

## Comment on “Topological invariants, instantons, and the chiral anomaly on spaces with torsion”

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In Riemann-Cartan spacetimes with torsion only its axial covector piece  $A$  couples to massive Dirac fields. Using renormalization group arguments, we show that in addition to the familiar Riemannian term only the Pontrjagin type four-form  $dA \wedge dA$  arises additionally in the chiral anomaly, but *not* the Nieh-Yan term  $d^*A$ , as has been claimed in a recent paper [Phys. Rev. D. **55**, 7580 (1997)].

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### I. INTRODUCTION

Quantum anomalies both in the Riemannian and in Riemann-Cartan spacetimes were calculated previously using different methods, see, e.g., Refs. [1,2]. However, recently [3] the completeness of these earlier calculations has been questioned, which demonstrated that the Nieh-Yan four-form [4] is irrelevant to the axial anomaly.

For the axial anomaly, we have a couple of distinguished features. Most prominent is its relation with the Atiyah-Singer index theorem. Also from the viewpoint of perturbative quantum field theory (QFT), the chiral anomaly has some features which signal its conceptual importance. For all topological field theories such as  $BF$  theories, Chern-Simons theories, and for all topological effects such as the anomaly, the remarkable fact holds that the relevant invariants do not renormalize — higher order loop corrections do not alter the one-loop value of the anomaly, for example. The fact that the anomaly is stable against radiative corrections guarantees that it can be given a topological interpretation. For the anomaly, this is the Adler-Bardeen theorem, while other topological field theories are carefully designed to have, among other properties, vanishing beta functions. Another feature is finiteness: in any approach, the chiral anomaly as a topological invariant is a finite quantity.

In a spacetime with torsion, Chandia and Zanelli [3] argue that the Nieh-Yan (NY) four-form  $d^*A$  will add to this quantity. As usual, they confront the fact that such a term, if it is generated at all, is ill defined, independent of the regularization. In their case, they use a Fujikawa-type approach and propose to absorb the regulator mass in a rescaled vierbein.

However, there is a severe misunderstanding in Ref. [3]. While there is no doubt that the NY term can possibly be generated, as demonstrated previously [5,2], this is not the end of the argument. In order to obtain a finite quantity, the tetrads have to be rescaled. While this might look like an innocent manipulation, this is not so. In rescaling the tetrad,

the authors of Ref. [3] ignore the presence of renormalization conditions and the generation of a scale upon renormalization. Rescaling the tetrad would ultimately change the wave function renormalization  $Z$  factor.

This factor creeps into the definition of the NY term at the quantum level, and thus a rescaling of the tetrad does not achieve the desired goals. This is not surprising: QFT demands a new  $Z$  factor for the NY term, in sharp contrast to proper topological invariants at the quantum level, which remain unchanged under renormalization.

With no renormalization condition available for the NY term, and other methods obtaining it as zero, we can only conclude that the response function of the quantum field theory to a gauge variation (this is the anomaly) delivers no NY term. Or, saying it differently, its finite value is zero after renormalization.

### II. GRAVITATIONAL CHERN-SIMONS AND PONTRJAGIN TERMS

In our notation, Clifford-algebra valued exterior forms [6], the constant Dirac matrices  $\gamma_\alpha$  obeying  $\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\delta_{\alpha\beta}$  are saturating the index of the orthonormal coframe one-form  $\vartheta^\alpha$  and its Hodge dual  $\eta^\alpha := * \vartheta^\alpha$  via  $\gamma := \gamma_\alpha \vartheta^\alpha$  and  $* \gamma = \gamma^\alpha \eta_\alpha$ . In terms of the connection  $\Gamma := (i/4) \Gamma^{\alpha\beta} \sigma_{\alpha\beta}$ , the  $SL(2, C)$ -covariant exterior derivative is given by  $D = d + \Gamma \wedge$ , where  $\sigma_{\alpha\beta} = (i/2)(\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha)$  are the Lorentz generators entering also in the Clifford-algebra valued two-form  $\sigma := (i/2) \gamma \wedge \gamma = \frac{1}{2} \sigma_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta$ .

Differentiation of these independent variables leads to the Clifford algebra-valued two-forms of torsion  $\Theta := D\gamma = T^\alpha \gamma_\alpha$  and curvature  $\Omega := d\Gamma + \Gamma \wedge \Gamma = (i/4) R^{\alpha\beta} \sigma_{\alpha\beta}$  of Riemann-Cartan (RC) geometry.

The Chern-Simons (CS) term [7] for the Lorentz connection  $C_{RR} := -\text{Tr}(\Gamma \wedge \Omega - \frac{1}{3} \Gamma \wedge \Gamma \wedge \Gamma)$  and its corresponding Pontrjagin term  $dC_{RR} = -\text{Tr}(\Omega \wedge \Omega) = \frac{1}{2} R^{\alpha\beta} \wedge R_{\alpha\beta}$  have the familiar form. Since the coframe is the translational part of the Cartan connection [7], there arises also the translational CS term [8]

$$C_{TT} := \frac{1}{8l^2} \text{Tr}(\gamma \wedge \Theta) = \frac{1}{2} (C_{RR} - \hat{C}_{RR}) \quad (2.1)$$

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which is related to the Nieh-Yan four-form [4]:

$$dC_{TT} = \frac{1}{2l^2} (T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta). \quad (2.2)$$

A fundamental length  $l$  unavoidably occurs here for dimensional reasons. This can also be motivated by a de Sitter type [9] approach, in which the  $\text{sl}(5, R)$ -valued connection  $\hat{\Gamma} = \Gamma + (1/l)(\vartheta^\alpha L_\alpha^4 + \vartheta_\beta L_4^\beta)$  is expanded into the dimensionless linear connection  $\Gamma$  plus the coframe  $\vartheta^\alpha = e_i^\alpha dx^i$  with canonical dimension [length]. The corresponding CS term  $\hat{C}_{RR}$  splits via  $\hat{C}_{RR} = C_{RR} - 2C_{TT}$  into the linear one and that of translations, see footnote 31 of Ref. [7]. This relation has recently been ‘‘recovered’’ by Chandia and Zanelli [3].

The one-form of axial vector torsion

$$A := \frac{1}{4} \text{Tr}(\check{\gamma})^* \Theta = \frac{1}{4} {}^* \text{Tr}(\gamma \wedge \Theta) = {}^*(\vartheta^\alpha \wedge T_\alpha) \quad (2.3)$$

is a conformal invariant under the combined transformation of classical Weyl rescalings of the coframe, in contrast to  ${}^*A = -2l^2 C_{TT}$ , see. Eqs. (3.14.1), (3.14.9) of Ref. [7].

### III. DIRAC FIELDS IN RIEMANN-CARTAN SPACETIME

The Dirac Lagrangian is given by the manifestly Hermitian four-form

$$\begin{aligned} L_D(\gamma, \psi, D\psi) &= \frac{i}{2} \{ \bar{\psi}^* \gamma \wedge D\psi + \overline{D\psi} \wedge {}^* \gamma \psi \} + {}^* m \bar{\psi} \psi \\ &= L(\gamma, \psi, D\psi) - \frac{1}{4} A \wedge \bar{\psi} \gamma_5^* \gamma \psi, \end{aligned} \quad (3.1)$$

for which  $\bar{\psi} := \psi^\dagger \gamma_0$  is the Dirac adjoint and  ${}^*m = m \eta$  the mass term, see Ref. [6]. The decomposed Lagrangian (3.1) leads to the following form of the Dirac equation:

$$i {}^* \gamma \wedge \check{D}\psi + {}^* m \psi = i {}^* \gamma \wedge \left[ D\psi + \frac{i}{4} m \gamma + \frac{i}{4} A \gamma_5 \right] \psi = 0 \quad (3.2)$$

in terms of the Riemannian connection  $\Gamma^\{\}$  with  $D\psi = 0$  and the irreducible piece (2.3) of the torsion. Hence, in a RC spacetime a Dirac spinor only feels the axial torsion one-form  $A$ . This can also be seen from the identity (3.6.13) of Ref. [7] which specializes here to the ‘‘on shell’’ commutation relation

$$[\check{D}, \check{D}] = \Omega^\{\} + \frac{i}{4} \gamma_5 dA - \frac{i}{8} m^2 \sigma. \quad (3.3)$$

In contrast to Ref. [3], Eq. (27), there arise in Eq. (3.3) no tensor or vector pieces of the torsion, because our operator  $\check{D}$  in Eq. (3.2) is the only possible result from the Lagrangian (3.1), which is Hermitian as required by QFT.

From the Dirac equation (3.2) and its adjoint one can readily deduce the well-known ‘‘classical axial anomaly’’  $dj_5 = d(\frac{1}{3} \bar{\psi} \sigma \wedge \gamma \psi) = 2miP = 2mi \bar{\psi} \gamma_5 \psi$  for massive Dirac

fields also in a RC spacetime. If we restore chiral symmetry in the limit  $m \rightarrow 0$ , this leads to classical conservation law of the axial current for massless Weyl spinors, or since  $dj = 0$ , equivalently, for the chiral current  $j_\pm := \frac{1}{2} \bar{\psi}(1 \pm \gamma_5) {}^* \gamma \psi = \bar{\psi}_{L,R}^* \gamma \psi_{L,R}$ .

The Einstein-Cartan-Dirac (ECD) theory of a gravitationally coupled spin  $\frac{1}{2}$  fermion field provides a dynamical understanding of the axial anomaly on a classical (i.e., not quantized) level. From Einstein’s equations  $-(1/2) \eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} = l^2 \Sigma_\alpha$  and the purely algebraic Cartan relation  $-(1/2) \eta_{\alpha\beta\gamma} \wedge T^\gamma = l^2 \tau_{\alpha\beta} = -(l^2/4) \eta_{\alpha\beta\gamma\delta} \check{\Psi} \gamma_5 \gamma^\delta \Psi \eta^\gamma$  one finds [6,10]

$$dj_5 \cong 4dC_{TT} = \frac{2}{l^2} (T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta) \quad (3.4)$$

which establishes a link to the NY four form [4], but only for massive fields [10]. However, if we restore chiral invariance for the Dirac fields in the limit  $m \rightarrow 0$ , we find within the dynamical framework of ECD theory that the NY four-form tends to zero ‘‘on shell,’’ i.e.,  $dC_{TT} \cong (1/4) dj_5 \rightarrow 0$ .

This is consistent with the fact that a Weyl spinor does not couple to torsion at all, because the remaining axial torsion  $A$  becomes a lightlike covector, i.e.,  $A_\alpha A^\alpha \eta = A \wedge {}^* A \cong (l^4/4) {}^* j_5 \wedge j_5 = 0$ . Here we implicitly assume that the light-cone structure of the axial covector  ${}^* j_5$  is not spoiled by quantum corrections, i.e. that no ‘‘Lorentz anomaly’’ occurs as in  $n = 4k + 2$  dimensions [11].

### IV. CHIRAL ANOMALY IN QFT

When quantum field theory (QFT) is involved, other boundary terms may arise in the chiral anomaly due to the nonconservation of the axial current, see Refs. [12,13]. Now, to approach the anomaly in the context of spacetime with torsion, we will proceed by switching off the curvature and concentrate on the last term in the decomposed Dirac Lagrangian (3.1).

Then, this term can be regarded as an external axial covector  $A$  [in view of Eq. (3.3) without Lorentz or ‘‘internal’’ indices] coupled to the axial current  $j_5$  of the Dirac field in an initially flat spacetime. By applying the result (11-225) of Itzykson and Zuber [14], we find that only the term  $dA \wedge dA$  arises in the axial anomaly, but *not* the NY type term  $d{}^*A \sim dC_{TT}$  as was recently claimed [3]. After switching on the curved spacetime of Riemannian geometry, we finally obtain for the axial anomaly

$$\langle dj_5 \rangle = 2m \langle iP \rangle + \frac{1}{24\pi^2} \left[ \text{Tr}(\Omega^\{\} \wedge \Omega^\{\}) - \frac{1}{4} dA \wedge dA \right]. \quad (4.1)$$

In addition to other perturbative methods such as point splitting, there is the further option to use dimensional regularization. If one adopts the  $\gamma_5$  scheme of Ref. [15], one immediately concludes that only the result (4.1) can appear. The only effect of the  $\gamma_5$  problem is the replacement of the usual trace by a noncyclic linear functional. The anomaly

appears as the sole effect of this noncyclicity. There is no room for other sources of noncyclicity apart from the very fermion loops which produce the result (4.1). The whole effect of noncyclicity is to have an operator  $\Delta$ ,  $\Delta^2=0$ , and the anomaly is in the image modulo the kernel of  $\Delta$ , which summarizes the fact that in this  $\gamma_5$  scheme no other anomalous contributions are possible in addition to Eq. (4.1).

But at this stage we have not discussed the possibility of a contorted spacetime which cannot be adiabatically deformed to the torsion-free case. In such a case it has been argued [3] that the boundary term  $dC_{TT}$  occurs, multiplied by a factor  $M^2$ . This factor  $M^2$  corresponds to a regulator mass in a Fujikawa type approach. For instance, in the heat kernel approach, the first nontrivial terms [5,2], which potentially could contribute to the axial anomaly, read

$$\begin{aligned} \text{Tr}(\gamma_5 K_2) &= -d^*A, \quad \mathcal{K} = *\check{D} \wedge *\check{D}^*A, \\ \text{Tr}(\gamma_5 K_4) &= \frac{1}{6} \left[ \text{Tr}(\Omega^{\flat} \wedge \Omega^{\flat}) - \frac{1}{4} dA \wedge dA + d\mathcal{K} \right]. \end{aligned} \quad (4.2)$$

However, there is an essential difference in the physical dimensionality of the terms  $K_2$  and  $K_4$ . Whereas in  $n=4$  dimensions the Pontrjagin type term  $K_4$  is dimensionless, the term  $K_2 \sim 2l^2 dC_{TT}$  carries dimensions. It can be consistently absorbed in a counterterm, and thus discarded from the final result for the anomaly.

This is also in agreement with the analysis in [16] where, in the framework of string theory, the chiral anomaly in the presence of torsion had a smooth adiabatic limit to the case of vanishing torsion. In contrast, in Ref. [3] it is argued that such contributions can be maintained by absorbing the divergent factor in a rescaled coframe  $\check{\mathcal{D}}^\alpha := M \vartheta^\alpha$  and proposed to consider the Wigner-Inönü contraction  $M \rightarrow \infty$  in the de Sitter gauge approach [9], with  $Ml$  fixed.

Apart from the fact that this would change also the dimension of  $\psi$ , in order to retain the physical dimension  $[\hbar]$  of the Dirac action, there are several points which seem unsatisfactory in such an argument.

(1) As the difference (2.1) of two Pontrjagin classes, the term  $dC_{TT}$  is a topological invariant after all. Now, it is actually *not* this term which appears as the torsion-dependent extra contribution to the anomaly, but more precisely  $-d^*A = 2l^2 dC_{TT}$ . Thus, measuring its proportion in units of the topological invariant  $dC_{TT}$ , we find that it vanishes when we consider the proposed limit  $M \rightarrow \infty$ , keeping  $Ml$  constant.

(2) Instead of rescaling the vierbein, it is consistent to compensate the ill-defined term by a counterterm. This implies that consistently a renormalization condition can be imposed which guarantees the anomaly to have the value (4.1). Even if one renders this extra term finite by a rescaling as in Ref. [3], one has to confront the fact that a (finite) renormalization condition can be imposed which settles the anomaly at this value. Further, if one were to keep this extra finite term, it would be undetermined, and is thus not related to the anomaly at all. Also, on-shell renormalization conditions adjust the wave function renormalization of the fermion propa-

gator to have unit residuum at the physical mass. Any rescaling of the tetrad cannot dispense for the fact that the NY term needs renormalization by itself, as it is proven by the very calculation of Ref. [3].

(3) From a renormalization group point of view, it is the scaling of the coupling which determines the scaling of the anomaly (regarded as a Green's function), a property which is desperately needed to maintain the validity, e.g., of the proof of the Adler-Bardeen theorem. Or, to put it otherwise, an anomaly is stable against radiative corrections for the reason that such corrections are compensated by a renormalization of the coupling. While, on the other hand, the topological invariant of Ref. [3] has no such property, its interpretation as an anomaly seems dubious to us. The only consistent way out is to impose a renormalization condition which adjust the finite value of the NY term to be zero, which is patently stable under radiative corrections.

(4) Finally, it is well known that usually the appearance of a chiral anomaly is deeply connected with the presence of a conformal anomaly [17–19]. This makes sense: usually, conformal invariance is lost due to the (dynamical) generation of a scale. But this is the very mechanism which destroys chiral invariance as well. Thus, one would expect any argument to fail trying to combine strict conformal invariance with a chiral anomaly.

## V. CONCLUSIONS

Our conclusion is that the NY term  $dC_{TT}$  does *not* contribute to the chiral anomaly in  $n=4$  dimensions, either classically or in quantum field theory, in sharp contrast to Ref. [3]. We once more stress the interrelation between the scale and chiral invariance [17–19]. Since renormalization amounts to a continuous scale deformation, only the Riemannian part of the Pontrjagin term contributes to the topology of the chiral anomaly.

The result of Chandia *et al.* is very different in spirit than a typical anomaly, where, in perturbative QFT (PQFT), the relevant Green's function is UV finite and only implicitly depends on the scale via the coupling. This contrasts the fact that the Chandia *et al.* term would depend explicitly on that scale.

Since the  $A$  is *not* a gauge field, one can also legitimately absorb the contribution from the axial torsion covector  $A$  into the redefined gauge-invariant physical current  $\hat{j}_5 := j_5 + (1/96\pi^2)A \wedge dA + M^2 *A$ , where the last term depends explicitly on the regulator mass  $M$ . It may arise from the counterterm  $M^2 A \wedge *A$  to the Dirac Lagrangian (3.2). Then, because of

$$\begin{aligned} \langle d\hat{j}_5 \rangle &= \langle dj_5 \rangle + (1/96\pi^2) dA \wedge dA + M^2 d^*A \\ &= 2m \langle iP \rangle + (1/24\pi^2) \text{Tr}(\Omega^{\flat} \wedge \Omega^{\flat}), \end{aligned} \quad (5.1)$$

only the *Riemannian* contribution remains for the axial anomaly of this new *physical* current. A consistent way to avoid regularization problems for  $M \rightarrow \infty$  is to assume that the ‘‘photon’’  $A$  is transverse, i.e.,  $d^*A = 0$ , which is just the vanishing of the NY term.

One would surmise that in  $n=2$  dimensional models only the term  $d^*A$  survives in the heat kernel expansion (4.2), since it then has the correct dimensions. However, it is well known [7] that in 2D the axial torsion  $A$  vanishes identically.

In general, the Pontrjagin topological invariant, as an integral over a *closed* four-form, depends also upon the second fundamental form of the embedding of the boundary  $\partial M$  into  $M$ . In Ref. [20] this is generalized to spaces with torsion supporting our view that the index shall be independent of regulator masses, hence excluding contributions from the NY term. Since the integral  $\int_M d^*A \equiv \int_{\partial M} *A$  vanishes identically for manifolds without boundary, the NY invariant would occur only for nonclosed manifolds, anyhow.

A situation where torsion is indeed realized in a discontinuous manner arises for the cosmic string solution within the EC theory [21–23]. We have shown in detail in Ref. [10] that the NY term (2.2) vanishes identically for this example

of a spinning cosmic string exhibiting a torsion line defect. The instantons of Ref. [3] with nonvanishing NY term are clearly detached from an Einstein-type dynamics, and also the recent analysis in Ref. [24] fails to substantiate the presence of the NY term.

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