Consistency relations for an implicit n-dimensional regularization scheme

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We extend an implicit regularization scheme to be applicable in the n-dimensional space-time. Within this scheme divergences involving parity violating objects can be consistently treated without recoursing to dimensional continuation. Special attention is paid to differences between integrals of the same degree of divergence, typical of one loop calculations which are, in principle, undetermined. We show how to use symmetries in order to fix these quantities consistently. We illustrate with examples in which regularization plays a delicate role in order to both corroborate and elucidate the results in the literature for the case of CPT violation in extended QED$_4$, topological mass generation in three-dimensional gauge theories, the Schwinger model, and its chiral version.

I. INTRODUCTION AND MOTIVATIONS

The renormalizability of the standard model (SM) of particle physics underpins its predictive power. Within perturbation theory, a formal proof of renormalizability entreats a gauge invariant regularization scheme. Whereas dimensional regularization (DR) is known as the most powerful and pragmatic method in the continuum space-time, care must be exercised when dealing with theories whose symmetry content depends on the space time dimension, such as chiral gauge theories and topological field theories. In other words, when parity violating objects such as γ_5 matrices or $\epsilon_{\mu_1\mu_2\cdots}$ tensors occur in the theory, an appropriate extension of DR must be performed since the properties of these objects clash with the idea of analytical continuation on the dimension of the space-time n. This is the case of the electroweak sector of the SM[1] as well as Chern-Simons (CS) and CS-matter type of theories. Although such an extension may be explicitly constructed, namely, the t' Hooft-Veltman dimensional continuation (tHVDC), it is not unique and several modifications were suggested [2]. This, in turn, may give rise to ambiguities (which we shall discuss throughout this paper) and the appearance of spurious anomalies. The latter is ultimately related to the asymmetric definition of γ_5 when the Dirac

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[1]The only consistent framework in which the renormalization of the electroweak SM in the continuum four-dimensional (4D) space-time can be carried out to all orders is algebraic renormalization within the Bogolubov-Parasiuk-Hepp-Zimmermann (BPHZ) formalism [1]. However for practical purposes it is rather involved as chiral and vector gauge symmetries are broken in intermediate stages which renders the calculations hard to handle.

algebra is extended to n dimensions [3]. Although on one hand, such shortcomings may in principle be controlled by imposing the Ward-Slavnov-Taylor identities order by order (and introducing new finite counterterms), on the other hand, this turns the calculations significantly cumbersome [4]. Some very interesting views on this subject have been recently presented [5].

Among the topological field theories in the three-dimensional (3D) space-time, the perturbative computation of (pure) CS theories have applications to both mathematics [6] and physics [7], even though some exact results can be drawn nonperturbatively [8]. When coupled to matter fields CS-matter theories are no longer exactly solvable in general. Notwithstanding, they have a wide range of applications in condensed matter physics (for a nice account see Refs. [9,10] and references therein). In either case, the regularization ambiguities stemming from their perturbative evaluation have been an everlasting matter of debate [11,12]. The third rank antisymmetric tensor in the CS Lagrangian is just the 3D analogue of the γ_5 in 4D theory. A naive DR cannot make the theory well defined [13], whereas a so-called dimensional reduction, that is the evaluation of the entire antisymmetric tensor algebra in 3D while the loop momentum integrations are performed in n dimensions, can be shown to be inconsistent [11]. The most accepted scheme is the Becchi–Rouet–Stora- (BRS)-invariant hybrid regularization comprising a high covariant derivative (HCD) term added in the Lagrangian (for instance, a Yang-Mills term $1/\Lambda^2 \operatorname{tr} F^2$) [14] and the tHVDC. The former renders the model power counting super-renormalizable and the remaining finite number of diagrams that are left unregularized must be regulated by the latter. The limits $n \to 3$ and $\Lambda \to \infty$ are well defined [15–17] and should be taken in the end. Some comments are in order. If we consider CS theory as a large topological mass limit of a topologically massive Yang-Mills theory, as conjectured

by Jackiw [18], then the Yang-Mills piece of the Lagrangian is the natural candidate for the HCD term. However, in principle one can construct higher covariant derivative terms at will using covariant derivatives [19]. In fact, as regards of the famous shift/nonshift of the CS parameter,[2] a class of (local, BRS-invariant) higher covariant derivative regularizations were used (see Giavarini *et al.* [12]). The shift depended on the large momentum leading term of the regularized action being parity even or parity odd.[3]

Another instance where the regularization ambiguities play a delicate role is in nonrenormalizable models, often used as effective theories of QCD. In such cases the regularization scheme is frequently defined as part of the model and any parameters introduced by a specific choice must be adjusted phenomenologically [20].

Given the scenario above and the need to go to higher orders in perturbation theory as the precision of the experiments increase, it would be desirable to find a regularization framework for diagrammatic computations which preserved the advantages of DR without the need to dimensionally continue γ_5 or the antisymmetric tensor and/or introduce HCD terms in the Lagrangian.

Recently, a step in this direction was taken. A technique was proposed for the manipulation and calculation of (4D) divergent amplitudes in a way that a regularization need only to be assumed implicitly [21,22]. The integrands are algebraically manipulated until the infinities are displayed in the form of basic divergent integrals in the loop momenta, which do not need to be explicitly evaluated in order to obtain the standard physical results (they can be fully absorbed in the definition of the renormalization constants). No dimensional continuation is involved and a regulator needs only implicitly to be assumed to mathematically justify the algebraic steps in the integrands of the divergent integrals.

An important ingredient of this technique is a set of consistency relations (CR) expressed by differences between divergent integrals of the same degree of divergence. In Ref. [21], it was shown that such CR should vanish in order to avoid ambiguities related to the various possible choices for the momentum routing in certain amplitudes involving loops, consistently with gauge invariance. This is an important feature of DR and it can be easily checked that the CR are readily fulfilled in the framework of DR. Alternatively and more generically, we can assign an arbitrary value to such CR and let gauge invariance to determine its value.

Our purpose in this contribution is twofold. First, to generalize this approach to be applicable for theories defined in any dimension n by deriving the corresponding

n-dimensional CR. This is very important in order to both treat ambiguities related to a particular choice of regularization and simplify the loop calculations in dimensions other than four; i.e., in the CS-matter theories. Second, for the purpose of illustration, we have selected examples where different regularization schemes have somewhat generated controversy in the literature.

This paper is organized as follows. In Sec. II we derive the CR for an n-dimensional regularization and then we proceed to illustrate in the context of theories defined in 4D in Sec. III for standard 4D (QED_4). In Sec. IV, we revisit a well-known example in 4D: the radiative generation of a CPT and Lorentz violating Chern–Simons-type term by introducing a term $\bar{\psi}\slashed{b}\gamma_5\psi$ in the fermionic sector of QED. In Sec. V, we study the topological mass generation in 3D QED and analyze, as an example, in 2D the Schwinger model and its chiral version in Sec. VI. Finally, we draw our conclusions and present some applications in which our scheme can be useful.

II. CONSEQUENCES OF MOMENTUM ROUTING INDEPENDENCE

Consider a one loop two-point function with two vertices Γ_i and Γ_j and let k be the momentum running in the loop. In each propagator that forms the loop we are allowed to add arbitrary 4-momenta, say k_1 and k_2, consistently with the momentum-energy conservation. In Ref. [21], it was shown for $\Gamma_i = \Gamma_j = 1$ (SS), $\Gamma_i = \Gamma_j = \gamma_5$ (PP), $\Gamma_i = \gamma_\mu$ and $\Gamma_j = \gamma_\nu$ (VV) and $\Gamma_i = \gamma_\mu \gamma_5$, and $\Gamma_j = \gamma_\nu \gamma_5$ (AA) that if these amplitudes were to be independent of the arbitrary momentum routing, that is to say, if they were translational invariant and consequently a shift the momentum integration variable was allowed, then a set of consistency relations (CR) between integrals of the same degree of divergence had to hold in the sense that the difference between the two integrals must vanish. Such a feature is manifested within DR (and hence DR obeys the CR) but may not be satisfied by other gauge invariant regularizations. The existence of at least two regularizations defined solely on the space time dimension of the theory that realize the CR was also shown in Ref. [21]. The same CR can be readily derived by imposing translational invariance on the (free) propagators of the theory. In this section, we derive the CR for an arbitrary space-time dimension n.

For definiteness, consider the free fermionic Green's function

$$S(x-x') = \int \frac{d^n k}{(2\pi)^n} \frac{e^{ik(x-x')}}{\slashed{k}-m} \tag{1}$$

and the corresponding "translated" Green's function (i.e., a different representation of the same object)

$$S_l(x-x') = \int \frac{d^n k}{(2\pi)^n} \frac{e^{i(k+l)(x-x')}}{\slashed{k}+\slashed{l}-m}, \tag{2}$$

where l is an arbitrary momentum. Since we are dealing with

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[2]A topological Ward identity constrains such shift to be an integer.

[3]This could well be one of the cases in which the radiative correction is finite but undetermined since, nonperturbatively, one can only assert that the β function vanishes to all orders what does not discard a finite correction.

distributions, the action of these objects require the definition of a set of test functions on which they act. It is straightforward to see that translational invariance implies that

$$\int S(x-x')j(x')d^nx' = \int S_l(x-x')j(x')d^nx'. \quad (3)$$

Thus we conclude that $S_l(x-x')$ should be independent of l:

$$\frac{d}{dl}\int S_l(x-x')\,\eta(x')d^nx' = 0. \quad (4)$$

This condition guarantees that the generating functional for the free theory $Z_0[\eta] = N\exp\{-i\int \bar\eta(x)S_l(x-y)\eta(y)d^nx d^ny\}$ does not depend on the particular Fourier representation that has been used, provided the test functions have the adequate physical behavior. The same will hold true for the generating functional of the interacting theory [21]. Let us take a closer look at the l dependence of the Green's function

$$\int S_l(x,y)\,\eta(y)d^ny$$

$$= \int d^ny \int \frac{d^np}{(2\pi)^n}\frac{e^{i(p+l)(x-y)}}{\not p + l - m}\int \frac{d^nq}{(2\pi)^n}e^{iqy}\,\eta(q). \quad (5)$$

One can now integrate over y to obtain

$$\int S_l(x,y)\,\eta(y)d^ny = \int \frac{d^np}{(2\pi)^n}\int \frac{d^nq}{(2\pi)^n}\frac{e^{i(p+l)x}}{\not p + l - m}$$

$$\times \eta(q)\delta^n(q-p-l)$$

$$= \int \frac{d^nq}{(2\pi)^n}\frac{e^{i(p+l)x}}{\not p + l - m}\,\eta(p+l), \quad (6)$$

which can also be conveniently rewritten in terms of the translation operator as

$$\int S_l(x,y)\,\eta(y)d^ny = \int \frac{d^np}{(2\pi)^n}e^{l^\mu\partial_\mu}\left(\frac{e^{ipx}}{\not p - m}\,\eta(p)\right)$$

$$= \int \frac{d^np}{(2\pi)^n}\left\{\frac{e^{ipx}}{\not p - m}\,\eta(p)\right\}$$

$$+ l^\mu\int \frac{d^np}{(2\pi)^n}\frac{\partial}{\partial p^\mu}\left\{\frac{e^{ipx}}{\not p - m}\,\eta(p)\right\}$$

$$+ \ldots . \quad (7)$$

The first term on the right-hand side is the result which expresses ''translational'' invariance as required by Eq. (3). All the other terms are surface terms which, provided $\eta(p)$ decays sufficiently fast as required on physical grounds, should

vanish identically. But on the improper integrals, S_l will act over a distribution,

$$\int S_l(x,y)D(x,y)d^ny d^nx, \quad (8)$$

typically the delta function or products of particle Green's function. So, we have

$$\int S_l(x,y)D(x,y)d^ny d^nx = \int \frac{d^np}{(2\pi)^n}e^{l^\mu\partial_\mu}\left(\frac{1}{\not p - m}D(p)\right). \quad (9)$$

For $D(x,y) = \delta(x-y)$, we have $D(p) = 1$, and, for instance, for the second term on the right-hand side

$$l^\mu\int^\Lambda \frac{d^np}{(2\pi)^n}\frac{\partial}{\partial p^\mu}\left\{\frac{1}{\not p} - m\right\}. \quad (10)$$

At this point, since the integral is divergent, some regulating procedure must be adopted. Assume that the ultraviolet divergent integrals in the momentum (say, k) are regulated by the multiplication of the integrand by a regularizing function $G(k^2,\Lambda_i)$,

$$\int_k f(k) \to \int \frac{d^nk}{(2\pi)^n}f(k)G(k^2,\Lambda_i) \equiv \int_k^\Lambda f(k), \quad (11)$$

where Λ_i are the parameters of a distribution G whose behavior for large k renders the integral finite. We shall only assume that such a regulator is even in k and that the connection limit $\lim_{\Lambda_i\to\infty}G(k^2,\Lambda_i) = 1$ is well defined. The latter will guarantee that the value of the finite amplitudes will not be affected by taking the limit. Now the integrand of Eq. (10) can be written as a difference of two divergent integrals of the same degree of divergence, namely

$$\int^\Lambda \frac{d^np}{(2\pi)^n}\frac{\partial}{\partial p^\mu}\left\{\frac{1}{\not p - m}\right\}$$

$$= \gamma_\nu\left\{\int^\Lambda \frac{d^np}{(2\pi)^n}\frac{g^{\mu\nu}}{p^2 - m^2} - \int^\Lambda \frac{d^np}{(2\pi)^n}\frac{2p^\mu p^\nu}{(p^2 - m^2)^2}\right\}. \quad (12)$$

If we vary the number of Lorentz indices in the integrals, we obtain, for a certain degree of divergence, other relations in the higher orders of the expansion (7). Moreover, the degree of divergence of the integrals depends on the dimension n.

1+1 Dimensions:

$$\Delta^0_{\mu\nu} \equiv \int^\Lambda \frac{d^2k}{(2\pi)^2}\frac{g_{\mu\nu}}{k^2 - m^2} - 2\int^\Lambda \frac{d^2k}{(2\pi)^2}\frac{k_\mu k_\nu}{(k^2 - m^2)^2}. \quad (13)$$

2+1 Dimensions:

$$\Xi^1_{\mu\nu} \equiv \int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{g_{\mu\nu}}{k^2-m^2} - 2\int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{k_\mu k_\nu}{(k^2-m^2)^2},$$
(14)

$$\Xi^1_{\mu\nu\alpha\beta} \equiv (g_{\mu\nu}g_{\alpha\beta} + g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})\int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{1}{k^2-m^2}$$

$$-8\int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{k_\mu k_\nu k_\alpha k_\beta}{(k^2-m^2)^3},$$
(15)

etc.

3+1 Dimensions:

$$\Upsilon^2_{\mu\nu} \equiv \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{g_{\mu\nu}}{k^2-m^2} - 2\int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2-m^2)^2},$$
(16)

$$\Upsilon^0_{\mu\nu} \equiv \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{g_{\mu\nu}}{(k^2-m^2)^2} - 4\int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2-m^2)^3},$$
(17)

$$\Upsilon^2_{\mu\nu\alpha\beta} \equiv (g_{\mu\nu}g_{\alpha\beta} + g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})\int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2}$$

$$-8\int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu k_\alpha k_\beta}{(k^2-m^2)^3},$$
(18)

$$\Upsilon^0_{\mu\nu\alpha\beta} \equiv (g_{\mu\nu}g_{\alpha\beta} + g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})$$

$$\times \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)^2} - 24\int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu k_\alpha k_\beta}{(k^2-m^2)^4},$$
(19)

etc. Hence, in order to assure momentum routing independence, we have to set the Δ's, Ξ's, and Υ's to vanish. A simple illustration of this feature will be drawn in Sec. III.

It is interesting to notice that precisely the same type of relations between divergent integrals may appear in an n-dimensional theory in connection with gauge invariance. In order to show this, let us consider a generic form for the polarization tensor:

$$\Pi_{\mu\nu}(k^2) = g_{\mu\nu}\Pi(0) + g_{\mu\nu}k^2\Pi_1(k^2) + k_\mu k_\nu\Pi_2(k^2).$$
(20)

Gauge invariance implies that

$$k^\mu\Pi_{\mu\nu}(k^2) = 0,$$
(21)

which is only true if $\Pi_{\mu\nu}(0) = 0$. We can write this, for the one loop calculation, as

$$\Pi_{\mu\nu}(0) = \int^\Lambda \frac{d^np}{(2\pi)^n} \frac{T_{\mu\nu}}{(p^2-m^2)^2},$$
(22)

where

$$T_{\mu\nu} = Ap^2g_{\mu\nu} + Bm^2g_{\mu\nu} + Cp_\mu p_\nu,$$
(23)

and A, B, and C are constants. Since $\Pi_{\mu\nu}(0) = 0$, we can suppose that the integrand is a total derivative, and investigate if there exist A, B, and C which satisfy the condition

$$\frac{T_{\mu\nu}}{(p^2-m^2)^2} = \frac{\partial}{\partial p^\mu}\left\{\frac{Dp_\nu}{p^2-m^2}\right\},$$
(24)

where D is also a constant. After a simple algebra, we conclude that $A = -B = D$ and $C = -2D$, so that

$$\Pi_{\mu\nu}(0) = D\left\{\int^\Lambda \frac{d^np}{(2\pi)^n} \frac{g_{\mu\nu}}{p^2-m^2} - 2\int^\Lambda \frac{d^np}{(2\pi)^n} \frac{p_\mu p_\nu}{(p^2-m^2)^2}\right\}$$

$$= 0.$$
(25)

In this case, we may say that the same condition is required to preserve both momentum routing independence and gauge invariance. However, in physical applications we should privilege the latter upon the former since there are examples in which gauge invariance can only be attained at the cost of adopting an especific momentum routing [23], namely, when one axial vertex is involved. We will come back to this issue in Sec. IV.

III. QED$_4$

In this section, we illustrate our regularization framework within QED in four dimensions in order to compare with well-known results, as well as to gain some insight especially in the role played by an arbitrary routing in the loop momentum of an amplitude in connexion with the CR.

Consider the vacuum polarization tensor to one loop order with arbitrary internal momentum routing

$$\Pi_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\{\gamma_\mu S(k+k_1)\gamma_\nu S(k+k_2)\},$$
(26)

where $S(k)$ is a usual half spin fermion propagator carrying momentum k. In order to make the arbitrary momentum dependence more explicit, Eq. (26) may be rewritten, after taking the trace over the Dirac matrices, as

$$\Pi_{\mu\nu} = 4\left(\int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{2k_\mu k_\nu}{[(k+k_1)^2-m^2][(k+k_2)^2-m^2]}\right.$$

$$+(k_1+k_2)_\nu \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{[(k+k_1)^2-m^2][(k+k_2)^2-m^2]}$$

$$+(k_2+k_1)_\mu \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{k_\nu}{[(k+k_1)^2-m^2][(k+k_2)^2-m^2]} + (k_{2\mu}k_{1\nu}+k_{1\mu}k_{2\nu})$$

$$\times \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{[(k+k_1)^2-m^2][(k+k_2)^2-m^2]}\right) - 2\,g_{\mu\nu}\left(\int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{[(k+k_1)^2-m^2]} + \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{[(k+k_2)^2-m^2]}\right.$$

$$\left.-(k_1-k_2)^2\int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{[(k+k_1)^2-m^2][(k+k_2)^2-m^2]}\right). \tag{27}$$

Now, we manipulate algebraically the integrands until the external momentum dependence appears solely in finite integrals by means of the identity

$$\frac{1}{[(k+k_i)^2-m^2]} = \sum_{j=0}^N \frac{(-1)^j(k_i^2+2k_i\cdot k)^j}{(k^2-m^2)^{j+1}}$$

$$+ \frac{(-1)^{N+1}(k_i^2+2k_i\cdot k)^{N+1}}{(k^2-m^2)^{N+1}[(k+k_i)^2-m^2]}, \tag{28}$$

$i=1,2$, and N is such that the last term in Eq. (28) is finite under integration over k [22]. After some straightforward algebra, we can cast Eq. (27) in the form

$$\Pi_{\mu\nu} = \widetilde{\Pi}_{\mu\nu} + 4\left(\Upsilon^2_{\mu\nu} - \frac{1}{2}(k_1^2+k_2^2)\Upsilon^0_{\mu\nu} + \frac{1}{3}(k_1^\alpha k_1^\beta + k_2^\alpha k_2^\beta\right.$$

$$\left. + k_1^\alpha k_2^\beta)\Upsilon^0_{\mu\nu\alpha\beta} - (k_1+k_2)^\alpha(k_1+k_2)_\mu \Upsilon^0_{\nu\alpha}\right), \tag{29}$$

where

$$\widetilde{\Pi}_{\mu\nu} = \frac{4}{3}((k_1-k_2)^2 g_{\mu\nu} - (k_1-k_2)_\mu(k_1-k_2)_\nu)$$

$$\times\left(I^\Lambda_{log}(m^2) - \frac{i}{(4\pi)^2}\left(\frac{1}{3} + \frac{(2m^2+(k_1-k_2)^2)}{(k_1-k_2)^2}\right.\right.$$

$$\left.\left.\times Z_0((k_1-k_2)^2;m^2)\right)\right), \tag{30}$$

and the Υ's are the CR defined in Eqs. (16)–(19),

$$I^\Lambda_{log}(m^2) = \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)^2}$$

and

$$Z_0(p^2;m^2) = \int_0^1 dz \ln\left(\frac{p^2z(1-z)-m^2}{-m^2}\right). \tag{31}$$

It is clear from Eq. (29) that in order to eliminate the ambiguous terms and to respect the Ward identities ($k_1 - k_2)^\mu\Pi_{\mu\nu} = (k_1-k_2)^\nu\Pi_{\mu\nu} = 0$, we must set all the Υ's to zero. Therefore, we obtain the usual result for the vacuum polarization tensor.

Now let us adopt the particular routing $k_1 = p$ and $k_2 = 0$ and hence let the value of the CR be arbitrary, namely

$$\Upsilon^0_{\mu\nu\alpha\beta} = c(g_{\mu\nu}g_{\alpha\beta} + g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\alpha\nu}),$$

$$\Upsilon^0_{\mu\nu} = a g_{\mu\nu}. \tag{32}$$

Thus we have

$$p^\mu\Pi = 4(p^\mu\Upsilon^2_{\mu\nu} + (c-2a)p^2p_\nu), \tag{33}$$

from which we see that gauge invariance is implemented for the choice

$$\Upsilon^2_{\mu\nu} = 0, \quad c = 2a. \tag{34}$$

This will be important for the discussions in Sec. IV.

Notice that at this point we can compare our result with any sound regularization procedure, for instance, DR by an explicit computation of $I^\Lambda_{log}(m^2)$. However, as far as the physical content is concerned, one need not do so. For instance, consider the calculation of the β function to one

loop order. We add the usual counterterm to define $\tilde{\Pi}_{\mu\nu R}$ $= \tilde{\Pi}_{\mu\nu} + (q_\mu q_\nu - q^2 g_{\mu\nu})(Z_3 - 1)$, $A^\mu = Z_3^{1/2} A_R^\mu$, and $q = k_1 - k_2$. The Callan-Symanzik β function can be written as

$$\beta = e_R \frac{\partial}{\partial \ln \Lambda} (\ln Z_3^{1/2}(e, \Lambda/m)). \quad (35)$$

We may choose the renormalization constant such that $(Z_3 - 1) = \frac{4}{3} i I_{\log}^\Lambda(m^2)$ (which amounts to a subtraction at $q = 0$) to get the well-known one loop result $\beta = 1/(12\pi^2)$ ($e_R = 1$), where we used that $\partial I_{\log}^\Lambda(m^2)/\partial m^2 = -i/((4\pi)^2 m^2)$. In Ref. [22] we also calculate the β function of φ_4^4 theory to two loop order within this approach.

IV. INDUCED LORENTZ AND *CPT* SYMMETRY BREAKING IN EXTENDED QED$_4$

While introducing a Chern-Simons term

$$\mathcal{L}_c = \frac{1}{2} c_\mu \epsilon^{\mu\nu\lambda\rho} F_{\nu\lambda} A_\rho, \quad c_\mu \text{ being a constant 4-vector,} \quad (36)$$

to violate Lorentz and *CPT* symmetries in conventional QED$_4$ [24] undergoes stringent theoretical and experimental bounds [25–27], there have been investigations on possible extensions of the standard model which could give rise to Lorentz and *CPT* violation [28]. A natural question is whether the term expressed in Eq. (36) could be generated radiatively when Lorentz and *CPT* violating terms occur in other parts of a larger theory. For instance, many authors have exploited the possibility of such terms being induced by introducing an explicit Lorentz and *CPT* violating term $b_\mu \bar{\psi} \gamma_\mu \gamma_5 \psi$ in the fermionic sector of standard QED$_4$ [29–33]. In fact, a meticulous work by Chen and Kunstatter [34] seems to rule out such particular extension by studying its effect on the calculation of the lambda-shift and on the anomalous magnetic moment. Hence it would not constitute a physically plausible source of radiatively induced terms like Eq. (36). However, since the issue here is the regularization dependence which is involved in the radiative correction, such calculation serves as a perfect laboratory for examining our framework.

Consider the modified fermionic sector of QED$_4$

$$\mathcal{L}_{\text{fermion}} = \bar{\psi}(i\slashed{\partial} - \slashed{A} - \slashed{b}\gamma^5 - m)\psi, \quad (37)$$

where b_μ is a constant 4-vector which selects a specific direction in space-time and therefore the gauge invariant term $\bar{\psi}\slashed{b}\gamma^5\psi$ explicitly violates *CPT* and Lorentz symmetries. The quantity of interest for deciding whether Eq. (36) is radiatively generated is the $O(A^2)$ part of the extented effective action

$$\Gamma_{\text{ext}}(A) = -i \ln \det(i\slashed{\partial} - \slashed{A} - \slashed{b}\gamma^5 - m), \quad (38)$$

from which the coefficient c_μ is determined from b_μ. To lowest order in b, this corresponds diagramatically to a tri-angle graph composed of two vector currents and one axial

vector current with zero-momentum transfer between the two vector gauge field vertices. Hence we can write generically that

$$\Gamma^2(A) \equiv \int \frac{d^4 p}{(2\pi)^4} A^\mu(-p) \Pi_{\mu\nu}(p) A^\nu(p), \quad (39)$$

with $\Pi_{\mu\nu}(p) \sim b_\alpha \Gamma^{\mu\nu\alpha}(p, -p)$. Now as it was discussed in Ref. [35], $\Gamma^{\mu\nu\alpha}(p, -p)$ is undetermined by an arbitrary parameter a, namely

$$\Gamma^{\mu\nu\alpha}(p, -p) \sim \Gamma^{\mu\nu\alpha}(p, -p) + 2ia\epsilon^{\mu\nu\alpha\beta} p_\beta, \quad (40)$$

which cannot be fixed by requiring transversality of $\Gamma^{\mu\nu\alpha}$. This is in contrast with the famous triangle anomaly and it is essentially because, in our case, the axial vector carries zero momentum. Moreover, there is no anomaly in the axial cur-rent conservation law in this case. The indetermination ex-pressed in Eq. (40) (and therefore in c_μ) is apparent in the b_μ perturbative approach [31,36]. However, following Jackiw [29] one can also carry out a nonperturbative calcu-lation by employing the b_μ-exact propagator

$$S'(k) = \frac{i}{i\slashed{k} - m - \slashed{b}\gamma_5}, \quad (41)$$

which appears to lead to a definite unambiguous result [29,32,33]. Before proceeding to study this problem within our approach, a few comments are in order following Ref. [29]. Because the axial current $j_\mu^5(x) \equiv \bar{\psi}(x)\gamma_\mu\gamma^5\psi(x)$ does not couple to any physical field but b_μ, physical gauge in-variance is achieved provided that j_μ^5 is gauge invariant at zero 4-momentum. This is equivalent to stating that it is the integrated quantity $\int d^4 x j_\mu^5(x)$ which is gauge invariant, in consonance with the fact that the induced quantity which we seek (36) is not gauge invariant while its spacetime integral is. Hence, according to Jackiw [35], any regularization which enforces gauge invariance at all momenta will render a van-ishing result for c_μ such as Pauli-Villars regularization [28]. As for DR, there is not a unique prescription to work within this scheme in the presence of a γ_5 matrix and one has as many results as alternative continuation prescriptions.

We believe that within our scheme, which preserves the characteristics of the theory as much as possible, one has a good setting to study this problem. For this purpose we il-lustrate it for both the nonperturbative and the perturbative in b_μ treatments. We start by calculating the induced term in the nonperturbative in b_μ scheme [29]. The exact propagator (41) can be separated as

$$S'(k) = S_F(k) + S_b(k), \quad (42)$$

where $S_F(k)$ is the usual free fermion propagator and

$$S_b(k) = \frac{1}{i\slashed{k} - m - \slashed{b}\gamma_5} \slashed{b}\gamma_5 S_F(k), \quad (43)$$

whereas the vacuum polarization tensor can be generically written as in [29]

$$\Pi^{\mu\nu} = \Pi_0^{\mu\nu} + \Pi_b^{\mu\nu} + \Pi_{bb}^{\mu\nu}. \tag{44}$$

We are concerned about the second term which is linearly divergent and thus it can be responsible for a momentum-routing ambiguity. Explicitly we have

$$\Pi_b^{\mu\nu}(p) = \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\{\gamma^\mu S_F(k)\gamma^\nu S_b(k+p)$$
$$+ \gamma^\mu S_b(k)\gamma^\nu S_F(k+p)\}. \tag{45}$$

To lowest order in b_μ, we can replace $S_b(k)$ with $-iS_F(k)\slashed{b}\gamma_5 S_F(k)$, so that $\Pi_b^{\mu\nu} = \Pi^{\mu\nu\alpha} b_\alpha$, with

$$\Pi^{\mu\nu\alpha}(p) = -i \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\{\gamma^\mu S(k)\gamma^\nu S(k+p)\gamma^\alpha\gamma_5 S(k+p)$$
$$+ \gamma^\mu S(k)\gamma^\alpha\gamma_5 S(k)\gamma^\nu S(k+p)\}$$
$$\equiv -\{I_1^{\mu\nu\alpha} + I_2^{\mu\nu\alpha}\}. \tag{46}$$

We shall calculate the two integrals separately, without doing a shift for the sake of clarity. The ambiguities in momentum routing discussed by Jackiw will be made explicit in the relations between divergent integrals that will appear. After taking the trace over the Dirac matrices we have

$$I_1^{\mu\nu\alpha} = \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{N}_1^{\mu\nu\alpha}}{[k^2-m^2][(k+p)^2-m^2]^2} \tag{47}$$

and

$$I_2^{\mu\nu\alpha} = \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{N}_2^{\mu\nu\alpha}}{(k^2-m^2)^2[(k+p)^2-m^2]}, \tag{48}$$

where

$$\mathcal{N}_1^{\mu\nu\alpha} = 4i\{\{[(k+p)^2-m^2]k_\beta - 2m^2 p_\beta\}\epsilon^{\mu\nu\alpha\beta}$$
$$- 2p_\sigma k_\beta k^\alpha \epsilon^{\mu\nu\sigma\beta}\} \tag{49}$$

and

$$\mathcal{N}_2^{\mu\nu\alpha} = 4i\{\{-[k^2-m^2](k+p)_\beta - 2m^2 p_\beta\}\epsilon^{\mu\nu\alpha\beta}$$
$$- 2p_\sigma k_\beta k^\alpha \epsilon^{\mu\nu\sigma\beta}\}. \tag{50}$$

Above, we only considered the terms which do not vanish after integration or because of symmetry properties in the Lorentz indices. After some straightforward algebra, we can write

$$I_1^{\mu\nu\alpha} = 4i\{[J_\beta^\Lambda - 2m^2 p_\beta J^1]\epsilon^{\mu\nu\alpha\beta} - 2p_\sigma g^{\alpha\lambda}\epsilon^{\mu\nu\sigma\beta}J_{\beta\lambda}^\Lambda\} \tag{51}$$

and

$$I_2^{\mu\nu\alpha} = 4i\{[-J_\beta^\Lambda - 2m^2 p_\beta J^1 - p_\beta J^\Lambda]\epsilon^{\mu\nu\alpha\beta}$$
$$- 2p_\sigma g^{\alpha\lambda}\epsilon^{\mu\nu\sigma\beta}J_{\beta\lambda}^\Lambda\}, \tag{52}$$

where we defined

$$J^1 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)^2[(k+p)^2-m^2]}, \tag{53}$$

$$J^\Lambda = \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)[(k+p)^2-m^2]}, \tag{54}$$

$$J_\beta^\Lambda = \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{k_\beta}{(k^2-m^2)[(k+p)^2-m^2]}, \tag{55}$$

and

$$J_{\beta\lambda}^\Lambda = \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{k_\beta k_\lambda}{(k^2-m^2)^2[(k+p)^2-m^2]}. \tag{56}$$

Among these integrals, the divergent are J, J_β, and $J_{\beta\lambda}$. We can manipulate them using Eq. (28) recursively to obtain

$$J^\Lambda = I_{log}^\Lambda(m^2) - \tilde{J}, \tag{57}$$

$$J_\beta^\Lambda = -2p^\rho\Theta_{\beta\rho}^\Lambda + \tilde{J}_\beta, \tag{58}$$

and

$$J_{\beta\lambda}^\Lambda = \Theta_{\beta\lambda}^\Lambda - \tilde{J}_{\beta\lambda}, \tag{59}$$

where

$$\Theta_{\alpha\beta}^\Lambda = \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{k_\alpha k_\beta}{(k^2-m^2)^3}, \tag{60}$$

$$\tilde{J} = \int \frac{d^4k}{(2\pi)^4} \frac{p^2+2p\cdot k}{[k^2-m^2]^2[(k+p)^2-m^2]}, \tag{61}$$

and

$$\tilde{J}_{\beta\lambda} = \int \frac{d^4k}{(2\pi)^4} \frac{(p^2+2p\cdot k)k_\beta k_\lambda}{[k^2-m^2]^3[(k+p)^2-m^2]}. \tag{62}$$

Now that we removed the external momentum dependence from the divergent integrals, we note that they cancel out in $I_1^{\mu\nu\alpha}$ unambiguously. It remains an undetermined finite term originated from a difference between divergent integrals in $I_2^{\mu\nu\alpha}$. Noting that $p_\beta\tilde{I} = 2\tilde{I}_\beta$ (which can be shown by partial integration), we get

$$I_1^{\mu\nu\alpha} = 4i\{[\tilde{J}_\beta - 2m^2 p_\beta J^1]\epsilon^{\mu\nu\alpha\beta} - 2p_\sigma g^{\alpha\lambda}\epsilon^{\mu\nu\sigma\beta}\tilde{J}_{\beta\lambda}\} \tag{63}$$

and

$$I_2^{\mu\nu\alpha} = I_1^{\mu\nu\alpha} + \frac{\phi}{2\pi^2} p_\beta\epsilon^{\mu\nu\alpha\beta}, \tag{64}$$

where the divergent integrals were combined, as in Eq. (17), to define

$$\Upsilon^0_{\alpha\beta} \equiv g_{\alpha\beta} I^\Lambda_{log}(m^2) - 4\Theta^\Lambda_{\alpha\beta} = \lambda g_{\alpha\beta}, \qquad (65)$$

in which λ is a dimensionless, finite parameter, and we have defined

$$\phi \equiv \frac{8\pi^2}{i}\lambda. \qquad (66)$$

The finite integrals can be readily solved using Feynman parameters, after which we can write

$$\Pi^{\mu\nu\alpha}_{\text{non-pert}} = \epsilon^{\mu\nu\alpha\beta} \frac{p_\beta}{2\pi^2}\left(\frac{\theta}{\sin\theta} - \phi\right), \qquad (67)$$

where $\theta = 2\arcsin(\sqrt{p^2}/(2m))$ and $p^2 < 4m^2$. The equation above is similar to the one encountered by Jackiw and Kostelecký [29], with our ϕ playing the role of their surface term. Now in order to arrive at their claimed unambiguous result within the nonperturbative approach, another information would have to be implemented. As it was discussed in Ref. [32] any regularization that had broken the spherical symmetry in their explicit integration would have altered their result. That is the case of DR which breaks the trace-lessness of the combination $k_\mu k_\nu - 1/4 g_{\mu\nu}k^2$ in the 4D space-time. In calculating the surface term, which is originated from the shift in the linearly divergent integral, one also makes use of such symmetry by performing a symmetric momentum limit $\lim_{k\to\infty}(k_\mu k_\nu)/k^2 = (g_{\mu\nu})/4$. Therefore, it is an easy matter to check that if we use symmetric integration in $\Theta_{\alpha\beta}$ in Eq. (65), then we will obtain that $\phi = 1/4$ in Eq. (67), to give in the limit of heavy fermion mass the result found in Refs. [29,32,33].

Now let us proceed to the perturbative in b computation. The relevant diagrams are the b_μ-linear one loop correction to the photon propagator in which a factor of $ib_\lambda \gamma^\lambda \gamma^5$ can be inserted in either of the two internal fermionic lines to render equal contributions. Thus the amplitude reads

$$\Pi^b_{\mu\nu} = 2(-i)b^\lambda \int \frac{d^4k}{(2\pi)^4} \text{tr}\gamma_\mu S_F(k-p)\gamma_\nu S_F(k)\gamma_\lambda \gamma^5 S_F(k)$$

$$\equiv 2b^\lambda \Pi_{\lambda\mu\nu}, \qquad (68)$$

where p is the external momentum. The integral above is just our $I^{\mu\nu\alpha}_2$ in the nonperturbative with $p \to -p$ and the μ, ν indices interchanged. Therefore, we can write, taking into account the change of signs,

$$\Pi^{\mu\nu\alpha}_{\text{pert}} = \epsilon^{\mu\nu\alpha\beta} \frac{p_\beta}{2\pi^2}\left\{\frac{\theta}{\sin\theta} - \phi'\right\}. \qquad (69)$$

The equation above is to be compared with Eq. (21) in Ref. [31]. Our undetermined parameter ϕ' is just their ratio $\ln(M1/M2)$. Thus we have achieved the same elegance as it is expressed within differential regularization with the advantage of working in the momentum space. This result was expected since we have not made use of an explicit regulator. We, too, have all the results obtained in other regularization schemes embodied in different choices for the parameter ϕ'

which is to be fixed on physical grounds either by symmetry requirements or a renormalization condition; all the possible ambiguities are expressed in terms of our so-called consistency relations which we left arbitrary until the final stage in this case. It is therefore not surprising that our approach achieved the same merits as those claimed within differential regularization.

It is interesting to observe that the indeterminacy expressed by our parameters ϕ and ϕ' are ultimately related to a nonvanishing value for the CR. In other words, the amplitudes considered in this section are not independent of the momentum routing in the loop. Would we have momentum routing independence then the parameter λ and consequently ϕ and ϕ' would be zero. Generically, we can state that the presence of an axial vertex has broken such momentum routing independence. In fact, this is not a new feature. It is well known that only a special routing of the integration momenta may result in a gauge invariant answer in the presence of axial vertices [29,23]. Notice also that total vacuum polarization amplitude (44) has a contribution $\Pi^0_{\mu\nu}$, which corresponds to pure QED_4. Would a particular routing choice violate gauge invariance in $\Pi^0_{\mu\nu}$? As we have seen, if an arbitrary momentum routing is taken, then the Υ's must be zero. On the other hand, it was shown in the last section that if we choose a particular routing, there is another possibility of mantaining gauge invariance namely by fixing the relative coefficients of one CR. Therefore, we can fix the momentum routing in $\Pi^b_{\mu\nu}$ and then adjust $c = 2a$ and $\Upsilon^2_{\mu\nu} = 0$, so as to respect gauge invariance.

V. TOPOLOGICAL MASS GENERATION IN THREE-DIMENSIONAL GAUGE THEORY

As we have seen in Sec. IV, CS terms can be induced by radiative quantum effects even if they are not present as bare terms in the original Lagrangian. In 3+1-dimensional space-time such terms could be induced by extending the fermionic sector of QED with an explicitly Lorentz and CPT violating axial-vector term. In 2+1 dimensions, however, such topological terms can naturally appear at quantum level without any extension in the classical Lagrangian [39]. Consider the QED_3 Lagrangian with fermions of mass m, $L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\Psi(i\not{D} + m)\Psi$. Now let us study the role played by a radiatively generated CS term in the sense of giving a mass for the gauge field.[4] In [37] it was shown that despite being all gauge invariant one could classify a set of regularizations in two groups: one in which the originally massless boson remained massless (such as Pauli-Villars regularization) and another in which it turned out to be massive (such as DR among others). Here we revisit this problem in the light of our framework. Consider an expansion in powers of A of the one loop effective action, namely

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[4]Please see [9] for a complete account on this matter.

$$\Gamma_{QED_3}[A] = \mathrm{tr}\ln\det(i\not\partial - m) + \mathrm{tr}\left(\frac{1}{i\not\partial - m}\not{A}\right)$$

$$+ \frac{1}{2}\mathrm{tr}\left(\frac{1}{i\not\partial - m}\not{A}\frac{1}{i\not\partial - m}\not{A}\right) + \ldots . \quad (70)$$

For an induced mass term, the relevant contribution is the one that is quadratic in A which we write generically as

$$\Gamma^2[A] = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}A^\mu(p)\Pi_{\mu\nu}(p)A^\nu(p), \quad (71)$$

where $\Pi_{\mu\nu}$ is the usual vacuum polarization tensor,

$$\Pi_{\mu\nu}(p) = \int\frac{d^3k}{(2\pi)^3}\mathrm{tr}\left(\gamma_\mu\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\gamma_\nu\frac{\not{k}+m}{k^2-m^2}\right)$$

$$= \int\frac{d^3k}{(2\pi)^3}\frac{\eta_{\mu\nu}}{((p+k)^2-m^2)(k^2-m^2)}, \quad (72)$$

with

$$\eta_{\mu\nu} = 2(2k_\mu k_\nu + 2p_\mu k_\nu - g_{\mu\nu}((k+p)\cdot k - m^2)$$
$$+ im\epsilon_{\mu\nu\alpha}p^\alpha), \quad (73)$$

where we used that in 3D $\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\alpha = -2i\epsilon^{\mu\nu\alpha}$. Thus we can write

$$\Pi_{\mu\nu} = 2(2I_{\mu\nu} + 2p_\mu I_\nu + im\epsilon_{\mu\nu\alpha}p^\alpha I - g_{\mu\nu}I^{(1)} - g_{\mu\nu}p^\alpha I_\alpha), \quad (74)$$

where

$$I, I_\mu, I_{\mu\nu} \equiv \int^\Lambda\frac{d^3k}{(2\pi)^3}\frac{1, k_\mu, k_\mu k_\nu}{((p+k)^2-m^2)(k^2-m^2)}$$

and

$$I^{(1)} \equiv \int^\Lambda\frac{d^3k}{(2\pi)^3}\frac{1}{(p+k)^2-m^2}. \quad (75)$$

Among the integrals defined above only I and I_μ are finite, whereas the others can be rewritten with Eq. (28) as

$$I_{\mu\nu} = \int^\Lambda\frac{d^3k}{(2\pi)^3}\frac{k_\mu k_\nu}{(k^2-m^2)^2}$$

$$- \int^\Lambda\frac{d^3k}{(2\pi)^3}\frac{(p^2+2p\cdot k)k_\mu k_\nu}{((p+k)^2-m^2)(k^2-m^2)}, \quad (76)$$

in which the second integral on the right-hand side is finite and

$$I^{(1)} = I_{lin}^\Lambda(m^2) - \int^\Lambda\frac{d^3k}{(2\pi)^3}\frac{p^2+2p\cdot k}{((p+k)^2-m^2)(k^2-m^2)}, \quad (77)$$

with

$$I_{lin}^\Lambda(m^2) \equiv \int^\Lambda\frac{d^3k}{(2\pi)^3}\frac{1}{k^2-m^2}. \quad (78)$$

However, the second integral on the right-hand side of Eq. (77) vanishes and therefore $I^{(1)} = I_{lin}^\Lambda(m^2)$. With the results given above, $\Pi_{\mu\nu}(p)$ reads

$$\Pi_{\mu\nu}(p) = 2\left(\Xi_{\mu\nu}^1 + F_1(p^2,m)\epsilon_{\mu\nu\alpha}p^\alpha + F_2(p^2,m)\right.$$

$$\left.\times\left(\frac{p_\mu p_\nu}{p^2} - g_{\mu\nu}\right)\right), \quad (79)$$

where $\Xi_{\mu\nu}^1$ is given as in Eq. (14). Generically, we can write $\Xi_{\mu\nu}^1 = \lambda g_{\mu\nu}$, on Lorentz invariance grounds where λ is a parameter (with dimension of mass) to be determined. In order to assure gauge invariance, we are led to set $\Xi_{\mu\nu}^1 = 0$ in this case.[5] This appears to be a natural choice as no parity violating objects appear in the vertex. Moreover, the finite coefficients F_1 and F_2 evaluate to

$$F_1(p^2,m) = \frac{i}{4\pi}\left[\frac{m}{\sqrt{p^2}}\ln\left(\frac{1+\sqrt{p^2/4m^2}}{1-\sqrt{p^2/4m^2}}\right)\right], \quad (80)$$

$$F_2(p^2,m) = \frac{1}{4\pi}\left[m - \frac{1}{4\sqrt{p^2}}(p^2+4m^2)\ln\left(\frac{1+\sqrt{p^2/4m^2}}{1-\sqrt{p^2/4m^2}}\right)\right], \quad (81)$$

which is just the result that is obtained in DR [37–40] . In the limit where $m\to\infty$ we obtain

$$\Pi_{m\to\infty}^{\mu\nu} = \frac{i}{4\pi}\frac{m}{|m|}\epsilon^{\mu\nu\alpha}p^\alpha, \quad (82)$$

which contributes to the one loop effective action with a term that in the coordinate space reads

$$\Gamma_{CS}^2 = -\frac{i}{2}\frac{1}{4\pi}\int d^3x\,\epsilon^{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha. \quad (83)$$

These results can be readily generalized to the non-Abelian case.

VI. SCHWINGER MODEL AND ITS CHIRAL VERSION

As an example in two dimensions, we study the Schwinger model (ScM) (QED_2 with massless fermions) and its chiral version (CScM). The ScM is exactly solvable [41] and has served as a good laboratory for both testing theoretical techniques and getting some insight in the vacuum structure of QCD_4. Several nontrivial features of the ScM and its massive and chiral version (such as massive physical states

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[5]A linearly divergent term $\propto\Lambda g_{\mu\nu}$ which would appear using an explicit cutoff calculation [38] does not appear in our case, as it would not appear in any gauge invariant regularization such as DR.

formed via chiral anomaly, instanton-like vacuum configurations labeled by an θ angle, etc.) have counterparts in more realistic theories [42]. In the ScM, the massless photon of the tree approximation acquires the mass e^2/π (e is the coupling constant) at the one loop level (which is exact in this case). Consider the effective action radiatively induced by fermions:

$$\Gamma_S = -i \ln \det(i\not{\partial} - e\not{A}). \qquad (84)$$

The mass generation is seen at order A^2 which, for this model, determines Eq. (84) completely. Hence all we need to do is to compute the vacuum polarization tensor

$$\Pi_S^{\mu\nu}(p) = i \operatorname{tr} \int \frac{d^2k}{(2\pi)^2} \gamma^\mu \frac{i}{\not{k}} \gamma^\nu \frac{i}{\not{k}+\not{p}}. \qquad (85)$$

After taking the traces, using Eq. (28) to write the divergence as a function of the loop momentum only and evaluating the finite integrals, we obtain (see, also, Ref. [35])

$$\Pi_S^{\mu\nu}(p) = \Pi_\infty^{\mu\nu} + \frac{1}{\pi}\left(\frac{g^{\mu\nu}}{2} - \frac{p^\mu p^\nu}{p^2}\right), \qquad (86)$$

where

$$\Pi_\infty^{\mu\nu} \equiv 2i \int^\Lambda \frac{d^2k}{(2\pi)^2} \frac{(-k^2 g^{\mu\nu} + 2k^\mu k^\nu)}{(k^2 - \mu^2)^2} \qquad (87)$$

and μ^2 is an infrared cutoff which is immaterial for the value of $\Pi_\infty^{\mu\nu}$. Some features are noteworthy. Notice that, in general, $\Pi_S^{\mu\nu}$ is not gauge invariant. Lorentz invariance tells us that $\Pi_\infty^{\mu\nu}$ should be proportional to $g_{\mu\nu}$ but the coefficient is, in principle, undetermined since the integral is divergent. Moreover, if $\Pi_\infty^{\mu\nu}$ assumes any value different from zero it would break the traceless of $\Pi_S^{\mu\nu}$ already manifest in its integral representation (85). However, Pauli-Villars or DR can be employed and gauge invariance restored within this schemes. DR, for instance, evaluates Eq. (87) to $(1/(2\pi))g^{\mu\nu}$ which gives for Eq. (86)

$$\bar{\Pi}_S^{\mu\nu}(p) = \frac{1}{\pi}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right). \qquad (88)$$

Now recall the CR (13), namely

$$\Delta_{\mu\nu}^0 = \int^\Lambda \frac{d^2k}{(2\pi)^2} \frac{g_{\mu\nu}}{k^2 - m^2} - 2 \int^\Lambda \frac{d^2k}{(2\pi)^2} \frac{k_\mu k_\nu}{(k^2 - m^2)^2}. \qquad (89)$$

The choice $\Delta_{\mu\nu}^0 = 0$ can be used in Eq. (87) to obtain a result which is just $(1/(2\pi))g^{\mu\nu}$ as it can be easily demonstrated. Hence gauge invariance is restored within our framework. This is close in spirit to Jackiw's approach in Ref. [35]. In other words, we can state that this particular value for $\Delta_{\mu\nu}^0$ is the one which restores gauge invariance, if we so wish. It plays the role of an undetermined local part in the quadratic term of the effective action. If the ScM really described a

physical particle, we could say that we had to choose $\Delta_{\mu\nu}^0$ to vanish so to explain the photon mass $m^2 = e^2/\pi$.

In order to gain some more intuition, let us make a similar analysis with the CScM. We simply substitute the vector interaction with a chiral interaction in Eq. (84):

$$\Gamma_\chi = -i \ln \det(i\not{\partial} - e(1 + \gamma_5)\not{A}), \qquad (90)$$

$\gamma_5 = \gamma_0 \gamma_1$. An analogous calculation leads us to the result

$$\Pi_\chi^{\mu\nu}(p) = \Pi_S^{\mu\nu}(p) + g_{\alpha\beta}(\epsilon^{\nu\alpha}\Pi_S^{\mu\beta}(p) + \epsilon^{\mu\alpha}\Pi_S^{\beta\nu}(p)) + \epsilon^{\mu\alpha}\epsilon^{\nu\beta}\Pi_S{}_{\alpha\beta}(p), \qquad (91)$$

where we used that

$$\gamma_5 \gamma_\mu = \epsilon^{\mu\nu}\gamma_\nu, \qquad (92)$$

and $\Pi_S^{\mu\nu}(p)$ is given as in Eq. (86). As it is well known [43], there occurs a chiral anomaly in this model: it cannot be made gauge invariant. This is a manifestation of the anomalous nonconservation of the chiral current in the ScM for

$$p_\nu \Pi_5^{\mu\nu} = -\frac{1}{\pi}\tilde{p}^\mu \rightarrow \partial^\nu j_\nu^5 = \frac{e}{\pi}\epsilon_{\nu\mu}\partial^\nu A^\mu, \qquad (93)$$

where $\Pi_5^{\mu\nu} = \epsilon^{\nu\kappa}(\bar{\Pi}_\kappa^\mu)_S$ because of Eq. (92) and $\tilde{p}^\nu = \epsilon^{\nu\alpha}p_\alpha$.[6]

Now let us write generically for the CR

$$\Delta_{\mu\nu}^0 = \frac{\lambda}{2\pi}g_{\mu\nu}$$

based on Lorentz invariance (λ is a dimensionless parameter). Thus

$$\Pi_\infty^{\mu\nu} = \left(\frac{\lambda+1}{2\pi}\right)g^{\mu\nu}, \qquad (94)$$

from which we see that the choice $\lambda = 0$ enforces gauge invariance on the ScM. We can rewrite the axial Ward identity (93) as a function of λ, namely

$$p_\nu \Pi_5^{\mu\nu} = -\frac{\lambda+2}{2\pi}\tilde{p}^\mu. \qquad (95)$$

Had we opted for preserving the AWI, we would have to set $\lambda = -2$. This, in turn, would transfer the anomaly to the VWI since

$$p_\mu \Pi_S^{\mu\nu}|_{\lambda=-2} = -\frac{1}{\pi}p^\nu, \qquad (96)$$

as expected.

On the other hand, for the CScM, Eq. (91) yields

$$\Pi_\chi^{\mu\nu}(p) = \frac{1}{\pi}\left((\lambda+2)g^{\mu\nu} - (g^{\mu\alpha} + \epsilon^{\mu\alpha})\frac{p_\alpha p_\beta}{p^2}(g^{\beta\nu} - \epsilon^{\beta\nu})\right). \qquad (97)$$

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[6]Notice that $(\bar{\Pi}_\mu^\mu)_S = 1/\pi$, which provides precisely the value of the anomaly.

Unlike the ScM, imposing gauge invariance does not fix the value of λ since

$$p_\mu \Pi_\chi^{\mu\nu}(p) = \frac{1}{\pi}((\lambda+1)p^\nu - \tilde{p}^\nu), \qquad (98)$$

which shows that the longitudinal part does not vanish for any value of λ. Despite the lack of gauge invariance and the arbitrary parameter λ, it constitutes a perfect sound theory [43]. It can be exactly solved to find that for $\lambda > -1$ it is a unitary and positive definite model, in which the photon acquires a mass

$$m^2 = \frac{e^2}{\pi} \frac{(\lambda+2)^2}{\lambda+1}. \qquad (99)$$

An equivalent formulation in a bosonized version of the CScM places λ as arising from ambiguities in the bosonization procedure. In fact the CScM can be formulated in a gauge invariant way in which a Wess-Zumino term[7] exactly cancels the variation of the original Lagrangian under a gauge transformation [44]. In addition, it was shown [45] that the anomalous formulation is nothing but a special gauge (unitary gauge $g=1$) of the gauge invariant formulation. Had we chosen the value $\lambda=0$ as we did for the ScM, we would obtain $m^2 = (4e^2)/\pi$. Curiously, this value has already been conjectured within another regularization (Faddeevian regularization) [46]; however, it turned out to be a special case of the CScM with a minimal Wess-Zumino term, with a restriction on an undetermined parameter correspondent to λ [47].

It is important to remark that there is no reason to impose $\lambda=0$ for the CScM as we did for the ScM. The best we could do, based on unitarity and positivity of the theory, was to establish a range of values for λ. This remains true in its gauge invariant formulation, since it obviously yields the same induced mass for the photon. Within our framework, we can somewhat generalize the ideas proposed by Jackiw [35] in the treatment of the ScM and the CScM to perturbative calculations in any quantum field theory where ultraviolet divergences appear. The latter can always be displayed either by basic divergent integrals or by differences between integrals of the same degree of divergence, whose value is finally fixed by imposing vital symmetries from the theory and/or by fitting with experimental data.

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[7]Such a term arises naturally by adopting the Faddeev-Popov trick for quantizing a theory with an anomaly.

VII. CONCLUDING REMARKS AND OUTLOOK

In this paper, we extended an implicit regularization scheme to be applicable in quantum field theories defined in n space-time dimensions. As we do not leave the integer dimension in which the theory is defined, parity violating objects present in chiral or topological field theories do not need to be dimensionally continued; therefore, we avoid well-known ambiguities involved in this procedure. Moreover, all the undeterminacies will be cast into a set of CR to be fixed on physical grounds either by imposing that the vital symmetries must not be violated or by experiment.

In this sense our framework is useful to simplify loop calculations in, for instance, Chern-Simons–matter theories [48]. This is because high-covariant-derivative regularizations make the calculations extremely lengthy (especially beyond one loop order) due to the complicated form that the gauge field propagator assumes. Moreover, there are cases where it seems to be possible to opt for an HCD or an extended DR; in Ref. [49] the one loop shift in noncommutative CS coupling depends on this choice. Therefore, even if one uses different regularizations that respect fundamental symmetries of a theory (such as gauge invariance), one may not get the same radiative correction. This is different from the situation when a theory possesses an intrinsic ambiguity whose value may have to be fixed only by experiment, even if the renormalization is finite [35]. As our framework does not modify or corrupt the underlying theory in consideration, it constitutes an ideal tool to study these problems.

Should any further constraint be imposed, such as (re)normalization conditions or some other physical requirement, they can be readily implemented within our framework [50]. Our main concern within this formulation was to keep the ambiguities to be fixed in the very final stage of the calculation.

When overlapping divergences occur, they are treated in a similar fashion [22]. Finally our approach may be generalized to multiloop calculations. The proof follows the same lines as the forest and skeleton construction in the BPHZ formulation [51].

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