seen that $R_1=R_2$, by substituting x' for $1-x'$ and x'' for $1-x''$.

This derivation of the zero-mode contribution shows clearly that a residue term which is derived by the contour method, may contain a spurious part that is not consistent with the requirement of Lorentz covariance. It is the zero-mode contribution that cancels this unphysical part of the residue term. This is an example of the deep connection between the zero-mode and the Lorentz invariance of the light-front formalism.

Next, we shall decompose the product of leptonic and hadronic tensors in the integrand of Eq. (B1) into products of light-front components. We use that $S^{(b)}_{\mu\lambda}=-S^{(a)}_{\mu\lambda}$, Eq. (3.20), and the result (3.22), and find

$$\begin{aligned}S^{(a)}_{\mu\lambda}L^{\mu\lambda}=2\,P'L\{&-2p_1'^-p_1''^-(k^+)^2\\&+p_1'^-[k^+(2m_\perp''^2+x''M_\pi^2)-(1-2x')M_\pi(2p''_\perp k_\perp+x''M_\pi k^+)]\\&-p_1''^-[k^+(2m_\perp'^2+x'M_\pi^2)-(1-2x'')M_\pi(2p'_\perp k_\perp+x'M_\pi k^+)]+\cdots\},\end{aligned} \tag{B10}$$

where we have omitted all those terms that are independent of $p_1'^-$ and $p_1''^-$.

If the decomposition (B10) and the basic integrals (B3)–(B5) are used to perform the integration of Eq. (B1) with respect to $p_1'^-$ and $p_1''^-$, it is obvious that the zero-mode contribution of Eq. (B5) vanishes, since the term that contains the product $p_1'^-p_1''^-$ is multiplied with k^+. The result for the momentum integral (B1) is given by the residues of the respective quark poles, and zero modes do not contribute:

$$T_3^{Feynman}=G\,V_{ud}\,P'L\,(Q_u+Q_d)\,\frac{\alpha N_c}{16\pi^5}\int_0^1 dx'\int d^2p'_\perp\int_{x'}^1 dx''\int d^2p''_\perp$$

$$\times\frac{R(x',x'')}{x'(1-x')x''(1-x'')(M_\pi^2-M_0'^2)(M_\pi^2-M_0''^2)\,k_>^4}\,\frac{M_W^2}{M_W^2-k_>^2+i\varepsilon},\tag{B11}$$

where we have used that the integrals for $k^+>0$ and $k^+<0$ are equal, and the function $R(x',x'')$ has been defined in Eq. (3.26).

We have proven that the covariant Feynman integral (B1) and the light-front integral (B11) are equal. Since the integrals are finite, this result can be verified by numerical calculation. At this stage we depart from the covariant (Feynman) perturbation theory and introduce phenomenological vertex functions, Eq. (2.18), into the two-loop light-front integrals. This step gives the light-front quark model expressions, Eqs. (3.28) and (3.29), for the amplitude T_3, corresponding to the diagrams of Fig. 3, in terms of a simple convolution of light-front vertex functions.

APPENDIX C: COVARIANCE PROPERTIES OF ONE- AND TWO-LOOP LIGHT-FRONT INTEGRALS

In order to treat the complete Lorentz structure of a hadronic matrix element the authors of Ref. [23] have developed a method to identify and separate spurious contributions and to determine the physical part of the hadronic matrix element. The formal aspects of this approach have been investigated in Ref. [24]. We have developed a basically different technique in Ref. [16] to deal with this problem. Both methods lead to the same 4-vector structure of the light-front integrals for the two-loop amplitudes $T_2^{(\gamma)}$ and T_3. In the first part of this appendix we shall use a method that has been proposed in Ref. [25] to determine the Lorentz invariant parts of the light-front integrals, that are free of spurious contributions; these are the LFQM expressions for the amplitudes $T_2^{(\gamma)}$, Eq. (3.13), and T_3, Eqs. (3.28) and (3.29).

The light-front integral for the amplitude $T_2^{(\gamma)}$, Eqs. (3.4) and (3.9), consists of three different terms which we write formally as

$$T_2^{(\gamma)}=\int_0^1 dx\int d^2p'_\perp\,\frac{h_0'^2}{(1-x)N_1'^2}\{f_1(p_1'^2)\,P'L+f_2(p_1'^2)\,p_1'L$$
$$+f_3(p_1'^2)\,p_1'L\cdot p_1'P'\},\tag{C1}$$

where p_1' is given in terms of the internal variables $x,p'_\perp$ by Eq. (2.17). In order to analyze the covariance properties of the amplitude $T_2^{(\gamma)}$ we shall use the procedure of Ref. [16] and shall show that the second term of Eq. (C1), which depends on the 4-vector $p'_{1\mu}$, and the third term of Eq. (C1), which depends on the tensor $p'_{1\mu}p'_{1\nu}$, generate 4-vector structures that are in general not covariant, since they contain a spurious dependence on the orientation of the light-front. The light-front is defined in terms of the lightlike 4-vector ω by the invariant equation $\omega x=0$. The special case $\omega=(2,0,0_\perp)$ corresponds to the light-front or null-plane $\omega x=x^+=0$.

We are only interested in the limit $q=P'-P''=0$. Therefore, the second term of Eq. (C1) can be decomposed with regard to P' and ω as

$$\int_0^1 dx\int d^2p'_\perp\,\frac{h_0'^2}{(1-x)N_1'^2}f_2(p_1'^2)\,p'_{1\mu}$$
$$=\int_0^1 dx\int d^2p'_\perp\,\frac{h_0'^2}{(1-x)N_1'^2}f_2(p_1'^2)\left\{xP'_\mu+C_1^{(1)}\frac{\omega_\mu}{2\,\omega P'}\right\}.\tag{C2}$$

Such an equation will be written in the following as a relation between integrands:

$$p'_{1\mu}\doteq xP'_\mu+C_1^{(1)}\frac{\omega_\mu}{2\,\omega P'}.\tag{C3}$$

The coefficient $C_1^{(1)}$ is given in [16], and we have shown there that the ω-term of Eq. (C3) is canceled exactly by the corresponding zero-mode contribution. Therefore, we can omit the ω-term and obtain a unique expression for the second term of Eq. (C1) by the replacement

$$p_1'L\to x\,P'L.\tag{C4}$$

The analysis of the third term of Eq. (C1) requires the tensor decomposition

$$p'_{1\mu}p'_{1\nu}\doteq-\tfrac{1}{2}\,p_\perp'^2\,g_{\mu\nu}+x^2\,P'_\mu P'_\nu$$
$$+\frac{1}{\omega P'}(P'_\mu\omega_\nu+\omega_\mu P'_\nu)B_1^{(2)}+\frac{\omega_\mu\omega_\nu}{4(\omega P')^2}C_2^{(2)},\tag{C5}$$

where

$$B_1^{(2)}=\frac{x}{2}[N_1'+(1-2x)M_\pi^2]+\tfrac{1}{2}p_\perp'^2.$$

The coefficient $C_2^{(2)}$ is given in [16] and will not be required here. We have argued in [16] that there is neither a zero-mode associated with the coefficient $B_1^{(2)}$, nor does the integrated $B_1^{(2)}$ vanish for the light-front vertex function h_0' given by Eq. (2.18). Moreover, $B_1^{(2)}$ contributes to the physical part of the amplitude and cannot be simply omitted. But even though $B_1^{(2)}$ is combined with ω_μ its physical contribution can be expressed in terms of a covariant structure. For its construction we shall adopt a method that has been proposed in Ref. [25] for a covariant calculation of the electromagnetic form factors of the nucleon. We shall split the $B_1^{(2)}$ term of Eq. (C5) into a part that is a linear combination of the 4-tensors $g_{\mu\nu}$ and $P'_\mu P'_\nu$, and a part that is orthogonal to $g_{\mu\nu}$ and $P'_\mu P'_\nu$:

$$\frac{1}{\omega P'}(P'_\mu\omega_\nu+\omega_\mu P'_\nu)B_1^{(2)}=B_{\mu\nu}B_1^{(2)}+R_{\mu\nu}B_1^{(2)}+\frac{\omega_\mu\omega_\nu}{(\omega P')^2}M_\pi^2 B_1^{(2)}. \quad \text{(C6)}$$

The orthogonality conditions are given by

$$P'_\mu P'_\nu R^{\mu\nu}=g_{\mu\nu}R^{\mu\nu}=0.$$

The contribution of the coefficient $B_1^{(2)}$ to the physical part of the amplitude is included correctly by the requirement

$$P'_\nu B^{\mu\nu}=P'^{\mu}.$$

The tensors $B_{\mu\nu}$ and $R_{\mu\nu}$ are then given by

$$B_{\mu\nu}=\frac{1}{3}g_{\mu\nu}+\frac{2}{3M_\pi^2}P'_\mu P'_\nu,$$

$$R_{\mu\nu}=\frac{1}{\omega P'}(P'_\mu\omega_\nu+\omega_\mu P'_\nu)-\frac{\omega_\mu\omega_\nu}{(\omega P')^2}M_\pi^2-B_{\mu\nu}. \quad \text{(C7)}$$

The construction (C6) contains an additional contribution proportional to $\omega_\mu\omega_\nu$ which can be combined with the $C_2^{(2)}$ term of Eq. (C5). The resulting expression for $p'_{1\mu}p'_{1\nu}$ is

$$p'_{1\mu}p'_{1\nu}\doteq\left(-\frac{1}{2}p'^2_\perp+\frac{1}{3}B_1^{(2)}\right)g_{\mu\nu}+\left(x^2+\frac{2}{3M_\pi^2}B_1^{(2)}\right)P'_\mu P'_\nu+B_1^{(2)}R_{\mu\nu}+\left(\frac{1}{4}C_2^{(2)}+M_\pi^2 B_1^{(2)}\right)\frac{\omega_\mu\omega_\nu}{(\omega P')^2}. \quad \text{(C8)}$$

It is clearly separated into a physical contribution (in terms of the 4-tensors $g_{\mu\nu}$ and $P'_\mu P'_\nu$) that does not depend on ω, the term $R_{\mu\nu}$ that is orthogonal to and independent of the physical part, and the term proportional to $\omega_\mu\omega_\nu$. The last term, which is expected to be canceled by the associated zero-mode contribution, and $R_{\mu\nu}$, which is expected to be canceled by higher order gluon exchange contributions, are spurious and will be omitted. Therefore, a unique expression for the third term of Eq. (C1) is obtained by the replacement

$$\begin{aligned} p'_1L\cdot p'_1P' &\to\left(-\frac{1}{2}p'^2_\perp+B_1^{(2)}+x^2M_\pi^2\right)P'L\\ &=\frac{x}{2}(N'_1+M_\pi^2)\,P'L\\ &=xp'_1P'\cdot P'L, \end{aligned} \quad \text{(C9)}$$

where Eqs. (2.16) and (2.17) have been used for the last step. In this manner we can rederive Eq. (3.13) for the amplitude $T_2^{(\gamma)}$. In principle, the coefficients of the 4-vectors and 4-tensors in Eqs. (C3) and (C8), and consequently the light-front integral (3.13), can also depend on ω, however, since $\omega^2=0$ the ω-dependence can enter only in the form of the ratio $\omega P'/\omega P''$ which is always unity from the condition $\omega q=0$, see Refs. [25] and [24].

In conclusion, this construction which basically consists of the separation and omission of the ω-dependent terms, leads to a Lorentz invariant expression for the amplitude $T_2^{(\gamma)}$ in terms of the light-front integral (3.13).

The same analysis can be carried out also for the amplitude T_3. It is given by Eq. (3.19) in terms of the product (3.22). If the vectors $p'_{1\mu}$ and $p''_{1\mu}$ are decomposed with regard to P'_μ and ω_μ as in Eq. (C3), we can get rid of the ω-dependent contributions by means of the replacements

$$\begin{aligned} p'_1L &\to x'\,P'L,\\ p''_1L &\to x''\,P'L. \end{aligned} \quad \text{(C10)}$$

These relations lead to Eq. (3.23) and we have shown in Appendix B, Eq. (B10), that there are no zero modes associated with the resulting expression for T_3, i.e., the light-front integral (3.29) for T_3 is invariant.

Note that our choice of the special representation (3.10) for the 4-vector L is equivalent to the replacements (C4), (C9) and (C10).

The developments of this Appendix enable us to resolve an inconsistency that is associated with the formalism of Ref. [16]. Based upon a special class of meson vertex functions we have shown in [16] that there exists an exact correspondence between the explicitly covariant 4-dimensional and the light-front calculations in one-loop order. In that framework we could analyze the role of zero modes, in particular we found that covariance requires the inclusion of the effect of zero modes, and the conditions for the exact disappearance of the spurious dependence on the orientation of the light-front have been derived [in terms of the "covariance conditions," Eq. (3.32) of Ref. [16]]. However, the vertex functions used in the approach [16] are not symmetric in the 4-momenta of the constituent quarks, and can hardly be considered a realistic approximation of the $q\bar{q}$ bound state. We have argued in [16] that mesons must be described in terms of light-front vertex functions which are symmetric in the variables of the constituent $q\bar{q}$ pair, and that it does not seem possible to establish an equally straightforward correspondence between the respective light-front approach and a manifestly covariant 4-dimensional formalism. This means in particular that the light-front expression of a one-loop matrix element for transitions between symmetric $q\bar{q}$ mesons contains in general a nonvanishing spurious contribution which violates Lorentz covariance. We have stated this fact in Ref. [16], but did not attempt to solve the problem.

In the covariant analysis of Ref. [16] we have calculated the form factors $F_1(q^2)$, $F_2(q^2)$, $g(q^2)$, $f(q^2)$ and $a_\pm(q^2)$ in terms of asymmetric meson vertex functions. For practical applications it is important to know, if it is consistent with the requirement of Lorentz covariance to use these formulas for form factors in combination with a general light-front vertex function (e.g. for a symmetric $q\bar{q}$ meson), or if it is necessary to modify the formulas.

A fully covariant approach to treat hadronic matrix elements in the light-front formalism can be established by a combination of the methods that have been developed in Refs. [23,25,24] and in Ref. [16], and it can be used as an alternative and more general method for a consistent evaluation of the form factors. This approach is valid for a general light-front vertex function, and we shall now compare the results obtained with this method, with those of Ref. [16].

For the calculation of F_1 and F_2 we require the decomposition of the tensor $p'_{1\mu}p'_{1\nu}$ with respect to the 4-vectors P', P'' and ω, under the condition $P'^2 \neq P''^2$. By a straightforward generalization of the covariant construction scheme, which we have proposed in this Appendix, we found the remarkable result that the formulas for F_1 and F_2 are the same as those given in [16], but expressed now in terms of general (e.g., symmetric) meson vertex functions.

For the calculation of g, f and $a_\pm$ we need the decomposition of the tensor $p'_{1\mu}p'_{1\nu}\epsilon p'_1$ with respect to the 4-vectors P', P'', ϵ and ω. The covariant construction of the form factors is again straightforward but requires lengthy algebraic manipulations. We found that the formulas for g and a_+ are reproduced, while f and a_- are modified with respect to the results of Ref. [16], if general meson vertex functions are used. In particular, Eq. (3.37) for $f(k^2;P'^2,P''^2)$ must be replaced by

$$f(k^2;P'^2,P''^2) \rightarrow f(k^2;P'^2,P''^2) + \Delta f(k^2;P'^2,P''^2),$$

where

$$\Delta f(k^2;P'^2,P''^2) = \frac{N_c}{8\pi^3}\int_0^1 dx \int d^2p'_\perp \frac{h_\pi(M'_0)h_\rho(M''_0)}{(1-x)x^2(P'^2-M'^2_0)(P''^2-M''^2_0)} \times \frac{1}{M''_0+2m}\left\{\frac{k^2 P^2}{kP}(B_1^{(3)}+B_2^{(3)})+O(k^4)\right\}. \tag{C11}$$

The functions $B_1^{(3)}$ and $B_2^{(3)}$ have been defined in Ref. [16] and are given as

$$B_1^{(3)}+B_2^{(3)} = \left(x-\frac{p'_\perp k_\perp}{k^2}\right)\left((1-x)P'^2 - xM'^2_0 + (k^2-kP)\frac{p'_\perp k_\perp}{k^2}\right) + x\frac{k^2+kP}{k^2}\left(p'^2_\perp + \frac{(p'_\perp k_\perp)^2}{k^2}\right). \tag{C12}$$

Note that Δf vanishes if Eq. (C11) is calculated with the asymmetric meson vertex functions of Ref. [16].

Since $\Delta f(0;P'^2,P''^2)=0$ the numerical results given in Eq. (3.40) are unchanged, except for Λ_f which is calculated, using Eqs. (3.37) and (C11), as $\Lambda_f = 1.80$ GeV. Consequently, the number given for the correction (3.41) has to be corrected in a minor way, i.e., $\delta^{(4)}_{axial} = -.00081$, while the final estimate (3.44) remains unchanged.

We have described in this Appendix how the approach of Ref. [16] can be combined with the approach of Refs. [23,25,24] in order to decompose a hadronic matrix element on the light-front into its Lorentz covariant parts. In this manner corresponding form factors can be determined that are consistent with the requirement of Lorentz covariance without imposing restrictions on the $q\bar{q}$ meson vertex function. We have calculated in this general framework the form factors F_1, F_2, g and a_+, and found that the formulas given in [16] are valid also in terms of general $q\bar{q}$ meson vertex functions. In contrast, the formulas for the form factors f and a_- are different. It is remarkable that neither the numerical results of the present work, that are based on Ref. [16], nor those of Ref. [16] itself are changed.

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