Exact black hole entropy bound in conformal field theory

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We show that a Rademacher expansion can be used to establish an exact bound for the entropy of black holes within a conformal field theory framework. This convergent expansion includes all subleading corrections to the Bekenstein-Hawking term.

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The statistical derivation of the Bekenstein-Hawking entropy formula for black holes in string theory [1] relies heavily on the Cardy formula for the asymptotic density of states in two-dimensional conformal field theory [2]. A particularly important example is the $(2+1)$-dimensional Bañados-Teitelboim-Zanelli (BTZ) black hole in anti–de Sitter space [3]. The asymptotic symmetry algebra in this case consists of a Virasoro algebra with both left-moving and right-moving sectors; this is known as the Brown-Henneaux algebra [4]. A simple application of the Cardy formula then yields the Bekenstein-Hawking entropy of the BTZ black hole [5,6]. The significance of this result follows from the fact that many black holes in string theory have a near-horizon structure containing the BTZ black hole. As a result, the derivation of the entropy in these examples is based essentially on the BTZ case [5,7–9].

In [10], it was pointed out that exact convergent expressions (Rademacher expansions [11]) exist for the Fourier coefficients of $SL(2,\mathbf{Z})$ modular forms. These were then used to investigate in detail the anti–de Sitter/conformal field theory (AdS/CFT) correspondence for type IIB string theory on $\mathrm{AdS}_3\times \mathrm{S}^3\times \mathrm{K3}$. One important example of such a Rademacher expansion is the exact formula for the partition function of an integer. Using this expansion, for example, the usually quoted asymptotic formula of Hardy-Ramanujan [12] can be simply derived. In this paper, we observe that a Rademacher expansion gives a precise way to determine the nature of all subleading corrections to the Bekenstein-Hawking black hole entropy. In effect, this generalizes the Cardy formula beyond the leading term. Furthermore, due to its convergence, the expansion leads directly to an exact entropy bound. We also point out that the logarithmic correction to the Bekenstein-Hawking entropy obtained recently within a conformal field theory framework [13] follows immediately from the Rademacher expansion. This logarithmic correction term first appeared in the quantum geometry formalism [14].

We are interested in how the microscopic degrees of freedom of a conformal field theory encode information about the entropy of a macroscopic black hole. The starting point is to consider the modular invariant partition function of a unitary conformal field theory defined on a two-torus, namely

$$Z(\tau)=\mathrm{Tr}\, e^{2\pi i(L_0-c/24)\tau}. \tag{1}$$

Here, τ is the modular parameter and c is the central charge of the conformal field theory. The Fourier expansion of the partition function takes the form

$$Z(\tau)=\sum_{n\geqslant 0} F(n)e^{2\pi i(n-c/24)\tau}. \tag{2}$$

The black hole entropy in this framework is given by $S=\ln F(n)$, for large n. Such a procedure is applicable to a wide variety of black hole geometries [1,5–9,15,16]. To study S, we use an exact convergent expansion, due to Rademacher [11], for the Fourier coefficients of a modular form of weight ω. This is given by [10]

$$\begin{aligned}F(n)=2\pi\sum_{m-c/24<0}\left(\frac{n-\frac{c}{24}}{\left|m-\frac{c}{24}\right|}\right)^{(\omega-1)/2} F(m)\\ \times\sum_{k=1}^{\infty}\frac{1}{k}Kl\left(n-\frac{c}{24},m-\frac{c}{24};k\right)\\ \times I_{1-\omega}\left(\frac{4\pi}{k}\sqrt{\left|m-\frac{c}{24}\right|\left(n-\frac{c}{24}\right)}\right).\end{aligned} \tag{3}$$

Here, $I_{1-\omega}$ is the standard Bessel function and $Kl(n,m;k)$ is a Kloosterman sum defined by [10]

$$Kl(n,m;k)=\sum_{d\in(\mathbf{Z}/k\mathbf{Z})^*}\exp\left[\frac{2\pi i}{k}(dn+d^{-1}m)\right]. \tag{4}$$

We are interested in the convergent expansion of $F(n)$ for $\omega=0$. For large n, this will correspond to the density of states in the conformal field theory for large eigenvalues of L_0. To begin, let us set $m=0$ and $F(0)=1$ in Eq. (3). This leads to the expression

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$$F(n)=\frac{c}{S_0}e^{S_0}\frac{\pi^2}{3}\sum_{k=1}^{\infty}\frac{1}{k}Kl\left(n-\frac{c}{24},-\frac{c}{24};k\right)\left[e^{-S_0}I_1\left(\frac{S_0}{k}\right)\right], \tag{5}$$

where

$$S_0=2\pi\sqrt{\frac{c}{6}\left(n-\frac{c}{24}\right)}. \tag{6}$$

Note that we have extracted a factor of e^{S_0} from the summation over k. The main point to recall is that $e^{-x}I_1(x)<1$, for all x [17]. Consequently, $e^{-x}I_1(x/k)<1$, for all x and for all k; in fact, one can show that $e^{-x}I_1(x/k)<1/k$, for all x and for all k. It is also known that the Kloosterman sum is bounded by $k^{1/2}$ [10,18]. It then follows that we can at least bound the argument of the summation in Eq. (5) by $1/k^{3/2}$. Thus, we obtain the following bound on $F(n)$:

$$F(n)<e^{S_0}\frac{c}{S_0}\frac{\pi^2}{3}\zeta\left(\frac{3}{2}\right), \tag{7}$$

where $\zeta(3/2)$ is the Riemann zeta function. The corresponding entropy $S=\ln F(n)$ is then bounded as follows:

$$S<S_0+\ln\left[\frac{c}{S_0}\frac{\pi^2}{3}\zeta(3/2)\right]. \tag{8}$$

At this point, we simply observe that the logarithmic term will yield a negative contribution if we impose the constraint

$$\frac{c}{S_0}<\frac{3}{\pi^2\zeta(3/2)}. \tag{9}$$

Subject to this constraint, the convergent Rademacher expansion leads directly to the exact entropy bound

$$S<S_0. \tag{10}$$

Below, we show that this constraint is naturally satisfied in many examples of interest. The entropy bound is exact in the sense that it is derived from an exact convergent Rademacher expansion.

We should remark on the restriction of the above analysis to the $m=0$ term in (3). For general m, the Bessel function term will be of the form $I_1[(S_0/k)\sqrt{|m-c/24|/(c/24)}]$. However, we can still extract a factor equal to the right-hand side of Eq. (7) from each of the terms in the m-summation. It follows that the terms with $m\neq 0$ are exponentially suppressed compared to the $m=0$ term. We then obtain a similar result to Eq. (8), except that the argument of the logarithmic term contains additional exponentially suppressed terms. Since we are interested in $\ln F(n)$ for large n, and hence for large S_0, these additional terms do not affect the bound (10).

As an explicit example, one can consider the BTZ black hole which is parametrized by its mass $M=(r_+^2+r_-^2)/8Gl^2$ and angular momentum $J=r_+r_-/4Gl$. Here, $\Lambda=-1/l^2$ is the cosmological constant, and $r_\pm$ denote the location of the inner and outer horizons. The Brown-Henneaux algebra relates the mass and angular momentum to the Virasoro generators by

$$Ml=L_0+\overline{L}_0-\frac{c}{12}, \quad J=L_0-\overline{L}_0. \tag{11}$$

Here, the normalization is that $L_0=\overline{L}_0=c/24$ corresponds to the zero mass black hole, and $L_0=\overline{L}_0=0$ corresponds to anti–de Sitter space. The central charge of the Virasoro algebra is $c=\overline{c}=3l/2G$.

Although we have derived the entropy bound for a single sector, one can consider the situation for a conformal field theory with both left-moving and right-moving sectors. However, for convenience, let us consider the extremal BTZ black hole, with $Ml=J$. Then, $\overline{L}_0=c/24$, and moreover

$$S_0=\frac{A}{4G}, \tag{12}$$

where $A=2\pi r_+$ is the length of the horizon. Note that since we are considering macroscopic black holes, the mass is large in Planck units, i.e., $r_+\gg l$. Thus, we see that the constraint (9) is automatically satisfied. As a result, the total entropy is bounded by the Bekenstein-Hawking term $A/4G$.

It is also of interest to examine the nature of the leading terms in the Rademacher expansion. From Eq. (5), we have

$$F(n)=\frac{(2\pi)^{3/2}}{12}c\,S_0^{-3/2}e^{S_0}\left[1-\frac{3}{8S_0}-\cdots\right]. \tag{13}$$

Here, we have written only the most dominant $k=1$ term. By comparison, the terms with $k>1$ are exponentially damped because of the $1/k$ factor in the Bessel function. We have also used the asymptotic expansion of the Bessel function $I_1(z)=(1/\sqrt{2\pi z})e^z[1-3/8z-\cdots]$, for $\mathrm{Re}(z)\to+\infty$. The leading terms in the black hole entropy are then given by

$$S=S_0-\frac{3}{2}\ln S_0+\ln c+\text{const.} \tag{14}$$

This reveals the presence of logarithmic corrections to the Bekenstein-Hawking entropy S_0; this is the usual correction term which arises from the power-like factor multiplying the asymptotic density of states [10]. As an example, for the extremal BTZ black hole considered above, one finds a logarithmic correction to the entropy of the form $-3/2\ln(A/4G)$. In [13], the original derivation of the Cardy formula was extended to include the first subleading correction. Indeed, Eq. (14) is in precise agreement with the formula derived in [13]. One interesting point to note is that the $-3/2\ln S_0$ term first appeared in the quantum geometry formalism [14].

However, the main point to stress here is that the Rademacher expansion is an exact convergent expression which determines all subleading corrections. In particular, we note that it is an expansion in which the central charge c and Virasoro generator L_0 always appear in the combination S_0 defined by Eq. (6), with one exception. There is a lone factor of the central charge c in Eq. (13), which leads to the $\ln c$ term in Eq. (14). Thus, within the conformal field theory framework, the coefficients of the logarithmic terms are fixed.

In [15,16], a conformal field theory approach to black hole entropy in arbitrary dimensions has been suggested. By treating the horizon as a boundary, one finds that with a suitable choice of boundary conditions the algebra of diffeomorphisms in the $(r-t)$-plane near the horizon is a Virasoro algebra. The central charge and Virasoro generator are given by

$$c=\frac{3A}{2\pi G}\frac{\beta}{\kappa}, \quad L_0=\frac{A}{16\pi G}\frac{\kappa}{\beta}. \tag{15}$$

Here, β is an arbitrary parameter, κ is the surface gravity, and A is the horizon area. We note the contrast with the BTZ black hole, where the curvature scale of anti–de Sitter space leads to a central charge $c=3l/2G$ which is independent of the area. In this case, we find the leading term reproduces the Bekenstein-Hawking entropy, i.e., $S_0=A/4G$, provided $\beta/\kappa \ll 1$. The constraint (9) is then automatically satisfied, and we again have the exact entropy bound $S<A/4G$.

In conclusion, we have used the convergent expansion of Rademacher to show that the Bekenstein-Hawking entropy provides an exact bound for the entropy of black holes within the two-dimensional conformal field theory framework.

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